Inventiones math. 28, 193–208 (1975)

Stratification of Real Analytic Mappings and Images

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Contents

1. Introduction

Here it is shown that the image I of a semianalytic set under a proper (real) analytic mapping of (real) analytic manifolds admits a locally-finite partition $\mathscr{P}$ into connected submanifolds P such that

$$Q \subset \operatorname{Clos} P \quad \text{and} \quad \dim Q < \dim P \qquad \text{whenever} \quad P \neq Q \in \mathscr{P} \quad \text{and} \quad Q \cap \operatorname{Clos} P \neq \varnothing .$$

Such a partition is called a *stratification* of I. A subset A of an analytic manifold M is called *semianalytic* if M can be covered by open sets U such that $U \cap A$ is a union of connected components of sets $g^{-1}\{0\} \sim h^{-1}\{0\}$ for g and h belonging to some finite family of real-valued functions analytic in U. Although semianalytic sets admit, by well-known arguments (3.2 or [6, pp. 150–153]), stratifications, the analytic image of even a compact analytic manifold may fail to be semianalytic ([6, p. 135]).

For any proper analytic mapping f: $M \to N$, there are, by 4.4, stratifications $\mathscr{S}$ of M and $\mathscr{T}$ of N so that for each $S \in \mathscr{S}, f(S) \in \mathscr{T}, f|S$ is one-one whenever $\dim f(S) = \dim S$, and $S \subset A$ whenever $S \cap A \neq \varnothing$. It follows from 4.0, 4.2, and 2.5 that the number of components of $A \cap f^{-1}\{y\}$ is locally bounded for $y \in N$. These stratifications may be refined to satisfy Whitney condition (B) ([8, § 3]); the proof, being completely analogous with Lojasiewicz's treatment of the semianalytic case [6, pp. 143–153] is omitted. In [18, 3.1, 3.3] such stratifications of images under consecutive projections in Euclidean space lead readily to CW decompositions.

Stratification theory has developed chiefly through the work of H. Whitney, R. Thom, and J. Mather. An excellent treatment and list of references is given in [8]. It is the goal of this paper to clear up the ideas of [11, III B–C] where many of the present results are stated without formal proof. Although we will, except in § 5, refer only to the classical local stratification of semianalytic sets as established in [4, § 11–15] or [6, § 13], other interesting facts about semianalytic sets are exposed in [1, 3.4.5–3.4.11], [2], [3, § 2], [4–7], [8, § 4], [12–18], [6] being the most basic.

* Research partially supported by National Science Foundation grant GP 29321.

After having written the manuscript, the author learned of the interesting work of Hironaka, [12] and [13] on subanalytic sets, which are sets that locally are analytic images of relatively compact semianalytic sets. Using his theory of the resolution of singularities, he establishes a Whitney stratification of subanalytic sets; because of differences in proofs as well as the discussions of § 2 and § 5, our article may be of independent interest.

Our notation is in accord with [1, pp. 669–671] with the following eight conventions.

For any family $\mathscr{A}$ of sets we let $\bigcup \mathscr{A} = \bigcup_{A \in \mathscr{A}} A$ and $\bigcap \mathscr{A} = \bigcap_{A \in \mathscr{A}} A$.

For any two families $\mathscr{A}$ and $\mathscr{B}$ of subsets of the same set, we say $\mathscr{A}$ is *compatible with* $\mathscr{B}$ if $A \subset B$ whenever $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $A \cap B \neq \varnothing$.

For any function f: $C \to D$, we identify f with the subset $\{(c, f(c))\colon c \in C\}$ of $C \times D$.

For any subset E of a topological space, we define the *frontier of* E, denoted Fron E, as (Clos E)$\sim E$.

All our manifolds are assumed to be paracompact.

By a submanifold F of a differentiable manifold we always mean a properly imbedded submanifold; hence, $F \cap \operatorname{Fron} F = \varnothing$.

For any differentiable mapping g: $M \to N$ of differentiable manifolds, we let rank g: $M \to \mathbf{Z}$,

$$(\operatorname{rank} g)(a) = \dim (Dg(a)[\operatorname{Tan}(M, a)]) \qquad \text{for } a \in M.$$

We let $\dim \varnothing = -1$, and, recall, for any nonempty subset G of a metric space, the *Hausdorff dimension* of G,

$$\dim G = \sup \{\rho\colon \mathscr{H}^\rho(G) > 0\} = \inf \{\sigma\colon \mathscr{H}^\sigma(G) = 0\},$$

where $\mathscr{H}^\rho$, for $\rho \geqq 0$, is ρ dimensional Hausdorff measure ([1, p. 171]). We also use the notation $\dim G$ whenever G is a nonempty subset of a differentiable manifold and the above formula does not depend on the choice of Riemannian metric. For example, if the function g above has rank $g \equiv k$, then $\dim g(M) = k$ because, by the rank theorem ([1, 3.1.18)], M may be covered by countably many open sets U so that $g(U)$ is a k dimensional submanifold; hence, $\mathscr{H}^\sigma[g(U)] = 0$ for $\sigma > k$ for any Riemannian metric on N.

2. Stratification and Mappings

Suppose M is a differentiable manifold. A *stratum in* M is a connected submanifold of M.

A locally-finite partition $\mathscr{S}$ of M is called a *stratification of* M if each S in $\mathscr{S}$ is a stratum such that

(*) $T \subset \operatorname{Fron} S$ and $\dim T < \dim S$ whenever $T \in \mathscr{S}$ and $T \cap \operatorname{Fron} S \neq \varnothing$.

For any mapping f: $M \to N$ of Hausdorff spaces, a *stratification of* f is a pair $(\mathscr{S}, \mathscr{T})$ such that $\mathscr{S}$ is a stratification of M, $\mathscr{T}$ is a stratification of N, and, for each $S \in \mathscr{S}$, $f(S) \in \mathscr{T}$, $f|S$ is differentiable, and rank $(f|S) \equiv \dim f(S)$.

A stratification $(\mathscr{S}, \mathscr{T})$ of f is said to be *one-one* if $f|S$ is one-one for all $S \in \mathscr{S}$ with $\dim f(S) = \dim S$.

