Numer. Math. 21, 317—327 (1973)

Finite Element Convergence for Singular Data

Ridgway Scott

Massachusetts Institute of Technology

Received April 4, 1973

Abstract. Convergence of the finite element solution u^h of the Dirichlet problem $\Delta u=\delta$ is proved, where δ is the Dirac δ-function (unit impulse). In two dimensions, the Green's function (fundamental solution) u lies outside H^1, but we are able to prove that $\|u-u^h\|_{L^2}=O(h)$. Since the singularity of u is logarithmic, we conclude that in two dimensions the function $\log r$ can be approximated in L^2 near the origin by piecewise linear functions with an error $O(h)$. We also consider the Dirichlet problem $\Delta u=f$, where f is piecewise smooth but discontinuous along some curve. In this case, u just fails to be in $H^{\frac{5}{2}}$, but as with the approximation to the Green's function, we prove the full rate of convergence: $\|u-u^h\|_1=O(h^{\frac{3}{2}})$ with, say, piecewise quadratics.

1. Introduction and Statement of Results

We consider the elliptic-boundary value problem:

$$\begin{aligned} Au&=f \quad \text{in} \quad \Omega \\ Bu&=0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1}$$

when f has two kinds of singularities:

i) f is the Dirac δ-function (unit impulse) at some point x_0 in Ω.

ii) f is piecewise smooth but discontinuous on some submanifold of Ω.

The first case, $f=\delta_{x_0}$, corresponds in physical problems to the idealization of a point load at x_0, and the second case corresponds to a discontinuous loading.

The finite element approximation u^h is defined as usual via the energy inner product $a(\cdot,\cdot)$ associated to A—for smooth f, the solution u to (1) satisfies:

$$a(u,v)=(f,v) \quad \text{for all } v \text{ in } V. \tag{2}$$

V is the completion in the Sobolev space $H^m(\Omega)$ of all smooth functions v that satisfy the boundary conditions $Bv=0$, where m is the highest order of derivatives occuring in $a(\cdot,\cdot)$ and half the order of A. Given a subspace S^h of V, we define u^h to be that function in S^h that satisfies:

$$a(u^h,v)=(f,v) \quad \text{for all } v \text{ in } S^h. \tag{3}$$

When S^h consists of continuous piecewise polynomials, (3) makes sense even when f is the δ-function. It becomes:

$$a(u^h,v)=v(x_0) \quad \text{for all } v \text{ in } S^h. \tag{3'}$$

However, the true solution u may be too singular for convergence in energy to occur. Indeed, if the dimension of Ω is n, then:

$$a(u, u) = \infty \quad \text{whenever} \quad n \geqq 2m.$$

This is the case for the problems of plane stress and plane strain: $m=1$ and $n=2$. However, we may ask if u^h converges to u even though the m-th derivatives cannot. That this is the case was first proved by I. Babuška [1], [2]. His results yield, for the case $m=1$, $n=2$:

$$\|u-u^h\|_0 \leqq c_\varepsilon h^{1-\varepsilon}, \quad \text{any} \quad \varepsilon > 0$$

if $\partial\Omega$ is smooth, and in case $\partial\Omega$ is only Lipschitzian, the exponent decreases to $\frac{1}{2}-\varepsilon$. We will deal only with the smooth case, where we are able to improve Babuška's estimate by removing the ε. And we obtain a similar result in any number of dimensions n, for elliptic equations of any order $2m$. For $\frac{n}{2} \geqq 2m$, the Green's function u does not lie in L^2, but we prove convergence in suitable negative norms.

When f is piecewise smooth, but discontinuous on some submanifold of Ω, f lies only in $H^{\frac{1}{2}-\varepsilon}(\Omega)$, any $\varepsilon > 0$. This allows for convergence in energy of u^h to u, but we would expect the rate to be $h^{m+\frac{1}{2}-\varepsilon}$ since u is in $H^{2m+\frac{1}{2}-\varepsilon}(\Omega)$. However, we prove in Section 3 that the ε again disappears in the error estimate:

$$\|u-u^h\|_m = O(h^{m+\frac{1}{2}}) \tag{5}$$

as long as S^h approximates to order $2m+1$.

