Z. Phys. D – Atoms, Molecules and Clusters 8, 315–327 (1988)

Atoms, Molecules
Zeitschrift für Physik D and Clusters

Continuum solutions of the Klein-Gordon equation

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Received 19 October 1987; final version 26 November 1987

We construct explicit solutions of the Klein-Gordon equation for continuum states. The role of the energy in the single-particle Klein-Gordon theory is elucidated. Special emphasis is laid on the determination of resonance states in the continuum for overcritical potentials. As examples for long-range interactions we depict solutions for the Coulomb potential of a point-like nucleus as well as an extended nucleus. The square-well potential and the exponential potential are treated to exemplify peculiarities of short-range interactions. We also derive continuum solutions for a scalar interaction of square-well type. Finally we discuss the behaviour of a spin-0 particle in an external homogeneous magnetic field.

PACS: 11.10.Q; 25.80H

1. Introduction

In this paper we examine stationary solutions of the Klein-Gordon equation for energies $|E|>mc^2$. We investigate the behaviour of unbound spin-0-particles in various long- and short-range potentials, respectively [1]. In particular we explore overcritical potentials in which the binding energy E_b of the strongest bound particle exceeds the pair-production threshold of twice the rest mass, i.e. $|E_b|>2mc^2$. The appearance of the bound state as a resonance in the negative frequency continuum is verified. This striking phenomenon exhibits a close analogy to the spontaneous positron production in overcritical Coulomb fields [2], which represents the underlying motivation for our studies. The continuum solutions for a nonsingular Coulomb potential may also be of considerable significance for the scattering of pions on finite-size nuclei. In addition we display distortion effects of spin-0-particles and -antiparticles in short-range potentials.

Our paper is organized as follows: In the premises we briefly review the standard representation and the Schrödinger representation. Furthermore we elaborate the difference between the separation constant describing the time-dependence of the wave function and the energy being defined as the volume integral over the T^{00}-component of the energy-momentum tensor. This quantity deserves special consideration within the framework of a relativistic wave equation for spin-0-particles. The next section deals with explicit solutions for a Coulomb potential. As examples for short-range potentials we treat the square-well potential and the continuous exponential potential. For the latter we restrict the evaluations to s-waves only. To demonstrate the consequences of a different type of interactions we study the behaviour of Klein-Gordon particles in a square-well potential coupled to the square of the rest mass in the differential equation. Finally we derive the stationary states of a Klein-Gordon particle in a homogeneous magnetic field.

2. Premises

2.1. Standard representation

The Klein-Gordon equation describes particles with spin zero. Taking electromagnetic interactions into account it is represented by

$$\left(\hat{p}^{\mu}-\frac{q}{c}A^{\mu}\right)\left(\hat{p}_{\mu}-\frac{q}{c}A_{\mu}\right)\psi=m_0^2c^2\psi, \tag{1}$$

where q is the charge and m_0 the rest mass of the particle and $\hat{p}^\mu = i\hbar\partial^\mu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\mathbf{\nabla}\right)$, $A^\mu = (A_0, \mathbf{A})$.

We now employ natural units ($\hbar = m_0 = c = 1$). Then (1) may be written more explicitly as

$$\left(i\frac{\partial}{\partial t} - qA_0\right)^2 \psi = ((-i\mathbf{\nabla} - q\mathbf{A})^2 + 1)\psi. \tag{2}$$

In the subsequent sections we neglect the vector potential and consider only spherical symmetric and time independent potentials

$$\mathbf{A}(\mathbf{r}, t) = 0, \quad qA_0(\mathbf{r}, t) = V(r). \tag{3}$$

The separation ansatz

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt} \tag{4}$$

leads to the stationary wave equation

$$(\mathbf{\nabla}^2 + \{E - V(r)\}^2 - 1)\phi(\mathbf{r}) = 0. \tag{5}$$

Except for plane wave solutions as well as for solutions for a homogeneous magnetic field we will separate the angular part of ϕ

$$\phi(\mathbf{r}) = v(r)\, Y_{lm}(\Theta, \Phi). \tag{6}$$

The radial component $v(r)$ satisfies the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right)v(r) = 0 \tag{7}$$

with

$$k^2 = \{E - V(r)\}^2 - 1. \tag{8}$$

Taking

$$v(r) = \frac{u(r)}{r} \tag{9}$$

we obtain

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right)u(r) = 0. \tag{10}$$

With the replacement $y_1 = u$ and $y_2 = du/dr$ the second-order differential equation (10) is easily transformed into two first-order coupled differential equations [1].

The four-current of a Klein-Gordon particle is given by

$$j^\mu = \frac{q}{2}(\psi^*\{i\partial^\mu - qA^\mu\}\psi - \psi\{i\partial^\mu + qA^\mu\}\psi^*). \tag{11}$$

The prefactor is chosen to yield the correct nonrelativistic limit [3, 4]. With $j^\mu = (\rho, \mathbf{j})$ and by using (3) and (4) this may be written as

$$\rho(\mathbf{r}) = q\{E - V(r)\}\phi^*\phi, \tag{12}$$

$$\mathbf{j}(\mathbf{r}) = q\frac{i}{2}(\phi\mathbf{\nabla}\phi^* - \phi^*\mathbf{\nabla}\phi). \tag{13}$$

2.2. *Schrödinger representation*

The Schrödinger representation transforms the second-order differential equation (2) in the time variable into two first-order equations. In this representation the Klein-Gordon equation reads [3, 5]

$$i\frac{\partial}{\partial t}\Psi = \hat{H}(q)\Psi \tag{14}$$

with

$$\hat{H}(q) = (\tau_3 + i\tau_2)\,\frac{1}{2}\left(\frac{1}{i}\mathbf{\nabla} - q\mathbf{A}\right)^2 + V + \tau_3 \tag{15}$$

and the Pauli-matrices in the standard representation [3]. The individual components of the two-component wave function

$$\Psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix} \tag{16}$$

are related to ψ from (2) via

$$\phi + \chi = \psi, \tag{17}$$

$$\phi - \chi = \left(i\frac{\partial}{\partial t} - V\right)\psi. \tag{18}$$

For the charge density and the current density one obtains

$$\rho = q\Psi^\dagger\tau_3\Psi = q(\phi^*\phi - \chi^*\chi), \tag{19}$$

$$\mathbf{j} = \frac{q}{2}\left(\Psi^\dagger\tau_3(\tau_3 + i\tau_2)\left(\frac{1}{i}\mathbf{\nabla}\Psi - q\mathbf{A}\Psi\right) - \left(\frac{1}{i}\mathbf{\nabla}\Psi^\dagger + q\mathbf{A}\Psi^\dagger\right)\tau_3(\tau_3 + i\tau_2)\Psi\right). \tag{20}$$

Using the potential (3) and the separation ansatz

$$\Psi(\mathbf{r}, t) = \Psi_0(\mathbf{r})e^{-iEt} = \begin{pmatrix}\phi_0(\mathbf{r})\\ \chi_0(\mathbf{r})\end{pmatrix}e^{-iEt} \tag{21}$$

we get the stationary equation

$$\hat{H}(q)\Psi_0 = E\Psi_0. \tag{22}$$

Finally we notice that if Ψ satisfies (14) the wave function

$$\Psi_c = \tau_1\Psi^* = \begin{pmatrix}\chi^*\\ \phi^*\end{pmatrix} \tag{23}$$

satisfies the equation

$$i\frac{\partial}{\partial t}\Psi_c = \hat{H}(-q)\Psi_c. \tag{24}$$

This will be discussed in more detail in Sect. 2.4.

2.3. Normalization and energy

At this point of the discussion we have to consider the relationship between the separation constant E for the time-dependence of the wave function, the normalization of the wave function and the energy p^0. We will also establish a relation between momentum and current of a Klein-Gordon particle.