2.1. Theorem. *Suppose $\mathcal{M}$ is a collection of differentiable manifolds, and for each $M \in \mathcal{M}$, $\mathcal{A}(M)$ is a family of subsets of M satisfying the following four conditions:*

(1) *The sets $\varnothing$, $\{x\}$, M, $\bigcup \mathcal{B}$, $\bigcap \mathcal{B}$, and $M \sim \bigcup \mathcal{B}$ belong to $\mathcal{A}(M)$ whenever $x \in M \in \mathcal{M}$ and $\mathcal{B} \subset \mathcal{A}(M)$ is locally-finite.*

(2) *If $M, N \in \mathcal{M}$, $C \in \mathcal{A}(M)$, $D \in \mathcal{A}(N)$, $E, F \in \mathcal{A}(M \times N)$, μ is the projection mapping of $M \times N$ onto M, and $\mu|\operatorname{Clos} E$ is proper, then $M \times N \in \mathcal{M}$,*

$$C \times D \in \mathcal{A}(M \times N), \qquad \mu^{-1}(D) \in \mathcal{A}(M \times N), \qquad \mu(E) \in \mathcal{A}(M),$$

and

$$(N \times M) \cap \{(y, x)\colon (x, y) \in F\} \in \mathcal{A}(N \times M).$$

(3) *If M, N and μ are as above and $G \in \mathcal{A}(M \times N)$ is a stratum, then*

$$G \cap \{a\colon \operatorname{rank}(\mu|G)(a) < \sup\{\operatorname{rank}(\mu|G)(b)\colon b \in G\}\}$$

is contained in some at most $(\dim G) - 1$ *dimensional member of $\mathcal{A}(M \times N)$.*

(4) *For any $M \in \mathcal{M}$ and locally finite $\mathcal{B} \subset \mathcal{A}(M)$ there is a stratification of M contained in $\mathcal{A}(M)$ and compatible with $\mathcal{B}$.*

Then for any $M, N \in \mathcal{M}$, any locally finite families $\mathcal{C} \subset \mathcal{A}(M)$ and $\mathcal{D} \subset \mathcal{A}(N)$, any continuous map $g\colon M \to N$, and any open $L \in \mathcal{A}(M)$ such that $g \in \mathcal{A}(M \times N)$ and $g|\operatorname{Clos} L$ is proper, there exists a stratification $(\mathcal{S}, \mathcal{T})$ of $f = g|L$ such that $\mathcal{S} \subset \mathcal{A}(M)$, $\mathcal{T} \subset \mathcal{A}(N)$, $\mathcal{S}$ is compatible with $\mathcal{C}$, and $\mathcal{T}$ is compatible with $\mathcal{D}$.

Proof. Let $\mu\colon M \times N \to M$ and $\nu\colon M \times N \to N$ be the projection maps, and let $l = \dim(M \times N)$. Since $f = g \cap \mu^{-1}(L) \in \mathcal{A}(M \times N)$, there is by (4) stratification $\mathcal{P}_l$ of $M \times N$ compatible with $\{f\} \cup \{\mu^{-1}(C)\colon C \in \mathcal{C}\}$. Having chosen stratifications $\mathcal{P}_k, \mathcal{P}_{k+1}, \ldots, \mathcal{P}_l \subset \mathcal{A}(M \times N)$ of $M \times N$ so that $\mathcal{P}_k$ is compatible with $\mathcal{P}_{k+1}$ and the functions $\operatorname{rank}(\mu|P)$ and $\operatorname{rank}(\nu|P)$ are constant whenever $P \in \mathcal{P}_k$ and $\dim P \geqq k+1$, we choose, for each $G \in \mathcal{P}_k$ with $\dim G = k$, an atmost $k-1$ dimensional set $Z_G \in \mathcal{A}(M \times N)$ containing

$$G \cap \{a\colon \operatorname{rank}(\mu|G)(a) < \sup\{\operatorname{rank}(\mu|G)(b)\colon\ b \in G\} \text{ or}$$

$$\operatorname{rank}(\nu|G)(a) < \sup\{\operatorname{rank}(\nu|G)(b)\colon\ b \in G\}\}$$

select a stratification $\mathcal{Q} \subset \mathcal{A}(M \times N)$ of $M \times N$ compatible with

$$\mathcal{P}_k \cup \{Z_G\colon\ G \in \mathcal{P}_k, \dim G = k\},$$

and obtain the stratification

$$\mathcal{P}_{k-1} = (\mathcal{P}_k \cap \{P\colon \dim P > k\}) \cup (\mathcal{Q} \cap \{Q\colon \dim Q \leqq k\}) \subset \mathcal{A}(M \times N) \qquad \text{of } M \times N$$

compatible with $\mathcal{P}_k$ such that the functions $\operatorname{rank}(\mu|P)$ and $\operatorname{rank}(\nu|P)$ are constant whenever $P \in \mathcal{P}_{k-1}$ and $\dim P \geqq k$.

For every $P \in \mathcal{P}_0$ with $P \subset f$, $\mu|P$ is a diffeomorphism, and $\operatorname{rank}(\nu|P)$ is constant. Moreover $\{\nu(P)\colon f \supset P \in \mathcal{P}_0\}$ is a locally-finite subfamily of $\mathcal{A}(N)$ because $\nu|\operatorname{Clos} f$ is proper. Selecting a stratification $\mathcal{T} \subset \mathcal{A}(N)$ of N compatible with

$$\mathcal{D} \cup \{\nu(P)\colon f \supset P \in \mathcal{P}_0\}$$

and letting $\mathscr{S}$ be the family of all connected components of $\mu[P\cap\nu^{-1}(T)]$ where $f\supset P\in\mathscr{P}_0$ and $\nu(P)\supset T\in\mathscr{T}$, we infer from (4) that $\mathscr{S}$ is locally-finite and from (1), (2), and the rank theorem ([1, 3.1.18]) that the pair $(\mathscr{S},\mathscr{T})$ satisfies the theorem.

2.2. Theorem. *Suppose $\mathscr{M}$ and $\mathscr{A}$ satisfy 2.1* (1) (2) (3) (4) *and the three additional conditions:*

(5) $\mathbf{R}\in\mathscr{M}$.

(6) *For any $M_1, M_2, M_3, \ldots, M_m\in\mathscr{M}$, $M_1\times\cdots\times M_m\in\mathscr{M}$, and the function which sends $(x_1,\ldots,x_m)\in M_1\times\cdots\times M_m$ onto*

$$((\cdots((x_1,x_2),x_3),\ldots),x_m)\in(\cdots((M_1\times M_2)\times M_3)\times\cdots)\times M_m$$

induces a bijection between

$$\mathscr{A}(M_1\times\cdots\times M_m)\quad\textit{and}\quad\mathscr{A}((\cdots((M_1\times M_2)\times M_3)\times\cdots)\times M_m).$$

(7) *For every $M\in\mathscr{M}$, $\{(x,x)\colon x\in M\}\in\mathscr{A}(M\times M)$ and there is a locally finite covering $\mathscr{K}$ of M by compact sets K in $\mathscr{A}(M)$ for which there is a continuous one-one map $\varphi\colon K\to\mathbf{R}^n$ for some n such that $\varphi\in\mathscr{A}(M\times\mathbf{R}^n)$.*

Then for L, M, N, $\mathscr{C}$, $\mathscr{D}$, f, and g as in 2.1, there exists a one-one stratification $(\mathscr{S},\mathscr{T})$ of f satisfying the conclusions of 2.1.