One consequence of our results is an approximation theorem for certain singular functions. For example, the singularity near x_0 of any solution of $\Delta u = \delta_{x_0}$ in two dimensions is logarithmic. Thus by (3), the function $\log r$ in the plane can be approximated by, say, continuous piecewise linear functions in L^2 with an error $O(h)$. This seems, at first, to violate a *saturation theorem*, since $\log r$ does not lie in H^1 near the origin. Such a theorem would state that if a function v in H^{m-1} can be approximated in the H^{m-1} norm by functions v^h in H^m with an $O(h)$ error, then v must be in H^m. This strong saturation theorem is false, but under the additional assumption that $\|v^h\|_m$ remains bounded as $h \to 0$, the theorem is true. (The proof is by showing that the difference quotients of v remain bounded in H^{m-1}.) This weak saturation theorem thus shows that the piecewise linear functions that approximate $\log r$ near the origin must be unbounded in H^1.

Isaac Fried [3] has considered a related approximation theorem for singular functions, calculating the error in interpolating the function r^α for non-integer α. The singularity in a crack problem behaves like $r^{\frac{1}{2}}$, and in [3] it is shown that $r^{\frac{1}{2}}$ differs from its interpolate by $O(h^{\frac{1}{2}})$ in the H^1 norm, even though $r^{\frac{1}{2}}$ lies outside $H^{\frac{3}{2}}$ near the origin. Thus the error in the finite element method is also $O(h^{\frac{1}{2}})$ in the energy norm for the crack problem. In Section 4, we study the approximation of singular functions in a general way, but restricted to a regular mesh (Sections 2 and 3 have no mesh restriction).

Our methods in Section 3 apply to the general case when f is globally C^{k-1} but fails to be C^k on some submanifold (which does not have to be smooth).

f lies in $H^{k+\frac{1}{2}-\varepsilon}(\Omega)$, but we can prove that the error in the finite element method behaves as if f were actually in $H^{k+\frac{1}{2}}(\Omega)$, without the ε. Because this general case is not as physically meaningful as a discontinuous f, we do not mention it further.

2. Convergence for the δ-Function

We imagine S^h to be a space of piecewise polynomials in a "triangulation" of Ω: $\Omega = \cup e_j$. Each function in S^h is to be continuous in Ω and a polynomial of degree $\leq K$ in each element e_j. For simplicity, we will assume that each e_j may be mapped by an affine transformation T_j onto a reference element e_{ref}, although our results apply when there is more than one reference element and when T_j is isoparametric. We will also need to assume that the elements are of comparable size and shape, which insure by demanding that entries in the Jacobian of T_j are bounded by ch^{-1}.

In addition to these assumptions on the triangulation and S^h, we will say that *S^h approximates to degree k in V* if for each smooth u in V:

$$\inf_{v\in S^h} \|u-v\|_s \leq c h^{k-s} \|u\|_k \quad \text{for} \quad 0 \leq s \leq m. \tag{6}$$

This approximation condition, aside from the difficulty the boundary imposes, simply means that in each element e_j, any polynomial of degree $\leq k-1$ can be represented by some function in S^h. We will ignore the boundary difficulties, for we feel that our results will apply to several different techniques that are used to handle curved boundaries: isoparametric elements, penalty functions, interpolated boundary conditions, etc.

Our basic assumption on the energy inner product $a(\cdot,\cdot)$ is that it be coercive over V:

$$|a(v,v)| \geq \gamma \|v\|_m^2 \quad \text{for all } v \text{ in } V \tag{7}$$

for some positive γ. When u is smooth, this inequality guarantees the convergence in $H^m(\Omega)$ of u^h to u. We recall this well-known fact in the following lemma.

Lemma 1. *Let the operator A in (1) be properly elliptic, and let $B = (b_1, \ldots, b_m)$ be normal, covering boundary conditions. Assume that $a(\cdot,\cdot)$ is coercive. Then if S^h approximates to degree k in V:*

$$\|u-u^h\|_m \leq c h^{s-m} \|f\|_{s-2m} \quad \textit{for} \quad 2m \leq s \leq k. \tag{10}$$

Under certain conditions on an adjoint problem to (1), it is known that u^h converges even faster to u in lower norms. To state these conditions precisely, we separate the boundary conditions B in (1) into the essential boundary conditions, E, and the natural conditions, N. (The essential boundary conditions are those in which the order of differentiation is less than $m = \frac{1}{2}$ order of A.) Consider now the adjoint problem to (2), to find u^* in V such that:

$$a(v, u^*) = (v, f) \quad \text{for all } v \text{ in } V. \tag{8}$$

The solution u^* satisfies an adjoint boundary problem to (1):

$$\begin{aligned} A^* u^* &= f \quad \text{in} \quad \Omega \\ B^* u^* &= (E, N^*) u^* = 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{9}$$

where the adjoint natural conditions $N^* u^* = 0$ are determined by an integration by parts.