In order to calculate energy and momentum of a Klein-Gordon particle we will follow the route of classical field theory. The Lagrange-density $\mathscr{L}_c$ of the combined Maxwell- and Klein-Gordon field is given by [6]

$$\mathscr{L}_c = \tfrac{1}{2}\{(-i\partial^\mu\psi^* - qA^\mu\psi^*)(i\partial_\mu\psi - qA_\mu\psi) - \psi^*\psi\} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{25}$$

where $F^{\mu\nu}$ is the field-strength tensor of the electromagnetic field. Since we are not interested in energy and momentum of the free electromagnetic field we consider only

$$\mathscr{L} = \mathscr{L}_c + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{26}$$

The energy-momentum tensor $T^{\mu\nu}$ is defined by [6]

$$T^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^*)}\partial^\nu\psi^* + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi - g^{\mu\nu}\mathscr{L}. \tag{27}$$

With the separation ansatz (4) and the potentials (3) we obtain for the energy-density

$$T^{00} = E(E-V)\phi^*\phi + \tfrac{1}{2}(\phi^*\phi + \nabla\phi^*\cdot\nabla\phi - (E-V)^2\phi^*\phi) \tag{28}$$

and for the momentum-density ($k=1, 2, 3$)

$$T^{0k} = \frac{i}{2}(E-V)(\phi^*\partial^k\phi - \phi\partial^k\phi^*). \tag{29}$$

For vanishing external potential T^{00} is always positive independent of E. Integration of expression (28) over all space yields

$$p^0 = \int T^{00}\,\mathrm{d}^3r = E\int(E-V)\phi^*\phi\,\mathrm{d}^3r + \frac{1}{2}\int_s \phi^*\nabla\phi\,\mathrm{d}\mathbf{f}. \tag{30}$$

To derive (30) Greens first identity and the Klein-Gordon equation were used. The remaining surface integral vanishes for localized wave functions. Let us now define

$$\varrho = \frac{\rho}{q}. \tag{31}$$

We will call ϱ the density. Then according to (12) the energy may be written as

$$p^0 = E\int\varrho(\mathbf{r})\,\mathrm{d}^3r. \tag{32}$$

In conclusion for strong external potentials the energy p^0 may become negative for $E<0$ and for states which are normalized to $+1$. However, bound states, which are normalized to -1 and which enter the bound state gap from the negative frequency continuum with $E<-1$, possess positive energy p^0.

The components of the momentum are obtained by integrating (29) over all space. In absence of magnetic fields we get

$$\mathbf{p} = \frac{1}{q}\int\{E - V(r)\}\mathbf{j}(\mathbf{r})\,\mathrm{d}^3r. \tag{33}$$

Here (13) has been inserted.
Bound state solutions of particles are normalized according the condition [1]

$$\int\rho(\mathbf{r},t)\,\mathrm{d}^3r = \pm q. \tag{34}$$

This normalization is not possible for continuum solutions if the integration domain extends to infinity. So analogous to the Schrödinger theory [7] the normalization is determined via

$$\int -\frac{q}{2}\left\{\psi_\mathbf{p}\left(i\frac{\partial}{\partial t} + V\right)\psi^*_{\mathbf{p}'} + \psi^*_{\mathbf{p}'}\left(-i\frac{\partial}{\partial t} + V\right)\psi_\mathbf{p}\right\}\mathrm{d}^3r = \pm q\delta(\mathbf{p}-\mathbf{p}'). \tag{35}$$

$\psi_\mathbf{p}$ denotes a wave function with fixed momentum $\mathbf{p}$. E.g., with this normalization the plane wave solutions of the free Klein-Gordon equation read ($E_p = +\sqrt{\mathbf{p}^2+1}$)

$$\psi_1(\mathbf{r},t) = \frac{1}{\sqrt{(2\pi)^2 E_p}}\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - E_p t)}, \tag{36}$$

$$\psi_2(\mathbf{r},t) = \frac{1}{\sqrt{(2\pi)^3 E_p}}\,\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} + E_p t)}. \tag{37}$$

But also the complex-conjugate of these two solutions are linear independent solutions of the free Klein-Gordon equation.

Since the spherical harmonics are normalized to one, the ansatz (6) leads to a normalization condition for the radial part $v(r)$

$$\int_0^\infty\{E - V(r)\}v^*_{p'}(r)v_p(r)r^2\,\mathrm{d}r = \pm\delta(p-p') \tag{38}$$

with

$$p = \sqrt{E^2-1}. \tag{39}$$

If we normalize the wave functions not on the momentum-scale, but on the energy-scale, i.e.,

$$\int_0^\infty\{E - V(r)\}v^*_{E'}(r)v_E(r)r^2\,\mathrm{d}r = \pm\delta(E-E'), \tag{40}$$

we employ

$$v_E(r)=\sqrt{\left|\frac{\mathrm{d}p}{\mathrm{d}E}\right|}\,v_p(r)=\sqrt{\frac{|E|}{p}}\,v_p(r). \tag{41}$$

In most cases we will normalize the wave functions on the energy-scale.

2.4. *Particle- and antiparticle-states*

In this section we reexamine the role of the negative frequency solutions of the Klein-Gordon equation. Let us first consider (32) which represents a relation between frequency, density and energy. In the case $|E|>m$ (continuum solutions) we obtain always two solutions, one with frequency $+|E|$ and positive norm

$$\int \varrho\, \mathrm{d}^3r=+1$$

and one with opposite sign of frequency and norm. In order to avoid the well known difficulties with expectation values of continuum solutions we regard here the particles as localized in a finite volume. The energy usually remains positive for unbound particles. The situation differs for particles bound in a strong potential. For example in a square well potential we may obtain solutions with negative frequency and positive norm [1]. It results that the energy in that case is negative. This is not surprising because it states only that the binding energy exceeds the rest mass. Solutions of a second type which appear for a sufficiently strong potential, however, yield a positive energy since they belong to a negative norm. What is the meaning of the second solutions? The usual answer is that they represent antiparticle-states [2, 11, 12]. We want to support this interpretation by some additional arguments.

Given a particle of charge $+q$ in an external field $A^\mu=A^\mu(\mathbf{r})$ the Klein-Gordon equation reads after separation of the time variable

$$(E-V)^2\phi=((-i\nabla-q\mathbf{A})^2+1)\phi. \tag{42}$$

Let us first consider unbound solutions. With the convention $E_p>+1$ we then obtain two equations reflecting the two possible values of $E=\pm E_p$

$$(E_p-V)^2\phi^{(1)}=((-i\nabla-q\mathbf{A})^2+1)\phi^{(1)},$$
$$(-E_p-V)^2\phi^{(2)}=((-i\nabla-q\mathbf{A})^2+1)\phi^{(2)}. \tag{43}$$

Analogous the equations for a particle with charge $-q$ in the same external field are

$$(E_p+V)^2\varphi^{(1)}=((-i\nabla+q\mathbf{A})^2+1)\varphi^{(1)},$$
$$(-E_p+V)^2\varphi^{(2)}=((-i\nabla+q\mathbf{A})^2+1)\varphi^{(2)}. \tag{44}$$

The solutions of these equations obviously have to fulfill the relations $\varphi^{(1)}=\phi^{(2)*}$, $\varphi^{(2)}=\phi^{(1)*}$. Our aim is to show that all expectation values of $\varphi^{(2)}$ are identical to those computed with $\phi^{(1)}$ and analogous for $\varphi^{(1)}$ and $\phi^{(2)}$. Because of the symmetry in the following considerations we restrict ourselves to the latter case.