Proof. Replacing $\mathscr{C}$ by $\mathscr{C}\cup\mathscr{K}$, we assume $\mathscr{K}\subset\mathscr{C}$. We will verify inductively that there exists, for each $k\in\{0,1,\ldots,\dim M\}$, a stratification $(\mathscr{S}_k,\mathscr{T}_k)$ of f such that $\mathscr{S}_k\subset\mathscr{A}(M)$ is compatible with $\mathscr{C}$, $\mathscr{T}_k\subset\mathscr{A}(N)$ is compatible with $\mathscr{D}$, and $f|S$ is one-one whenever $S\in\mathscr{S}_k$ and $\dim f(S)=\dim S\leq k$. With $(\mathscr{S},\mathscr{T})$ as in 2.1, let $\mathscr{T}_0$ be a stratification of N compatible with $\mathscr{T}\cup\{g(\operatorname{Fron}S)\colon S\in\mathscr{S}\}$ and $\mathscr{S}_0$ be the family of components of $S\cap f^{-1}(T)$ for $S\in\mathscr{S}$ and $T\in\mathscr{T}_0$. Let

$$\mathscr{R}=\mathscr{S}_0\cap\{S\colon\dim f(S)=\dim S=k\}.$$

For each $R\in\mathscr{R}$, $(f|R)\colon R\to f(R)$ is a differentiable covering map because $g(\operatorname{Fron}R)$, having dimension less than k, does not intersect $f(R)$. Moreover, $\mathscr{R}\cap\{S\colon f(S)=f(R)\}$ is a finite family, and there is an integer $j(R)$ so that

$$\operatorname{card}(R\cap f^{-1}\{y\})=j(R)\qquad\text{whenever } y\in f(R).$$

Letting $p_R\colon M^{j(R)}\to M$, $p_R(x_1,\ldots,x_{j(R)})=x_1$ for $(x_1,\ldots,x_{j(R)})\in M^{j(R)}$, $f_R=f\circ p_R$, and

$$G_R=R^{j(R)}\cap\{(x_1,\ldots,x_{j(R)})\colon f(x_1)=\cdots=f(x_{j(R)})\text{ and } x_h\neq x_i\text{ for } 1\leq h<i\leq j(R)\},$$

we infer that $f(R)=f_R(G_R)$, $p_R|\operatorname{Clos}G_R$ is proper, and $G_R\in\mathscr{A}(M^{j(R)})$ by (6), (7), and 2.1 (1) (2). By (7) we may choose an integer $n(R)$ and a continuous one-one map $\varphi_R\colon\operatorname{Clos}R\to\mathbf{R}^{n(R)}$ with $\varphi_R\in\mathscr{A}(M\times\mathbf{R}^{n(R)})$. For integers $n\in\{1,2,\ldots,n(R)\}$, $i\in\{1,\ldots,j(R)\}$, and $j\in\{1,\ldots,j(R)\}$, we let $\varphi^R_{n,i,j}\colon R^{j(R)}\to\mathbf{R}$ be given by

$$\varphi^R_{n,i,j}(x_1,\ldots,x_{j(R)})=\mathbf{e}_n\cdot[\varphi_R(x_i)-\varphi_R(x_j)]\qquad\text{for } (x_1,\ldots,x_{j(R)})\in R^{j(R)};$$

hence $\varphi^{R-1}_{n,i,j}\{0\}\in\mathscr{A}(M^{j(R)})$ by (5), (6), (7), and 2.1 (1) (2). Let $\mathscr{Q}_R\subset\mathscr{A}(M^{j(R)})$ be a stratification of $M^{j(R)}$ compatible with

$$\{G_R\}\cup\{\varphi^{R-1}_{n,i,j}\{0\}\colon n\in\{1,\ldots,n(R)\};\, i,j\in\{1,\ldots,j(R)\}\}.$$

Then $\mathscr{E}=\{p_R(Q): R\in\mathscr{R}, G_R\supset Q\in\mathscr{Q}_R\}$ is a locally finite subset of $\mathscr{A}(M)$, and we may, by induction, choose a stratification $(\mathscr{S}_k, \mathscr{T}_k)$ of f such that $\mathscr{S}_k\subset\mathscr{A}(M)$ is compatible with $\mathscr{S}_0\cup\mathscr{E}$ (and hence with $\mathscr{C}$), $\mathscr{T}_k\subset\mathscr{A}(N)$ is compatible with $\mathscr{D}$, and

$$f|S \text{ is one-one whenever } S\in\mathscr{S}_k \text{ and } \dim f(S)=\dim S\leqq k-1.$$

Suppose $S\in\mathscr{S}_k$ and $\dim f(S)=\dim S=k$. Then $S\subset R\cap p_R(Q)$ for some $R\in\mathscr{R}$ and $G_R\supset Q\in\mathscr{Q}_R$. To show that $f|S$ is one-one, and complete the proof, we verify that $f_R|Q$ is one-one.

If not, there is a point $y\in f(R)$, a set $R\cap f^{-1}\{y\}=\{x_1, \ldots, x_{j(R)}\}$, and a permutation σ of $\{1, 2, \ldots, j(R)\}$ different from the identity so that $(x_1, \ldots, x_{j(R)})$ and $(x_{\sigma(1)}, \ldots, x_{\sigma(j(R))})$ both belong to Q. Accordingly $\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(j)})-\varphi_R(x_j)]\neq 0$ for some $j\in\{1, \ldots, j(R)\}$ and $n\in\{1, \ldots, n(R)\}$. Defining

$$I=\{1, \ldots, j(R)\}\cap\{i: \operatorname{sign}\mathbf{e}_n\cdot[\varphi_R(x_i)-\varphi_R(x_j)]=\operatorname{sign}\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(j)})-\varphi_R(x_j)]\},$$

we observe that $j\notin I$, that $\sigma(j)\in I$, and that for each $i\in\{1, \ldots, j(R)\}$,

$$\operatorname{sign}\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(i)})-\varphi_R(x_{\sigma(j)})]=\operatorname{sign}\mathbf{e}_n\cdot[\varphi_R(x_i)-\varphi_R(x_j)]$$

because Q is connected and $\{Q\}$ is compatible with $\{\varphi_{n,i,j}^{R-1}\{0\}\}$. Using the equation

$$\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(i)})-\varphi_R(x_j)]=\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(i)})-\varphi_R(x_{\sigma(j)})]+\mathbf{e}_n\cdot[\varphi_R(x_{\sigma(j)})-\varphi_R(x_j)]$$

for every $i\in I$, we infer that $\sigma(I)\subset I$, hence $\sigma(I)=I$, which contradicts that $\sigma(j)\in I\sim\sigma(I)$.

2.3. *Applications.* By [10, Theorem 1], Theorem 2.2 may be used with $\mathscr{M}$ equal to the class of Euclidean spaces and $\mathscr{A}(M)$ equal to the family of semialgebraic subsets of M.

By [9, p. 67, Theorem 1] and the proper mapping theorem ([9, p. 129, Theorem 2]) we may apply 2.1 with $\mathscr{M}$ being the class of paracompact complex manifolds and $\mathscr{A}(M)$ being the smallest family of sets satisfying 2.1 (1) and containing each connected component of the regular points of every holomorphic subvariety of M. However, here the conclusion of Theorem 2.2 fails because holomorphic mappings may not admit one-one stratifications in $\mathscr{A}(M)$. For example, since the complement of any two dimensional strata S in $\mathscr{A}(\mathbf{C})$ is a discrete set, the restriction to S of the function which maps z to z^2 is not one-one.

