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Newton's Method for Nonlinear Inequalities*

James W. Daniel

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Abstract. A Newton-type algorithm has been presented elsewhere for solving nonlinear inequalities of the form $f(x) \leqq 0$, $g(x) = 0$, and quadratic convergence has been proved under very strong hypotheses. In this paper we show that the same results hold under a considerable weakening of the hypotheses.

1. Introduction

In [Robinson (1972c)], a Newton-type algorithm was presented and its convergence analyzed for the problem of finding an x^* in a reflexive real Banach space X solving $f(x^*) \in K$ where K is a closed convex cone in a real Banach space Y and f maps X into Y. When one specializes these results to the case in which X and Y are finite dimensional and K is the negative orthant, it appears that the hypotheses needed for these results are very strong; it is assumed, for example, that the rows of the Frechet derivative matrix $f'(x)$ of f at x are nonnegatively linearly independent, which means that the inequality $f'(x)\, h \leqq b$ must be solvable for every b and which, when f is linear, rules out the possibility that the region described by the inequalities is bounded. This difficulty arises because the results are based on the main theorem of [Robinson (1972b)] which concerns perturbations of sets of linear inequalities; it is this latter theorem which uses the above restrictive hypothesis. In [Daniel (1972)], however, we presented perturbation results which allowed us to eliminate this restrictive hypothesis; by using this new result here, we can strengthen the result of [Robinson (1972c)] on Newton's method for inequalities.

A somewhat similar algorithm to that studied here was presented for the finite dimensional case in [Pshenichnyi (1970)]; the hypotheses used to deduce convergence are quite different, however.

We shall now consider the problem of finding an x^* which solves

$$f(x) \leqq 0 \qquad g(x) = 0 \tag{1.1}$$

where f and g map $\mathbb{R}^{n'}$ into $\mathbb{R}^{m'}$ and $\mathbb{R}^{r'}$ respectively, for some integers n', m', and r', and have continuous Frechet derivatives given by the matrices $f'(x)$ and

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$g'(x)$, respectively. For brevity, we shall no longer remind the reader of the dimensions of various vectors and matrices; the dimensions are always such that the indicated operations are well-defined. For convenience, we use a fixed norm, $\|\cdot\|$, chosen as the Euclidean norm: $\|x\|^2 = \sum_i |x_i|^2$.

As in [Robinson (1972c)] and as is usual in the analysis of Newton methods, we shall be able to give a Kantorovich-type theorem, which says that, based on local information at some point x_0, we can determine that a solution to Eq. (1.1) exists and that a sequence we generate converges to such a solution. In the present analysis the constants in such a theorem become very complicated and only obscure the trend of the arguments. For convenience, then, we shall be somewhat casual in our statement of the main result; the interested and patient reader can keep track of all the various constants himself if he wishes to see a more precise statement.

In the next section we shall describe the algorithm and begin its analysis, assuming that certain linearizations of the system in Eq. (1.1) can be solved. Section 3 treats the question of the solvability of these linear systems, while Section 4 presents the main result. No numerical examples will be found in this paper; two examples were presented in [Robinson (1972c)].

2. The Algorithm

We now describe the algorithm as presented in [Robinson (1972c)] for solving Eq. (1.1).

Let x_0 be given. Having found x_n, for $n \geqq 0$, choose x_{n+1} to

$$\text{minimize } \|x - x_n\| \tag{2.1}$$

over the set S_n of x solving

$$\begin{aligned} f(x_n) + f'(x_n)(x - x_n) &\leqq 0 \\ g(x_n) + g'(x_n)(x - x_n) &= 0. \end{aligned} \tag{2.2}$$

Since the set S_n defined by Eq. (2.2) is a convex polyhedron, clearly a unique x_{n+1} exists provided that S_n is nonempty. Actually, other norms, such as the l_1 or l_∞ norms, could be used as well here since linear programming could be used to provide a particular solution; we adhere to our choice of the Euclidean norm, however.