First we notice that the Lagrangian (26) is invariant against the substitutions $q\to -q$ and $\phi^{(2)}\to\varphi^{(1)}$. We use

$$\psi^{(1)}=\varphi^{(1)}\exp\{-\mathrm{i}E_pt\},\quad \psi^{(2)}=\phi^{(2)}\exp\{\mathrm{i}E_pt\}. \tag{45}$$

This leads to

$$\begin{aligned}\mathscr{L}(-q,\psi^{(1)})&=\tfrac{1}{2}\{(-i\partial^\mu\psi^{(1)*}+qA^\mu\psi^{(1)*})\\&\quad\times(i\partial_\mu\psi^{(1)}+qA_\mu\psi^{(1)})-\psi^{(1)*}\psi^{(1)}\}\\&=\tfrac{1}{2}\{(i\partial_\mu\psi^{(2)}-qA_\mu\psi^{(2)})\\&\quad\times(-i\partial^\mu\psi^{(2)*}-qA^\mu\psi^{(2)*})-\psi^{(2)*}\psi^{(2)}\}\\&=\mathscr{L}(+q,\psi^{(2)}).\end{aligned} \tag{46}$$

In a complete analogous manner it follows $j^\mu(-q,\psi^{(1)})=j^\mu(+q,\psi^{(2)})$, and similarly $T^{\mu\nu}(-q,\psi^{(1)})=T^{\mu\nu}(+q,\psi^{(2)})$. In conclusion energy, momentum, charge and current display the indicated symmetries between particle- and antiparticle-states.

In order to extend this to general expectation values we may derive an interesting relation. Given a hermitian operator $\hat{O}(+q)$ belonging to charge $+q$ and the corresponding operator $\hat{O}(-q)$ belonging to charge $-q$ we consider the expectation value [3, 5] in the Schrödinger representation

$$O(-q,\Phi^{(1)})=\int \mathrm{d}^3r\,\Phi^{(1)\dagger}\tau_3\hat{O}(-q)\Phi^{(1)}. \tag{47}$$

Because of $\tau_3\tau_1=-\tau_1\tau_3$ and $\Phi^{(1)}=\tau_1\Phi^{(2)*}$ we obtain

$$\begin{aligned}O(-q,\Phi^{(1)})&=\int \mathrm{d}^3r\,\Phi^{(2)T}\tau_1\tau_3\hat{O}(-q)\tau_1\Phi^{(2)*}=\\&=-\left(\int \mathrm{d}^3r\,\Phi^{(2)\dagger}\tau_3(\tau_1\hat{O}^*(-q)\tau_1)\Phi^{(2)}\right)^*.\end{aligned} \tag{48}$$

$\Phi^{(2)}$ corresponds to Ψ_c in (23). If the operator now fulfills the relation

$$\tau_1\hat{O}^*(-q)=-\hat{O}(+q)\tau_1 \tag{49}$$

we finally obtain

$$\begin{aligned}O(-q,\Phi^{(1)})&=\left(\int \mathrm{d}^3r\,\Phi^{(2)\dagger}\tau_3\hat{O}(+q)\Phi^{(2)}\right)^*\\&=O^*(+q,\Phi^{(2)}).\end{aligned} \tag{50}$$

So for hermitian operators the expectation values are the same. It is easy to verify that relation (49) is fulfilled by the charge operator qI_2 and the Hamiltonian

$$\tau_1\hat{H}^*(-q)=\tau_1(\tau_3+i\tau_2)\,\frac{1}{2}\left(-\frac{1}{i}\nabla+q\mathbf{A}\right)^2$$
$$+\tau_1(-V)+\tau_1\tau_3=-\hat{H}(+q)\tau_1. \tag{51}$$

So we may demand that meaningful operators in the framework of the Klein-Gordon theory should fulfill relation (49) in addition of being hermitian. Under these circumstances it doesn't make any difference whether we use the second solution of the 'particle-equation' or the first solution of the 'antiparticle-equation'. The sign of the charge is a degree of freedom in the Klein-Gordon theory.

3. The Coulomb potential of a point-like nucleus

The attractive Coulomb potential is given by

$$V(r)=-\frac{Z\alpha}{r}, \tag{52}$$

where Z is the nuclear charge number and $\alpha \simeq \frac{1}{137}$ the fine structure constant. It is known [1–4] that (10) yields bound state solutions only for

$$Z\alpha \leqslant \left(l+\frac{1}{2}\right). \tag{53}$$

The energy eigenvalue of a Klein-Gordon particle bound in potential (52) always remains positive [1–4].

Now we consider the continuum solutions. The physical solution being regular at the origin ($r \to 0$) is given by ($p=\sqrt{E^2-1}$)

$$u(r)=NM_{-i\delta,\mu}(2ipr) \tag{54}$$

with the Whittaker function [8]

$$M_{\kappa,\mu}(z)=e^{-z/2}z^{\frac{1}{2}+\mu}\,{}_1F_1(\tfrac{1}{2}+\mu-\kappa,1+2\mu,z) \tag{55}$$

and

$$\mu=\sqrt{(l+\tfrac{1}{2})^2-(Z\alpha)^2}, \quad \delta=\frac{Z\alpha E}{p}. \tag{56}$$

${}_1F_1(a,b,z)$ denotes the confluent hypergeometric function [8]. N is the normalization constant. Again Z is restricted by (53) in order to yield real energy eigenvalues. Formally there exists a second solution of the Klein-Gordon equation for the Coulomb potential of a point nucleus in which the sign of μ in (54) is reversed. The same problem already appears in the discussion of the bound state spectrum. Here it can be demonstrated, that the solution with negative sign of μ would lead to a divergent value for the integral over the energy density T^{00}. In conclusion the second solution, which displays a stronger singular dependence close to the origin, cannot be accepted. Based on similar arguments we also discard this second solution for continuum states.

Using the way of normalizing wave functions by their asymptotic expansions [7] we can determine the remaining constant N. Normalization according to (40) yields

$$N_E=\frac{1}{\sqrt{2\pi p}}e^{\frac{\pi}{2}\delta}\frac{|\Gamma(\frac{1}{2}+\mu+i\delta)|}{\Gamma(1+2\mu)} \tag{57}$$

and according to (38), $N_p=\sqrt{p/|E|}\,N_E$. The normalization on the energy scale is used in the figures. In the nonrelativistic limit the density ϱ for a Klein-Gordon particle agrees completely with the corresponding expression for a Dirac particle [14]. In Fig. 1 we display the radial density times r^2 for different positive and negative frequencies. All units are natural units, the radial density ϱ is defined via the radial charge density (12) and by (31).

Due to the fact that the given potential is repulsive for an antiparticle the density belonging to a negative frequency becomes very small in the vicinity of the nucleus for energies near the rest mass.

In order to demonstrate the influence of the centrifugal term $l(l+1)/r^2$ we display in Fig. 2 the radial density for a fixed energy but different angular momenta. Figure 3 also indicates that for negative frequencies the radial density becomes positive in the region of the nucleus. This holds true for all negative frequencies and may be deduced from the relation for the radial density

$$\varrho(r)r^2=\left(E+\frac{Z\alpha}{r}\right)|u(r)|^2 \overset{r\to 0}{=} Er^{1+2\mu}+Z\alpha r^{2\mu}. \tag{58}$$

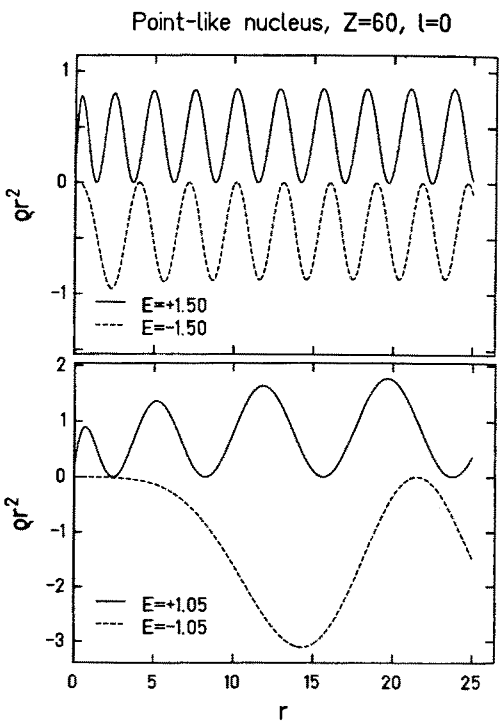

Fig. 1. Radial densities for different frequencies in the continuum of the Klein-Gordon equation for a Coulomb potential. The positive densities belong to particle states, the negative to antiparticle states. Natural units are employed, ϱ is defined by (12) and (31)