Next suppose $\mathscr{M}$ is the class of paracompact real analytic manifolds. Inasmuch as the projection of a compact semianalytic set may fail to be semianalytic ([6, p. 135]), we may not use Theorem 2.1 with $\mathscr{A}(M)$ being the class $\mathscr{S}(M)$ of semianalytic subsets of M. In §4 we will verify 2.1 (3) (4) for the smallest collection of families $\mathscr{P}(M)\supset\mathscr{S}(M)$ for $M\in\mathscr{M}$ satisfying 2.1 (1) (2).

2.4. Lemma. *Suppose M is a connected m dimensional Riemannian manifold, $0\leqq k\leqq m-1$, E is a closed $\mathscr{H}^k$ null subset of M, $\mathscr{F}$ is a finite disjointed family of k dimensional connected submanifolds in $M\sim E$ whose frontiers lie in E, and $\mathscr{G}$ is the family of components of $M\sim(E\cup\bigcup\mathscr{F})$. Then* $\operatorname{card}\mathscr{G}\leqq\sup\{1, 2\operatorname{card}\mathscr{F}\}$, *and $F\subset\operatorname{Clos}G$ whenever $F\in\mathscr{F}$, $G\in\mathscr{G}$, and $F\cap\operatorname{Clos}G\neq\varnothing$.*

Proof. First note that if X is a closed $\mathscr{H}^{m-1}$ null subset of M, then any two points a, b in $M\sim X$ may be joined by a curve in $M\sim X$. In fact, if M is an open ball in Euclidean space and C is an open ball about b in $M \sim X$, then there is, by the argument of [3, 2.7, p. 83] a closed half-line outside of X which joins a with C. In general there are open coordinate balls $U_1, U_2, \ldots, U_n$ in M with $a \in U_1$, $U_1 \cap U_2 \neq \varnothing$, $U_2 \cap U_3 \neq \varnothing, \ldots, U_{n-1} \cap U_n \neq \varnothing$, and $b \in U_n$.

In case $k < m-1$, it follows that $M \sim (E \cup \bigcup \mathscr{F})$ is connected. Moreover any $F \in \mathscr{F}$, being a submanifold of $[M \sim (E \cup \bigcup \mathscr{F})] \cup F$, lies in $\operatorname{Clos}[M \sim (E \cup \bigcup \mathscr{F})]$.

In case $k = m-1$, it suffices to modify the argument of [3, 2.7] by replacing $\mathbf{R}^{t+1}, f(A) \sim X$, and X by M, $\bigcup \mathscr{F}$, and E and noting in the proof of (2) that, instead of a half line L, there is a curve in $M \sim E$ joining b with C.

2.5. Theorem. *If $\mathscr{M}$ and $\mathscr{A}$ are as in 2.2, $M, N \in \mathscr{M}$, g: $M \to N$ is continuous, $g \in \mathscr{A}(M \times N)$, and $A \in \mathscr{A}(M)$ is relatively compact, then the number of components of $A \cap g^{-1}\{y\}$ is bounded for $y \in N$.*

Proof. We use induction on $\dim A$. In case $\dim A \leqq 0$, A is finite by 2.1 (4). We now assume $l = \dim A \geqq 1$ and the corollary is true for dimensions less than l. Choose, by 2.1, a stratification $(\mathscr{S}, \mathscr{T})$ of $g|\operatorname{Clos} A$ so that $\mathscr{S} \subset \mathscr{A}(M)$ is compatible with $\{A\}$. By induction it now suffices to show that the number of components of $S \cap g^{-1}\{y\}$ is bounded for $y \in N$ whenever $A \supset S \in \mathscr{S}$ and $\dim S = l$. Let $k = \dim g(S)$.

Suppose for contradiction that there is a countable subset Y of $g(S)$ so that the number of components of $S \cap g^{-1}\{y\}$ is unbounded for $y \in Y$. We may assume, by 2.2 (7), that M and N are Euclidean spaces and, by replacing M, S, and g by $M \times N$, $\{(x, g(x)): x \in S\}$, and the projection of $M \times N$ onto N, that g is an orthogonal projection of $\mathbf{R}^m$ onto $\mathbf{R}^n$ for some m and n.

Let p: $\mathbf{R}^n \to \mathbf{R}^k$ be an orthogonal projection so that $p(\operatorname{Tan}[g(S), y]) = \mathbf{R}^k$ whenever $y \in Y$. Since, by the rank theorem, the points of Y have disjoint neighborhoods relative to $g(S)$ which are mapped homeomorphically by p, the number of components of $S \cap (p \circ g)^{-1}\{p(y)\}$ is unbounded for $y \in Y$.

Fix a point $x \in S \cap g^{-1}(Y)$. Since $(p \circ g)[\operatorname{Tan}(S, x)] = \mathbf{R}^k$, there are orthogonal projections h: $\mathbf{R}^m \to \mathbf{R}^l$ and q: $\mathbf{R}^l \to \mathbf{R}^k$ so that $q \circ h = p \circ g$ and $h[\operatorname{Tan}(S, x)] = \mathbf{R}^l$. By 2.1 (3), 2.2, there is a stratification $(\mathscr{Q}, \mathscr{R})$ of h so that $\mathscr{Q}$ is contained in $\mathscr{A}(\mathbf{R}^m)$, $\mathscr{Q}$ is compatible with $\{S\}$, and $h|Q$ is one-one whenever $Q \in \mathscr{Q}$ and $\dim Q = l$. Then, for some $S \supset Q \in \mathscr{Q}$, the number of components of

$$Q \cap (q \circ h)^{-1}\{p(y)\} = Q \cap (p \circ g)^{-1}\{p(y)\}$$

is unbounded for $y \in Y$. By induction, $l = \dim Q = \dim h(Q)$, and the number of components of $h(Q) \cap q^{-1}\{p(y)\}$ is also unbounded for $y \in Y$. If $\mathscr{B}$ is a stratification of $\operatorname{Fron} h(Q)$ so that $\operatorname{rank}(q|B)$ is constant for all $B \in \mathscr{B}$, then we may, by the rank theorem, apply, for each $y \in Y$, 2.4 with $M = q^{-1}\{p(y)\}$,

$$\mathscr{F} = \{\text{components of } B \cap q^{-1}\{p(y)\}\colon B \in \mathscr{B} \text{ and } \dim B - \operatorname{rank}(q|B) = l - k - 1\},$$

and $E = \operatorname{Fron} h(Q) \sim \bigcup \mathscr{F}$; we conclude that the number of components of $B \cap q^{-1}\{p(y)\}$ for $y \in Y$ is unbounded for at least one $B \in \mathscr{B}$, contradicting the inductive assumption, with A and g replaced by B and q.

3. Semianalytic Sets

3.1. Lemma. *Suppose $\mathscr{A}$ is a family of subsets of a locally compact space M which is closed under finite union and finite intersection and $\mathscr{S}$ is the family of all S*

such that M can be covered by open sets U so that $U \cap S$ is a union of connected components of sets $A \sim B$ for A and B belonging to some finite subfamily of $\mathscr{A}$. If $\mathscr{B} \subset \mathscr{S}$ is locally-finite, then $\bigcup \mathscr{B} \in \mathscr{S}$, $\bigcap \mathscr{B} \in \mathscr{S}$, and $M \sim \bigcup \mathscr{B} \in \mathscr{S}$.