As has been kindly observed by the referee, one can modify the results of [Robinson (1972c)] in a straightforward manner so as to weaken the so-called GLI hypothesis in [Robinson (1972c)] that requires $u^T f'(x_0) + v^T g'(x_0) = 0$ with $u \geqq 0$ to imply $u = 0$ and $v = 0$. One could divide the constraints defined by f into two classes (say considering which of the linearized constraints at x_0 are satisfied as equalities at x_1) I and J such that those in I satisfy the GLI hypothesis while those in J are satisfied in a sufficiently large ball about x_0 or x_1. The results of [Robinson (1972c)] would then show that the method generates a sequence converging to an x^* satisfying those constraints in I while those in J would be satisfied since x^* could be assumed to lie in the aforementioned ball associated with J. Thus the results of [Robinson (1972c)] can be extended in this straightforward fashion. The difficulty with this approach lies in the definitions of the sets I and J. Using only local information at x_0 or x_1 we must somehow select I so as to contain all those constraints which might be meaningfully active at the unknown solution x^* or at any intermediate step of the iteration, and we must assume the GLI hypothesis for I. In what follows we shall take a different approach to removing the GLI hypothesis on f, yielding a convergence theorem different from that of [Robinson (1972c)] and from that just outlined; we now proceed to this analysis.

For the moment let us blithely assume that S_n is nonempty; later we shall be able to guarantee this. To analyze the convergence of our method we need to estimate $\|x_{n+1}-x_n\|$, the minimum in Eq. (2.1). It is clear that x_n solves the system

$$\begin{aligned} f'(x_n)\,(x-x_n) &\leqq 0 \\ g'(x_n)\,(x-x_n) &= 0 \end{aligned} \tag{2.3}$$

and that the system of Eq. (2.2) is just a perturbation of this system with both being perturbations of the fixed system defining S_0; any results available on the perturbation of such systems should enable us to give bounds on $\|x_{n+1}-x_n\|$. Before we can apply the results of this type from [Daniel (1972)], we need additional hypotheses.

Assume that

$$\begin{aligned} &\text{there exists } \hat{x}_0 \text{ in } S_0 \text{ such that} \\ &f(x_0)+f'(x_0)\,(\hat{x}_0-x_0)\leqq -p, \quad \text{with} \quad p>0. \end{aligned} \tag{2.4}$$

For any vector y, we denote by y^+ that vector each of whose components is given by the maximum of zero and that same component of y. By Theorem 4.16 of [Daniel (1972)], there exist constants $\bar{c}$ and $\varepsilon_0>0$, depending on $f'(x_0)$, $g'(x_0)$, and p, such that there exists x'_{n+1} in S_n satisfying

$$\|x'_{n+1}-x_n\|\leqq \bar{c}\{\|g(x_n)\|+\|[f(x_n)]^+\|\}(1+\|x_n\|) \tag{2.5}$$

whenever the following three statements hold:

$$S_n \text{ is nonempty}, \tag{2.6}$$

$$\operatorname{rank}[g'(x_n)]=\operatorname{rank}[g'(x_0)], \tag{2.7}$$

$$\begin{aligned} (1+\|x_n\|)\{&\|f'(x_n)-f'(x_0)\|+\|g'(x_n)-g'(x_0)\|+\|f'(x_0)\,x_0-f(x_0)+f(x_n) \\ &-f'(x_n)\,x_n\|+\|g'(x_0)\,x_0-g(x_0)+g(x_n)-g'(x_n)\,x_n\|\}\leqq \varepsilon_0. \end{aligned} \tag{2.8}$$

We wish to see what the estimate of Eq. (2.5) gives us. If f' and g' satisfy the Lipschitz conditions

$$\|f'(x)-f'(y)\|\leqq L\|x-y\|, \qquad \|g'(x)-g'(y)\|\leqq L\|x-y\| \tag{2.9}$$

then we have, from the definition of x_n, that

$$\|g(x_n)\|=\|g(x_n)-[g(x_{n-1})+g'(x_{n-1})\,(x_n-x_{n-1})]\|\leqq \tfrac{1}{2}L\|x_n-x_{n-1}\|^2,$$

and

$$\|[f(x_n)]^+\| = \|\{f(x_n) - [f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})] + [f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})]\}^+\| \leqq \|\{f(x_n) - [f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})]\}^+\| + \|[f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})]^+\| \leqq \|f(x_n) - [f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})]\| \leqq \tfrac{1}{2} L \|x_n - x_{n-1}\|^2.$$