Lemma 2. *Suppose that the adjoint problem* (8) *is regular in the sense that:* $\|u^*\|_{s+2m} \leqq c_s \|f\|_s$ *for* $s \geqq 0$. *Then for* $2m - k \leqq s \leqq m$:

$$\|u - u^h\|_s \leqq c h^{m-s} \|u - u^h\|_m. \tag{11}$$

Both lemmas will be used in proving convergence for singular data in the following theorem.

Theorem 1. *Under the hypotheses in Lemmas* 1 *and* 2, *if* u *solves* (1) *in the sense of distributions for* $f = \delta_{x_0}$ *and* u^h *solves* (3′), *then:*

$$\|u - u^h\|_s \leqq c(x_0)\, h^{2m - \frac{n}{2} - s} \quad \text{for} \quad 2m - k \leqq s < 2m - \frac{n}{2} \tag{4'}$$

if $k \geqq 2m$.[1] *The constant* $c(x_0)$ *goes to* $+\infty$ *as* x_0 *approaches* $\partial\Omega$.

Proof. We begin by approximating δ_{x_0} by a sequence δ_h with the following properties:

a) for each v in S^h, $(\delta_h, v) = v(x_0)$

b) $\|\delta_h\|_0 = O(h^{-n/2})$

c) $\|\delta - \delta_h\|_{-r} = O(h^{r-n/2})$, $\frac{n}{2} < r \leqq k$

d) $\delta_h = 0$ outside the element e in which x_0 lies.

Postponing the construction of δ_h, we define an intermediate function U^h as the solution to (1) for $f = \delta_h$. In variational form, U^h satisfies:

$$a(U^h, v) = (\delta_h, v) \quad \text{for all } v \text{ in } V. \tag{12}$$

Condition a) states that δ and δ_h are identical linear functionals on S^h, so we may view u^h also as the finite element approximation to U^h. Thus by Lemma 1 and 2:

$$\|U^h - u^h\|_s \leqq c h^{2m-s} \|\delta_h\|_0 = O\left(h^{2m - \frac{n}{2} - s}\right).$$

We now estimate $u - U^h$. Our immediate impulse is to appeal to elliptic regularity theory to conclude:

$$\|u - U^h\|_s \leqq c_s \|\delta - \delta_h\|_{s-2m}. \tag{13}$$

That is, we would hope that the solution u to (1) is related to the data f by:

$$\|u\|_s \leqq c_s \|f\|_{s-2m}. \tag{13'}$$

Inequality (13′), however, is not true for $s \leqq m - \frac{1}{2}$.

We will show that (13) is indeed true, but with the constant c_s depending on the distance of x_0 to $\partial\Omega$. Once we prove (13), the theorem is completed using the triangle inequality:

$$\|u - u^h\|_s \leqq \|u - U^h\|_s + \|U^h - u^h\|_s$$

---

1 For $2m - \frac{n}{2} \leqq k < 2m$, we have to include an ε in the final error estimate.

and condition c):

$$\|\delta-\delta_h\|_{s-2m}=O\left(h^{2m-\frac{n}{2}-s}\right).$$

We turn now to the proof of (13).

In solving elliptic problems with singular data, one must resort to more arcane data spaces than the negative Sobolev spaces. The approach suggested by Lions and Magenes [4] is to introduce the space

$$\Xi^r=\left\{v\in L^2\colon \|v\|_{\Xi^r}=\sum_{|\alpha|\leqq r}\|\varrho^{|\alpha|}D^\alpha v\|_0<\infty\right\}$$

where ϱ is a smooth function such that:

$$c_1\leqq\frac{\varrho(x)}{d(x,\partial\Omega)}\leqq c_2\qquad\text{for all } x \text{ in } \Omega.$$

$d(x,\partial\Omega)$ denotes the distance from x to $\partial\Omega$. Ξ^{-r}, for $r\geqq 0$, is defined to be the dual of Ξ^r with the dual norm. We quote the following theorem from [4] (Theorem 6.3 of Chapter 2):

