Appl. Math. Optim. 7, 295–308 (1981)

Applied Mathematics and Optimization

Farkas–Minkowski Systems in Semi-Infinite Programming

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Communicated by J. Stoer

Abstract. The Farkas–Minkowski systems are characterized through a convex cone associated to the system, and some sufficient conditions are given that guarantee the mentioned property. The role of such systems in semi-infinite programming is studied in the linear case by means of the duality, and, in the nonlinear case, in connection with optimality conditions. In the last case the property appears as a constraint qualification.

1. Introduction

Let $\{a_t'x \geqq \beta_t,\ t \in T\}$ be a linear infinite system in $\mathbb{R}^n$. We will suppose in the next two sections that this system is consistent. Two systems are equivalent if they have the same solutions.

The relation $a'x \geqq \beta$, with associated vector $\begin{bmatrix} a \\ \beta \end{bmatrix} \in \mathbb{R}^{n+1}$, is a consequence relation of the system if every solution of it satisfies the relation.

The consistent system $\{a_t'x \geqq \beta_t,\ t \in T\}$ satisfies the property of Farkas–Minkowski (F-M) if every consequence relation of the system is a consequence of a finite subsystem.

The system $\{a_t'x \geqq \beta_t,\ t \in T\}$ is canonically closed (CC) if the following conditions hold:

(1) There is an algebraic interior point, i.e., for some $x^0 \in \mathbb{R}^n$,

$$a_t'x^0 > \beta_t, \qquad \text{for all } t \in T.$$

(2) There is a function $\alpha_t : T \to]0, +\infty[$ such that the set

$$\left\{ \alpha_t \begin{bmatrix} a_t \\ \beta_t \end{bmatrix},\ t \in T \right\} \text{ is compact.}$$

0095-4616/81/0007-0295$02.80

Symmetric definitions are to be considered for a system of type $\{a_t'x \leqq \beta_t, t \in T\}$.

The convex cone of generalized positive finite sequences will be denoted by $\mathbb{R}_+^{(T)}$: $\mathbb{R}_+^{(T)} = \{\alpha : T \to \mathbb{R}_+ / \alpha_t = 0, \forall t \in T \sim T_\alpha, T_\alpha \text{ finite set}\}$.

Given a nonempty set $\mathbb{C} \subset \mathbb{R}^p$, $\langle \mathbb{C} \rangle$ denotes the convex hull of $\mathbb{C}$, and $K(\mathbb{C})$ the convex cone generated by $\mathbb{C}$.

By $\bar{y}$ we represent a vector in $\mathbb{R}^{n+1}$, with $\bar{y} = \begin{bmatrix} y \\ y_{n+1} \end{bmatrix}$, $y \in \mathbb{R}^n$, $y_{n+1} \in \mathbb{R}$.

Concepts and notation about functions are to be found in [11].

2. Characterization of Consequence Relations

The well-known generalized Farkas lemma—which establishes that "$a'x \geqq 0$ is a consequent relation of $\{a_t'x \geqq 0, t \in T\}$ if, and only if, $a \in \overline{K}\{a_t, t \in T\}$"—allows us to characterize the consequence relations of a consistent nonhomogeneous linear system.

The convex cone generated by $\left\{ \begin{bmatrix} a_t \\ \gamma_t \end{bmatrix}, \gamma_t \leqq \beta_t, t \in T \right\}$ will be denoted by K_c.

Theorem 2.1. *$a'x \geqq \beta$ is a consequence relation of the consistent system $\{a_t'x \geqq \beta_t, t \in T\}$ if, and only if, $\begin{bmatrix} a \\ \beta \end{bmatrix} \in \overline{K}_c$.*

Proof. Let $\begin{bmatrix} a \\ \beta \end{bmatrix} \in \overline{K}_c$. Then

$$\begin{bmatrix} a \\ \beta \end{bmatrix} = \lim_k \begin{bmatrix} b_k \\ \delta_k \end{bmatrix}, \quad \begin{bmatrix} b_k \\ \delta_k \end{bmatrix} = \sum_{i=1}^{r_k} \lambda_{k_i} \begin{bmatrix} a_{t_{k_i}} \\ \gamma_{t_{k_i}} \end{bmatrix},$$

$$\lambda_{k_i} \geqq 0, t_{k_i} \in T, \gamma_{t_{k_i}} \leqq \beta_{t_{k_i}}, i = 1, \ldots, r_k, \ r_k \in N.$$

If x^0 is a solution of the system, then $a_{t_{k_i}}'x^0 - \gamma_{t_{k_i}} \geqq a_{t_{k_i}}'x^0 - \beta_{t_{k_i}} \geqq 0$. Hence $b_k'x^0 - \delta_k \geqq 0$, k$=1,2,\ldots$. Then $a'x^0 - \beta \geqq 0$. This means that $a'x \geqq \beta$ is a consequence of the system.

We suppose now that $a'x \geqq \beta$ is a consequence relation of $\{a_t'x \geqq \beta_t, t \in T\}$. First, we prove that $[a' \quad \beta] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geqq 0$ (I) is consequent of the following system in $\mathbb{R}^{n+1}$:

$$\left\{ [a_t' \quad \gamma_t] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geqq 0, \ \gamma_t \leqq \beta_t, t \in T \right\} \text{ (II)}.$$

For this purpose it is enough to prove that the solutions of the latter system of the form

$$\begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix}, \begin{bmatrix} \tilde{x} \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix},$$

are solutions of the relation (I).

If $\begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix}$ is a solution of (II), then $a_t'\tilde{x}-\beta_t \geqq 0$, for all $t \in T$. In such case $a'\tilde{x} \geqq \beta$, i.e., $\begin{bmatrix} a' & \beta \end{bmatrix}\begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix} \geqq 0$.