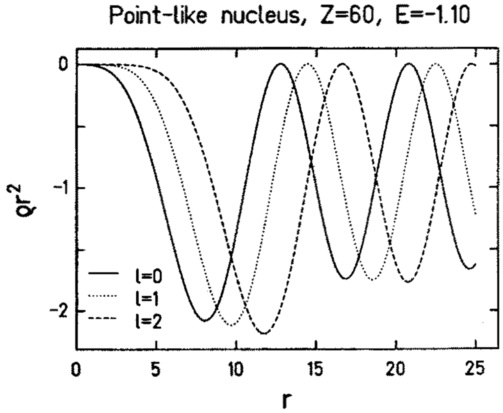

Fig. 2. Radial densities for states with different angular momenta for the Coulomb potential of a point-like nucleus. Continuum states with negative frequencies are considered

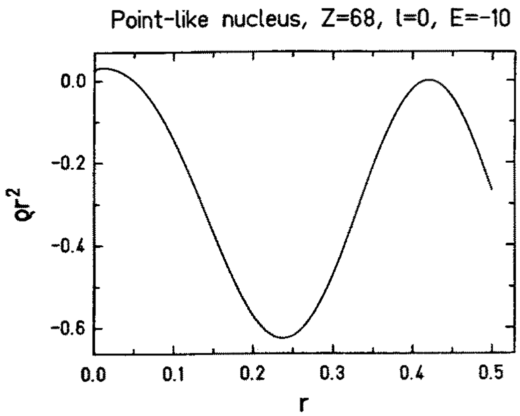

Fig. 3. Radial density of an antiparticle in an s-state of a point-like nucleus with $Z=68$. The values are chosen to demonstrate the positive density close to the origin ($r \to 0$)

4. The potential of a finite-size nucleus

The nucleus is assumed to be a homogeneously charged sphere of radius R. The potential is given by

$$V(r) = -\frac{Z\alpha}{2R}\left(3-\frac{r^2}{R^2}\right) \quad \text{for } r \leqslant R,$$

$$V(r) = -\frac{Z\alpha}{r} \qquad \text{for } r > R. \tag{59}$$

Solving (10) with this potential for the inner region ($r \leqslant R$) yields

$$u_i(r) = D r^{l+1} \mathscr{B}(r) \tag{60}$$

with D as a normalization constant and

$$\mathscr{B}(r) \overset{\text{def}}{=} \sum_{n=0}^{\infty} b_n r^{2n}. \tag{61}$$

The coefficients of $\mathscr{B}(r)$ are given by the general formula

$$b_n = \frac{-Bb_{n-1} + 2ACb_{n-2} - C^2 b_{n-3}}{4nl + 2n(2n+1)} \tag{62}$$

for $n \geqslant 1$ with the convention $b_{-1} = b_{-2} = 0$ and $b_0 = 1$. We have used the abbreviations

$$A = E + \frac{3Z\alpha}{2R}, \quad B = A^2 - 1, \quad C = \frac{Z\alpha}{2R^3}. \tag{63}$$

For the outer region we obtain a similar expression as (54)

$$u_0(r) = N(\cos\eta f_1(r) + \sin\eta f_2(r)) \tag{64}$$

with

$$f_1(r) = (2ip)^{-1/2-\mu} M_{-i\delta,\mu}(2ipr),$$
$$f_2(r) = (2ip)^{-1/2+\mu} M_{-i\delta,-\mu}(2ipr) \tag{65}$$

and with the same meaning of μ, p and δ. But now the solution is not restricted to that part which is regular at $r=0$. N denotes the general normalization constant and η a phase, which will be derived from the matching condition. The constants D, N and η are determined by the conditions

$$(i)\ u_i(R) = u_0(R),$$
$$(ii)\ \frac{du_i}{dr}(R) = \frac{du_0}{dr}(R), \tag{66}$$
$$(iii)\ u(r) \xrightarrow{r\to\infty} \sqrt{\frac{2}{\pi p}} \sin(pr + \phi(r)),$$

in which $\phi(r)$ has the form $\delta \ln r + \phi_0$. Condition (*iii*) is equivalent to normalization according to (40).

In the determination of the constants D, N and η we first consider the case $Z\alpha \leqslant l + \frac{1}{2}$, i.e. μ is real. First we realize that $f_1(r)$ and $f_2(r)$ are real. This can be verified by the definition (65) and by using the Kummer transformation [8] of the confluent hypergeometric function.

According to condition (*i*) we simply obtain

$$D = \frac{N}{R^{l+1}\mathscr{B}(R)} (\cos\eta\, f_1(R) + \sin\eta\, f_2(R)). \tag{67}$$

To fulfill condition (*ii*) we first form the derivative [8]

$$\frac{\partial}{\partial r} M_{-i\delta,\pm\mu}(2ipr) = \frac{1}{r}(ipr + i\delta) M_{-i\delta,\pm\mu}(2ipr)$$
$$+ \frac{1}{r}\left(\frac{1}{2} \pm \mu - i\delta\right) M_{-i\delta+1,\pm\mu}(2ipr). \tag{68}$$

We introduce the abbreviations

$$g(r)=\frac{\mathscr{B}'(r)}{\mathscr{B}(r)}+\frac{l+1}{r}, \tag{69}$$

$$B_1^{\pm}(r)=M_{-i\delta,\,\pm\mu}(2ipr), \tag{70}$$

$$B_2^{\pm}(r)=\frac{1}{r}(ipr+i\delta)M_{-i\delta,\,\pm\mu}(2ipr), \tag{71}$$

$$B_3^{\pm}(r)=\frac{1}{r}\left(\frac{1}{2}\pm\mu-i\delta\right)M_{-i\delta+1,\,\pm\mu}(2ipr). \tag{72}$$

For the logarithmic derivative at $r=R$ it follows

$$g(r)=\frac{\cos\eta(B_2^+(R)+B_3^+(R))+\sin\eta(2ip)^{2\mu}(B_2^-(R)+B_3^-(R))}{\cos\eta B_1^+(R)+\sin\eta(2ip)^{2\mu}B_1^-(R)}. \tag{73}$$

This may be resolved in an elementary manner to yield

$$\tan\eta=-\frac{g(R)B_1^+(R)-B_2^+(R)-B_3^+(R)}{g(R)B_1^-(R)-B_2^-(R)-B_3^-(R)}(2ip)^{-2\mu}. \tag{74}$$

To fulfill condition (*iii*) we consider the asymptotic of $u_o(r)$

$$u_o(r)\overset{r\to\infty}{=}2N\{\cos\eta|\xi_1|\cos(pr+\delta\ln r-\arg\xi_1)$$
$$+\sin\eta|\xi_2|\cos(pr+\delta\ln r-\arg\xi_2)\} \tag{75}$$

with

$$\xi_1=\frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-i\delta)}(-2ip)^{-(\frac{1}{2}+\mu+i\delta)}, \tag{76}$$

$$\xi_2=\frac{\Gamma(1-2\mu)}{\Gamma(\frac{1}{2}-\mu-i\delta)}(-2ip)^{-(\frac{1}{2}-\mu+i\delta)}. \tag{77}$$

Now we identify

$$2N\{\cos\eta|\xi_1|\cos(\arg\xi_1)+\sin\eta|\xi_2|\cos(\arg\xi_2)\}$$
$$=\sqrt{\frac{2}{\pi p}}\sin\phi_0, \tag{78}$$

$$2N\{\cos\eta|\xi_1|\sin(\arg\xi_1)+\sin\eta|\xi_2|\sin(\arg\xi_2)\}$$
$$=\sqrt{\frac{2}{\pi p}}\cos\phi_0, \tag{79}$$

which represents the condition to determine N and ϕ_0. The above relationship may be summarized as

$$N\cos\eta\,\xi_1+N\sin\eta\,\xi_2=\frac{i}{\sqrt{2\pi p}}e^{-i\phi_0}. \tag{80}$$

Finally we obtain

$$\tan\phi_0=\frac{\mathrm{Re}\,\xi_1+\tan\eta\,\mathrm{Re}\,\xi_2}{\mathrm{Im}\,\xi_1+\tan\eta\,\mathrm{Im}\,\xi_2}, \tag{81}$$

$$N=\frac{1}{\sqrt{2\pi p}}[\cos^2\eta|\xi_1|^2+\sin^2\eta|\xi_2|^2$$
$$+2\cos\eta\sin\eta(\mathrm{Re}\,\xi_1\mathrm{Re}\,\xi_2+\mathrm{Im}\,\xi_1\mathrm{Im}\,\xi_2)]^{-1/2}. \tag{82}$$

This yields the desired asymptotic behaviour for $u_o(r)$.