Lemma 3. *Let the operator A in (1) be properly elliptic, and let $B=(b_1,\ldots,b_m)$ be normal, covering boundary conditions for A.*[2] *Then for any f in Ξ^{-r}, $r\geqq 0$, there is a distribution solution u to (1) which satisfies:*

$$\|u\|_{2m-r}\leqq c\|f\|_{\Xi^{-r}}.$$

We now apply Lemma 3 to $u-U^h$, and we are left to estimate $\|\delta-\delta_h\|_{\Xi^{s-2m}}$. But if $\Omega_1\subset\subset\Omega$, then for distributions with support in Ω_1, the $H^{-r}(\Omega)$ and Ξ^{-r} norms are equivalent. In fact, $\|v\|_{\Xi^{-r}}\leqq c\,d(\Omega_1,\partial\Omega)^{-r}\|v\|_{-r}$ if supp $v\subset\Omega_1$, where c depends only on Ω and the choice of ϱ. By d), the support of δ_h is contained in some $\Omega_1\subset\subset\Omega$ for h sufficiently small, and thus (10) is proved. The constant $c(x_0)$ in the theorem can be estimated by $c(x_0)=O\left(d(x_0,\partial\Omega)^{s-2m}\right)$.

We now construct δ_h. Let e be an element containing x_0 (there may be more than one if x_0 lies on an edge or is a vertex). Then T maps e to the reference element e_{ref}. Let φ be the polynomial of degree K such that:

$$\int_{e_{\text{ref}}}P(y)\,\varphi(y)=P\left(T(x_0)\right)$$

for all polynomials P of degree K. The mapping $T(x_0)\to\varphi$ is continuous, so we conclude that:

$$\sup_{y\in e_{\text{ref}}}|\varphi(y)|\leqq C$$

regardless of the position of x_0 in e. Define δ_h by:

$$\delta_h(x)=\begin{cases}|T|\,\varphi(Tx) & \text{in } e\\ 0 & \text{outside } e\end{cases}$$

where $|T|$ is the determinant of the Jacobian of T. Changing variables, we have:

$$\int_\Omega\delta_h(x)\,v(x)\,dx=\int_{e_{\text{ref}}}\varphi(y)\,v(T^{-1}y)\,dy=v(x_0)$$

2 We also assume that (1) has a unique solution for every smooth f to simplify the exposition.

and condition a) is proved. Our assumption on the Jacobian of T says that:

$$\sup_{x\in\Omega} |\delta_h(x)| \leqq C|T| = O(h^{-n}).$$

Squaring and integrating over e, we prove b).

To prove c), we consider the inner product $(\delta - \delta_h, \psi)$ for ψ in $\mathring{H}^r(\Omega)$. By the Bramble-Hilbert Lemma, there is a polynomial P of degree $k-1$ such that:

$$\|\psi - P\|_{0,e} \leqq c h^r \|\psi\|_{r,e}, \quad r \leqq k$$

$$\sup_{x\in e} |\psi(x) - P(x)| \leqq c h^{r-\frac{n}{2}} \|\psi\|_{r,e}, \quad r > \frac{n}{2} \tag{14}$$

where the subscript e indicates that the integrations take place over e only. Thus:

$$(\delta - \delta_h, \psi) = \psi(x_0) - P(x_0) - \int_e \delta_h(\psi - P)$$

and since $\|\delta_h\|_0 = O\left(h^{-\frac{n}{2}}\right)$, we conclude from (14) that:

$$|(\delta - \delta_h, \psi)| \leqq c h^{r-\frac{n}{2}} \|\psi\|_r, \quad \frac{n}{2} < r \leqq k$$

and this proves c).

Remark 1. The contrast between the Ξ^{-r} and H^{-r} norms is exemplified by δ_x as we let $x \to \partial\Omega$. Indeed

$$\|\delta_x\|_{-r} \leqq c_1 d(x, \partial\Omega)^{r-\frac{n}{2}}$$

$$\|\delta_x\|_{\Xi^{-r}} \geqq c_2 \log\left(d(x, \partial\Omega)\right).$$

In solving (1) for $f = \delta_x$, it is the Sobolev norm that predicts what happens as $d(x, \partial\Omega) \to 0$: using Poisson's formula to solve $\Delta u = \delta_x$ in the unit disc, $u = 0$ on the unit circle, one finds that the solution u_x *does* converge to zero as $d(x, \partial\Omega) \to 0$.