Using these estimates and the fact that x_{n+1} minimizes $\|x - x_n\|$ over S_n, we get

$$\|x_{n+1} - x_n\| \leqq \bar{c} L \|x_n - x_{n-1}\|^2 (1 + \|x_n\|). \tag{2.10}$$

For this estimate to be valid for all n, we need Eqs. (2.6), (2.7), and (2.8) to hold for all n; since we shall have to assume Eq. (2.7) and since we shall address ourselves to Eq. (2.6) in the next section, we turn to Eq. (2.8). The expression there can be easily bounded by $(1 + \|x_n\|) \|x_n - x_0\| [2L + 2L\|x_n\| + L\|x_n - x_0\|]$; thus, to satisfy Eq. 2.8), we need a uniform bound on $\|x_n\|$ and a sufficiently small uniform bound on $\|x_n - x_0\|$. We can derive such bounds by induction. Since we have already assumed that S_0 is nonempty, we know that x_1 exists, allowing us to start an induction based on Eq. (2.10). If we define

$$B = \|x_0\| + 2\|x_1 - x_0\| \quad \text{and} \quad t = \bar{c} L \|x_1 - x_0\| (1 + B), \tag{2.11}$$

then we have

$$\|x_2 - x_1\| \leqq \bar{c} L \|x_1 - x_0\|^2 (1 + \|x_1\|) \leqq \bar{c} L \|x_1 - x_0\|^2 (1 + B) \leqq t \|x_1 - x_0\| \tag{2.12}$$

if S_2 is nonempty and if $(1 + \|x_1\|) \|x_1 - x_0\| [2L + 2L\|x_1\| + L\|x_1 - x_0\|] \leqq \varepsilon_0$. Thus, if $\|x_1 - x_0\|$ is small enough, in particular so that $t \leqq \frac{1}{2}$, then Eq. (2.12) is valid and hence also

$$\|x_2\| \leqq \|x_0\| + (1 + t)\|x_1 - x_0\| \leqq \|x_0\| + 2\|x_1 - x_0\| = B \tag{2.13}$$

and

$$\|x_2 - x_0\| \leqq (1 + t)\|x_1 - x_0\| \leqq \frac{\|x_1 - x_0\|}{1 - t}. \tag{2.14}$$

Thus we have the bounds we need for $n = 2$. Assuming S_n to be nonempty for all n, the obvious induction yields more generally the crude estimates

$$\|x_{n+1} - x_n\| \leqq t^{2^n - 1} \|x_1 - x_0\|, \tag{2.15}$$

$$\|x_{n+1}\| \leqq \|x_0\| + \frac{1 - t^{n+1}}{1 - t} \|x_1 - x_0\| \leqq \|x_0\| + \frac{\|x_1 - x_0\|}{1 - t} \leqq \|x_0\| + 2\|x_1 - x_0\| = B, \tag{2.16}$$

$$\|x_{n+1} - x_0\| \leqq \frac{1 - t^{n+1}}{1 - t} \|x_1 - x_0\| \leqq \frac{\|x_1 - x_0\|}{1 - t}, \tag{21.7}$$

provided that

$$(1 + B) \frac{\|x_1 - x_0\|}{1 - t} \left[2L + 2LB + L \frac{\|x_1 - x_0\|}{1 - t}\right] \leqq \varepsilon_0. \tag{2.18}$$

Thus, by assuming that $\|x_1 - x_0\|$ is small enough, we conclude that Eq. (2.10) holds for all n. We have essentially proved

(2.19) **Proposition.** Assume, in addition to the general hypotheses, that Eqs. (2.4), (2.6), (2.7), (2.9), and (2.11) hold. If $\|x_1 - x_0\|$ is so small that Eq. (2.18) holds, then we have $\|x_{n+1} - x_n\| \leqq \bar{c} L \|x_n - x_{n-1}\|^2 (1 + \|x_n\|)$.

Remark. We note that the Lipschitz condition in Eq. (2.9) need only hold in the ball of radius $\|x_1 - x_0\|/(1 - t)$ about x_0.

We must now consider the crucial question of whether or not S_n can be empty; we address this in the next section.