Next we consider the alternative case $Z\alpha>l+\frac{1}{2}$ and we substitute

$$\mu\to i\tilde{\mu}=i\sqrt{(Z\alpha)^2-(l+\tfrac{1}{2})^2}. \tag{83}$$

Now we have to evaluate the corresponding constants D, N and η, which in principle may take on complex values. The quantities indicated by a tilde, e.g. $\tilde{\xi}_1$, are understood to be the same as without tilde, except for the obvious replacement (83). First we notice that

$$f_1^*(r)=f_2(r) \tag{84}$$

with

$$f_1(r)=(2ip)^{-1/2-i\tilde{\mu}}M_{-i\delta,i\tilde{\mu}}(2ipr),$$
$$f_2(r)=(2ip)^{-1/2+i\tilde{\mu}}M_{-i\delta,-i\tilde{\mu}}(2ipr). \tag{85}$$

This leads to the ansatz

$$u_o(r)=N(e^{i\eta}f_1(r)+e^{-i\eta}f_2(r)). \tag{86}$$

Again for D we simply obtain

$$D=\frac{N}{R^{l+1}\mathscr{B}(R)}(e^{i\eta}f_1(R)+e^{-i\eta}f_2(R)). \tag{87}$$

The logarithmic derivative at $r=R$ yields

$$g(R)=\frac{e^{i\eta}f_1'(R)+e^{-i\eta}f_2'(R)}{e^{i\eta}f_1(R)+e^{-i\eta}f_2(R)} \tag{88}$$

with $f_i'(r)=\frac{\partial}{\partial r}f_i(r)$ and $f_1^*=f_2$. This, immediately, leads to

$$e^{-2i\eta}=\frac{f_1'-gf_1}{gf_2-f_2'}=-\frac{f_1'-gf_1}{(f_1'-gf_1)^*}. \tag{89}$$

Again we use the derivative (68) with the substitution (83). We thus get explicitly

$$e^{-2i\eta}=-\frac{g(R)\tilde{B}_1^+(R)-\tilde{B}_2^+(R)-\tilde{B}_3^+(R)}{g(R)\tilde{B}_1^-(R)-\tilde{B}_2^-(R)-\tilde{B}_3^-(R)}(2ip)^{-2i\tilde{\mu}}. \tag{90}$$

The new asymptotic reads

$$u_o(r)\overset{r\to\infty}{=}N\{(e^{i\eta}\tilde{\xi}_1+e^{-i\eta}\tilde{\xi}_2)\,e^{-i(pr+\delta\ln r)}$$
$$+(e^{i\eta}\tilde{\xi}_2^*+e^{-i\eta}\tilde{\xi}_1^*)\,e^{i(pr+\delta\ln r)}\} \tag{91}$$

and finally with condition (*iii*) it results

$$N=\frac{1}{\sqrt{2\pi p}}[|\tilde{\xi}_1|^2+|\tilde{\xi}_2|^2+2\,\mathrm{Re}(e^{-2i\eta}\tilde{\xi}_1^*\tilde{\xi}_2)]^{-1/2}. \quad (92)$$

Since the conditions to determine the constants D, N and η lead to awkward expressions we decided to integrate the radial differential equations for a finite-size nucleus numerically. For this purpose we used the following method. From the series expansion of the radial wave function inside the nucleus we evaluated the fraction $F=u_i(r_0)/u_i'(r_0)$ for a relative small value r_0. Then we set with an arbitrary initial value c, $u_i(r_0)=c$. It follows $u_i'(r_0)=c/F$. With these initial values we integrated the radial equation numerically using the computer program [9] up to a large value of r. The correct normalization of the wave function is then obtained from condition (*iii*). The nuclear radius R can be taken fixed or can be linked with the charge number Z, e.g. via $R=r_0(2.5\,Z)^{1/3}$ with $r_0=1.2$ fm.

In the Klein-Gordon equation we inserted the pion rest mass. For bound states [1] the energy eigenvalue reaches -1 in natural units for $Z_{cr}(1s)=3287$ and $Z_{cr}(2p)=3444$, respectively, if R depends on Z as indicated above. For the fixed value of $R=10$ in natural units the same happens for $Z_{cr}(1s)=1986$ and $Z_{cr}(2p)=2095$ [1].

We investigated the wave functions with R fixed to 10 (in natural units), $l=0$ and $Z=2400$. In order to find the value of E for which the continuum state resembles a bound state we computed the value of ϱr^2 at $r=3$ for several energies (Fig. 4). We found a very sharp peak at the resonance position of $E=-1.4361129191$. If E is greater or smaller than this value by 10^{-10} the value of ϱr^2 changes by an order of magnitude. The situation is demonstrated in Fig. 4.

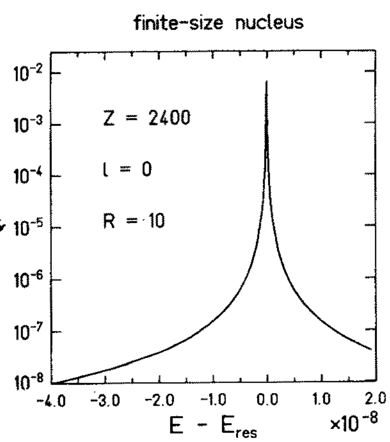

Fig. 4. Radial density at $r=3$ in natural units of a particle in an s-state in the Coulomb potential of a finite-size nucleus with $Z=2400$. For the nuclear charge distribution a homogeneously charged sphere is assumed with radius $R=10$ in natural units. ϱr^2 is plotted on a logarithmic scale versus the quantity $E-E_{res}$ with $E_{res}=-1.4361129191$. The resonance behaviour is obvious

The radial density for this energy and slightly different energies is plotted in Fig. 5a and b. For $r\leqslant 20$ the shape of the density is quite similar to that of the bound state just above the border to the continuum [1]. From Fig. 5b it can be deduced that the wave functions for $r\geqslant 20$ display the typical oscillatory pattern of a continuum state. The asymptotic behaviour ($r\to\infty$) of the various radial densities in Fig. 5a is almost identical. The narrow resonance depicted in Figs. 4 and 5 reflects the strongly suppressed tunnel probability of a pion as massive particle through the effective Coulomb barrier [2].