Remark 2. We showed that the finite element approximation u^h to a singular function u may be viewed as the finite element approximation to a smoother function U^h. Thus if our problem were to investigate for a singularity, we would be fooled by the smoothness of u^h if h were not small enough. This suggests the importance of an *a priori* knowledge, through error estimates, of the order of magnitude of the error $u - u^h$.

3. Convergence Rates for Piecewise Constant Data

Suppose f is a smooth function except on the set $\omega \subset \Omega$ where it is discontinuous. ω need not be smooth; in practice, it will be piecewise smooth, but our only assumption is that its dimension be $n-1$, namely, we demand that the set

$$\omega^h = \{x:\ d(x, \omega) < h\}$$

have measure $O(h)$. We require f and its derivatives to be bounded in $\Omega - \omega$, and this places f in $H^{\frac{1}{2}-\varepsilon}(\Omega)$. However, when f is truly discontinuous, it can not be in $H^{\frac{1}{2}}(\Omega)$. Thus the solution u to (1) for discontinuous f can not be in $H^{2m+\frac{1}{2}}(\Omega)$. However, we have the following theorem for the convergence of the finite element solution u^h to u. Since we only consider convergence in energy, we just assume that $a(\cdot, \cdot)$ is coercive and that (1) is regular as stated in Lemma 1. Under these conditions, we prove:

Theorem 2. *Suppose S^h approximates to degree $2m+1$ in V. If f is smooth in $\Omega-\omega$ and $|\omega^h|=O(h)$, then:*

$$\|u-u^h\|_m \leqq c h^{m+\frac{1}{2}} \sup_{x\in\Omega} |f(x)| + c' h^{m+1} \|f\|_{1,\Omega-\omega}. \tag{15}$$

Remark. We conjecture that a more careful analysis can reduce the right side of estimate (15) to: $ch^{m+\frac{1}{2}}\left(\sup_{x\in\Omega} |f(x)| + \|f\|_{\frac{1}{2},\Omega-\omega}\right)$. However, we do not know (but would like to) if the required approximation order $2m+1$ can be reduced to $2m+\frac{1}{2}$. For example, the space of piecewise quadratics that are zero outside a polygon inscribed in Ω approximate to order $\frac{5}{2}$ in $\mathring{H}^1(\Omega)$, but we are unable to prove $\|u-u^h\|_1 = O(h^{\frac{3}{2}})$ for discontinuous f using these elements.

Proof of the Theorem. As in the previous section, we approximate f by a sequence f_h with the properties:

a) $(f, v) = (f_h, v)$ for every v in S^h.

b) $\|f_h\|_1 \leqq c h^{-\frac{1}{2}} \sup_{x\in\Omega} |f(x)| + c' \|f\|_{1,\Omega-\omega}$.

c) $\|f-f_h\|_0 \leqq c h^{\frac{1}{2}} \sup |f(x)| + c' h \|f\|_{1,\Omega-\omega}$.

Define U^h as the solution to (1) with data f_h. Using the variational form of (1) and a), we again find that:

$$a(U^h - u^h, v) = 0 \quad \text{for all } v \text{ in } S^h,$$

so that u^h is also the finite element approximation to U^h. Thus by Lemma 1:

$$\|U^h - u^h\|_m \leqq c h^{m+1} \|f_h\|_1. \tag{16}$$

Using the coerciveness of $a(\cdot, \cdot)$, we have:

$$\gamma \|u - U^h\|_m^2 \leqq |a(u-U^h, u-U^h)| = |(f-f_h, u-U^h)|.$$

By a), $(f-f_h, u-U^h) = (f-f_h, u-U^h-v)$ for any v in S^h. Using the approximation assumption only to order $2m$, we find:

$$|(f-f_h, u-U^h)| \leqq c h^m \|f-f_h\|_0 \|u-U^h\|_m.$$

Dividing by $\|u-U^h\|_m$, we get:

$$\|u-U^h\|_m \leqq c h^m \|f-f_h\|_0. \tag{17}$$

In view of properties b) and c) and the triangle inequality, (16) and (17) combine to yield the theorem.

The construction of f_h begins with a smoothing. We let η be a positive C^∞ function with support in $|x| \leqq 1$ such that $\int_{R^n} \eta = 1$. We extend f outside Ω by zero, and define an intermediate $\tilde{f}_h$ by convolution:

$$\tilde{f}_h = f * \eta^h.$$

The notation η^h means $\eta^h(x) = h^{-n}\eta\left(\frac{x}{h}\right)$. We will first show that $\tilde{f}_h$ satisfies b) and c), although it does not satisfy a). We will have to perturb it slightly to satisfy a).

