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# Inequalities for a Pair of Maps $S \times S \to S$ with $S$ a Finite Set

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## 1. Introduction, Basic Definitions and Results

Let $S^n$ be the family of all subsets of the set $\{1, 2, \ldots, n\}$. It was shown by the second author in [3] that

(1.1) $|A|\,|B| \leqq |A \vee B|\,|A \wedge B|$ for all $A, B \subset S^n$,

where $A \vee B = \{a \cup b\colon a \in A, b \in B\}$ and $A \wedge B = \{a \cap b\colon a \in A, b \in B\}$.

The present investigation started with the discovery that instead of the pair of Boolean operations union, intersection $(\cup, \cap)$ one can also use symmetric difference, intersection $(\triangle, \cap)$, so

(1.2) $|A|\,|B| \leqq |A \triangle B|\,|A \wedge B|$ for all $A, B \subset S^n$,

where $A \triangle B = \{a \triangle b\colon a \in A, b \in B\}$.

It was then natural to look for all pairs of Boolean operations for which inequalities of the above type hold. It turns out that up to simple isomorphies, explained in Sect. 2, the two inequalities above are the only non-trivial ones. Again up to isomorphies the trivial ones are

(1.3) $|A|\,|B| \leqq |A|\,|A \triangle B| \leqq |A \triangle B|\,|A \triangle B|$ for all $A, B \subset S^n$.

A more fruitful and challenging investigation started from the following two facts:

1) The proof in [3] of (1.1) and our proof for (1.2) were quite different, and we felt the need for a unified approach.

2) Our 4-weights inequality of [1] has far reaching consequences, as explained in Sect. 9. However that inequality for the pair $(\cup, \cap)$ does not hold for the pair $(\triangle, \cap)$.

In order to analyze and understand those facts, we consider more general maps $\psi\colon S \times S \to S$, where $S$ is now an arbitrary finite set, and we introduce several notions of *expansiveness* for pairs of such maps $(\varphi, \psi)$. Our studies are centered around the problem of how those notions behave under direct products of two pairs of maps. As a result we find new lattice inequalities.

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*Basic Definitions*

Before we list our definitions of expansiveness and establish relations between them, we explain our terminology.

Throughout $\mathbb{R}$ denotes the non-negative reals, and $S$, $T$ finite sets.

For $\varphi: S \times S \to S$ and $A, B \subset S$ we write

$$\varphi(A,B) = \{\varphi(a,b): \ a \in A, b \in B\}.$$

If $\varphi, \psi$: $S \times S \to S$ and $E_1$, $E_2 \subset S \times S$, then

$$\varphi\psi(E_1, E_2) = \{\varphi(a_1, b_2), \psi(a_2, b_1): \ (a_1, b_1) \in E_1, (a_2, b_2) \in E_2\} \subset S \times S,$$

and when $E_1 = E_2 = E$ we write $\varphi\psi(E)$ for $\varphi\psi(E,E)$. Note that when $E = A \times B$, then $\varphi\psi(E) = \varphi(A,B) \times \psi(A,B)$. For $\alpha$: $S \to \mathbb{R}$ and $A \subset S$ put $\alpha(A) = \sum_{a \in A} \alpha(a)$. The map $\alpha: S \to \mathbb{R}$ with $\alpha(a) = 1$ for all $a \in S$ is denoted by $\mathbb{1}$, and for this map $\alpha(A)$ is the cardinality $|A|$ of $A$. The set of all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of maps $\alpha, \beta, \gamma, \delta$: $S \to \mathbb{R}$ is denoted by $\mathfrak{M}_S$. Frequently we write $\mathfrak{M}$ instead of $\mathfrak{M}_S$ if no misunderstanding is possible. By $\mathfrak{C}$ we denote a subset of $\mathfrak{M}$ containing $(\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})$. Lastly $X + Y$ denotes the union of the sets $X$, $Y$ and says that $X \cap Y = \varnothing$.

For two pairs of maps $\varphi_S, \psi_S$: $S \times S \to S$ and $\varphi_T, \psi_T$: $T \times T \to T$ define the *direct product* $\varphi_{ST}, \psi_{ST}$: $(S \times T) \times (S \times T) \to S \times T$ by

(1.4) $\varphi_{ST}((s_1, t_1), (s_2, t_2)) = (\varphi_S(s_1, s_2), \varphi_T(t_1, t_2))$
$\psi_{ST}((s_1, t_1), (s_2, t_2)) = (\psi_S(s_1, s_2), \psi_T(t_1, t_2))$
for all $s_1, s_2 \in S$ and all $t_1, t_2 \in T$.

For $\varphi, \psi$: $S \times S \to S$ define the *square functions* $\varphi^2, \psi^2$: $S^2 \times S^2 \to S$ by

(1.5) $\varphi^2((a,b),(c,d)) = (\varphi(a,d), \psi(c,b))$
$\psi^2((a,b),(c,d)) = (\varphi(c,b), \psi(a,d))$ for all $a, b, c, d \in S$.

*Kinds of Expansion*

$(\varphi, \psi)$ is *expansive* if

(1.6) $|A|\,|B| \leqq |\varphi(A,B)|\,|\psi(A,B)|$ for all $A, B \subset S$

$(\varphi, \psi)$ is *set-expansive* if

(1.7) $|E| \leqq |\varphi\psi(E)|$ for all $E \subset S \times S$.

$(\varphi, \psi)$ is *partition-expansive* if there are two partitions $S \times S = D_1 + \ldots + D_z = D_1^* + \ldots + D_z^*$, $|D_i| = |D_i^*|$ $(1 \leqq i \leqq z)$, such that for each $i$, $1 \leqq i \leqq z$, and all $E_1, E_2 \subset D_i$ with $|E_1| = |E_2|$ we have both

(1.8) $\varphi\psi(E_1, E_2) \subset D_i^*$

and

(1.9) $|E_1| \leqq |\varphi\psi(E_1, E_2)|$.

Since $|\varphi\psi(E_1, E_2)| = |\psi\varphi(E_2, E_1)|$, it is clear that (1.9) can be replaced by

(1.10) $|E_1| \leqq |\psi\varphi(E_1, E_2)|$.

$(\alpha, \beta, \gamma, \delta) \in \mathfrak{C}$ is *compatible* with $(\varphi, \psi)$, if

(1.11) $\alpha(a)\,\beta(b) \leqq \gamma(\varphi(a, b))\,\delta(\psi(a, b))$ for all $a, b \in S$.

$(\varphi, \psi)$ is *$\mathfrak{C}$-expansive* if for any $(\alpha, \beta, \gamma, \delta) \in \mathfrak{C}$ which is *compatible* with $(\varphi, \psi)$ we have

(1.12) $\alpha(A)\,\beta(B) \leqq \gamma(\varphi(A, B))\,\delta(\psi(A, B))$ for all $A, B \subset S$.

$(\varphi, \psi)$ is *$\mathfrak{C}$-set-expansive* if for any compatible $(\alpha, \beta, \gamma, \delta) \in \mathfrak{C}$ we have

(1.13) $\sum\limits_{(a,b) \in E} \alpha(a)\,\beta(b) \leqq \sum\limits_{(a,b) \in \varphi\psi(E)} \gamma(a)\,\delta(b)$ for all $E \subset S \times S$.

The most important special case is obtained by choosing $\mathfrak{C} = \mathfrak{M}$ in the preceding definitions. We then say that $(\varphi, \psi)$ is *$\mathfrak{M}$-expansive* resp. *$\mathfrak{M}$-set-expansive.*

Finally, we call $(\varphi, \psi)$ *explosive* if

(1.14) $|E|\,|F| \leqq |\varphi\psi(E, F)|\,|\varphi\psi(F, E)|$ for all $E, F \subset S \times S$

and *$\mathfrak{M}$-explosive* if given any $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \omega \colon S \to \mathbb{R}$ such that

(1.15) $\alpha(a)\,\beta(b)\,\gamma(c)\,\delta(d) \leqq \lambda(\varphi(a, d))\,\mu(\psi(c, b))\,\nu(\varphi(c, b))\,\omega(\psi(a, d))$
for all $a, b, c, d \in S$,

we have

(1.16) $(\sum_E)(\sum_F) \leqq (\sum_{EF}) \cdot (\sum_{FE})$ for all $E, F \subset S \times S$
where

$\sum_E = \sum((a, b) \in E)\,\alpha(a)\,\beta(b)$

$\sum_F = \sum((a, b) \in F)\,\gamma(a)\,\delta(b)$

$\sum_{EF} = \sum((a, b) \in \varphi\psi(E, F))\,\lambda(a)\,\mu(b)$

$\sum_{FE} = \sum((a, b) \in \varphi\psi(F, E))\,\nu(a)\,\omega(b)$.

There are some obvious relations between those concepts:

(1.17) $\mathfrak{M}$-explosive $\Rightarrow$ $\mathfrak{C}$-set-expansive $\Rightarrow$ $\mathfrak{C}$-expansive $\Rightarrow$ expansive

(1.18) $\mathfrak{M}$-explosive $\Rightarrow$ explosive $\Rightarrow$ set-expansive $\Rightarrow$ expansive

(1.19) $\mathfrak{C}$-set-expansive $\Rightarrow$ set-expansive

(1.20) partition-expansive $\Rightarrow$ set-expansive.