Proof. Clearly $\bigcup \mathscr{B} \in \mathscr{S}$. To see that $\bigcap \mathscr{B} \in \mathscr{S}$, we observe that if E is a component of $A \sim B$ and F is a component of $C \sim D$, then $E \cap F$ is open and closed relative to – hence a union of connected components of – $(A \sim B) \cap (C \sim D) = (A \cap C) \sim (B \cup D)$. Moreover,

$$M \sim E = (M \sim A) \cup B \cup \bigcup \{\text{components of } A \sim B \text{ other than } E\}.$$

Thus $M \sim \bigcup \mathscr{B} \in \mathscr{S}$.

3.2. Inasmuch as the product and the sum of the squares of two real-valued analytic functions is analytic, hypothesis 2.1 (1) holds, by 3.1, for the class $\mathscr{S}(M)$ of semianalytic subsets of M. Since the cartesian product and composition of analytic mappings is analytic, 2.2 (5) (6) (7) and 2.1 (2) – excluding that $\mu(E) \in \mathscr{S}(M)$ – are easily verified for $\mathscr{S}(M)$. Moreover 2.1 (3) (4) for $\mathscr{S}(M)$ will follow from 3.2 (3) (5) below.

Recall from [3, §2] that a subset G of an m dimensional analytic manifold, which is a connected component of $g^{-1}\{0\} \sim h^{-1}\{0\}$ for some $\mathbf{R}^{m-l}$-valued function g and $\mathbf{R}$-valued function h analytic in a neighborhood of $\operatorname{Clos} G$ such that $(\operatorname{rank} g) | G \equiv m - l$ is called an l dimensional *analytic block* in M.

We will need the following five facts which are consequences of Lojasiewicz's local decomposition of semianalytic sets ([4, §15] or [6, §13]).

(1) *If A is a semianalytic subset of M, then*

$$\dim A = \sup\{k\colon \text{there is an open subset } U \text{ of } M \text{ so that } U \cap A \text{ is a } k \text{ dimensional analytic submanifold}\},$$

and $\operatorname{Fron} A$ *is semianalytic with* $\dim \operatorname{Fron} A \leq (\dim A) - 1$.

(2) *If M is connected and E is a proper analytic subset of M, then* $\dim E \leq (\dim M) - 1$.

In fact, if $\dim E = \dim M$, then $\operatorname{Int} E \neq \varnothing$ by (1) and $E = M$ by analytic continuation.

(3) *If M and N are analytic manifolds, $\mu\colon M \times N \to M$ is the projection map, and G is a connected analytic submanifold of $M \times N$, then*

$$Z_G = G \cap \{a\colon \operatorname{rank}(\mu|G)(a) < \sup\{\operatorname{rank}(\mu|G)(b)\colon b \in G\}\}$$

is a proper analytic subset of G. Moreover, if G is an analytic block in $M \times N$, then Z_G is semianalytic.

In fact both statements follow from the argument of [3, 2.9].

(4) *If A and B are semianalytic strata in an analytic manifold M with $B \subset \operatorname{Fron} A$, then there is a closed, at most* $(\dim B) - 1$ *dimensional semianalytic set Z_B^A so that every point $b \in B \sim Z_B^A$ has arbitrarily small neighborhoods W such that $W \cap B$ is contained in the closure of each component of $W \cap A$.*

In fact, if A and B are members of a Lojasiewicz normal decomposition ([6, §13]), then this condition holds at every $b \in B$. In general M may be covered by a locally-finite family $\mathscr{U}$ of open semianalytic sets U for which there is a normal decomposition $\mathscr{D}_U$ of U compatible with $\{A, B\}$; hence $U \cap B$ is the union of an atmost $(\dim B)-1$ dimensional semianalytic set Y_U and $\dim B$ dimensional members $\Gamma \in \mathscr{D}_U$ with $\Gamma \subset B \cap \operatorname{Fron} \Delta$ for some $A \supset \Delta \in \mathscr{D}_U$.

(5) *For any locally-finite family $\mathscr{A}$ of semianalytic sets in M, there is a stratification of M into relatively compact analytic blocks which is compatible with $\mathscr{A}$.*

In fact, it will be sufficient to prove, by induction on $\dim \bigcup \mathscr{A}$, that there is a stratification $\mathscr{S}$ of M compatible with $\mathscr{A}$ so that each $S \in \mathscr{S}$ with $\dim S \leqq \dim \bigcup \mathscr{A}$ is a relatively compact analytic block in M; for we may then replace $\mathscr{A}$ by $\mathscr{A} \cup \{M\}$. In case $\dim \bigcup \mathscr{A} = -1$; hence $\mathscr{A} = \varnothing$, let $\mathscr{S} = \{M\}$. We now assume that $0 \leqq l = \dim \bigcup \mathscr{A} \leqq m = \dim M$, and, by (1), that $\bigcup \mathscr{A}$ is closed.

For each $a \in \bigcup \mathscr{A}$, there are, by [4, p. 34] or [6, p. 68], an open semianalytic neighborhood U_a of a and analytic functions $g_a\colon U_a \to \mathbf{R}^{m-l}$ and $d_a\colon U_a \to \mathbf{R}$ so that $U_a \cap \bigcup \mathscr{A} \subset g_a^{-1}\{0\}$, $(\operatorname{rank} g_a)(x) = m - l$ whenever $d_a(x) \neq 0$, $\dim(d_a^{-1}\{0\} \cap \bigcup \mathscr{A}) < l$, and the collection $\mathscr{C}_a$ of components of $U_a \cap [(\bigcup \mathscr{A}) \sim d_a^{-1}\{0\}]$ is finite and compatible with $\mathscr{A}$. By [3, 2.2 (7)], there are, for $i \in \{1, 2, \ldots\}$, $a_i \in M$ and $r_i > 0$ so that $V_i = \mathbf{U}(a_i, r_i)$ has compact closure in U_{a_i}, the dimension of the frontier of

$$R_C = C \cap \left(V_i \sim \bigcup_{j=1}^{i-1} \operatorname{Clos} V_j \right)$$

is less than l whenever $C \in \mathscr{C}_{a_i}$, and $\{V_i\colon i = 1, 2, \ldots\}$ is a locally-finite cover of M. Choosing, by induction, a semianalytic stratification $\mathscr{T}$ of M compatible with

$$\{A \cap U_{a_i} \cap d_{a_i}^{-1}\{0\}\colon A \in \mathscr{A},\ i = 1, 2, \ldots\} \cup \{\operatorname{Fron} R_C\colon C \in \mathscr{R}_{a_i},\ i = 1, 2, \ldots\}$$

so that every $T \in \mathscr{T}$ with $\dim T < l$ is a relatively compact analytic block, it suffices, by 2.4, to let

$$\mathscr{R} = \{R_C\colon C \in \mathscr{C}_{a_i} \text{ and } i = 1, 2, \ldots\},$$

$$\mathscr{S} = [\mathscr{T} \cap \{T\colon T \subset (\bigcup \mathscr{A}) \sim \bigcup \mathscr{R}\}] \cup \mathscr{R} \cup \{\text{connected components of } M \sim \bigcup \mathscr{A}\}.$$