Let $\Omega^h = \{x \in \Omega : d(x, \partial\Omega) \geqq h\}$. Then because convolution with η^h reproduces constants:

$$\|f - \tilde{f}_h\|_{s, \Omega^h - \omega^h} \leqq c h^{1-s} \|f\|_{1, \Omega - \omega}, \quad \text{for} \quad 0 \leqq s \leqq 1. \tag{18}$$

To get an estimate in the complement of $\Omega^h - \omega^h$, observe that:

$$\sup_{y \in \Omega} |f * \eta^h(y)| \leqq \sup_{x \in \Omega} |f(x)|.$$

Squaring and integrating over $\Omega - \Omega^h \cup \omega^h$ we thus have:

$$\|f - \tilde{f}_h\|_{0, \Omega - \Omega^h \cup \omega^h} \leqq c h^{\frac{1}{2}} \sup_{x \in \Omega} |f(x)| \tag{19}$$

because the area of $\Omega - \Omega^h \cup \omega^h$ is $O(h)$. To estimate derivatives in $\Omega - \Omega^h \cup \omega^h$, we write:

$$D^\alpha f_h = f * D^\alpha \eta^h = f * \frac{1}{h} (D^\alpha \eta)^h$$

where $(D^\alpha \eta)^h(x) = h^{-n} D^\alpha \eta\left(\frac{x}{h}\right)$. Thus:

$$\|f_h\|_{1, \Omega - \Omega^h \cup \omega^h} \leqq c h^{-\frac{1}{2}} \sup_{x \in \Omega} |f(x)|. \tag{20}$$

(18), (19), and (20) prove that $\tilde{f}_h$ satisfies b) and c). Now we modify it to satisfy a).

We want a function g_h such that

a′) $(f - \tilde{f}^h, v) = (g_h, v)$ for v in S^h.

b′) $\|g_h\|_0 \leqq c h^{+\frac{1}{2}} \sup_{x \in \Omega} f(x) + c' h \|f\|_{1, \Omega - \omega}$.

c′) $\|g_h\|_1 \leqq c h^{-\frac{1}{2}} \sup_{x \in \Omega} f(x) + c' \|f\|_{1, \Omega - \omega}$.

We define g_h on each element e_j and then sum, making g_h vanish at the edges of e_j so that this sum remains smooth. $g_h | e_j$ will lie in $\mathring{H}^1(e_j)$, so we can use Poincare's inequality:

$$\|g_h\|_0 \leqq c h \|g_h\|_1$$

to prove b′) from c′). It is obvious that $f_h = \tilde{f}_h + g_h$ satisfies a)–c). The construction of g_h is via the following lemma:

Lemma 4. *Suppose F is a bounded function on e_{ref}. Then we may find two functions φ^1, φ^2 in $\mathring{H}^1(e_{\mathrm{ref}})$ such that:*

$$\sup_{\substack{x \in e_{\mathrm{ref}} \\ |\alpha| \leqq 1}} |D^\alpha \varphi^1(x)| \leqq C^1 \sup_{x \in e_{\mathrm{ref}}} |F(x)|, \tag{21}$$

$$\|\varphi^2\|_{1, e_{\mathrm{ref}}} \leqq C^2 \|F\|_{1, e_{\mathrm{ref}}} \tag{22}$$

and such that $\int_{e_{\mathrm{ref}}} \varphi^i P = \int_{e_{\mathrm{ref}}} F P$ for any polynomial P of degree K.

Proof. Let E^1 be the completion of $C_0^\infty(e_{\text{ref}})$ in the C^1 norm given in (21), and let E^2 be $\mathring{H}^1(e_{\text{ref}})$. E^1 is contained in $\mathring{H}^1(e_{\text{ref}})$. Let $\mathscr{P}_K = \{\text{polynomials of degree } K\}$, and define:

$$N^i = \{\varphi \in E^i \colon \int \varphi P = 0 \text{ for all } P \text{ in } \mathscr{P}_K\}.$$

The function F defines a linear form on $\mathscr{P}_K$:

$$P \to \int_{e_{\text{ref}}} FP$$

and this gives a (continuous) mapping of $L^\infty(e_{\text{ref}})$ onto $\mathscr{P}_K^*$. Taking any φ in E^i, we may similarly define a linear form:

$$P \to \int_{e_{\text{ref}}} \varphi P,$$

and this gives a (continuous, open) mapping of E^i onto $\mathscr{P}_K^*$.