*The Results*

**Theorem 1.** *The direct product* $((\varphi_S, \varphi_T), (\psi_S, \psi_T))$ *of an* $\mathfrak{M}_S$*-expansive pair of maps* $(\varphi_S, \psi_S)$ *with an* $\mathfrak{M}_T$*-expansive pair of maps* $(\varphi_T, \psi_T)$ *is* $\mathfrak{M}_{S\times T}$*-expansive.*

The power of the Theorem lies in the fact that one can apply it iteratively. This leaves us of course with the task to decide whether a component pair $(\varphi, \psi)$ is $\mathfrak{M}$-expansive. We completely settle this for all Boolean $\varphi, \psi$: $\{0, 1\}^2 \to \{0, 1\}$ in Sect. 2. All expansive Boolean pairs $(\varphi, \psi)$ fall into equivalence classes (in a sense made precise there). The representatives are

$$(h_1(x, y), w_1(x, y)) = (y, x); \qquad (h_2(x, y), w_2(x, y)) = (x \cup y, x \cap y);$$
$$(h_3(x, y), w_3(x, y)) = (x \triangle y, x); \qquad (h_4(x, y), w_4(x, y)) = (x \triangle y, x \triangle y);$$
$$(h_5(x, y), w_5(x, y)) = (x \triangle y, x \cap y).$$

Among those only $(h_5, w_5)$ is *not* $\mathfrak{M}$-expansive.

**Corollary 1.** *Let* $\alpha, \beta, \gamma, \delta$: $\{0, 1\}^n \to \mathbb{R}$ *and let* $(\varphi, \psi) \in \{(h_i, w_i): 1 \leqq i \leqq 4\}$ *be compatible with* $(\alpha, \beta, \gamma, \delta)$, *then*

$$\alpha(A)\,\beta(B) \leqq \gamma(\varphi(A, B))\,\delta(\psi(A, B)) \qquad \textit{for all } A, B \subset \{0, 1\}^n.$$

The fact that not all expansive pairs are $\mathfrak{M}$-expansive naturally leads to a more general *question:*

(1.21) *If* $(\varphi_S, \psi_S)$ *is* $\mathfrak{C}_S$*-expansive,* $\mathfrak{C}_S \subset \mathfrak{M}_S$, *and if* $(\varphi_T, \psi_T)$ *is* $\mathfrak{C}_T$*-expansive,* $\mathfrak{C}_T \subset \mathfrak{M}_T$, *for which* $\mathfrak{C} \subset \mathfrak{M}_{S\times T}$ *is* $((\varphi_S, \varphi_T), (\psi_S, \psi_T))$ $\mathfrak{C}$*-expansive?*

A first result in this direction is

**Theorem 2.** *If* $(\varphi_S, \psi_S)$ is $\mathfrak{M}$*-expansive and* $(\varphi_T, \psi_T)$ is *expansive, then the direct product* $((\varphi_S, \varphi_T), (\psi_S, \psi_T))$ *is expansive.*

We firmly believe, but have not been able to prove it, that a stronger statement is true.

**Conjecture 1.** *The direct product of expansive pairs of maps is expansive.*

This is probably the most outstanding problem in this context. Since $(\triangle, \cap)$ is not $\mathfrak{M}$-expansive Theorem 2 also does not say anything about this case. In order to get results for this case and also to have a unified proof for both (1.1) and (1.2) we have introduced the notions of set-expansive and partition-expansive. From Lemma 4 in Sect. 2 we know that *all* expansive Boolean pairs are also partition-expansive and hence set-expansive. Therefore the following result gives *all* inequalities in the Boolean case for $(\alpha, \beta, \gamma, \delta) = (\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})$.

**Theorem 3.** *If* $(\varphi_s, \psi_s)$ *is set-expansive and* $(\varphi_T, \psi_T)$ *is partition-expansive, then the product is set-expansive.*

Next we investigate for which weights $(\alpha, \beta, \gamma, \delta)$ on $\{0, 1\}^n$ is $(\triangle, \cap)$ expansive. Lemma 3 in Sect. 2 exhibits some cases for $n = 1$. For general $n$ we have in so far the following result.

**Theorem 4.** *Let $L$ be a sublattice of $\{0,1\}^n$ and let $\alpha, \beta, \gamma, \delta$: $L \Rightarrow \mathbb{R}$ with $\delta$ a monotone increasing function on $L$.*

*If $(\triangle, \cap)$ is compatible with $(\alpha, \beta, \gamma, \delta)$, then*

$$\alpha(A)\,\beta(B) \leqq \gamma(A \triangle B)\,\delta(A \triangle B) \quad \text{for all } A, B \subset L.$$

The proof makes use of special properties of $(\triangle, \cap)$. Another result for this pair is in the spirit of question (1.21).

**Theorem 5.** *Let $L$ be a sublattice of $\{0,1\}^n$, let $S$ be a finite set, and let $(\varphi_S, \psi_S)$ be $\mathfrak{M}_S$-expansive. Let $(\varphi, \psi)$ be the direct product of $(\varphi_S, \psi_S)$ and $(\triangle, \cap)$ on $L$.*

*Suppose $\alpha, \beta, \gamma, \delta$: $S \times L \to \mathbb{R}$ are compatible with $(\varphi, \psi)$ and that $\delta = \delta(s, l)$ is a monotone increasing function in $l \in L$, then*

$$\alpha(A)\,\beta(B) \leqq \gamma(\varphi(A,B))\,\delta(\psi(A,B)) \quad \text{for all } A, B \subset S \times L.$$

Our last result originated with the idea of finding weight inequalities for arbitrary subsets $E \subset S \times S$ rather than just the usual $E = A \times B \subset S \times S$. This led us to introduce the notions of explosive and $\mathfrak{M}$-explosive pairs $(\varphi, \psi)$ and to study square functions $(\varphi^2, \psi^2)$.

**Theorem 6.** *The following are equivalent:*

(a) $(\varphi, \psi)$ *$\mathfrak{M}$-expansive,*
(b) $(\varphi^2, \psi^2)$ *$\mathfrak{M}$-expansive,*
(c) $(\varphi, \psi)$ *$\mathfrak{M}$-explosive.*

*Furthermore* (a′) *implies* (b′) *implies* (c′) *for the statements*

(a′) *the direct product $((\varphi, \varphi), (\psi, \psi))$ expansive,*
(b′) $(\varphi^2, \psi^2)$ *expansive,*
(c′) $(\varphi, \psi)$ *explosive.*

Since we know that in the Boolean case $((\varphi, \varphi), (\psi, \psi))$ is expansive iff $(\varphi, \psi)$ is expansive, we have the

**Corollary 2.** *A Boolean pair is expansive iff it is explosive.*

There are all kinds of special cases of Theorem 6. For instance we know that $(\cup, \cap)$ is $\mathfrak{M}$-expansive on $\{0,1\}$, hence $\mathfrak{M}$-expansive on $\{0,1\}^n$ by Theorem 1 and therefore $\mathfrak{M}$-explosive on $\{0,1\}^n$. Since this implies $\mathfrak{M}$-set-expansive we have the

**Corollary 3.** *Let $\alpha, \beta, \lambda, \mu$: $\{0,1\}^n \to \mathbb{R}$ be compatible with $(\cup, \cap)$, then for $E \subset \{0,1\}^n \times \{0,1\}^n$,*

$$F = \{(a_1 \cup b_2,\ a_2 \cap b_1\!:\ (a_1, b_1), (a_2, b_2) \in E\}$$

$$\sum_{(a,b) \in E} \alpha(a)\,\beta(b) \leqq \sum_{(a,b) \in F} \lambda(a)\,\mu(b).$$

This is the kind of inequality we were looking for. The same holds for any sublattice $L \subset \{0,1\}^n$, for if $\alpha(a)\,\beta(b) \leqq \lambda(a \cup b)\,\mu(a \cap b)$ for all $a, b \in L$, just define $\alpha(c) = \beta(c) = \lambda(c) = \mu(c) = 0$ for all $c \notin L$. Then the hypothesis holds on all of $\{0,1\}^n$. It easily can be shown by standard techniques that $L$ can be replaced by any (finite or infinite) distributive lattice. Now if we restrict $E$ to be of form $E=A\times B$ with $A, B\subset L$, then we get our old inequality of [1].

## 2. A Classification of all Boolean Pairs of Maps

We adress ourselves in this Section to the Boolean case $S=\{0,1\}$. The 16 binary Boolean functions $f\colon S\times S\to S$ are

(2.1) $0,\ x\cap y,\ x\setminus y,\ y\setminus x,\ x,\ y,\ x\triangle y,\ x\cup y;$
$1,\ \overline{x\cap y},\ \overline{x\setminus y},\ \overline{y\setminus x},\ \bar{x},\ \bar{y},\ \overline{x\triangle y},\ \overline{x\cup y}.$

Let us use the notation $\bar{A}=\{\bar{a}\colon a\in A\}$ and $\bar{\varepsilon}(a)=\varepsilon(\bar{a})$ for $\varepsilon\colon S\to\mathbb{R}$ and $a\in S$. We call $(\varphi,\psi)$ *$(\alpha,\beta,\gamma,\delta)$-expansive* iff

(2.2) $\alpha(A)\,\beta(B)\leqq\gamma(\varphi(A,B))\,\delta(\psi(A,B))$ for all $A, B\subset S$.

Obviously the following rule is valid.