4. Semianalytic Shadows

A subset C of an analytic manifold M is called a *semianalytic shadow* if M can be covered by open sets U such that $U \cap C$ is a union of sets $\mu(A) \sim \mu(B)$ for some analytic manifold N with projection map $\mu\colon M \times N \to M$ and A and B belonging to some finite family of relatively compact semianalytic subsets of $M \times N$. (Using a resolution of singularities, H. Hironaka has shown in [12] that any semianalytic shadow is locally the projection of some relatively compact semianalytic set; but we will not need this interesting result.) Noting that

$$\mu(A) \cup \mu'(A') = \mu''[(M \times N \times N') \cap \{(x, y, z)\colon (x, y) \in A \text{ or } (x, z) \in A'\}],$$

$$\mu(A) \cap \mu'(A') = \mu''[(M \times N \times N') \cap \{(x, y, z)\colon (x, y) \in A \text{ and } (x, z) \in A'\}]$$

whenever $A \subset M \times N$, $A' \subset M \times N'$, and

$$\mu\colon M \times N \to M, \quad \mu'\colon M \times N' \to M, \quad \text{and} \quad \mu''\colon M \times N \times N' \to M$$

are the projection maps, we infer from 3.1 and 3.2 hypothesis 2.1 (1) for the class $\mathscr{P}(M)$ of semianalytic shadows in M. Moreover using 3.2 and the equality

$$\mu'^{-1}[\mu(A)]=\rho[(M\times N\times N')\cap\{(x,y,z):(x,y)\in A\}]$$

where $\rho: M\times N\times N'\to M\times N'$ is the projection mapping, we readily verify 2.1 (2) and 2.2 (5) (6) (7) for $\mathscr{P}(M)$.

4.1. Theorem. *For any analytic manifold and finite family $\mathscr{C}\subset\mathscr{P}(M)$ with* Clos $\bigcup\mathscr{C}$ *compact, there are an analytic manifold N with projection $\mu: M\times N\to N$ and a subfamily $\mathscr{A}$ of some semianalytic stratification of $M\times N$ with $\bigcup\mathscr{A}$ compact so that*

(1) *Each $A\in\mathscr{A}$ is an analytic block with rank $(\mu|A)$ constant.*

(2) *Each $C\in\mathscr{C}$ is a union of connected components of sets $\mu(A)\sim\mu(B)$ for some $A, B\in\mathscr{A}$.*

(3) *For each pair $A, B\in\mathscr{A}$ with $B\subset\operatorname{Fron}A$, every point b in B has arbitrarily small neighborhoods W such that $W\cap B$ is contained in the closure of each component of $W\cap A$.*

Proof. There are a finite open cover $\mathscr{U}$ of Clos$\mathscr{C}$ and, for each $C\in\mathscr{C}$ and $U\in\mathscr{U}$, an analytic manifold N_U^C along with the projection mapping $\mu_U^C: M\times N_U^C\to M$ and finite family $\mathscr{A}_U^C$ of relatively compact semianalytic subsets of $M\times N_U^C$ so $U\cap C$ is a union of components of $\mu_U^C(A)\sim\mu_U^C(B)$ for some $A, B\in\mathscr{A}_U^C$. Letting $N=\prod_{C\in\mathscr{C}, U\in\mathscr{U}} N_U^C$ and, for each $C\in\mathscr{C}$ and $U\in\mathscr{U}$, $\pi_U^C: M\times N\to M\times N_U^C$ be the projection mapping and $\mathscr{B}_U^C=\{\pi_U^{C-1}(A): A\in\mathscr{A}_U^C\}$, we obtain (2) with $\mathscr{A}$ replaced by

$$\mathscr{B}=\{B: B\in\mathscr{B}_U^C, C\in\mathscr{C}, U\in\mathscr{U}\}.$$

To obtain a suitable stratification of $M\times N$ we use 3.2 (3) (4) (5) and downward induction. Specifically with $l=\dim(M\times N)$, we first choose a stratification $\mathscr{S}_{l+1}$ of $M\times N$ into relatively compact analytic blocks which is compatible with $\mathscr{B}$. Having chosen stratifications $\mathscr{S}_{l+1}, \mathscr{S}_l, \ldots, \mathscr{S}_{i+1}$ of $M\times N$, we select a stratification $\mathscr{T}_i$ of $M\times N$ into semianalytic blocks which is compatible with

$$\begin{aligned}\mathscr{Z}_i=\mathscr{S}_{i+1}&\cup\{Z_B: B\in\mathscr{S}_{i+1} \text{ and } \dim B=i\}\\ &\cup\{Z_B^A: A\in\mathscr{S}_i, B\in\mathscr{S}_i, B\subset\operatorname{Fron}A, \text{ and } \dim B=i\},\end{aligned}$$

and let

$$\begin{aligned}\mathscr{S}_i=&[\mathscr{T}_i\cap\{T: T\subset(\textstyle\bigcup\mathscr{Z}_i)\cup(\bigcup\mathscr{S}_{i+1}\cap\{S:\dim S<i\})\}]\\ &\cup\{\text{components of } B\sim Z_B\sim\textstyle\bigcup\{Z_B^A: A\in\mathscr{S}_i, B\subset\operatorname{Fron}A\}: B\in\mathscr{S}_i, \dim B=i\}\\ &\cup[\mathscr{S}_{i+1}\cap\{S:\dim S>i\}].\end{aligned}$$

Then $\mathscr{A}=\mathscr{S}_0\cap\{S: S\subset\operatorname{Clos}\bigcup\mathscr{B}\}$ satisfies the theorem.

4.2. Theorem. *For any real analytic manifold M and locally finite family $\mathscr{C}$ of semianalytic shadows in M, there is a stratification $\mathscr{S}$ of M compatible with $\mathscr{C}$ such that each S in $\mathscr{S}$ is a semianalytic shadow contained in $\mu_S(A_S)$ for some analytic manifold N_S, projection $\mu_S: M\times N_S\to M$, and relatively compact semianalytic stratum A_S in $M\times N_S$ with* rank $(\mu_S|A_S)\equiv\dim S$.