Taking quotients, we have

$$E^i/N^i \cong \mathscr{P}_K^*.$$

Composing, we have a continuous map $F \to \overline{\varphi}^i$ of $L^\infty(e_{\text{ref}})$ onto E^i/N^i, where for any $\varphi^i \in \overline{\varphi}^i$:

$$\int_{e_{\text{ref}}} FP = \int_{e_{\text{ref}}} \varphi^i P \quad \text{for all } P \text{ in } \mathscr{P}_K.$$

The norm of $\overline{\varphi}^i$ in E^i/N^i is simply the infimum, over all of φ^i in $\overline{\varphi}^i$, of the norm of φ^i. Since E^i is complete, this infimum is taken on by some φ^i, and this φ^i satisfies the requirements of the lemma.

To construct g_h on e_j, we set $F = (f - \tilde{f}_h) \circ T_j^{-1}$ and

$$g_h = \begin{cases} \varphi^1 \circ T_j & \text{if} \quad e_j \cap (\Omega - \Omega^h \cup \omega^h) \neq \emptyset \\ \varphi^2 \circ T_j & \text{if} \quad e_j \subset \Omega^h - \omega^h. \end{cases}$$

In elements e_j touching $\Omega - \Omega^h \cup \omega^h$, we have:

$$\sup_{\substack{x \in e_j \\ |\alpha| \leq 1}} |D^\alpha g_h(x)| \leq c h^{-1} \sup_{x \in e_j} |f(x) - \tilde{f}_h(x)| \leq c' h^{-1} \sup_{x \in \Omega} |f(x)|,$$

using (21). (The Jacobian of T_j introduces the factor ch^{-1}.) Away from ω and $\partial\Omega$, we use (22):

$$\|g_h\|_{1, e_j} \leq C^2 \|f - \tilde{f}_h\|_{1, e_j}.$$

The area covered by elements near $\partial\Omega$ and ω is $O(h)$, thus:

$$\begin{aligned} \|g_h\|_1 &\leq c h^{-\frac{1}{2}} \sup_{x \in \Omega} |f(x)| + c' \|f - f_h\|_{1, \Omega^h - \omega^h} \\ &\leq c h^{-\frac{1}{2}} \sup_{x \in \Omega} |f(x)| + c' \|f\|_{1, \Omega - \omega} \end{aligned}$$

and this proves c′).

4. Approximation of Singular Functions

In this section we will prove a general theorem for spline approximation that implies the results in Sections 2 and 3 for a regular mesh. We hope this theorem will make clear the "reason" that an $\varepsilon > 0$ does not appear in those error estimates. The "reason" becomes clearer after a Fourier transformation. If, for example, u solves $\Delta u = \delta$, then its Fourier transform $\hat{u}(\xi)$ behaves like $|\xi|^{-2}$ at infinity. Thus u has only $2 - \frac{n}{2} - \varepsilon$ Sobolev derivatives, even though the behavior of $\hat{u}$ at infinity is fairly regular. We introduce a new norm to reflect this type of behavior more accurately. Let

$$[u]_s^2 = \sup_{\xi \in R^n} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^{s + \frac{n}{2}}.$$

Notice that for any $\varepsilon > 0$, $\|u\|_{s-\varepsilon} \leqq c_\varepsilon [u]_s$, but $c_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We will say that a function Φ *generates splines of degree* k if for all $|\alpha| < k$:

$$\sum_{j \in \mathscr{Z}^n} j^\alpha \Phi(x - j) = x^\alpha. \tag{23}$$

If we define S^h to be functions of the form:

$$v(x) = \sum_{j \in \mathscr{Z}^n} c_j \Phi\left(\frac{x}{h} - j\right)$$

then there is a well-known approximation theory for S^h in terms of Sobolev norms (e.g., [5] on which the ideas of this section are based). However, we will derive a different approximation theorem for S^h in terms of our new norm, namely:

Theorem 3. *If $\Phi \in H^s(R^n)$ has compact support and generates splines of degree k, then:*

$$\inf_{v \in S^h} \|u - v\|_s \leqq c_s h^{r-s} [u]_r \tag{24}$$

as long as $s < r \leqq k - \frac{n}{2}$.