If $\begin{bmatrix} \tilde{x} \\ 1 \end{bmatrix}$ is a solution of (II), then $a_t'\tilde{x}+\beta_t-r \geqq 0$, for all $(t,r) \in T \times N$. This is a contradiction.

If $\begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}$ is a solution of (II), this means that $a_t'\tilde{x} \geqq 0$, for all $t \in T$. Let x^0 be a solution of $\{a_t'x \geqq \beta_t,\ t \in T\}$. Clearly, $x^0+\lambda\tilde{x}$ is a solution of the last system, for all $\lambda > 0$. Hence, $a'(x^0+\lambda\tilde{x}) \geqq \beta$, $\forall \lambda > 0$. Therefore, $a'\tilde{x} \geqq 0$, i.e., $\begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}$ satisfies (I).

By the generalized Farkas lemma $\begin{bmatrix} a \\ \beta \end{bmatrix} \in \overline{K}\left\{\begin{bmatrix} a_t \\ \gamma_t \end{bmatrix}, \gamma_t \leqq \beta_t, t \in T\right\}$. □

This result can also be deduced from Theorem 2 of K. Fan [3].

3. Systems of Farkas–Minkowski

It is clear that the system $\{a_t'x \geqq \beta_t,\ t \in T\}$ is a F-M system if, and only if,

$$\begin{cases} a_t'x \geqq \beta_t, & t \in T \\ \vec{0}'x \geqq -1 & \text{(trivial constraint)} \end{cases}$$

is a F-M system. The last one will be called "extended system."

We shall denote by $\tilde{K}_c$ the convex cone associated, according to section 2, to the extended system. This cone allows us to give a characterization of the F-M systems.

Theorem 3.1. *The system $\{a_t'x \geqq \beta_t,\ t \in T\}$ satisfies the property of* F-M *if, and only if, $\tilde{K}_c$ is closed.*

Proof. To simplify the proof, we will assume that $\{a_t'x \geqq \beta_t,\ t \in T\}$ contains the trivial constraint $\vec{0}'x \geqq -1$. Hence, $\tilde{K}_c = K_c$. If K_c is closed and $a'x \geqq \beta$ is a consequence relation, then $\begin{bmatrix} a \\ \beta \end{bmatrix} \in K_c$ (Theorem 2.1), and we can write:

$$\begin{bmatrix} a \\ \beta \end{bmatrix} = \sum_{i=1}^{r} \lambda_i \begin{bmatrix} a_{t_i} \\ \gamma_{t_i} \end{bmatrix}, \qquad \gamma_{t_i} \leqq \beta_{t_i}, \quad t_i \in T, \quad \lambda_i \geqq 0, i = 1, \ldots, r$$

We can choose $r \leqq n+1$, by applying Caratheodory's theorem. With a new application of Theorem 2.1, we conclude that $a'x \geqq \beta$ is a consequence of the finite subsystem $\{a_{t_i}'x \geqq \beta_{t_i},\ i=1,\ldots,r\}$.

Let us now suppose that the system satisfies the F-M property, being $a'x \geqq \beta$ a consequence relation of the system. Then $a'x \geqq \beta$ will be a consequence of a finite subsystem $\{a_{t_i}'x \geqq \beta_{t_i},\ i=1,\ldots,r\}$. Theorem 2.1 gives

$$\begin{bmatrix} a \\ \beta \end{bmatrix} = \sum_{i=1}^{r} \alpha_i \begin{bmatrix} a_{t_i} \\ \beta_{t_i} \end{bmatrix} + \lambda \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}, \qquad t_i \in T, \quad \alpha_i \geqq 0, \quad i = 1, \ldots, r, \quad \lambda \geqq 0.$$

If $\alpha_1 = \cdots = \alpha_r = 0$, $\begin{bmatrix} a \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix} \in K_c$.

Let us assume $\alpha_1 \neq 0$. Then

$$\begin{bmatrix} a \\ \beta \end{bmatrix} = \alpha_1 \begin{bmatrix} a_{t_1} \\ \beta_{t_1} - \dfrac{\lambda}{\alpha_1} \end{bmatrix} + \sum_{i=2}^{r} \alpha_i \begin{bmatrix} a_{t_i} \\ \beta_{t_i} \end{bmatrix} \in K_c.$$

This proves that K_c is closed. □

Note that $K_c \subset \tilde{K}_c \subset \bar{K}_c$. Furthermore,

$$\tilde{K}_c = K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T, \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}\right\} = K_c \cup K\left\{\begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}\right\}.$$

The condition "$\tilde{K}_c$ is closed" can be reformulated in the following way: "K_c contains all its boundary points except, perhaps, the points of the half-line $\lambda \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}$, $\lambda \geqq 0$."

In fact, let x be a boundary point of K_c, $x \notin K\left\{\begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}\right\}$. Since $\tilde{K}_c$ is closed and $K_c \subset \tilde{K}_c$, $x \in \tilde{K}_c$. Hence, $x \in K_c$. Conversely,

$$\bar{\tilde{K}}_c = \bar{K}_c \cup K\left\{\begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}\right\} = K_c \cup K\left\{\begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}\right\} = \tilde{K}_c.$$

Corollary 3.1.1. *If the consistent system* $\{a_t'x \geqq \beta_t,\ t \in T\}$ *satisfies one of the following conditions, then it is a* F-M *system*:

(i) K_c *is closed.*
(ii) $K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T\right\}$ *is closed.*
(iii) *The system is canonically closed.*
(iv) *T is a compact convex set in* $\mathbb{R}^m$, *a_t and β_t are linear functions and* $T \cap Ker \begin{bmatrix} a_t \\ \beta_t \end{bmatrix} = \varnothing$.
(v) *T is a finitely generated cone in* $\mathbb{R}^m$ *and a_t, β_t are homogeneous linear functions.*

Proof.