Proof. Suppose that for $i \in \{1, 2, \ldots\}$ U_i and V_i are open semianalytic subsets of some coordinate neighborhood with $\operatorname{Clos} V_i$ being a compact subset of U_i and $\bigcup_{i=1}^{\infty} V_i = M$. If one lets $\mathscr{S}_0 = \{M\}$ and finds inductively a stratification $\mathscr{S}_i \subset \mathscr{P}(M)$ of M compatible with

$$\{\operatorname{Fron} V_i\} \cup \{C \cap \operatorname{Clos} V_i : C \in \mathscr{C} \cup \mathscr{S}_{i-1}\}$$

for $i = 1, 2, \ldots$, then

$$\mathscr{S} = \bigcup_{i=1}^{\infty} \mathscr{S}_i \cap \left\{S : S \subset V_i \sim \bigcup_{j=1}^{i-1} V_j\right\}$$

is a stratification of M in $\mathscr{P}(M)$ compatible with $\mathscr{C}$. Thus we may now assume $\operatorname{Clos} \bigcup \mathscr{C}$ is compact, hence $\mathscr{C}$ is finite, and $M = \mathbf{R}^m$, and we need no longer insist that A_S be relatively compact, only that $\mu_S | \operatorname{Clos} A_S$ be proper.

Choosing N, μ, and $\mathscr{A}$ as in 4.1, we note that any stratification of M compatible with $\{\mu(A) : A \in \mathscr{A}\}$ is compatible with $\mathscr{C}$. In case $\dim \mu(\bigcup \mathscr{A}) \leqq 0$, $\mu(\bigcup \mathscr{A})$ is finite, and it suffices to let

$$\mathscr{S} = \{\{x\} : x \in \mu(\bigcup \mathscr{A})\} \cup \{\text{connected components of } M \sim \mu(\bigcup \mathscr{A})\}$$

and, by 3.2, for each $S \in \mathscr{S}$, N_S be any compact analytic manifold and $A_S = \mu_S^{-1}(S)$.

Let k be a positive integer. Assuming inductively that the theorem is true whenever there exist N, μ, and $\mathscr{A}$ as in 4.1 with $\dim \mu(\bigcup \mathscr{A}) < k$, we now suppose $\dim \mu(\bigcup \mathscr{A}) = k$.

Let $\mathscr{B} = \mathscr{A} \cap \{A : \dim \mu(A) < k\}$. Using the projection

$$\delta : (M \times N)^2 \to M, \qquad \delta(a, b) = \mu(a) \qquad \text{for } (a, b) \in (M \times N)^2,$$

and 3.2 (3) (5) we select a semianalytic stratification $\mathscr{E}$ of $(M \times N)^2$ compatible with

$$\mathscr{F} = \{(A \times B) \cap \{(a, b) : \mu(a) = \mu(b)\} : A, B \in \mathscr{A} \sim \mathscr{B}\}$$

so that rank $(\delta | E)$ is constant whenever $E \in \mathscr{E}$. Let

$$\mathscr{D} = \mathscr{E} \cap \{D : D \subset \bigcup \mathscr{F} \text{ and } \dim \delta(D) < k\}.$$

By induction *there is a stratification $\mathscr{T} \subset \mathscr{P}(M)$ of M, along with suitable N_T, μ_T, A_T for $T \in \mathscr{T}$, so that $\mathscr{T}$ is compatible with*

$$\mathscr{G} = \{\mu(A) \cap \mu(B) : A \in \mathscr{A}, B \in \mathscr{B}\} \cup \{\mu(A) \cap \delta(D) : A \in \mathscr{A}, D \in \mathscr{D}\}.$$

In fact, with $P = (M \times N) \times (M \times N) \times (M \times N)^2$, $q : M \times P \to M$ being the projection mapping, and $\mathscr{P}$ being a stratification of $M \times P$ into analytic blocks as in 4.1 compatible with

$$\begin{aligned} \mathscr{Q} = &\{(M \times [A \times B \times (M \times N)^2]) \cap \{(x, (a, b, c)) : x = \mu(a) = \mu(b)\} : A \in \mathscr{A}, B \in \mathscr{B}\} \\ &\cup \{(M \times [A \times (M \times N) \times D]) \cap \{(x, (a, b, c)) : x = \mu(a) = \delta(c)\} : A \in \mathscr{A}, D \in \mathscr{D}\} \end{aligned}$$

so that rank $(q | P)$ is constant whenever $P \in \mathscr{P}$, it suffices to apply induction with $\mathscr{C}$, N, μ, and $\mathscr{A}$ replaced by $\mathscr{G} = \{q(Q) : Q \in \mathscr{Q}\}$, P, q, and $\mathscr{P} \cap \{P : P \subset \bigcup \mathscr{Q}\}$.

It follows that $\dim \mu(\operatorname{Clos} B) = \dim \operatorname{Clos} \mu(B) < k$ and

$$\dim \delta(\operatorname{Clos} D) = \dim \operatorname{Clos} \delta(D) < k$$

whenever $B \in \mathscr{B}$ and $D \in \mathscr{D}$; thus, $\bigcup \mathscr{B}$, $\bigcup \mathscr{D}$, and hence $\bigcup \mathscr{G} = \mu(\bigcup \mathscr{B}) \cup \delta(\bigcup \mathscr{D})$ are compact. With

$$\mathscr{U} = \mathscr{T} \cap \{T\colon T \subset \bigcup \mathscr{G}\},$$
$$\mathscr{V} = \{\text{connected components of } \mu(A) \sim \bigcup \mathscr{G}\colon A \in \mathscr{A} \sim \mathscr{B}\},$$
$$\mathscr{W} = \{\text{connected components of } M \sim \mu(\bigcup \mathscr{A})\},$$

$\mu_V = \mu$ and $A_V \in \mathscr{A} \sim \mathscr{B}$ chosen so that $V \subset \mu(A_V)$ for $V \in \mathscr{V}$, and $\mu_W = \mu$ and A_W equalling $M \times \{\text{point in } N\}$ for $W \in \mathscr{W}$ we complete the proof by showing, in the following four steps, that the family $\mathscr{S} = \mathscr{U} \cup \mathscr{V} \cup \mathscr{W}$ of semianalytic shadows is a stratification of M compatible with $\{\mu(A)\colon A \in \mathscr{A}\}$.

Step I. *$\mathscr{V}$ is a disjointed family of k dimensional analytic submanifolds of M.*

In fact, by induction on $l = \dim \bigcup \mathscr{A}$, it suffices to show that

$$H = \mu(\bigcup \mathscr{A}) \sim \mu(\bigcup \mathscr{A} \cap \{B\colon \dim B < l\}) \sim \bigcup \mathscr{G}$$

is a k dimensional analytic submanifold of M because

$$\operatorname{Fron}[\mu(A) \sim \bigcup \mathscr{G}] \subset \mu(\operatorname{Fron} A) \cup \bigcup \mathscr{G} \subset \mu(\bigcup \mathscr{A} \cap \{B\colon \dim B < l\}) \cup \bigcup \mathscr{G}$$

whenever $A \in \mathscr{A} \sim \mathscr{B}$ and $\dim A = l$.