Figure 6 is very similar to Fig. 4 except for the artificial insertion of the electron rest mass into the

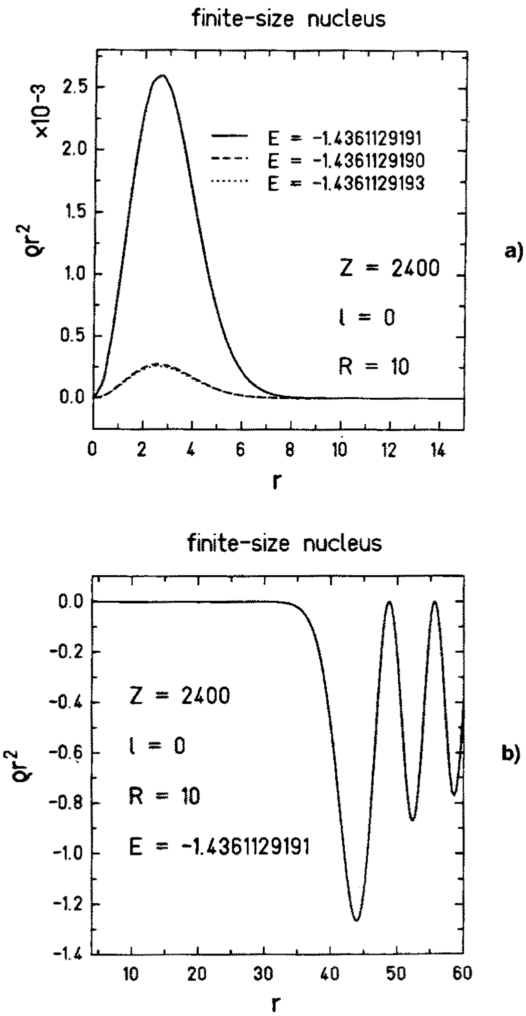

Fig. 5a and b. Radial densities for continuum states with negative frequencies in the Coulomb potential of a finite-size nucleus with $Z=2400$. The nuclear radius is $R=10$. ϱr^2 is plotted versus r for the resonance state with $E=-1.4361129191$ inside the nucleus a as well as outside the nucleus b to demonstrate the asymptotic oscillatory pattern. Figure 5a also contains the radial density for states with slightly different energies to indicate the resonance behaviour

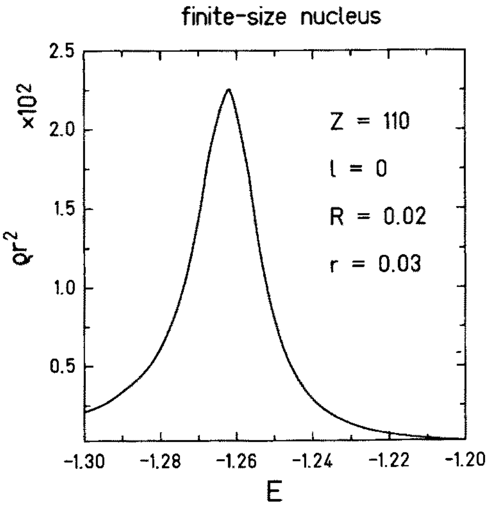

Fig. 6. Radial density at $r=0.03$ in natural units of a particle in an s-state in the Coulomb potential of a finite-size nucleus with $Z=110$ (cf. Fig. 4). For the nuclear charge distribution a homogeneously charged sphere is assumed with radius $R=0.02$ in natural units. ϱr^2 is plotted on a linear scale versus E. At $E=-1.263$ a s-state resonance appears

Klein-Gordon equation. This calculation is basically suited for illustrative purposes to exemplify the consequences of overcritical electric fields within the framework of this relativistic equation of motion. But it might be also of some relevance in connection with the possible production of a new elementary particle in collisions of very heavy ions. A new light boson could be created spontaneously in the strong external field of two heavy nuclei. It would preferentially decay into an monoenergetic electron-positron pair. The critical value ($E_{1s}=-1$) for the $1s$-bound state here appears at $Z_{cr}(1s)\simeq 108$. For $Z_{cr}=110$ we plot the radial density at $r=0.03$ versus the eigenvalue E. The resonance at $E=-1.263$, obviously, is much broader as for a Klein-Gordon particle with the pion rest mass. The radial densities of the resonance state as well as for continuum states with slightly modified energies are presented in Fig. 7a and b. In contrast to Fig. 5b the depicted states exhibit different phases in the asymptotic domain.

5. The square-well potential

The square-well potential is given by

$$V(r)=-V_0\Theta(R-r), \tag{93}$$

where $\Theta(R-r)$ denotes the Heaviside unit step function. In order to link the potential depth with a certain nuclear charge number we set

$$V_0=\frac{Z\alpha}{R}. \tag{94}$$

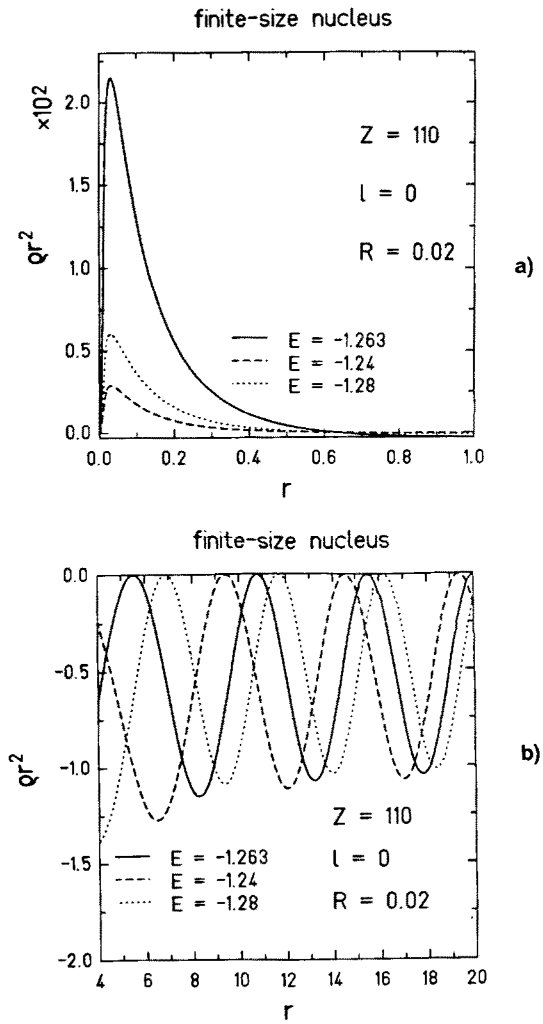

Fig. 7a and b. Radial densities for continuum states with negative frequencies in the Coulomb potential of a finite-size nucleus with $Z=110$. The nuclear radius is $R=0.02$. ϱr^2 is plotted versus r for the resonance state with $E=-1.263$ as well as for slightly different energies (cf. Fig. 5)

If the radius is assumed to depend on Z via $R=r_0(2.5\,Z)^{1/3}$ with $r_0=1.2$ fm, one obtains [1] the energy eigenvalue of a bound state as a function of Z for the $1s$-state, which reaches the negative continuum at $Z_{cr}(1s)=5626$. If the radius is fixed to $R=1.5$ fm the binding is stronger and E reaches -1 at $Z_{cr}(1s)=420$ and at $Z_{cr}(1p)=599$ for the $1p$-state. Here we adopted the nuclear physics convention for classifying the bound states. We now investigate the continuum solutions of (7) with the potential (93). In the inner region ($r\leqslant R$) $v(r)$ is given by

$$v_i(r)=Aj_l(k_i r) \tag{95}$$

with

$$k_i=\sqrt{(E+V_0)^2-1}. \tag{96}$$

j_l signifies the spherical Bessel function of the first kind [8] and A is a constant. For $r>R$ a linear combination of two independent solutions of (7) has to be considered. This may be written as

$$v_o(r)=N(\cos\delta_l\, j_l(kr)-\sin\delta_l\, y_l(kr)) \tag{97}$$

with

$$k=\sqrt{E^2-1}=p. \tag{98}$$

y_l denotes the spherical Bessel function of second kind. The three constants A, N and δ_l are connected by the matching conditions at $r=R$ which yields

$$\frac{A}{N}=\frac{\cos\delta_l\, j_l(kR)-\sin\delta_l\, y_l(kR)}{j_l(k_i R)} \tag{99}$$

The phase δ_l is obtained from the logarithmic derivative

$$\frac{k_i j_l'(k_i R)}{j_l(k_i R)}=k\frac{\cos\delta_l\, j_l'(kR)-\sin\delta_l\, y_l'(kR)}{\cos\delta_l\, j_l(kR)-\sin\delta_l\, y_l(kR)}, \tag{100}$$

which leads to

$$\tan\delta_l=\frac{kj_l'(kR)j_l(k_iR)-k_ij_l'(k_iR)j_l(kR)}{ky_l'(kR)j_l(k_iR)-k_ij_l'(k_iR)y_l(kR)}. \tag{101}$$

Here j_l' and y_l' mean derivatives with respect to the argument of the corresponding Bessel function.