(i) It follows immediately from $K_c \subset \tilde{K}_c \subset \bar{K}_c$.

(ii) We will prove that $\tilde{K}_c$ is closed.

For the sake of brevity we represent the cone $K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T\right\}$ by M. Let x be $x = \lim x^r$, with $x^r \in \tilde{K}_c$, $r = 1, 2, \ldots$. We can write $x^r = y^r + \lambda^r \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}$, $y^r \in M$, $\lambda^r \geqq 0$, $r = 1, 2 \ldots$. If $\{\lambda^r\}$ is not bounded, there is a subsequence $\{\lambda^{r_k}\}$ such that

$\lim \lambda^{r_k} = +\infty$. But $\frac{1}{\lambda^{r_k}} x^{r_k} = \frac{1}{\lambda^{r_k}} y^{r_k} + \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}$ and $\{x^{r_k}\}$ is bounded. Hence $\begin{bmatrix} \vec{0} \\ 1 \end{bmatrix} = \lim \frac{1}{\lambda^{r_k}} y^{r_k} \in M \subset \bar{K}_c$. This means that $\vec{0}'x \geqq 1$ is a consequent relation of the system, i.e., it is not consistent.

Since $\{\lambda^r\}$ is bounded, there is a subsequence $\{\lambda^{r_k}\}$ such that $\lim \lambda^{r_k} = \lambda^0$. The equality $x^{r_k} = y^{r_k} + \lambda^{r_k} \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}$ gives $x = y^0 + \lambda^0 \begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}$, for some $y^0 \in M$, i.e., $x \in \tilde{K}_c$.

(iii) Let $\left\{ \alpha_t \begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T \right\}$ be compact, $\alpha_t > 0$, $t \in T$, with an algebraic interior point. By a well-known property of cones (c.f. [6], p. 203), $K\left\{ \alpha_t \begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T \right\} = M$ is a closed cone and (ii) holds.

A known property of compact-convex sets (c.f. [12], p. 79) shows that M is closed if (iv) holds, while, assuming (v), M is a finitely generated cone. □

Condition (ii) of Corollary 3.1.1 is given in [4], where M is called the "moment cone" of the system. Condition (iii) is exactly the nonhomogeneous theorem of Haar (see [1]).

Every closed convex set in $\mathbb{R}^n$ can be represented through a linear system which satisfies (i) and (ii). In fact, let S be a nonempty closed convex set and $T = \left\{ \begin{bmatrix} a \\ \beta \end{bmatrix} \in \mathbb{R}^{n+1} / a'x \geqq \beta, \forall x \in S \right\}$. Then $\left\{ a'x \geqq \beta, \begin{bmatrix} a \\ \beta \end{bmatrix} \in T \right\}$ satisfies (i) and (ii) by Theorem 2.1.

The three sentences "K_c is closed," "$\tilde{K}_c$ is closed," and "M is closed" are independent, as the following examples prove:

In $\{tx \geqq e^{-t}, t \in [0, +\infty[\}$ (i) holds and (ii) does not.

In $\{tx \geqq 1, t \in [1,2]\}$ (ii) holds and (i) does not.

Condition (v) of Corollary 3.1.1, "*T is a finitely generated cone*," can not be substituted by "*T is a closed convex cone*," as the following counterexample proves:

Let $T = K_1 \times K_2$ where $K_2 = K\{(0, -1, 0)'\}$ and

$$K_1 = \{(\lambda\rho\cos\theta, \lambda, -\lambda(1+\rho\,\mathrm{sen}\,\theta))'/\lambda \geqq 0, \rho \in [0,1], \theta \in [-\pi, \pi[\}.$$

Let $a_t = (t_1 + t_4, t_2 + t_5)'$, $\beta_t = t_3 + t_6$, $t \in T \subset \mathbb{R}^6$. K_1 and K_2 are closed convex cones. So is T. But $M = K_1 + K_2$ is not closed. Furthermore, $\tilde{K}_c = K_c = M$ and $\{a_t'x \geqq \beta_t, t \in T\}$ is not a F-M system.

Finally it is curious to observe that the very strong condition "the system has a finite equivalent system" is even not sufficient to guarantee the property of F-M, as the following example shows:

$$\{tx \geqq 1, t \in]1,2]\} \quad \text{and} \quad \{x \geqq 1\} \text{ are equivalent.}$$

While the second system is, obviously, a F-M system, the first is not.

Similar results can be easily established by setting $\leqq$ instead of $\geqq$, and conversely, in sections 2 and 3.

4. The Farkas–Minkowski Systems in Linear Semi-Infinite Programming

In this section we prove the perfect duality theorem ([9], [10], and [4]) by means of Theorem 2.1. Its particularization, when the system of constraints for the primal program is a F-M one, constitutes the Haar's duality theorem for the consistent case ([2]).

The perfect dual programs are established as follows:

Primal program (PP)	*Dual program* (DP)
$\operatorname{Min} c'x$	$\operatorname{Max} y_{n+1}$
s.t. $a_t'x \geqq \beta_t,\ t \in T$	s.t. $\bar{y} \in \bar{K}_c,\ y = c$

Theorem 4.1. *Programs PP and DP are in perfect duality, because:*

(I) *If one program is consistent and has finite value, the other is also consistent, and*

(II) *If both programs are consistent, both have the same finite value.*

Proof.