*Rule I. Equivalent are:*

(a) $(\varphi,\psi)$ *$(\alpha,\beta,\gamma,\delta)$-expansive,*
(b) $(\varphi,\bar{\psi})$ *$(\alpha,\beta,\gamma,\bar{\delta})$-expansive,*
(c) $(\psi,\varphi)$ *$(\alpha,\beta,\delta,\gamma)$-expansive.*

Now we establish a similar rule for partition-expansive.

For $F\subset S\times S$ write $F=F(0)\times\{0\}+F(1)\times\{1\}$ and define $F'=F(0)\times\{1\}+F(1)\times\{0\}$. Clearly

(2.3) $|F'|=|F|$ and $F\subset G$ iff $F'\subset G'$.

If $(\varphi,\psi)$ is partition-expansive, there exist partitions

$$S\times S=D_1+\ldots+D_z=D_1^*+\ldots+D_z^*$$

such that for all $E_1, E_2\subset D_i$, $|E_1|=|E_2|$, $\varphi\psi(E_1,E_2)\subset D_i^*$ and $|E_1|\leqq|\varphi\psi(E_1,E_2)|$.

Since $\varphi\bar{\psi}(E_1,E_2)=(\varphi\psi(E_1,E_2))'$ and since $(D_1^*)'+\ldots+(D_z^*)'=S\times S$ it suffices to use the partitions $S\times S=D_1+\ldots+D_z=(D_1^*)'+\ldots+(D_z^*)'$ to see that $(\varphi,\bar{\psi})$ is also partition-expansive. By (1.10) also $(\psi,\varphi)$ is partition-expansive. Therefore we have

*Rule II. Equivalent are:*

(a) $(\varphi,\psi)$ *partition-expansive,*
(b) $(\varphi,\bar{\psi})$ *partition-expansive,*
(c) $(\psi,\varphi)$ *partition-expansive.*

It is clear from Rules I, II that in studying expansive or diagonal-expansive or $\mathfrak{M}$-expansive maps $(\varphi,\psi)$ we can limit ourselves to the pairs described by the following triangular configuration.

**Table 1**

| $\psi$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x \cup y$ | | | | | | | | $*$ |
| $x \Delta y$ | | | | | | | $Q$ | $P$ |
| $y$ | | | | | | $*$ | $Q$ | $*$ |
| $x$ | | | | | $*$ | $Q$ | $Q$ | $*$ |
| $y-x$ | | | | $*$ | $*$ | $*$ | $P$ | $*$ |
| $x-y$ | | | $*$ | $Q$ | $*$ | $*$ | $P$ | $*$ |
| $x \wedge y$ | | $*$ | $*$ | $*$ | $*$ | $*$ | $P$ | $Q$ |
| $0$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| | $0$ | $x \wedge y$ | $x-y$ | $y-x$ | $x$ | $y$ | $x \Delta y$ | $x \cup y$ $\varphi$ |

$*$ = *not*-expansive
$P$ = partition-expansive
$Q$ = $\mathfrak{M}$-expansive *and* partition-expansive

The justification of Table 1 goes in several steps. First one decides whether a $(\varphi, \psi)$ is expansive or not. This can be done case by case with the help of the following

**Table 2**

| | | |
|---|---|---|
| 1 | $\psi(0,1)$ / $\varphi(0,1)$ | $\psi(1,1)$ / $\varphi(1,1)$ |
| 0 | $\psi(0,0)$ / $\varphi(0,0)$ | $\psi(1,0)$ / $\varphi(1,0)$ |
| | 0 | 1 |

On easily verifies that $(\varphi, \psi)$ is expansive iff in Table 2: a) adjacent squares are not equal, and b) $\varphi$ and $\psi$ are *not* constant.

Thus one obtains that exactly the following pairs are expansive: $(h_i, w_i)$, $1 \leqq i \leqq 5$, defined in Sect. 1, and

$$(h_6(x,y), w_6(x,y)) = (x \Delta y, x-y); \quad (h_7(x,y), w_7(x,y)) = (x \Delta y, y-x);$$
$$(h_8(x,y), w_8(x,y)) = (x \vee y, x \Delta y); \quad (h_9(x,y), w_9(x,y)) = (y-x, x-y);$$
$$(h_{10}(x,y), w_{10}(x,y)) = (x \Delta y, y).$$

Using Rules I and the fact that $\bar{a}$ ranges over $S$ if a does leads us to the following relations:

(2.4) *Equivalent are:*

(a) $(h_5, w_5)$ $(\alpha, \beta, \gamma, \delta)$-*expansive*,
(b) $(h_6, w_6)$ $(\alpha, \beta, \bar{\gamma}, \delta)$-*expansive*,
(c) $(h_7, w_7)$ $(\alpha, \beta, \bar{\gamma}, \delta)$-*expansive*,
(d) $(h_8, w_8)$ $(\alpha, \beta, \bar{\delta}, \gamma)$-*expansive.*

(2.5) *Equivalent are:*

(e) $(h_2, w_2)$ $(\alpha, \beta, \gamma, \delta)$-*expansive*,
(f) $(h_9, w_9)$ $(\alpha, \beta, \bar{\gamma}, \delta)$-*expansive*.

(2.6) *Equivalent are:*

(g) $(h_3, w_3)$ $(\alpha, \beta, \gamma, \delta)$-*expansive*,
(i) $(h_{10}, w_{10})$ $(\beta, \alpha, \gamma, \delta)$-*expansive*.

By the forgoing it suffices for the study of $\mathfrak{M}$-expansiviness to consider the representatives $(h_i, w_i)$, $1 \leqq i \leqq 5$.

If we use Rule II instead of Rule I we get the same classification with respect to partition-expansive.

In the remainder of this Section we show that all 5 representatives are partition-expansive and that with the only exception of $(x \triangle y, x \cap y)$ they are all $\mathfrak{M}$-expansive. The following result helps in deciding whether a pair $(\varphi, \psi)$ which is expansive is also $\mathfrak{M}$-expansive

**Lemma 1.** *Let* $\varphi, \psi$: $S \times S \to S = \{1, \ldots, t\}$ *be an expansive pair of maps. Let* $\alpha, \beta, \gamma, \delta$: $S \to \mathbb{R}$ *satisfy*

(2.7) $\alpha(a)\beta(b) \leqq \gamma(\varphi(a,b))\,\delta(\psi(a,b))$ *for all* $a, b \in S$.

*Then*

$$\alpha(A)\beta(B) \leqq \gamma(\varphi(A,B))\,\delta(\psi(A,B)) \quad \text{for all } A, B \subset S \text{ with } |A|=1 \text{ or } |B|=1.$$

*Proof.* Suppose $A=\{a\}$ and $B=\{b_1, \ldots, b_r\}$, if $1 \leqq i < j \leqq r$ and $B'=\{b_i, b_j\}$ then by expansiveness

$$2=|A|\,|B'| \leqq |\varphi(a,b_i) \cup \varphi(a,b_j)|\,|\psi(a,b_i) \cup \psi(a,b_j)|,$$

so $(\varphi(a,b_i), \psi(a,b_i)) \neq (\varphi(a,b_j), \psi(a,b_j))$. Hence the set $\{(\varphi(a,b_i), \psi(a,b_i)) \colon 1 \leqq i \leqq r\}$ consists of $r$ distinct points of $\varphi(a,B) \times \psi(a,B)$, and it follows that

$$\sum_{i=1}^{r} \gamma(\varphi(a,b_i))\,\delta(\psi(a,b_i)) \leqq \gamma(\varphi(a,B))\,\delta(\psi(a,B)).$$

The result now follows by summing (2.7) with $b=b_i$ over $1 \leqq i \leqq r$.

**Lemma 2.** *Let* $S=\{0,1\}$: $(x \triangle y, x \wedge y)$ *is not* $\mathfrak{M}$-*expansive, all other Boolean representatives* $(h_i, w_i)$, $1 \leqq 1 \leqq 4$, *are* $\mathfrak{M}$-*expansive.*

*Proof.* From Lemma 1 we know that we have to consider only the case $|A|=|B|=2$, that is, $A=B=S$.

*Case* $(h_1, w_1)$: Here

$$X_1=\alpha(0)\beta(0) \leqq \gamma(0)\delta(0)$$
$$X_2=\alpha(0)\beta(1) \leqq \gamma(1)\delta(0)$$
$$X_3=\alpha(1)\beta(0) \leqq \gamma(0)\delta(1)$$
$$X_4=\alpha(1)\beta(1) \leqq \gamma(1)\delta(1)$$

and therefore

$$\sum X_i = \alpha(A)\,\beta(B) \leqq \gamma(\varphi(A,B))\,\delta(\psi(A,B)).$$

*Case* $(h_2, w_2)$: This was already proved in [1].