The normalization constant N can be obtained from the asymptotic behaviour

$$v_o \overset{r\to\infty}{\equiv} \frac{N}{kr}\sin(kr-\tfrac{1}{2}l\pi+\delta_l), \tag{102}$$

which is easily deduced from the asymptotic form of the spherical Bessel function [15]. For the normalization on the energy-scale N is given by

$$N_E=\sqrt{\frac{2k}{\pi}}. \tag{103}$$

while normalization on the momentum-scale leads to $N_k=\sqrt{k/|E|}\,N_E$.

In particular we examined the behaviour of the radial density $\varrho\cdot r^2$ in the neighbourhood of the point at which the energy eigenvalue reaches the negative frequency continuum. For example we considered the parameters $l=1$, $Z=5800$ and $R=20.58$ in natural units. We plot $\varrho\cdot r^2$ versus r (in natural units) for different energies (cf. Fig. 8). There is an obvious enhancement for $E\simeq -1.007$. We also investigated the continuum states keeping the radius R fixed to 1.06 in natural units. In particular we consider the point $Z=400$, $l=0$, which is an almost critical value for the 1s-state [1]. Figure 9 shows $\varrho\cdot r^2$ versus r for $E=1.1$ and -1.1, respectively. The discontinuities at the potential radius R are caused by the definition (12) of the Klein-Gordon density.

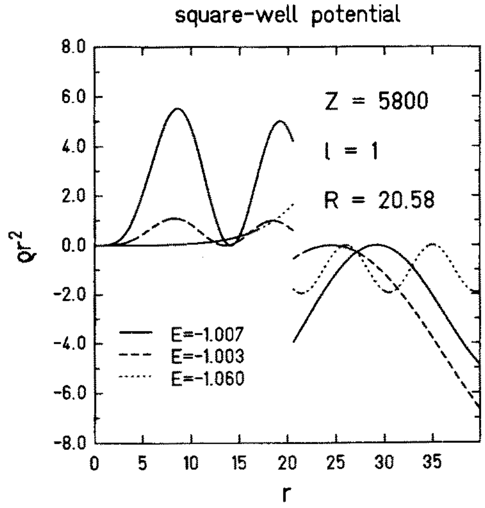

Fig. 8. Radial densities of a Klein-Gordon particle in a p-state with the indicated negative frequencies in a square-well potential

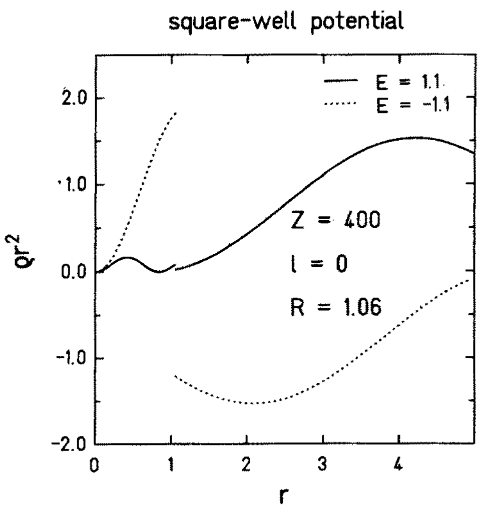

Fig. 9. Radial density of a Klein-Gordon particle in a s-state with $E=1.1$. and -1.1, respectively, in a square-well potential

6. The exponential potential

The attractive exponential potential has the simple form

$$V(r)=-ae^{-br}. \tag{104}$$

It is an example for a continuous short-range potential. Compared with the square-well potential it has the advantage to be analytical for all spatial values. But due to its short-range character we expect a similar behaviour of the solutions for both potentials.

Bawin and Lavine demonstrated that the Klein-Gordon equation with the potential (104) can be solved analytically for s-waves [13]. We start with the separation ansatz

$$u(r)=e^{\frac{b}{2}r}f(t) \tag{105}$$

and the substitution

$$t=\frac{2ia}{b}e^{-br}. \tag{106}$$

The insertion of this ansatz and the potential (104) into (10) for $l=0$ leads to

$$\frac{d^2f(t)}{dt^2}+\left(-\frac{1}{4}+\frac{\lambda}{t}+\left(\frac{1}{4}-\kappa^2\right)\frac{1}{t^2}\right)f(t)=0. \tag{107}$$

We have used the abbreviations

$$\lambda=-i\frac{E}{b},\quad \kappa=i\frac{\sqrt{E^2-1}}{b}=i\frac{p}{b}. \tag{108}$$

Equation (107) is the Whittaker differential equation. The general solution of this equation is given by [8]

$$f(t)=C_1M_{\lambda,\kappa}(t)+C_2M_{\lambda,-\kappa}(t). \tag{109}$$

$M_{\lambda,\kappa}(t)$ again denotes the Whittaker function. With the condition that $u(r)$ has to vanish at the origin $(r=0)$

$$u(0)=C_1M_{\lambda,\kappa}(\tfrac{2ia}{b})+C_2M_{\lambda,-\kappa}(\tfrac{2ia}{b})=0, \tag{110}$$

we obtain as solution

$$u(r)=N e^{\frac{b}{2}r}(M_{\lambda,-\kappa}(\tfrac{2ia}{b})M_{\lambda,\kappa}(\tfrac{2ia}{b}e^{-br})$$
$$-M_{\lambda,\kappa}(\tfrac{2ia}{b})M_{\lambda,-\kappa}(\tfrac{2ia}{b}e^{-br})). \tag{111}$$

N is a normalization constant. Using the relation (55) between the Whittaker functions and the confluent hypergeometric functions as well as the Kummer transformation [8] we can derive another expression for $u(r)$,

$$u(r)=N\frac{2ia}{b}\{\zeta^* e^{-i\frac{a}{b}\exp(-br)}$$
$$\times {}_1F_1(\tfrac{1}{2}+\kappa-\lambda, 1+2\kappa, \tfrac{2ia}{b}e^{-br})e^{-ipr}-\zeta e^{i\frac{a}{b}\exp(-br)}$$
$${}_1F_1^*(\tfrac{1}{2}+\kappa-\lambda, 1+2\kappa, \tfrac{2ia}{b}e^{-br})e^{ipr}\} \tag{112}$$

with

$$\zeta=e^{-i\frac{a}{b}}{}_1F_1(\tfrac{1}{2}+\kappa-\lambda, 1+2\kappa, \tfrac{2ia}{b}). \tag{113}$$

Normalization on the energy-scale according to (40) by using the asymptotic expansion of $u(r)$ yields

$$N_E=\frac{b}{2a\sqrt{2\pi p}\,|e^{-i\frac{a}{b}}{}_1F_1(\frac{1}{2}+\kappa-\lambda, 1+2\kappa, \frac{2ia}{b})|} \tag{114}$$

and on the momentum-scale according to (38) $N_p=\sqrt{p/|E|}\,N_E$.

In Fig. 10 we display the radial density ϱr^2 of various continuum states for an exponential potential. The coupling strength of the potential is determined via $a=Z\alpha$. The range of the potential follows from $b=1/R$. Strong distortion effects are visible. Especially, the repulsion acting on the negative frequency states is obvious.