(I) Let us suppose DP consistent and with finite value. Let $\bar{y}^0 \in \mathbb{R}^{n+1}$ such that $\bar{y}^0 \in \bar{K}_c$, $y^0 = c$. If PP is inconsistent, the following system has no solution:

$$\left\{ \begin{matrix} a_t'x + \beta_t x_{n+1} \geqq 0,\ t \in T \\ x_{n+1} < 0 \end{matrix} \right\}.$$

Therefore $x_{n+1} \geqq 0$ is a consequence relation of the system $\{a_t'x + \beta_t x_{n+1} \geqq 0,\ t \in T\}$, and, by the generalized Farkas lemma,

$$\begin{bmatrix} \vec{0} \\ 1 \end{bmatrix} \in \bar{K}\left\{ \begin{bmatrix} a_t \\ \beta_t \end{bmatrix},\ t \in T \right\} \subset \bar{K}_c.$$

It is easy to see that, for all $\lambda \geqq 0$, $\bar{y}^0 + \lambda \begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}$ are feasible points for DP. However, the objective function of DP is not bounded from above over this half-line. Hence PP is consistent.

Let now PP be consistent and let $\alpha \in \mathbb{R}$ be its value. The relation $c'x \geqq \alpha$ is obviously a consequence of the system $\{a_t'x \geqq \beta_t,\ t \in T\}$, and, by Theorem 2.1, $\begin{bmatrix} c \\ \alpha \end{bmatrix}$ belongs to $\bar{K}_c$, which proves that DP is consistent.

(II) If both programs are consistent, it is easy to prove that $c'x \geqq y_{n+1}$, being x and $\bar{y} = \begin{bmatrix} c \\ y_{n+1} \end{bmatrix}$ feasible points for PP and DP, respectively ("*weak duality*"). As we have already shown, $\begin{bmatrix} c \\ \alpha \end{bmatrix} \in \bar{K}_c$ for $\alpha = \inf. PP$. Consequently, $\alpha = \text{supr. } DP$. □

Corollary 4.1.1. *If the system of constraints for the PP in Haar's duality satisfies the* F-M *property, the PP is consistent, and*: (I) inf. $PP = -\infty$ *if, and only if, DP is not consistent.* (II) inf. $PP > -\infty$ *if, and only if, DP is consistent and* inf. $PP =$ max. DP.

Proof. The dual problems of Haar's duality

$$\begin{array}{ll} PP & DP \\ \text{Min. } c'x & \text{Max. } \sum_{t\in T}\lambda_t\beta_t \\ \text{s.t. } a_t'x\geqq\beta_t,\ t\in T & \text{s.t. } \sum_{t\in T}\lambda_t a_t=c,\ (\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}. \end{array}$$

are equivalent to the following pair:

$$\begin{array}{ll} \widetilde{PP} & \widetilde{DP} \\ \text{Min. } c'x & \text{Max. } -\mu+\sum_{t\in T}\lambda_t\beta_t \\ \text{s.t. } a_t'x\geqq\beta_t,\ t\in T & \text{s.t. } \sum_{t\in T}\lambda_t a_t=c,\ (\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)},\ \mu\geqq 0. \\ \quad\ \vec{0}'x\geqq -1. & \end{array}$$

As has already been said, the extended system of constraints for the $\widetilde{PP}$ is also a F-M one, being $\tilde{K}_c$ closed. $\widetilde{DP}$ can be rewritten as follows:

$$\begin{array}{l} \text{Max. } y_{n+1} \\ \text{s.t. } \bar{y}\in\tilde{K}_c,\ y=c. \end{array}$$

Hence $\widetilde{PP}$ and $\widetilde{DP}$ are in perfect duality and we finish by applying Theorem 4.1. □

5. Systems of Farkas–Minkowski in Nonlinear Semi-Infinite Programming

The well-known theory of the Lagrangian saddle points in nonlinear programming can be extended through suitable generalization of the concepts. The sufficient condition for optimum point is completely general, while to obtain necessary conditions we must introduce new hypotheses related with the property of F-M.

Let us consider the general problem of SIP, called P:

$$\text{Min. } \{\varphi(x)/x\in S\}$$

$$S=\{x\in C/f_t(x)\geqq 0,\ t\in T\},$$

where $C\subset\mathbb{R}^n$ is the supporting set of the functions and T is an infinite set.

The *Lagrangian function* associated with the problem is:

$$\Psi(x,\lambda)=\varphi(x)+\sum_{t\in T}\lambda_t f_t(x),\qquad x\in C,\lambda\in\mathbb{R}_+^{(T)}.$$

As in the finite case, $(\bar{x},\bar{\lambda})$ will be a saddle point for $\Psi(x,\lambda)$ if $\Psi(\bar{x},\lambda)\leqq\Psi(\bar{x},\bar{\lambda})\leqq\Psi(x,\bar{\lambda})$, for all $x\in C$ and $\lambda\in\mathbb{R}_+^{(T)}$.

Theorem 5.1. *If $(\bar{x}, \bar{\lambda}) \in C \times \mathbb{R}_+^{(T)}$ is a saddle point for $\Psi(x, \lambda)$, then $\bar{x}$ is an optimal solution of P.*

Proof. For every $\tilde{t} \in T$ we define $\tilde{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$\tilde{\lambda}_t = \begin{cases} \bar{\lambda}_t, & t \neq \tilde{t} \\ \bar{\lambda}_{\tilde{t}} + 1, & t = \tilde{t} \end{cases}$$

and, by the condition of saddle point, $f_{\tilde{t}}(\bar{x}) \leqq 0$. This proves that $\bar{x}$ is a feasible point. Obviously $\sum_{t \in T} \bar{\lambda}_t f_t(\bar{x}) \leqq 0$, which combined with the condition of saddle point for $\lambda = 0$ gives $\sum_{t \in T} \bar{\lambda}_t f_t(\bar{x}) = 0$ (condition of complementarity). Then, if $x \in S$, $\varphi(\bar{x}) \leqq \varphi(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x) \leqq \varphi(x)$. □

In order to obtain a converse theorem, we have to impose restrictions to the functions. From now on, we shall suppose, by reasons of simplicity, that $C = \mathbb{R}^n$ in problem P.