3. The Solvability of the Linearized System

The fundamental estimate of Eq. (2.10) was derived on the assumption that S_n was nonempty; we must now guarantee this by showing that the system in Eq. (2.2) is solvable. To do so, we require additional hypotheses. First we assume that the linearizations of the equality constraints are not degenerate; more precisely, we assume that

(3.1) $$g'(x_n) \text{ is of full rank for all } n \geqq 0.$$ [1]

Thus we know that the systems $g'(x_n) h = b$ are always solvable; one may take, for example, $h = [g'(x_n)]^{\#} b$, where $A^{\#}$ denotes the Moore-Penrose pseudo-inverse of the matrix A. In particular, let

(3.2) $$h_n = [g'(x_n)]^{\#} b_n$$

solve

(3.3) $$g'(x_n) h_n = b_n$$

where

(3.4) $$b_n = -g(x_n) - g'(x_n)(\hat{x}_0 - x_n)$$

and $\hat{x}_0$ is as defined in Eq. (2.4). Since $g(x_0) + g'(x_0)(\hat{x}_0 - x_0) = 0$, we have

$$\begin{aligned} b_n &= b_n + g(x_0) + g'(x_0)(\hat{x}_0 - x_0) \\ &= [g(x_0) - g(x_n)] + [g'(x_0) - g'(x_n)](\hat{x}_0 - x_0) + g'(x_n)(x_n - x_0). \end{aligned}$$

Assuming then that S_i is nonempty for $1 \leqq i \leqq n-1$ we can use Eqs. (2.15), (2.16), and (2.17) with n replaced by $n-1$ to conclude that

(3.5) $$\|b_n\| \leqq c \|x_1 - x_0\| (1 + \|\hat{x}_0 - x_0\|)$$

for some constant c. This then gives

(3.6) $$\|h_n\| \leqq \|[g'(x_n)]\|^{\#} c \|x_1 - x_0\| (1 + \|\hat{x}_0 - x_0\|).$$

Since $g'(x_n)$ and $g'(x_0)$ have equal rank and since

$$\|g'(x_n) - g'(x_0)\| \leqq L \|x_n - x_0\| \leqq \frac{L\|x_1 - x_0\|}{1 - t},$$

the pseudo-inverse of $g'(x_n)$ is near that of $g'(x_0)$ for small $\|x_1 - x_0\|$. Thus, for small enough $\|x_1 - x_0\|$, we have

(3.7) $$\|h_n\| \leqq c \|x_1 - x_0\| (1 + \|\hat{x}_0 - x_0\|) \quad \text{for some constant } c.$$

1 As the referee has observed, the hypothesis can rarely be verfied a priori.

From the definition of h_n, we see that $\hat{x}_0 + h_n$ solves

(3.8) $$g(x_n) + g'(x_n)[(\hat{x}_0 + h_n) - x_n] = 0.$$

Moreover,

$$\begin{aligned} f(x_n) + f'(x_n)[(\hat{x}_0 + h_n) - x_n] &= f(x_0) + f'(x_0)(\hat{x}_0 - x_0) + [f(x_n) - f(x_0)] \\ &\quad + [f'(x_n) - f'(x_0)](\hat{x}_0 - x_0) + f'(x_n)(x_0 - x_n) + f'(x_n) h_n \\ &\leqq -p + \{[f(x_n) - f(x_0)] + [f'(x_n) - f'(x_0)](\hat{x}_0 - x_0) + f'(x_n)(x_0 - x_n) \\ &\quad + f'(x_n) h_n\}. \end{aligned}$$

Assuming again that S_i is nonempty for $1 \leqq i \leqq n-1$ we again use Eqs. (2.15), (2.16), and (2.17) and also Eq. (3.7) to find that the expression above in curly brackets can be bounded by $c_0 \|x_1 - x_0\|(1 + \|\hat{x}_0 - x_0\|)$ for some constant c_0 determined by f, g, and x_0. Thus $\hat{x}_0 + h_n$ will satisfy Eq. (2.2), that is, S_n will be nonempty, provided that $\min_i p_i \geqq c_0 \|x_1 - x_0\|(1 + \|\hat{x}_0 - x_0\|)$, where the p_i are the components of the vector p, and the induction can be continued. We have essentially proved

(3.9) **Proposition.** Assume, in addition to the general hypotheses, that Eqs. (2.4), (2.9), and (3.1) are valid. Then there exist constants c_0 and ε_1 depending on f, g, and x_0, such that S_n is nonempty for all n provided that $\|x_1 - x_0\| \leqq \varepsilon_1$ and $\min_i p_i \geqq c_0 \|x_1 - x_0\|(1 + \|\hat{x}_0 - x_0\|)$.