*Case* $(h_3, w_3)$

$$X_1 \leqq \gamma(0)\,\delta(0), \qquad X_2 \leqq \gamma(1)\,\delta(0)$$
$$X_3 \leqq \gamma(1)\,\delta(1), \qquad X_4 \leqq \gamma(0)\,\delta(1).$$

Again

$$\sum X_i \leqq (\gamma(0)+\gamma(1)(\delta(0)+\delta(1))$$
$$= (\gamma(\varphi(A,B))\,\delta(\psi(A,B)).$$

*Case* $(h_4, w_4)$

$$X_1 \leqq \gamma(0)\,\delta(0) = s(0)^2, \qquad X_2 \leqq \gamma(1)\,\delta(1) = s(1)^2$$
$$X_3 \leqq \gamma(1)\,\delta(1) = s(1)^2, \qquad X_4 = \gamma(0)\,\delta(0) = s(0)^2$$

where

$$s(0) = \sqrt{\gamma(0)\,\delta(0)}, \qquad s(1) = \sqrt{\gamma(1)\,\delta(1)}.$$

We show first that $\sum X_i \leqq (s(0)+s(1))^2$ and then we apply the inequality

$$(\sqrt{\gamma(0)\,\delta(0)} + \sqrt{\gamma(1)\,\delta(1)})^2 \leqq (\gamma(0)+\gamma(1))(\delta(0)+\delta(1)),$$

which is a simple consequence of the arithmetic-geometric mean inequality, to complete the proof. Notice that

$$X_1 X_4 = X_2 X_3 \leqq \min(s(0)^4, s(1)^4)$$

and therefore

$$\min(X_1, X_4) \leqq \min(s(0)^2, s(1)^2), \qquad \min(X_2, X_3) \leqq \min(s(0)^2, s(1)^2).$$

This implies

$$\sum X_i \leqq \min((3\,s(0)^2 + s(1)^2), \quad (s(0)^2 + 3\,s(1)^2)) \leqq (s(0)+s(1))^2.$$

*Case* $(h_5, w_5)$: Choose $\alpha(0)=\beta(0)=\gamma(0)=\delta(0)=2$, $\delta(1)=\frac{1}{2}$, and $\alpha(1)=\beta(1)=\gamma(1)=1$ to see that this pair is *not* $\mathfrak{M}$-expansive.

Even though $(\triangle, \cap)$ is not $\mathfrak{M}$-expansive it is still $\mathfrak{C}$-expansive for interesting classes $\mathfrak{C}$. Clearly $\mathfrak{C}'$-expansive and $\mathfrak{C}$-expansive imply $(\mathfrak{C} \cup \mathfrak{C}')$-expansive. We describe now some interesting classes

**Lemma 3.** $(\triangle, \cap)$ *is* $\mathfrak{C}$*-expansive for*

(a) $\mathfrak{C} = \{(\alpha, \beta, \gamma, \delta)\colon\ \delta(1) \geqq \delta(0)\}$,

(b) $\mathfrak{C} = \{(\alpha, \beta, \gamma, \delta)\colon\ \delta \equiv \gamma \ \text{or}\ \delta \equiv \beta \ \text{or}\ \delta \equiv \alpha\}$,

(c) $\mathfrak{C} = \{(\alpha, \beta, \gamma, \delta)\colon\ \alpha \equiv \beta \equiv 1\}$,

(d) $\mathfrak{C} = \{(\alpha, \beta, \gamma, \delta)\colon\ \gamma(0) \geqq \gamma(1) \ \text{and}\ \alpha(1) \geqq \alpha(0)\}$.

*Proof.* By hypothesis

(2.8) $X_1 \leqq \gamma(0)\,\delta(0), \quad X_2 \leqq \gamma(1)\,\delta(0),$
$X_3 \leqq \gamma(1)\,\delta(0), \quad X_4 \leqq \gamma(0)\,\delta(1).$

(a) Since $\delta(1) \geqq \delta(0)$, $\sum X_i \leqq (\gamma(0)+\gamma(1))(\delta(0)+\delta(1))$.

(b) Let $\alpha \equiv \delta$ ($\beta \equiv \delta$ is symmetrically the same). From (2.8) we conclude $\beta(0) \leqq \gamma(0)$ and $\beta(1) \leqq \gamma(1)$ and therefore

$$(\alpha(0)+\alpha(1))(\beta(0)+\beta(1)) \leqq (\gamma(0)+\gamma(1))(\delta(0)+\delta(1)).$$

If $\gamma \equiv \delta$ we can normalize such that $\gamma(0)=\delta(0)=1$, $\gamma(1)=\delta(1)=s$. Then $X_1 \leqq 1$; $X_2, X_3, X_4 \leqq s$. If $s \geqq 1$, then $s^2 \geqq s$ and $(\alpha(0)+\alpha(1))(\beta(0)+\beta(1)) \leqq (1+s)^2$ as wanted. Assume therefore $s^2 \leqq s \leqq 1$ and choose $\varepsilon$ such that $\alpha(1)\,\beta(1) = s^2+\varepsilon \leqq s \leqq 1$, $\varepsilon > 0$.

$$X_1 = \frac{X_2 X_3}{X_4} \leqq \frac{s^2}{s^2+\varepsilon},$$

therefore

$$\sum X_i \leqq 2s + \frac{s^2}{s^2+\varepsilon} + s^2 \leqq 2s+1+s^2$$

as wanted.

(c) From (2.8) $\delta(0) \geqq \dfrac{1}{\gamma(1)}$, $\gamma(0) \geqq \dfrac{1}{\delta(1)}$ and therefore

$$\delta(0)+\delta(1)\,\gamma(0)+\gamma(1) \geqq \left(\frac{1}{\gamma(1)'}+\delta(1)\right)\left(\frac{1}{\delta(1)}+\gamma(1)\right) \geqq 4 = \sum X_i.$$

(d) Since $\alpha(1) \geqq \alpha(0)$, $X_1 \leqq \alpha(1)\,\beta(0) \leqq \gamma(1)\,\delta(0)$ and $X_2 \leqq \alpha(1)\,\beta(1) \leqq \gamma(0)\,\delta(1)$. Now $\gamma(0) \geqq \gamma(1)$ and using (a) we can also assume $\delta(0) \geqq \delta(1)$. This implies $\gamma(0)\,\delta(0)+\gamma(1)\,\delta(1) \geqq \gamma(1)\,\delta(0)+\gamma(0)\,\delta(1) \geqq X_3+X_4$.

The results of Lemma 3 are for $n=1$ and they do not necessarily extend to general $n$. We give are counterexample for the class described in (b).

*Example* ($n=3$)

| | 000 | 010 | 101 | 111 | 100 | 011 | 110 |
|---|---|---|---|---|---|---|---|
| $\alpha$ | 4 | 1 | 2 | 1 | 0 | 0 | 0 |
| $\beta$ | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\gamma$ | 4 | 1 | 1 | $\frac{1}{2}$ | 0 | 0 | 0 |

One now checks that both, $(\alpha, \beta, \gamma, \alpha)$ and $(\alpha, \beta, \gamma, \gamma)$, are compatible with $(\triangle, \cap)$. However, for $A=\{101, 111\}$, $B=\{000, 010\}$

$$\alpha(A)=\beta(B)=3, \qquad \alpha(A \wedge B)=\gamma(A \wedge B)=5, \qquad \gamma(A \triangle B)=\tfrac{3}{2}.$$

Therefore

$$\alpha(A)\,\beta(B) \nleqq \gamma(A \triangle B)\,\alpha(A \wedge B)$$
$$\alpha(A)\,\beta(B) \nleqq \gamma(A \triangle B)\,\gamma(A \wedge B).$$

Thus the example serves simultaneously as counterexample in case $\delta \equiv \alpha$ and in case $\delta \equiv \gamma$.

We tend to believe that the following is true for general $n$:

**Conjecture 2.** *If* $(\alpha, \beta, \alpha, \beta)$ *is compatible with* $(\triangle, \wedge)$, *then*

$$\alpha(A)\,\beta(B) \leqq \alpha(A \triangle B)\,\beta(A \wedge B) \quad \textit{for all } A, B \subset \{0,1\}^n.$$

A special case would be $\alpha \equiv \beta$.

**Lemma 4.** *All Boolean representatives* $(h_i, w_i)$, $1 \leqq i \leqq 5$, *are partition-expansive and therefore also set-expansive.*

*Proof. Case* $(h_1, w_1)$: Partition $S \times S = D_1 + \ldots + D_4 = D_1^* + \ldots + D_4^*$ such that $|D_i| = |D_i^*| = 1$ and $D_i^* = \{(a,b)\}$ if $D_i = \{(b,a)\}$, $1 \leqq i \leqq 4$.

In all other cases we can choose $D_i = D_i^*$, $1 \leqq i \leqq z$.

*Case* $(h_2, w_2)$: Choose $D_1 = \{00\}$, $D_2 = \{11\}$, $D_3 = \{01, 10\}$. Since $h_2 w_2(D_i) = D_i$, $1 \leqq i \leqq 3$, we are done if $E_1 = E_2 = D_i$. Otherwise $|E_1| = |E_2| = 1$ and obviously $|E_1| \leqq |h_2 w_2(E_1, E_2)|$.

*Case* $(h_3, w_3)$: Choose $D_1 = \{00\}$, $D_2 = \{01\}$, $D_3 = \{10, 11\}$. Again $h_2 w_2(D_i) = D_i$, $1 \leqq i \leqq 3$, and the previous argument applies.

*Case* $(h_4, w_4)$: Choose $D_1 = S^2$. For $E_1, E_2 \subset S^2$, $|E_1| = |E_2|$, and any $(a,b) \in E_2$

$$|\{(\varphi(a_1, b_2), \psi(a_2, b_1)) \colon (a_1, b_1) \in E_1, (a_2, b_2) \in E_2\}|$$
$$\geqq |\{(\varphi(a_1, b), \psi(a, b_1)) \colon (a_1, b_1) \in E_1\}| = |E_1|.$$

*Case* $(h_5, w_5)$: Choose $D_1 = S^2$. One verifies that in the table

| $a_2 b_2$ \ $a_1 b_1$ (entries $a_1 \triangle b_2 / a_2 \cap b_1$) | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 11 | 1/0 | 1/1 | 0/0 | 0/1 |
| 10 | 0/0 | 0/1 | 1/0 | 1/1 |
| 01 | 1/0 | 1/0 | 0/0 | 0/0 |
| 00 | 0/0 | 0/0 | 1/0 | 1/0 |

every $s \times s$-minor has at least $s$ different elements $x/y$.