The Constraint Qualification Q

The system $\{f_t(x) \leqq 0,\ t \in T\}$ satisfies the constraint qualification Q if the linear system $\{f_t(y) + \xi'(x-y) \leqq 0,\ (t, y) \in T \times \mathbb{R}^n,\ \xi \in \partial f_t(y)\}$ is a F-M *system, where $\partial f_t(y)$ denotes the subdifferential of f_t in y.*

Obviously, if f_t is convex and differentiable in y, $\partial f_t(y) = \{\nabla f_t(y)\}$. More particularly, if the constraints of problem P are linear functions, i.e., $f_t = a_t'x - \beta_t$, then Q holds if, and only if, $\{a_t'x \leqq \beta_t,\ t \in T\}$ satisfies the property of Farkas–Minkowski.

We must emphasize that the linear problem plays a more crucial role in semi-infinite programming than in finite programming, because a wide class of SIP problems can be reformulated as a linear ones (See [5].)

Theorem 5.2. *If* (1) *all the functions of problem P are convex, and* (2) *the constraint qualification Q holds, then, if $\bar{x}$ is an optimum solution of P, there is some $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of $\Psi(x, \lambda)$.*

Proof. Let $\tilde{S}$ be the set of solutions of the system

$$\{f_t(y) + \xi'(x-y) \leqq 0, (t, y) \in T \times \mathbb{R}^n, \xi \in \partial f_t(y)\}.$$

Note that the convexity of f_t guarantees that $\partial f_t(y) \neq \varnothing$, for all $y \in \mathbb{R}^n$. We shall prove first that $S = \tilde{S}$.

Let $x^0 \in S$. Then $f_t(x^0) \leqq 0$, $\forall t \in T$, and given $y \in \mathbb{R}^n$, $\xi \in \partial f_t(y)$, we have $0 \geqq f_t(x^0) \geqq f_t(y) + \xi'(x^0 - y)$, i.e., $x^0 \in \tilde{S}$. Conversely, given $x^0 \in \tilde{S}$, we consider some of the linear inequalities, corresponding to $t \in T$ and x^0, $f_t(x^0) + \xi'(x - x^0) \leqq 0$. Then $f_t(x^0) \leqq 0$ for all $t \in T$, and hence $x^0 \in S$. Consequently, the system above is a linear representation of S, which satisfies the F-M property.

Since $\bar{x}$ is an optimum solution of Min. $\varphi(x)$ on S, there is a subgradient $\xi \in \partial\varphi(\bar{x})$ such that $\xi'(x-\bar{x}) \geqq 0$, for all $x \in S$. In other words, the relation $-\xi' x \leqq -\xi'\bar{x}$ is a consequence of the system, and there are parameters $\bar{\lambda}_i \geqq 0$, $i=1,\dots,q$, $\mu \geqq 0$, such that

$$-\begin{bmatrix} \xi \\ \xi' x \end{bmatrix} = \sum_{i=1}^{q} \bar{\lambda}_i \begin{bmatrix} \xi^i \\ \xi^{i\prime} \cdot y^i - f_{t_i}(y^i) \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \mu \end{bmatrix}.$$

Multiplying by $\begin{bmatrix} x \\ -1 \end{bmatrix}$, we have for any $x \in \mathbb{R}^n$

$$-\xi' x + \xi'\bar{x} = \sum_{i=1}^{q} \bar{\lambda}_i \left[f_{t_i}(y^i) + \xi^{i\prime}(x-y^i) \right] - \mu \leqq \sum_{i=1}^{q} \bar{\lambda}_i \left[f_{t_i}(y^i) + \xi^{i\prime}(x-y^i) \right].$$

But $\xi_i \in \partial f_{t_i}(y_i)$ means that $f_{t_i}(y^i) + \xi^{i\prime}(x-y^i) \leqq f_{t_i}(x)$ and therefore $\xi'\bar{x} \leqq \xi' x + \sum_{i=1}^{q} \bar{\lambda}_i f_{t_i}(x)$, for all $x \in \mathbb{R}^n$. Similarly, $\varphi(\bar{x}) + \xi'(x-\bar{x}) \leqq \varphi(x)$, which combined with the above inequality gives $\varphi(x) + \sum_{i=1}^{q} \bar{\lambda}_i f_{t_i}(x) \geqq \varphi(\bar{x})$, for all $x \in \mathbb{R}^n$.

Since $\bar{x}$ is a feasible point, we have

$$\varphi(\bar{x}) + \sum_{t \in T} \lambda_t f_t(\bar{x}) \leqq \varphi(\bar{x}), \forall \lambda \in \mathbb{R}_+^{(T)}.$$

Defining $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ as

$$\bar{\lambda}_t = \begin{cases} \bar{\lambda}_i, \; t = t_i \\ 0, \; t \neq t_i, \; i=1,\dots,q, \end{cases}$$

we obtain

$$\varphi(\bar{x}) + \sum_{t \in T} \lambda_t f_t(\bar{x}) \leqq \varphi(\bar{x}) \leqq \varphi(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x),$$

for all $\lambda \in \mathbb{R}_+^{(T)}$ and for all $x \in \mathbb{R}^n$. The specification of the double inequality for $x = \bar{x}$ and $\lambda = \bar{\lambda}$ gives $\sum_{t \in T} \bar{\lambda}_t f_t(\bar{x}) = 0$, which completes the proof. □

A more operative result can be obtained for differentiable constraints. The proof of the first part is based on the fact that the constraint qualification Q holds. We need a previous lemma, in which S^b denotes the set of boundary points of S.