Proof. Eq. (2.7) holds because of Eq. (3.1). The induction argument sketched above, combined with the result of Proposition (2.19), completes the proof. Q.E.D.

4. The Main Result

It is now a simple matter to prove the main result.

(4.1) **Theorem.** In addition to the general hypotheses, assume that Eqs. (2.4), (2.9), and (3.1) are satisfied. Starting with x_0, let the algorithm described by Eqs. (2.1) and (2.2) be applied. Let a c_0 and ε_1 be determined as in Proposition (3.9) and suppose that $\|x_1 - x_0\| \leqq \varepsilon_1' \leqq \varepsilon_1$ for a certain ε_1' determined from f, g, and x_0 and that $\min_i p_i \geqq c_0 \|x_1 - x_0\|(1 + \|\hat{x}_0 - x_0\|)$. Then the algorithm is well defined and generates a sequence $\{x_n\}$ converging to a point x^* solving Eq. (1.1). The estimate of Eq. (2.15) is valid, and moreover there are constants c and $q < 1$ such that the error estimate $\|x_n - x^*\| \leqq c q^{2^n}$ holds.

Proof. Under our hypotheses we know that the conclusions of Proposition (2.19) and Proposition (3.9) are valid, so that Eq. (2.15) is valid, and we may take ε_1' and thereby $\|x_1 - x_0\|$ so small that $t < 1$, where t is defined in Eq. (2.11). Then, with $q = t$, we have $\|x_{n+1} - x_n\| \leqq \frac{\|x_1 - x_0\|}{t} q^{2^n}$ by Eq. (2.15). The usual argument shows

$$\|x_{n+m} - x_n\| \leqq \frac{\|x_1 - x_0\|}{t} q^{2^n} \sum_{i=n}^{\infty} q^{2^i - 2^n} \leqq \frac{\|x_1 - x_0\|}{t(1-q)} q^{2^n},$$

which implies that $\{x_n\}$ is a Cauchy sequence and hence converges to some x^*; the error estimate follows by letting m tend to infinity. That x^* solves Eq. (1.1) follows from the continuity of f, g, f', and g' and the fact that x_{n+1} solves Eq. (2.2). Q.E.D.

Remark. As is usual with Kantorovich-type results, this theorem roughly states that if x_0 comes close enough to solving our problem, where "close enough" is a complicated condition involving $\|x_1 - x_0\|$, then the algorithm will work and converge to a solution at least quadratically; the condition on $\|x_1 - x_0\|$ is natural, since x_0 would in fact be a solution to the basic problem should $x_1 = x_0$.

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References

1. Daniel, J. W.: On perturbations in systems of linear inequalities. SIAM J. Numer. Anal. **10**, 299—307 (1973). Also appeared as CNA-50, Center for Numerical Analysis, UT-Austin (1972)
2. Pshenichnyi, B. N.: Newton's method for the solution of systems of equalities and inequalities. Mat. Zametki **8**, 635–640; translated in Math. Notes **8**, 827–830 (1970)
3. Robinson, S. M.: Bounds for error in the solution set of a perturbed linear program. J. Lin. Alg. Appl. **6**, 69–82 (1972a)
4. Robinson, S. M.: Normed convex processes. Trans. Am. Math. Soc. **174**, 127–140 (1972b). Also appeared as Tech. Sum. Report 1135, Math. Res. Center, U. of Wisconsin, 1971
5. Robinson, S. M.: Extension of Newton's method to nonlinear functions with values in a cone. Numer. Math. **19**, 341–347 (1972c). Also appeared as Tech. Sum. Report 1161, Math. Res. Center, U. of Wisconsin, 1971

Prof. Dr. James W. Daniel
Dept of Mathematics
University of Texas at Austin
Austin, Texas 78712/USA