In conclusion let us mention that we tend to believe that also in the non-Boolean case

**Conjecture 3.** *Expansive is the same as set-expansive.*

Furthermore, it is conceivable that also in general expansive is the same as partition-expansive. This and Theorem 3 would imply Conjecture 1.

## 3. Proofs of Theorems 1 and 2

*Proof of Theorem 1*

We are given $\mathfrak{M}$-expansive pairs of maps $(\varphi_S, \psi_S)$, $(\varphi_T, \psi_T)$ and their direct product $(\varphi, \psi)$ on $S \times T$. Let $\alpha, \beta, \gamma, \delta$: $S \times T \to \mathbb{R}$ satisfy

(3.1) $\alpha(a)\,\beta(b) \leqq \gamma(\varphi(a,b))\,\delta(\psi(a,b)) \qquad$ for all $a, b \in S \times T$.

Let $A, B$ be arbitrary *fixed* subsets of $S \times T$. We must show that

(3.2) $\alpha(A)\,\beta(B) \leqq \gamma(\varphi(A,B))\,\delta(\psi(A,B))$.

We define now marginal weights which do depend on $A, B$. Define $\alpha_S, \beta_S, \gamma_S, \delta_S$: $S \to \mathbb{R}$ by

(3.3)
$$\begin{aligned}
\alpha_S(s) &= \textstyle\sum (t \in T, (s,t) \in A)\,\alpha(s,t)\\
\beta_S(s) &= \textstyle\sum (t \in T, (s,t) \in B)\,\beta(s,t)\\
\gamma_S(s) &= \textstyle\sum (t \in T, (s,t) \in \varphi(A,B))\,\gamma(s,t)\\
\delta_S(s) &= \textstyle\sum (t \in T, (s,t) \in \psi(A,B))\,\delta(s,t).
\end{aligned}$$

Then

(3.4) $\alpha(A) = \sum_{(s,t)\in A} \alpha(s,t) = \sum_{s \in S} (\sum (t \in T, (s,t) \in A)\,\alpha(s,t)) = \sum_{s\in S} \alpha_S(s) = \alpha_S(S)$.

Similarly $\beta(B) = \beta_S(S)$.

(3.5)
$$\begin{aligned}
\gamma(\varphi(A,B)) &= \textstyle\sum ((s,t) \in \varphi(A,B))\,\gamma(s,t)\\
&= \sum_{s \in \varphi_S(S,S)} (\textstyle\sum (t \in T, (s,t) \in \varphi(A,B))\,\gamma(s,t))\\
&= \sum_{s \in \varphi_S(S,S)} \gamma_S(s) = \gamma_S(\varphi_S(S,S)).
\end{aligned}$$

Similarly $\delta(\psi(A,B)) = \delta_S(\psi_S(S,S))$.

(Since $(\varphi_S, \psi_S)$ is $\mathfrak{M}$-expansivs it is expansive and so

$$|S|\,|S| \leqq |\varphi_S(S,S)|\,|\psi_S(S,S)|.$$

Therefore $S = \varphi_S(S,S) = \psi_S(S,S)$, but this is not used.)

Assume for the moment that

(3.6) $\alpha_S(s_1)\,\beta_S(s_2) \leqq \gamma_S(\varphi_S(s_1,s_2))\,\delta_S(\psi_S(s_1,s_2) \qquad$ for all $s_1, s_2 \in S$.

Since $(\varphi_S, \psi_S)$ is $\mathfrak{M}$-expansive this together with (3.4) and (3.5) implies

$$\begin{aligned}(3.7)\quad \alpha(A)\,\beta(B)&=\alpha_S(S)\,\beta_S(S)\leqq\gamma_S(\varphi_S(S,S))\,\delta_S(\psi_S(S,S))\\ &=\gamma(\varphi(A,B))\,\delta(\psi(A,B))\end{aligned}$$

which is (3.2) as required. Thus it remains to prove (3.6).
Let $s_1$, $s_2$ be fixed arbitrarily, and put $s_3=\varphi_S(s_1,s_2)$ and $s_4=\psi_S(s_1,s_2)$. Define $\alpha_T$, $\beta_T$, $\gamma_T$, $\delta_T$: $T\to\mathbb{R}$ by

$$(3.8)\quad \alpha_T(t)=\begin{cases}\alpha(s_1,t) & \text{if } (s_1,t)\in A\\ 0 & \text{otherwise}\end{cases}$$

$$\beta_T(t)=\begin{cases}\beta(s_2,t) & \text{if } (s_2,t)\in B\\ 0 & \text{otherwise}\end{cases}$$

$$\gamma_T(t)=\begin{cases}\gamma(s_3,t) & \text{if } (s_3,t)\in\varphi(A,B)\\ 0 & \text{otherwise}\end{cases}$$

$$\delta_T(t)=\begin{cases}\delta(s_4,t) & \text{if } (s_4,t)\in\psi(A,B)\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$(3.9)\quad \alpha_S(s_1)=\sum(t\in T,(s_1,t)\in A)\,\alpha(s,t)=\sum_{t\in T}\alpha_T(t)=\alpha_T(T).$$

Similarly,

$$\beta_S(s_2)=\beta_T(T)$$

$$\begin{aligned}(3.10)\quad \gamma_S(s_3)&=\sum(t\in T,(s_3,t)\in\varphi(A,B))\,\gamma(s_3,t)\\ &=\sum(t\in\varphi_T(T,T),(s_3,t)\in\varphi(A,B))\,\gamma(s_3,t)\\ &=\sum(t\in\varphi_T(T,T))\,\gamma_T(t)=\gamma_T(\varphi_T(T,T)).\end{aligned}$$

Similarly $\delta_S(s_4)=\delta_T(\psi_T(T,T))$.
   Assume for the moment that

$$(3.11)\quad \alpha_T(t_1)\,\beta_T(t_2)\leqq\gamma_T(\varphi_T(t_1,t_2))\,\delta_T(\psi_T(t_1,t_2))\qquad\text{for all } t_1,t_2\in T.$$

Then by hypothesis on $(\varphi_T,\psi_T)$ we have

$$(3.12)\quad \alpha_T(T)\,\beta_T(T)\leqq\gamma_T(\varphi_T(T,T))\,\delta_T(\psi_T(T,T)),$$

in other words (3.6) follows.

To complete the proof we must show (3.11). The left hand side of (3.11) is zero unless $(s_1,t_1)\in A$ and $(s_2,t_2)\in B$, in which case it is $\alpha(s_1,t_1)\,\beta(s_2,t_2)$. Furthermore in this case

$$\varphi((s_1,t_1),(s_2,t_2))=(\varphi_S(s_1,s_2),\varphi_T(t_1,t_2))=(s_3,\varphi_T(t_1,t_2))\in\varphi(A,B)$$
$$\psi((s_1,t_1),(s_2,t_2))=(\psi_S(s_1,s_2),\psi_T(t_1,t_2))=(s_4,\psi_T(t_1,t_2))\in\psi(A,B)$$

so the right hand side of (3.11) is $\gamma(\varphi((s_1,t_1),(s_2,t_2))\,\delta(\psi((s_1,t_1),(s_2,t_2))$ and (3.11) follows from (3.1). Q.E.D.

*Proof of Theorem 2*

We just write out the proof of Theorem 1 again with $\alpha=\beta=\gamma=\delta=\mathbb{1}$, except that we cannot use the $\mathfrak{M}$-expansiveness of $(\varphi_T, \psi_T)$ to go from (3.11) to (3.12). Instead we put

$$T_1=\{t\colon\ \alpha_T(t)=1\}, \qquad T_2=\{t\colon\ \beta_T(t)=1\},$$
$$T_3=\{t\colon\ \gamma_T(t)=1\}, \qquad T_4=\{t\colon\ \delta_T(t)=1\},$$

then

$$\alpha_T(T)=|T_1|, \quad \beta_T(T)=|T_2|, \quad \gamma_T(T)=|T_3|, \quad \delta_T(T)=|T_4|.$$

Since (3.11) holds again this implies for $t_1 \in T_1$ and $t_2 \in T_2$ that $\varphi_T(t_1, t_2) \in T_3$ and $\psi_T(t_1, t_2) \in T_4$.

Therefore $\varphi_T(T_1, T_2) \subset T_3$ and $\psi_T(T_1, T_2) \subset T_4$. Since $(\varphi_T, \psi_T)$ is expansive

$$\alpha_T(T)\,\beta_T(T)=|T_1|\,|T_2| \leqq |\varphi_T(T_1, T_2)|\,|\psi_T(T_1, T_2)| \leqq |T_3|\,|T_4| = \gamma_T(T)\,\delta_T(T)$$

so (3.12) holds as required. Q.E.D.

## 4. Proof of Theorem 3

We want to show that $(\varphi_S, \psi_S)$ set-expansive, $(\varphi_T, \psi_T)$ partition-expansive implies that the direct product $(\varphi, \psi)=((\varphi_S, \varphi_T), (\psi_S, \psi_T))$ is set-expansive. Given $G \subset (S \times T) \times (S \times T)$ we must show that

(4.1) $|G| \leqq |\varphi\psi(G)|$.