Lemma 5.3. *Let $S \subset \mathbb{R}^n$ be a closed convex set and $\{c_t' x \leqq \delta_t, \; t \in T\}$ a system such that*: (I) *every point of S is a solution*; (II) *there is a $x^0 \in S$ such that $c_t' x^0 < \delta_t$, $t \in T$; and* (III) *given any $y \in S^b$, there is some $t \in T$ such that $c_t' y = \delta_t$. Then $S = \{x \in \mathbb{R}^n / c_t' x \leqq \delta_t, \; t \in T\}$.*

Proof. Let us suppose $z \notin S$ such that z satisfies the system. If we consider the segment between x^0 and z, there is only one $\bar{\lambda} \in]0,1[$ such that $\bar{\lambda}z+(1-\bar{\lambda})x^0=y \in S^b$. We choose $\bar{t} \in T$ such that $c'_{\bar{t}}\bar{y}=\delta_{\bar{t}}$. Nevertheless $c'_{\bar{t}}x^0<\delta_{\bar{t}}$ and $c'_{\bar{t}}z \leqq \delta_{\bar{t}}$, and therefore $c'_{\bar{t}}\bar{y}<\delta_{\bar{t}}$. This means that every solution of the system belongs to S. □

Theorem 5.4. *If* (I) $T \subset \mathbb{R}^m$ *is a compact set,* (II) *all the functions of problem P are convex,* (III) *the constraints are differentiable and the functions* $f_t(x)$ *and* $\nabla f_t(x)$ *are continuous in both variables, and* (IV) *there is a point* x^0 *such that* $f_t(x^0)<0$ *for all* $t \in T$, *then, if* $\bar{x}$ *is an optimum solution of P, there is some* $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ *such that* $(\bar{x}, \bar{\lambda})$ *is a saddle point of* $\Psi(x,\lambda)$.

Proof. For the demonstration we distinguish two steps, assuming in the first one, (i), S is a compact set.

(i) We shall prove that Q holds.

Let us observe first that $S^b=\{y \in S / f_t(y)=0 \text{ for some } t \in T\}$. In fact, if $x \in S^b$, there is a sequence $\{x^r\}$ in $\mathbb{R}^n \sim S$ such that $\lim_r x^r=x$. Let $t^r \in T$ be such that $f_{t^r}(x^r)>0$. Since T is compact, there is a subsequence $\{t^{r_k}\}$ such that $\lim_k t^{r_k}=t^0 \in T$. By continuity, $f_{t^0}(x) \geqq 0$. But $x \in S$ and hence $f_{t^0}(x)=0$. Conversely, if $y \in S$ and $f_t(y)=0$, $t \in T$, we define $y^r=x^0+\left(\frac{1+r}{r}\right)(y-x^0)$, $r=1,2,\ldots$. Since $f_t(x^0)<0$ and f_t is convex, $f_t(y^r)>0$. Then $y^r \in \mathbb{R}^n \sim S$, $r=1,2,\ldots$ and $\lim_r y^r=y$, i.e., $y \in S^b$.

We can prove now that the following system is a linear representation of S:

$$\{f_t(y)+\nabla f_t(y)'(x-y) \leqq 0,\ (t,y) \in T \times S^b\}.$$

Let $\hat{S}$ be the set of solutions of the last system. By convexity, $S \subset \hat{S}$.

Applying the axiom of choice we can choose for every $y \in S^b$ an index $t(y) \in T$ such that $f_{t(y)}(y)=0$.

Since $\hat{S} \subset \{x \in \mathbb{R}^n / f_{t(y)}(y)+\nabla f_{t(y)}(y)'(x-y) \leqq 0,\ y \in S^b\}$, if we prove that S is the set of solutions of this subsystem, then $S=\hat{S}$. But $\{f_{t(y)}(y)+\nabla f_{t(y)}(y)'(x-y) \leqq 0,\ y \in S^b\}$ satisfies all the assumptions of Lemma 5.3. The linear system $\{\nabla f_t(y)'x \leqq \nabla f_t(y)'y-f_t(y),\ (t,y) \in T \times S^b\}$ is a linear representation of S which is canonically closed: $T \times S^b$ is a compact set and, consequently, the following is also a compact set in $\mathbb{R}^{n+1}$:

$$\left\{\begin{bmatrix} \nabla f_t(y) \\ \nabla f_t(y)'y-f_t(y) \end{bmatrix},\ (t,y) \in T \times S^b\right\};$$

furthermore

$$f_t(y)+\nabla f_t(y)'(x^0-y) \leqq f_t(x^0)<0,\ (t,y) \in T \times S^b.$$

The property of Farkas–Minkowski is guaranteed by Corollary 3.1.1.

We conclude the first step by proving that Q holds. In fact, the system $\{f_t(y)+\nabla f_t(y)'(x-y)\leqq 0,\ (t,y)\in T\times\mathbb{R}^n\}$ is a F-M system since it is obtained by addition of consequent relations to a F-M system. We must consider that, by convexity, both systems have S as the set of solutions. Theorem 5.2 can be applied to obtain the intended result.