Proof. Following [5], we choose coefficients q_α such that[3]:

$$\sum_{|\alpha| < k} q_\alpha \xi^\alpha \hat{\Phi}(\xi) = 1 + O(|\xi|^k) \quad \text{near} \quad \xi = 0.$$

If $v(x) = \sum c_j \Phi\left(\frac{x}{h} - j\right)$, then:

$$\hat{v}(\xi) = (h^n \sum c_j e^{-i\xi \cdot jh}) \hat{\Phi}(\xi).$$

To define the approximation to u, we set:

$$h^n \sum c_j e^{-i\xi \cdot jh} = \hat{u}(\xi) \sum_\alpha q_\alpha (h\xi)^\alpha \quad \text{in} \quad \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n,$$

and let $v(x) = \sum c_j \Phi\left(\frac{x}{h} - j\right)$. We claim that v satisfies (24).

If C/h denotes the cube $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n$, then we have to estimate three integrals:

$$I_1 = \int_{C/h} |\hat{u}(\xi) - \hat{v}(\xi)|^2 (1 + |\xi|^2)^s d\xi$$

$$I_2 = \int_{R^n - C/h} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi$$

$$I_3 = \int_{R^n - C/h} |\hat{v}(\xi)|^2 (1 + |\xi|^2)^s d\xi.$$

3 We will, for simplicity, assume that $\hat{\Phi}(0) \neq 0$.

Beginning with the first, we have (c denotes different constants):

$$\begin{aligned} I_1 &= \int_{C/h} |u(\xi)|^2 |1-\sum q_\alpha (h\xi)^\alpha \Phi(h\xi)|^2 (1+|\xi|^2)^s d\xi \\ &\leqq c \int_{C/h} |\hat{u}(\xi)|^2 |h\xi|^{2(r-s)+n} (1+|\xi|^2)^s d\xi \\ &\leqq c h^{2(r-s)+n} \int_{C/h} |u(\xi)|^2 (1+|\xi|^2)^{r+n/2} d\xi \\ &\leqq c h^{2(r-s)+n} [u]_r^2 \int_{C/h} d\xi \leqq c h^{2(r-s)} [u]_r^2. \end{aligned}$$

For the second, assuming $h \leqq 1$, we have:

$$I_2 \leqq c [u]_r^2 \int_{R^n - C/h} |\xi|^{-2(r-s)-n} d\xi.$$

Introducing polar coordinates (ϱ, θ) we find:

$$\int_{R^n - C/h} |\xi|^{-2(r-s)-n} d\xi \leqq c \int_{h^{-1}}^{\infty} \varrho^{-2(r-s)-1} d\varrho = \frac{c}{2(r-s)} h^{2(r-s)} \quad \text{if} \quad r > s.$$

Thus, $I_2 \leqq c h^{2(r-s)} [u]_r^2$.

The estimate of I_3 is more subtle. By a change of variables,

$$I_3 = \int_{C/h} |u(\xi) \sum q_\alpha (h\xi)^\alpha|^2 \sum_{j \neq 0} |\Phi(h\xi + 2\pi j)|^2 \left(1 + \left|\xi + \frac{2\pi j}{h}\right|^2\right)^s d\xi.$$

In [5], it is shown that

$$\sum_{j \neq 0} |\hat{\Phi}(h\xi + 2\pi j)|^2 |h\xi + 2\pi j|^{2s} = O(|h\xi|^{2k}).$$

Therefore:

$$I_3 \leqq c h^{2(r-s)+n} \int_{C/h} |\hat{u}(\xi)|^2 |\xi|^{2r+n} \leqq c h^{2(r-s)} [u]_r^2.$$

Summing these estimates and taking a square root yields the theorem.

Bibliography

1. Babuska, I.: Error bounds for the finite element method. Numer. Math. **16**, 322–333 (1971)
2. Babuska, I., Aziz, A. K.: Survey lectures on the mathematical foundations of the finite element method. Section 6.3.6, Proc. of the Conf. on the Mathematical Foundations of the Finite Element Method. New York: Academic Press 1973
3. Fried, Isaac, Yang, Shok Keng: Best finite element distribution around a singularity. AIAA Journal **10**, 1244–1246 (1972)
4. Lions, J. L., Magenes, E.: Problemes aux limites non homogenes et applications, vol. 1. Paris: Dunod 1968
5. Strang, G., Fix, G.: A fourier analysis of the finite element method. Proc. of the CIME summer school (1971), to appear

Ridgway Scott
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
USA