7. The scalar square-well potential

The radial Klein-Gordon equation for scalar spherical symmetric interactions coupled to the square of the mass reads

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+E^2-1-W(r)\right)u(r)=0. \tag{115}$$

We consider the scalar square-well potential

$$W(r)=-W_0\Theta(R-r). \tag{116}$$

The calculations are identical to those presented in Chap. 6 except for the obvious replacement

$$k_i=\sqrt{E^2-1+W_0}. \tag{117}$$

In consequence we may take over all formulae of Chap. 6 including the normalization constant. This differs from the situation for bound state solutions [1]. For continuum states the normalization constant is determined by the asymptotic behaviour ($r\to\infty$) of the wave function, i.e. outside the attractive potential. Of course, in the definition of the density we have to set V

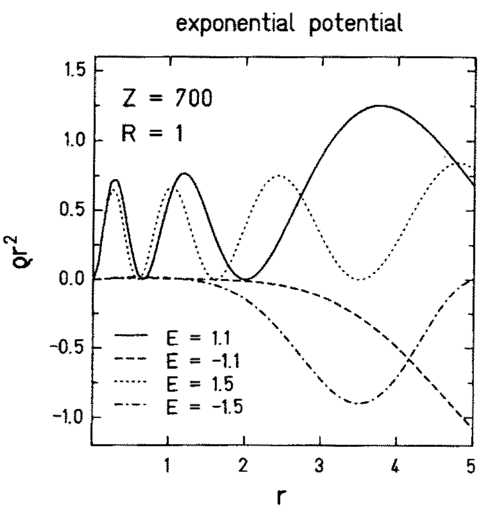

Fig. 10. Radial density of particle and antiparticle continuum states in an exponential potential. For the determination of the coupling strength and the range of the potential we used $a=Z\alpha$ and $b=1/R$, respectively

$=0$. Particle and antiparticle states exhibit the same density except for a different overall sign.

In Fig. 11 we depict the radial density of a Klein-Gordon particle in a scalar square-well potential. The potential depth is fixed through $W_0 = Z\alpha/R$ with $R=1$ in natural units. Continuum states with $l=0$ and $E=1.1$, 1.5 and 2. are considered.

8. The homogeneous magnetic field

This section deals with the behaviour of a Klein-Gordon particle in an external homogeneous magnetic field. The magnetic field strength $\mathbf{B}$ is assumed to point in z-direction, thus $B_x = B_y = 0$ and $B = B_z$. For the vector potential $\mathbf{A}$ we choose the gauge

$$A_x = -By, \quad A_y = A_z = 0. \tag{118}$$

In contrast to the previous sections the zeroth component of the four-potential vanishes: $V(\mathbf{r})=0$. The close relationship of the Klein-Gordon equation to the Schrödinger equation allows to follow completely the route of nonrelativistic quantum mechanics [16] in order to determine the wave function of a spinless relativistic particle in an external magnetic field. The corresponding result for a Dirac particle in a homogeneous magnetic field, e.g., may be deduced from [17].

Taking the potential as given above the stationary Klein-Gordon equation reads

$$\left[-\nabla^2 + q^2B^2y^2 - 2iqBy\frac{\partial}{\partial x} + 1\right]\phi(\mathbf{r}) = E^2\phi(\mathbf{r}). \tag{119}$$

With the separation ansatz

$$\phi(x, y, z) = e^{i(p_x x + p_z z)}\varphi(y) \tag{120}$$

it follows

$$\left[-\frac{\partial^2}{\partial y^2} + q^2B^2y^2 + 2p_x qBy\right]\varphi(y) =$$
$$[E^2 - 1 - p_x^2 - p_z^2]\varphi(y). \tag{121}$$

The transformation

$$\xi = \sqrt{|qB|}\left(y + \frac{p_x}{qB}\right) \tag{122}$$

leads to the oscillator equation. Thus we obtain the solution

$$\varphi_n(\xi) = e^{-\xi^2/2}H_n(\xi) \tag{123}$$

with the Hermite polynomials

$$H_n(\xi) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\, e^{\xi^2}\frac{d^n e^{-\xi^2}}{d\xi^n} \tag{124}$$

For E it results

$$E = \pm\sqrt{|qB|(2n+1) + 1 + p_z^2} \tag{125}$$

with $n = 0, 1, 2, 3 \ldots$ The normalization condition

$$\int_{-\infty}^{+\infty} EN^2\varphi_n^2(y)\,dy = \pm 1 \tag{126}$$

leads to the normalized eigenfunctions

$$\phi(x, y, z) = \frac{1}{2\pi}\sqrt{\frac{\sqrt{|qB|}}{|E|}}\, e^{i(p_x x + p_z z)} e^{-\xi^2/2} H_n(\xi). \tag{127}$$

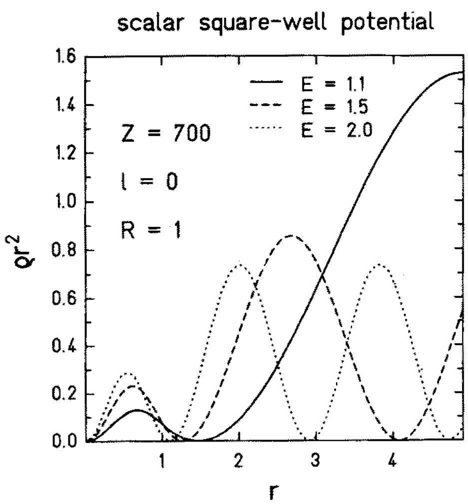

Fig. 11. Radial density of a Klein-Gordon particle in a s-state with $E=1.1$, 1.5 and 2.0, respectively, in a scalar square-well potential

9. Conclusions

The occurrence of states with negative energy has been demonstrated for massive spin-0 particles bound in external potentials. In overcritical potentials bound states may enter as resonance the negative frequency continuum. We explicitly verified the existence of this resonance phenomenon. The continuum spectrum of the Klein-Gordon equation was determined for various long- and short-range interactions within an analytical framework. Sign changes in the associated charge densities were depicted. In particular we derived the stationary solution for the Coulomb potential of an extended nucleus which could be of relevance for pion scattering on nuclei.

We are grateful for enlightening discussions with Prof. B. Müller.

References

1. Fleischer, W., Soff, G. Z. Naturforsch, **39a**, 703 (1984)[1]
2. Greiner, W., Müller, B., Rafelski, J.: Quantum electrodynamics of strong fields, Berlin, Heidelberg, New York: Springer 1985
3. Dawydow, A.S.: Quantenmechanik. Berlin: VEB Deutscher Verlag der Wissenschaften 1967
4. Flügge, S.: Lehrbuch der Theoretischen Physik, Band IV: Quantentheorie I, Berlin, Göttingen, Heidelberg, New York: Springer 1964
5. Feshbach, H., Villars, F.: Rev. Mod. Phys. **30**, 24 (1958)
6. Schmutzer, E.: Grundprinzipien der klassischen Mechanik und der klassischen Feldtheorie. Berlin: VEB Deutscher Verlag der Wissenschaften 1973
7. Landau, L.D., Lifschitz, E.M.: Lehrbuch der Theoretischen Physik III: Quantenmechanik. Berlin: Akademie-Verlag 1974
8. Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions. New York: Dover 1965
9. Shampine, L.F., Gordon, M.K.: Computer solution of ordinary differential equations: The initial value problem. San Francisco: Freeman 1975
10. Pauli, W., Weisskopf, V.: Helv. Phys. Acta. **7**, 709 (1934)
11. Schiff, L.I., Snyder, H., Weinberg, J.: Phys. Rev. **57**, 315 (1940)
12. Fulling, S.A.: Varieties of instability of a boson field in an external potential and black hole Klein paradoxes. Report 1975
13. Bawin, M., Lavine, J.P.: Nuovo Cimento **23A**, 311 (1974)
14. Rose, M.E.: Relativistic electron theory. New York: Wiley 1961
15. Arfken, G.: Mathematical methods for physicists. New York: Academic Press 1970
16. Blochinzew, D.I.: Grundlagen der Quantenmechanik. Frankfurt: Deutsch 1966
17. Schlüter, P., Soff, G., Wietschorke, K.-H., Greiner, W.: J. Phys. **B18**, 1685 (1985)

[1] There are a few misprints in Ref. 1: In (2.1) replace $\mathbf{V}^2$ by $\mathbf{V}$. In the figure caption of Fig. 12 delete the denotations '$\pi^- \rightarrow$' and '$\pi^+ \rightarrow$'