(ii) Let us now assume that S is not bounded. We define a new problem $\tilde{P}$ by addition of a constraint to problem P in such a way that $\bar{x}$ remains an optimum solution of $\tilde{P}$. We shall distinguish the elements of the new problem of those of P through the sign $\sim$. Let t_0 be a point of $\mathbb{R}^m\sim T$. We define $\tilde{T}=T\cup\{t_0\}$, which is a compact set in $\mathbb{R}^m$.

We associate to t_0 the constraint

$$f_{t_0}(x)\equiv x'x-2x'\bar{x}+\bar{x}'\bar{x}-\rho^2\leqq 0,$$

with $\rho>d(\bar{x},x^0)$ arbitrarily chosen. Consequently $\tilde{S}=\{x\in\mathbb{R}^n/f_t(x)\leqq 0,\ t\in\tilde{T}\}=\{x\in S/\|x-\bar{x}\|\leqq\rho\}$ is a compact set. The problem $\tilde{P}$ satisfies all the conditions of this theorem (i) since $d(t_0,T)>0$. Then, there is some $\bar{\lambda}\in\mathbb{R}_+^{(\tilde{T})}$ such that

$$\varphi(\bar{x})+\sum_{t\in T}\lambda_t f_t(\bar{x})-\lambda_{t_0}\rho^2\leqq\varphi(\bar{x})\leqq\varphi(x)+\sum_{t\in T}\bar{\lambda}_t f_t(x)+\bar{\lambda}_{t_0}f_{t_0}(x),$$

for all $\lambda\in\mathbb{R}_+^{(\tilde{T})}$, and for all $x\in\mathbb{R}^n$.

Obviously, the constraint $f_{t_0}(x)\leqq 0$ is not active in $\bar{x}$. Hence, the condition of complementarity implies $\bar{\lambda}_{t_0}=0$. We obtain $\varphi(\bar{x})\leqq\varphi(x)+\sum_{t\in T}\bar{\lambda}_t f_t(x)$, for all $x\in\mathbb{R}^n$, while $\varphi(\bar{x})+\sum_{t\in T}\lambda_t f_t(\bar{x})\leqq\varphi(\bar{x})$ for all $\lambda\in\mathbb{R}_+^{(T)}$ since $\bar{x}$ is a feasible point. This means that $(\bar{x},\bar{\lambda})$ is a saddle point of $\Psi(x,\lambda)$. □

Example. Let us consider the problem P: Min $\varphi(x)$, subject to

$$(x_1)^2+(x_2)^2-2x_1+2(t-3)x_2+\left(\tfrac{5}{9}t^2-2t+1\right)\leqq 0,\qquad t\in[0,3]$$

with

$$\varphi(x)=\begin{cases}(x_2)^2+2x_2, & x_2\geqq 0\\ 2x_2, & x_2<0.\end{cases}$$

The point $\bar{x}=(1,0)$ satisfies all the conditions of Theorem 5.4, choosing, for instance, $x^0=(1,\frac{1}{2})$. Our purpose is to find out $\lambda\in\mathbb{R}_+^{(T)}$ such that $(\bar{x},\bar{\lambda})$ is a saddle point.

The condition of complementarity $\sum_{t\in T}\bar{\lambda}_t\left(\frac{5}{9}t^2-2t\right)=0$ gives $\bar{\lambda}_t=0$ for all $t\in]0,3]$.

The "right" condition of saddle point gives for $x_2\geqq 0$, $3\lambda_0-1\leqq 0$, and for $x_2<0$, $3\lambda_0-1\geqq 0$. It can be shown that $\bar{\lambda}$, defined as

$$\bar{\lambda}_t=\begin{cases}\frac{1}{3}, & t=0\\ 0, & t\neq 0\end{cases},$$

is the only one (in $\mathbb{R}_+^{(T)}$) satisfying the requested condition.

The property of F-M is not only involved in the theory of nonlinear SIP through the constraint qualification Q, but also in connection with some useful concepts, such as those of "regular point" and a "Kuhn–Tucker type" condition.

Corollary 5.4.1. *If the conditions of Theorem 5.4 hold and $\varphi(x)$ is differentiable in $\bar{x}$, which is an optimum solution of problem P, then there are nonnegative real numbers λ_i and indices $t_i \in T$, $i=1,\ldots,r$ such that*

$$\nabla \varphi(\bar{x}) + \sum_{i=1}^{r} \lambda_i \nabla f_{t_i}(\bar{x}) = \vec{0} \qquad \text{(K-T condition in SIP)}.$$

Proof. If $(\bar{x}, \bar{\lambda})$ is a saddle point of $\Psi(x, \lambda)$, then

$$\varphi(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t f_t(\bar{x}) \leqq \varphi(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x),$$

for all $x \in \mathbb{R}^n$.

Let us assume that $\bar{\lambda}_t = 0$, $t \neq t_i$, $i=1,\ldots,r$. Obviously the function $\varphi(x) + \sum_{i=1}^{r} \bar{\lambda}_{t_i} f_t(x)$ attains in $\bar{x}$ its minimum value on $\mathbb{R}^n$ and, consequently, $\nabla \varphi(\bar{x}) + \sum_{i=1}^{r} \bar{\lambda}_{t_i} \nabla f_t(\bar{x}) = \vec{0}$. □

The following concepts can be found in [6] and [8].

Let $\mathbb{R}^n$ be the support set of the problem P, and $\bar{T}$ that of the index set corresponding to the active constraints in $\bar{x}$, where all the functions are supposed to be differentiable.

We associate to the point $\bar{x}$ the following linear system:

$$\{\nabla f_t(\bar{x})' h \leqq 0, t \in \bar{T}\}$$

called "system of tangential constraints."