Let $T \times T = D_1 + \ldots + D_z = D_1^* + \ldots + D_z^*$ be the two partitions with the properties as required in the definition of partition-expansive ((1.8), (1.9)). Partition $G$ as $G=G_1+\ldots+G_z$ by the rule: $((s,t),(s',t'))$ of $G$ goes in $G_i$ iff $(t,t') \in D_i$. Replacing $D_i$ by $D_i^*$ we get similarly $\varphi\psi(G)=H_1+\ldots+H_z$.

If for $i$, $1 \leqq i \leqq z$, $((s_1,t_1),(s_1',t_1')),((s_2,t_2),(s_2',t_2')) \in G_i$, then this implies that $(t_1,t_1'),(t_2,t_2') \in D_i$ and hence by hypothesis on $(\varphi_T, \psi_T)$: if $(t_3, t_4) = (\varphi_T(t_1, t_2'), \psi(t_2, t_1'))$, then $(t_3, t_4) \in D_i^*$. Therefore,

$$(\varphi((s_1,t_1),(s_2',t_2')),(\psi((s_2,t_2),(s_1',t_1'))=((\varphi_S(s_1,s_2'),t_3),(\psi_S(s_2,s_1'),t_4)) \in H_i.$$

Thus we have shown that

(4.2) $\varphi\psi(G_i) \subset H_i \quad$ for $1 \leqq i \leqq z$.

Suppose for the moment that

(4.3) $|G_i| \leqq |\varphi\psi(G_i)| \quad$ for $1 \leqq i \leqq z$,

then $|G| = \sum |G_i| \leqq \sum |\varphi\psi(G_i)| \leqq \sum |H_i| = |\varphi\psi(G)|$ as required.

So it remains to prove (4.3). We choose a value of $i$ and fix it. Then for *convenience* we write $J$ instead of $G_i$, $D$ instead of $D_i$, $D^*$ instead of $D_i^*$, $H$ instead of $H_i$. For $k=1,2,\ldots$ and $U \subset (S \times T) \times (S \times T)$ let $U_k$ be the set of all points $(s,s')$ of $S \times S$ for which there exist at least $k$ distinct points $(t_1,t'_1),\ldots,(t_k,t'_k)$ of $T \times T$ such that

$$((s,t_1),(s',t'_1)),\ldots,((s,t_k),(s',t_k),(s',t'_k)) \in U.$$

Notice that when $U=J$ then $(t_1,t'_1),\ldots,(t_k,t'_k)$ necessarily lie in $D$. By hypothesis on $S$ we have

$$|J_k| = |\varphi_S \psi_S(J_k)| \qquad \text{for } k=1,2,\ldots.$$

Let $x$ be a point of $\varphi_S \psi_S(J_k)$. This means that there are $(s_1,s'_1),(s_2,s'_2) \in J_k$ with $x = (\varphi_S(s_1,s'_2), \psi_S(s_2,s'_1))$.

Further for $j=1,2$ there exist distinct

$$\{((s_j,t_{j1}),(s'_j,t_{j1})),\ldots,((s_j,t_{jk}),(s'_j,t'_{jk}))\} \subset J.$$

By hypothesis on $T$ we have

$$k \leqq |\{\varphi_T(t_{1i},t'_{2j}), \psi_T(t_{2j},t'_{1i})) : 1 \leqq i,j \leqq k\},$$

and hence $x \in (\varphi\psi(J))_k$ giving $\varphi_S \psi_S(J_k) \subset (\varphi\psi(J))_k$.

Finally,

$$|J| = \sum |J_k| \leqq \sum |\varphi_S \psi_S(J_k)| \leqq \sum |(\varphi\psi(J))_k| = |\varphi\psi(J)|.$$

This establishes (4.3) and completes the proof.

## 5. Proof of Theorem 4

Let $L$ be a finite distributive lattice, which always can be viewed as sublattice of $\{0,1\}^n$. Let $\alpha$, $\beta$, $\gamma$, $\delta$: $L \to \mathbb{R}$ with $\delta$ a monotone *function* ($\delta(a) \leqq \delta(b)$ for $a \leqq b$) satisfy

$$\alpha(a)\,\beta(b) \leqq \gamma(a \triangle b)\,\delta(a \wedge b) \qquad \text{for all } a,\, b \in L$$

then we have to show that

$$\alpha(A)\,\beta(B) \leqq \gamma(A \triangle B)\,\delta(A \wedge B) \qquad \text{for all } A,\, B \subset L.$$

We make use of the elementary

**Lemma.** *If for* $s,t \in L$ $C(s,t) = \{(a,b) : a,b \in L; a \triangle b = s, a \wedge b = t\}$ *then* $(a,b)$, $(a',b') \in C(s,t)$ *implies* $(a \vee a', b \wedge b') \in C(s,t)$.

Clearly, the $C(s,t)$ partition $L \times L$:

$$L \times L = \sum_{(s,t) \in L \times L} C(s,t).$$

Since $\vee$, $\wedge$ are symmetric operations

$$C^*(s,t)=\{a:(a,b)\in C(s,t)\}$$

equals

$$^*C(s,t)=\{b:(a,b)\in C(s,t)\}$$

and by the Lemma this set is closed under $\vee$, $\wedge$, that is a sublattice of $L$.

Now $a\triangle b=a\triangle b'$ iff $b=b'$ and therefore $C^*(s,t)\cap C^*(s,t')=\varnothing$. For all $s,t\in L$ define $D(s,t)=\{(s,d):\ d\in C^*(s,t)\}$. Then clearly $(s,t)\neq(s',t')$ implies $D(s,t)\cap D(s',t')=\varnothing$. Since $|D(s,t)|=|C^*(s,t)|=|C(s,t)|$ and the $C(s,t)$ partition $L\times L$, the $D(s,t)$ in fact partition $L\times L$ also.

Assume for the moment that

$$(5.1)\quad |E_{st}|\leqq|F_{st}|\quad \text{for all } s,t\in L \text{ and } A,B\subset L,$$

where $E_{st}=\{A\times B\}\cap C(s,t)$ and $F_{st}=\{(A\triangle B)\times(A\wedge B)\}\cap D(s,t)$. In other words $E_{st}$ is the set of all $(a,b)\in A\times B$ with $a\triangle b=s$ and $a\wedge b=t$ while $F_{st}$ is the set of all $(c,d)\in(A\triangle B)\times(A\wedge B)$ with $c=s$ and $d\in C^*(s,t)$. Then

$$\begin{aligned}\alpha(A)\,\beta(B)&=\sum_{(a,b)\in A\times B}\alpha(a)\,\beta(b)=\sum_{s,t\in L}\sum_{(a,b)\in E_{st}}\alpha(a)\,\beta(b)\\&\leqq\sum_{s,t\in L}\sum_{(a,b)\in E_{st}}\gamma(a\triangle b)\,\delta(a\wedge b)\\&=\sum_{s,t\in L}\sum_{(a,b)\in E_{st}}\gamma(s)\,\delta(t)\\&\leqq\sum_{s,t\in L}\sum_{(c,d)\in F_{st}}\gamma(s)\,\delta(t)\\&\leqq\sum_{s,t\in L}\sum_{(c,d)\in F_{st}}\gamma(c)\,\delta(d)\\&=\sum_{(c,d)\in(A\triangle B)\times(A\wedge B)}\gamma(c)\,\delta(d)=\gamma(A\triangle B)\,\delta(A\wedge B).\end{aligned}$$

Now $E_{st}$ is of the form $E=\{(e_1,\pi e_1),\dots,(e_k,\pi e_k)\}$ where $\pi$ *denotes complementation in the sublattice* $C^*(s,t)$, so

$$\begin{aligned}&|F_{st}|=|\{(s,d):d=e_i\wedge\pi e_j;\,1\leqq i,j\leqq k\}|\\&\quad\{e_i-e_j:\,1\leqq i,j\leqq k\}\subset C^*(s,t)=|\{e_i-e_j:\,1\leqq i,j\leqq k\}|,\end{aligned}$$

hence $|E_{st}|\leqq|F_{st}|$ by the Marica-Schönheim Theorem ([9]) and our proof is complete.

## 6. Proof of Theorem 5

The proof is *literally* the same as the proof of Theorem 1 with *one exception:* Since $(\varphi_T,\psi_T)$ is not $\mathfrak{M}$-expansive we have to show how (3.12) follows from (3.11).