It can be shown that every sequential tangent vector satisfies the system above; the converse is not always valid. Let H be the cone of tangent vectors.

The point $\bar{x}$ is said to be *regular* if:

(I) the set of solutions of the system of tangential constraints is H, and

(II) the cone $K\{\nabla f_t(\bar{x}), t \in \bar{T}\}$ is closed.

Theorem 5.5. *The point $\bar{x}$ is regular if, and only if, the set of solutions of the system of tangential constraints is H, and such system satisfies the property of Farkas–Minkowski.*

Proof. The homogeneous system $\{\nabla f_t(\bar{x})' h \leqq 0, t \in \bar{T}\}$ is a F-M system if, and only if $\tilde{K}_c = K\left\{\begin{bmatrix} \nabla f_t(\bar{x}) \\ 0 \end{bmatrix}, t \in T, \begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}\right\} = K\{\nabla f_t(\bar{x}), t \in T\} \times \mathbb{R}_+$ is closed, i.e., $K\{\nabla f_t(\bar{x}), t \in T\}$ is closed. □

By using the generalized Farkas lemma it is possible to prove directly that, if $\bar{x}$ is a regular point and local minimum, then $\exists \lambda \in \mathbb{R}_+^{(\bar{T})}$ such that $\nabla \varphi(\bar{x}) + \sum_{t \in T} \lambda_t \nabla f_t(\bar{x}) = \vec{0}$ ([6], Theorem 11.1).

Let us associate to the point $\bar{x}$ three sets, being P and P_0 cones:

$$P = \{h \in \mathbb{R}^n / \nabla \varphi(\bar{x})'h \leqq 0, \ \nabla f_t(\bar{x})'h \leqq 0, t \in \bar{T}\}$$

$$P_0 = \{h \in P / \nabla \varphi(\bar{x})'h = 0\}$$

$$T_S = \{t \in \bar{T} / \nabla f_t(\bar{x})'h = 0, \forall h \in P\}$$

In [8] the following statement is proved: $P = \{\vec{0}\}$ is a sufficient condition for $\bar{x}$ to be a local minimum (strict).

A more direct proof can be given by means of Theorem 4.6.2 of [6].

Theorem 5.6. *If the system of tangential constraints satisfies the property of Farkas–Minkowski and $P = P_0$, then there is some $\lambda \in \mathbb{R}_+^{(T_S)}$ such that*

$$\nabla \varphi(\bar{x}) + \sum_{t \in T_S} \lambda_t \nabla f_t(\bar{x}) = \vec{0}.$$

Proof. If $P = P_0$, the system $\{h' \nabla \varphi(\bar{x}) < 0, \ h' \nabla f_t(\bar{x}) \leqq 0, t \in \bar{T}\}$ has no solution. Then $h'[-\nabla \varphi(\bar{x})] \leqq 0$ is a consequence of the system $\{h' \nabla f_t(\bar{x}) \leqq 0, \ t \in \bar{T}\}$. By the generalized Farkas lemma, we have $-\nabla \varphi(\bar{x}) \in \bar{K}\{\nabla f_t(\bar{x}), \ t \in \bar{T}\} = K\{\nabla f_t(\bar{x}), t \in \bar{T}\}$, this equality being a consequence of the F-M property. Hence there is a $\lambda \in \mathbb{R}_+^{(\bar{T})}$ such that $-\nabla \varphi(\bar{x}) = \sum_{t \in \bar{T}} \lambda_t \nabla f_t(\bar{x})$ (i). If $\lambda_t > 0$, $t \in \bar{T} \sim T_S$, then for some $\bar{h} \in P$, $\nabla f_t(\bar{x})'\bar{h} < 0$. The product of the equality (i) by $\bar{h}$ gives a contradiction. This means that $\nabla \varphi(\bar{x}) + \sum_{t \in T_S} \lambda_t \nabla f_t(\bar{x}) = \vec{0}$. □

The system of tangential constraints is a F-M system either if the active constraints in $\bar{x}$ are finite or if $\bar{x}$ is a regular point. In the finite case this result is correctly given in [7], while in [8] a proof is proposed by means of the Motzkin's theorem of alternative, i.e., the same reasoning followed in [7], under the only assumption $P = P_0$. The following counterexample shows that our condition is not superfluous, even in the case of $\bar{x}$ being a global optimum point.

Counterexample. Let us consider the problem $\text{Min}\, \varphi(x) = x_1 + (x_2)^2$,

$$\text{subject to } f_t(x) = tx_1 + t^2 x_2 \leqq 0, \qquad t \in [-1, 1].$$

The feasible set is $S = \{(0, x_2)/x_2 \leqq 0\}$.

Clearly $\bar{x} = (0,0)$ is the minimum point of $\varphi(x)$ over S. In this case $\bar{T} = [-1, 1]$ and $P = P_0 = S$. Nevertheless, if $\bar{\lambda} \nabla \varphi(\bar{x}) + \sum_{t \in \bar{T}} \lambda_t \nabla f_t(\bar{x}) = \vec{0}$, for some $\lambda \in \mathbb{R}_+^{(\bar{T})}$, i.e., $\bar{\lambda}\begin{bmatrix}1\\0\end{bmatrix}+\sum_{t\in\bar{T}}\lambda_t\begin{bmatrix}t\\t^2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, we have $\sum_{t\in\bar{T}}\lambda_t t^2=0$. This means that $\lambda_t=0$, $t\in]0,1]$. The first component gives finally that $\bar{\lambda}=0$. In fact, it can be shown that the system of tangential constraints is not a F-M system.

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Received October 14, 1980; accepted for publication November 7, 1980