We wish to show that

$$(6.1) \quad \alpha_T(T)\,\beta_T(T) \leqq \gamma_T(\varphi_T(T,T))\,\delta_T(\psi_T(T,T)) \\ = \gamma_T(T)\,\delta_T(T).$$

The equality holds here because $(\varphi_T, \psi_T) = (\triangle, \wedge)$ is expansive. Let

$$T_A = \{t : (s_1, t) \in A\}$$
$$T_B = \{t : (s_2, t) \in B\}$$
$$T_\varphi = \{t : (s_3, t) \in \varphi(A, B)\}$$
$$T_\psi = \{t : (s_4, t) \in \psi(A, B)\}.$$

Then

$$T_A \triangle T_B = \{t_1 \triangle t_2 : t_1 \in T_A, t_2 \in T_B\}$$
$$= \{t_1 \triangle t_2 : (s_1, t_1) \in A, (s_2, t_2) \in B\}$$
$$\subset \{t : (s_3, t) \in \varphi(A, B)\} = T_\varphi,$$

$$T_A \cap T_B = \{t_1 \cap t_2 : t_1 \in T_A, t_2 \in T_B\}$$
$$= \{t_1 \cap t_2 : (s_1, t_1) \in A, (s_2, t_2) \in B\}$$
$$\subset \{t : (s_4, t) \in \psi(A, B)\} = T_\psi.$$

Next define $\alpha_{(T)}, \beta_{(T)}, \gamma_{(T)}, \delta_{(T)} : T \to \mathbb{R}_+$ by

$$\alpha_{(T)}(t) = \alpha(s_1, t)$$
$$\beta_{(T)}(t) = \beta(s_2, t)$$
$$\gamma_{(T)}(t) = \gamma(s_3, t)$$
$$\delta_{(T)}(t) = \delta(s_4, t)$$

so $\delta_{(T)}$ increases with $t$. Then

$$\alpha_{(T)}(T_A) = \sum_{t \in T_A} \alpha_{(T)}(t) = \sum_{t \in T_A} \alpha(s_1, t) = \sum_{\substack{t \in T \\ (s_1, t) \in A}} \alpha(s_1, t) = \alpha_T(T)$$

$$\beta_{(T)}(T_B) = \sum_{t \in T_B} \beta_{(T)}(t) = \sum_{t \in T_B} \beta(s_2, t) = \sum_{\substack{t \in T \\ (s_2, t) \in B}} \beta(s_2, t) = \beta_T(T)$$

$$\gamma_{(T)}(T_\varphi) = \sum_{t \in T_\varphi} \gamma_{(T)}(t) = \sum_{t \in T_\varphi} \gamma(s_3, t) = \sum_{\substack{t \in T \\ (s_3, t) \in \varphi(A, B)}} \gamma(s_3, t) = \gamma_T(T)$$

$$\delta_{(T)}(T_\psi) = \sum_{t \in T_\psi} \delta_{(T)}(t) = \sum_{t \in T_\psi} \delta(s_4, t) = \sum_{\substack{t \in T \\ (s_4, t) \in \psi(A, B)}} \delta(s_4, t) = \delta_T(T).$$

Assume for the moment that

$$(6.2) \quad \alpha_{(T)}(t_1)\,\beta_{(T)}(t_2) \leqq \gamma_{(T)}(t_1 \triangle t_2)\,\delta_{(T)}(t_1 \wedge t_2) \qquad \text{for all } t_1, t_2 \in T.$$

Then by Theorem 4

$$\alpha_{(T)}(T_A)\,\beta_{(T)}(T_B) \leqq \gamma_{(T)}(T_A \triangle T_B)\,\delta_{(T)}(T_A \wedge T_B) \leqq \gamma_{(T)}(T_\varphi)\,\delta_{(T)}(T_\psi)$$

which is (6.1).

But (6.2) simply says

$$\begin{aligned}\alpha(s_1, t_1)\,\beta(s_2, t_2) &\leqq \gamma(\varphi_S(s_1, s_2), t_1 \triangle t_2)\,\delta(\psi_S(s_1, s_2), t_1 \cap t_2)\\ &= \gamma(s_3, t_1 \triangle t_2)\,\delta(s_4, t_1 \cap t_2)\end{aligned}$$

and this is true by our original hypothesis.

## 7. Proof of Theorem 6

Recall Definitions (1.4) and (1.5) for the direct product and for square functions.

(a) $\Rightarrow$ (b)

The key idea in this proof is to express the square functions as direct products of $\mathfrak{M}$-expansive pairs.

For $z=(a,b)\in S^2$ and $Z\subset S^2$ define

(7.1) $\quad z^*=(b,a), \qquad Z^*=\{(b,a):(a,b)\in Z\}$

and for $\eta\colon S^2 \to \mathbb{R}$ define

(7.2) $\quad \eta^*(b,a)=\eta(a,b) \qquad$ for all $(a,b)\in S^2$.

Put $\varphi_S=\varphi$, $\psi_S=\psi$, $\varphi_T=\psi^*$, $\psi_T=\varphi^*$.

Then for all $X$, $Y\subset S^2$ we have the identities

(7.3) $\quad \varphi^2(X,Y)=\varphi_{ST}(X,Y^*)$

(7.4) $\quad \psi^2(X,Y)=(\psi_{ST}(X,Y^*))^*$.

This can be justified as follows:

for $x=(a,b), \quad y=(c,d)\in S^2$

$$\begin{aligned}\varphi^2(x,y)=\varphi^2((a,b),(c,d)) &= (\varphi(a,d), \psi(c,b))\\ &= (\varphi_S(a,d), \varphi_T(b,c))\\ &= \varphi_{ST}((a,b),(d,c)) = \varphi_{ST}(x,y^*)\end{aligned}$$

and

$$\begin{aligned}\psi^2(x,y)=\psi^2((a,b),(c,d)) &= (\varphi(c,b), \psi(a,d))\\ &= (\psi_T(b,c), \psi_S(a,d)) = (\psi_S(a,d), \psi_T(b,c))^*\\ &= (\psi_{ST}((a,b),(d,c)))^* = (\psi_{ST}(x,y^*)).\end{aligned}$$

By hypothesis

(7.5) $\alpha(x)\,\beta(y)\leqq\gamma(\varphi^2(x,y))\,\delta(\psi^2(x,y))$ for all singletons $x,y\in S\times S$.

This is equivalent to

(7.6) $\alpha(x)\,\beta^*(y^*)\leqq\gamma(\varphi_{ST}(x,y^*))\,\delta((\psi_{ST}(x,y^*))^*)$ for all $x,y\in S\times S$

and with $z=y^*$ equivalent to

(7.7) $\alpha(x)\,\beta^*(z)\leqq\gamma(\varphi_{ST}(x,z))\,\delta^*(\psi_{ST}(x,z))$ for all $x,z\in S\times S$.

Since $(\varphi,\psi)$ is $\mathfrak{M}$-expansive, by the definition of $\varphi_{ST}$, $\psi_{ST}$ and Theorem 1 also $(\varphi_{ST},\psi_{ST})$ is $\mathfrak{M}$-expansive. Therefore (7.7), and hence also (7.6) and (7.5) hold for sets $X$, $Y\subset S^2$.

(b) $\Rightarrow$ (c)
Given $\alpha,\beta,\dots,\omega$ satisfying (1.15), define $\alpha^2$, $\beta^2$, $\lambda^2$, $\mu^2$: $S\times S\to\mathbb{R}$ by

$$\alpha^2(a,b)=\alpha(a)\,\beta(b)\qquad \beta^2(a,b)=\gamma(a)\,\delta(b)$$
$$\lambda^2(a,b)=\lambda(a)\,\mu(b)\qquad \mu^2(a,b)=\nu(a)\,\omega(b).$$

Then (1.15) says

$$\begin{aligned}\alpha^2(a,b)\,\beta^2(c,d)&\leqq\lambda^2(\varphi(a,d),\psi(c,b))\,\mu^2(\varphi(c,b),\psi(a,d))\\&=\lambda^2(\varphi^2((a,b),(c,d)))\,\mu^2(\psi^2((a,b),(c,d))).\end{aligned}$$

By hypothesis $(\varphi^2,\psi^2)$ is $\mathfrak{M}$-expansive, so for all $E$, $F\subset S\times S$ we have

$$\alpha^2(E)\,\beta^2(F)\leqq\lambda^2(\varphi^2(E,F))\,\mu^2(\psi^2(E,F))\quad\text{or}\quad \sum_E\sum_F\leqq\sum_{EF}\sum_{FE}.$$

(c) $\Rightarrow$ (a)
We know already that $\mathfrak{M}$-explosive implies $\mathfrak{M}$-expansive.

(a′) $\Rightarrow$ (b′) This is the same as the proof for (a) $\Rightarrow$ (b) except that now every weight is the unit weight and we use now the hypothesis $((\varphi,\varphi),(\psi,\psi))$ is expansive instead of Theorem 1.

(b′) $\Rightarrow$ (c′) Specialize the proof of (b) $\Rightarrow$ (c) to unit weights.

## 8. Remarks on Contractions

We say $\varphi$, $\psi$: $S\times S\to S$ is *contractive* if $\varphi$ and $\psi$ are surjective and if

(8.1) $|\varphi(A,B)|\,|\psi(A,B)|\leqq|A|\,|B|$ for all $A,B\subset S$.

Further let $\mathfrak{M}$-contractive be defined in the obvious way.

*Remark 1.* One easily checks in Table II that in the Boolean case $S=\{0,1\}$ only $(x,y)$ and the 7 equivalent to it $((\bar{x},y),(x,\bar{y}),(\bar{x},\bar{y}),(y,x),\dots)$ are contractive.

*Remark* 2. Some information about the structure of contractive pairs is readily obtained. If for $a, b, c \in S$, $b \neq c$, $\varphi(a,b)=\varphi(a,c)$ then necessarily $\psi(a,b)=\psi(a,s)$ for all $s \in S$. It follows that for every row of the table for $(\varphi, \psi)$ corresponding to Table II either $\varphi$ is constant when we say "top" or $\psi$ is constant when we say "bot". The same applies to columns.

*Type 1*. Every row for instance has "bot". Then no column has "bot" for otherwise $\psi$ would be constant. In this case *equality* holds in (8.1). The pair is both $\mathfrak{M}$-contractive and $\mathfrak{M}$-expansive.

*Type 2*. Every row and every column has both a "top" and a "bot". If $|S|=2$ then it can be shown that those pairs are all expansive *and* contractive. Therefore, since proper expansion occurs for $n>1$, the direct product of contractive pairs is in general *not* contractive. If $|S|>2$ we get at least one strict inequality in (8.1).

*Remark* 3. In case $|S| \geqq 2$ a $(\varphi, \psi)$ of Type 2 is *never* $\mathfrak{M}$-contractive. This can be seen as follows. Let $t$ = common "top" value, $b$ = common "bot" value.

Since $(\varphi, \psi)$ is surjective there exist $b_0 \neq b$, $t_0 \neq t$ and part of their table is

| | top | bot | |
|---|---|---|---|
| $b_2$ | $t$ / $b_0$ | $t$ / $b$ | top |
| $b_1$ | $t$ / $b$ | $t_0$ / $b$ | bot |
| | $a_1$ | $a_2$ | |

Choose

$$\gamma(a)=\begin{cases}\frac{1}{3} & \text{if } a=t\\ \frac{1}{2} & \text{if } a=t_0\\ 0 & \text{otherwise}\end{cases} \qquad \delta(a)=\begin{cases}3 & \text{if } a=b_0\\ 2 & \text{if } a=b\\ 0 & \text{otherwise}\end{cases}$$

$$\alpha(a_1)=\alpha(a_2)=\beta(b_1)=\beta(b_2)=1, \qquad \alpha(\;)=\beta(\;)=7 \quad \text{elsewhere.}$$

Then $\gamma(\varphi(a,b))\,\delta(\psi(a,b)) \leqq \frac{1}{2}\cdot 3 \leqq 1\cdot 7 \leqq \alpha(a)\,\beta(b)$.

However if $A=\{a_1, a_2\}$, $B=\{b_1, b_2\}$ then

$$\tfrac{25}{6}=(\tfrac{1}{3}+\tfrac{1}{2})(2+3)=\gamma(\varphi(A,B))\,\delta(\psi(A,B)) \nleqq 4=2\cdot 2=\alpha(A)\,\beta(B).$$

## 9. Consequences of the 4-weight Inequality of [1]

We first restate the inequality.

Let $L \subset \{0,1\}^n$ be a sublattice of $\{0,1\}^n$ and let $\alpha$, $\beta$, $\gamma$, $\delta$: $L \to \mathbb{R}$, then

(9.1) $\quad \alpha(a)\,\beta(b) \leqq \gamma(a \cup b)\,\delta(a \cap b) \quad$ for all $a, b \in L$

implies

(9.2) $\quad \alpha(A)\,\beta(B) \leqq \gamma(A \vee B)\,\delta(A \wedge B) \quad$ for all $A, B \subset L$.

If $\alpha=\beta=\gamma=\delta=\mathbb{1}$ one gets

I. Daykin [3]

(9.3) $|A|\,|B| \leqq |A \vee B|\,|A \wedge B|$ *for all* $A, B \subset L$.

This inequality implies several known inequalities. We list here the following ones

a) Marica-Schönheim [9]

(9.4) $|A| \leqq |A \setminus A|$ *for all* $A \subset L$.

*Proof.*

$$|A|\,|B| = |A|\,|\bar{B}| \leqq |A \vee \bar{B}|\,|A \wedge \bar{B}|$$
$$= |\overline{A \vee \bar{B}}|\,|A \wedge \bar{B}| = |\bar{A} \wedge B|\,|A \wedge \bar{B}| = |A \setminus B|\,|B \setminus A|.$$

Choose now $B = A$.

b) Daykin, Kleitman, West [5]

(9.5) $|A|\,|B| \leqq |A \vee B|\,|L|$ *for all* $A, B \subset L$.

*Proof.* Clearly $|A \wedge B| \leqq |L|$.

c) Kleitman [8]

(9.6) $|U \cap D|\,|L| \leqq |U|\,|D|$ *for $U$ an upset and $D$ a downset of $L$.*

*Proof.* Put $A = U \cap D$ and notice that $L \vee A \subset U$, $L \wedge A \subset D$. Then

$$|A|\,|L| \leqq |L \vee A|\,|L \wedge A| \leqq |U|\,|D|.$$

d) Seymour [11]

(9.7) $|U_1|\,|U_2| \leqq |U_1 \cap U_2|\,|L|$ *for upsets* $U_1, U_2 \subset L$.

*Proof.* Notice that $U_1 \cap U_2 = U_1 \vee U_2$.

II. Chebychev

*Let* $\alpha_0, \alpha_1, \ldots, \alpha_n;\ \beta_0, \ldots, \beta_n \in \mathbb{R}_+$.
*Put* $\gamma_k = \max\{\alpha_i \beta_k, \alpha_k \beta_i : 0 \leqq i \leqq k\}$, $0 \leqq k \leqq n$, *then*

(9.8) $(\sum \alpha_i)(\sum \beta_j) \leqq (n+1) \sum \gamma_k$.

*In particular if* $0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_n$ *and* $0 \leqq \beta_1 \leqq \ldots \leqq \beta_n$, *then* $\gamma_k = \alpha_k \beta_k$ *and we get Chebychev's inequality*

(9.9) $(\sum \alpha_i)(\sum \beta_j) \leqq (n+1) \sum_{k=0}^{n} \alpha_k \beta_k$.

*Proof.* Let $\alpha, \beta$: $\{0,1\}^n \to \mathbb{R}_+$ and define

$$\gamma(c) = \max_{c=a\cup b} \alpha(a)\,\beta(b) \quad \text{for all } c \in \{0,1\}^n.$$

Then

$$\alpha(a)\,\beta(b) \leqq \gamma(a \cup b) \cdot 1, \quad \delta = \mathbb{1}, \quad \text{and hence}$$
$$\alpha(A)\,\beta(B) \leqq \gamma(A \vee B)\,|A \wedge B|.$$

Choose now $\alpha, \beta$: $\{0,1\}^n \to \mathbb{R}_+$ such that

$$\alpha(\{1,2,\ldots,k\}) = \alpha_k, \quad \beta(\{1,2,\ldots,k\}) = \beta_k, \quad 0 \leqq k \leqq n, \quad \text{and}$$

$$\alpha = \beta = 0 \quad \text{otherwise.}$$

For $A = B = \{\{1,2,\ldots,k\} : 0 \leqq k \leqq n\}$ $A \vee B = A \wedge B = A$, $|A| = n+1$, and therefore $\alpha(A) = \sum \alpha_k$, $\beta(B) = \sum \beta_k$, and $(\sum \alpha_k)(\sum \beta_k) \leqq (n+1) \sum \gamma_k$.

III. Holley [7]

*If* $\alpha, \beta$: $\{0,1\}^n \to \mathbb{R}_+$ *and*

(9.10) $\alpha(a)\,\beta(b) \leqq \alpha(a \cup b)\,\beta(a \cap b)$ *for all* $a, b \in \{0,1\}^n$

*then for an upset* $U$ *and* $L = \{0,1\}^n$

(9.11) $\alpha(L)\,\beta(U) \leqq \alpha(U)\,\beta(L)$.

*Proof.* Choose $\gamma = \alpha$, $\delta = \beta$, $A = L$, $B = U$. Then $L \vee U = U$, $L \wedge U = L$ and hence the inequality.

More generally, for a monotone function $f$: $\{0,1\}^n \to \mathbb{R}_+$ rather than just the characteristic function of an up-set, Holley's inequality says that under hypothesis (9.10) one has

(9.12) $\alpha(L)(\sum_{p \in L} f(p)\,\beta(p)) \leqq (\sum_p f(p)\,\alpha(p))\,\beta(L)$.

This follows immediately from (9.11) by writing $f$ as

$$f = \sum \lambda_i I_{u_i} \quad \text{with } \lambda_i \geqq 0 \quad \text{suitable.}$$

IV. Fortuin, Kasteleyn, Ginibre [6]

*Suppose that for* $\alpha$: $\{0,1\}^n \to \mathbb{R}_+$

(9.13) $\alpha(a)\,\alpha(b) \leqq \alpha(a \cup b)\,\alpha(a \cap b)$ for all $a, b \in L = \{0,1\}^n$

*then for two up-functions* $f$, $g$

(9.14) $(\sum_{p \in L} \alpha(p)\,f(p))(\sum_{p \in L} \alpha(p)\,g(p)) \leqq (\sum_{p \in L} \alpha(p)\,f(p)\,g(p))(\sum_{p \in L} \alpha(p))$.

*In particular if* $U$, $V$ *are up-sets* $f = I_U$ *and* $g = I_V$, *then* (9.14) *says that*

(9.15) $\quad \alpha(U)\,\alpha(V) \leqq \alpha(U \wedge V)\,\alpha(L)$.

*Proof.* Since $U \cap V = U \vee V$, (9.15) follows immediately from our inequality. Notice also that we actually get the sharper estimate

$$\alpha(U)\,\alpha(V) \leqq \alpha(U \vee V)\,\alpha(U \wedge V).$$

The derivation of (9.14) from (9.15) is standard, one just writes $f$ and $g$ as

$$f(p) = \sum_i \lambda_i I_{U_i}(p), \qquad g(p) = \sum_j \mu_j I_{V_j}(p)$$

and calculates the expressions in (9.14).

We suggest that extensions of our results be found for the case of non-discrete sets $S$ (in the spirit of Preston's generalization of Holley's inequality [10]).

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