Every $a \in (\bigcup \mathscr{A}) \sim \mu^{-1}(\bigcup \mathscr{G})$ belongs to some member A_a of $\mathscr{A} \sim \mathscr{B}$, and there is, by the rank theorem [1, 3.1.18], a connected, relatively open neighborhood R_a of a in A_a so that $\mu(R_a)$ is a k dimensional analytic submanifold of M. For any $x \in \mu(\bigcup \mathscr{A}) \sim \bigcup \mathscr{G}$ and any finite subset F of $\mu^{-1}\{x\} \cap \bigcup \mathscr{A}$, there is an open neighborhood U of x in $M \sim \bigcup \mathscr{G}$ so that $U \cap \mu(R_a)$ is connected and $U \cap \operatorname{Fron} \mu(R_a) = \varnothing$ whenever $a \in F$. Moreover if $a, b \in F$, then $U \cap \mu(R_a) = U \cap \mu(R_b)$. In fact, otherwise $U \cap \mu(R_a) \cap \mu(R_b)$, being a proper analytic subset of $U \cap \mu(R_a)$, would have Hausdorff dimension less than k by 3.2(3). If $(a, b) \in D \in \mathscr{E}$, then

$$D \subset (A_a \times A_b) \cap \{(y, z)\colon \mu(y) = \mu(z)\},$$

and there is a relatively open neighborhood $D_{a,b}$ of (a, b) in

$$D \cap ([A_a \cap \mu^{-1}(U)] \times [A_b \cap \mu^{-1}(U)])$$

so that $\delta(D_{a,b})$ is a $\dim \delta(D)$ dimensional analytic submanifold of M in $U \cap \mu(A_a) \cap \mu(U_b)$. Thus $\dim \delta(D) < k$ and $x \in \delta(D) \subset \bigcup \mathscr{G}$, a contradiction.

For every x belonging to H (which equals $\mu(\bigcup \mathscr{A}) \sim \bigcup \mathscr{G}$ in case $l = k$), the set

$$\bigcup \{\mu^{-1}\{x\} \cap A\colon A \in \mathscr{A} \sim \mathscr{B} \text{ and } \dim A = l\} = \mu^{-1}\{x\} \cap \bigcup \mathscr{A}$$

is compact, and the sets F and U above may be chosen so that

$$U \cap \mu(\bigcup \mathscr{A}) = U \cap \bigcup_{a \in F} \mu(R_a).$$

Hence, H is a k dimensional analytic submanifold of M.

Step II. *$\mathscr{S}$ is compatible with $\{\mu(A): A\in\mathscr{A}\}$.*

In fact, suppose $S\in\mathscr{S}$, $A\in\mathscr{A}$, and $S\cap\mu(A)\neq\varnothing$. Then $S\notin\mathscr{W}$. We will show that $S\subset\mu(A)$ in the three remaining cases.

Case 1. $S\in\mathscr{U}$. Here $S\subset\mu(\bigcup\mathscr{B})\cup\delta(\bigcup\mathscr{D})$ and either $S\cap\mu(A)\cap\mu(B)\neq\varnothing$ for some $B\in\mathscr{B}$, hence $S\subset\mu(A)\cap\mu(B)$, or $S\cap\mu(A)\cap\delta(D)\neq\varnothing$ for some $D\in\mathscr{D}$, hence $S\subset\mu(A)\cap\delta(D)$.

Case 2. $S\in\mathscr{V}$ *and* $S\cap\mu(\text{Fron }A)=\varnothing$. Here $A\in\mathscr{A}\sim\mathscr{B}$, and S is a connected component of $\mu(B)\sim\bigcup\mathscr{G}$ for some $B\in\mathscr{A}\sim\mathscr{B}$. Let U be an open neighborhood of S so that $U\cap\mu(B)=U\cap S$. Since $\mu(A)\sim\mu(\text{Fron }A)$ is a k dimensional analytic submanifold of M, $S\cap\mu(A)$ is an analytic subset of S. Moreover $S\cap\mu(A)$ is not a proper subset of S; otherwise, $\dim[S\cap\mu(A)]<k$, and, as before,

$$S\cap\mu(A)=U\cap\mu(A)\cap\mu(B)\subset\delta(\bigcup\mathscr{D})\subset\bigcup\mathscr{G},$$

an impossibility.

Case 3. $S\in\mathscr{V}$ *and* $S\cap\mu(\text{Fron }A)\neq\varnothing$. Here we use induction on $\dim A$. Choosing $B\in\mathscr{A}$ with $B\subset\text{Fron }A$ and $S\cap\mu(B)\neq\varnothing$, we apply Case 2 if $S\cap\mu(\text{Fron }B)=\varnothing$ or induction if $S\cap\mu(\text{Fron }B)\neq\varnothing$ to infer that $S\subset\mu(B)$.

Let $x\in S$. To see that $x\in\mu(A)$, choose, since $\dim S=k\geqq 1$, a one-dimensional analytic submanifold L of S passing through x so that $L\sim\{x\}$ has two components. Let b be an element of $B\cap\mu^{-1}\{x\}$, B' be the connected component of $B\cap\mu^{-1}(L)$ containing b, and A' be a component of $A\cap\mu^{-1}(L)$ whose closure contains b. By the rank theorem, A' and B' are analytic submanifolds, $\mu(B')$ is a relative neighborhood of x in L, and $B'\subset\text{Clos }A'$ because $\mu^{-1}(L)\cap B\cap\text{Clos }A$ is open and closed relative to $\mu^{-1}(L)\cap B$. Let λ be a continuous function on L which is positive on one component of $L\sim\{x\}$ and negative on the other. Then $\lambda\circ\mu$, being both positive and negative on B', and hence on A', vanishes somewhere on A'. Thus $\{x\}=\lambda^{-1}\{0\}\subset\mu(A')\subset\mu(A)$.

Step III. *Every member of $\mathscr{S}$ satisfies the frontier property* (*) *of* 2.0.

In fact, suppose $R\in\mathscr{S}$, $S\in\mathscr{S}$, and $R\cap\text{Fron }S\neq\varnothing$. We will verify that $R\subset\text{Fron }S$ and $\dim R<\dim S$. The cases $R\in\mathscr{V}$ and $S\in\mathscr{U}$ or $R\in\mathscr{W}$ and $S\in\mathscr{U}\cup\mathscr{V}\cup\mathscr{W}$ are eliminated because $\bigcup\mathscr{U}$ is closed, $\bigcup(\mathscr{U}\cup\mathscr{V})$ is closed, and $\bigcup\mathscr{W}$ is open. We examine the four remaining possibilities.

Case 1. $R\in\mathscr{U}$ *and* $S\in\mathscr{U}$. Here $R\subset\text{Fron }S$ and $\dim R<\dim S$ by induction because R and S belong to $\mathscr{T}$.

Case 2. $R\in\mathscr{V}$ *and* $S\in\mathscr{V}$. Here $\dim R=k=\dim S$, and we will derive a contradiction. For each $Q\in\mathscr{V}$, let

$$d_Q=\inf\{\dim A:\ A\in\mathscr{A}\ \text{ and }\ Q\cap\mu(A)\neq\varnothing\}.$$

Suppose $Q\in\mathscr{V}$, $A\in\mathscr{A}$, $Q\cap\mu(A)\neq\varnothing$, and $\dim A=d_Q$. Then $A\in\mathscr{A}\sim\mathscr{B}$ and $Q\subset\mu(A)$. Moreover if $P\in\mathscr{V}$ and $P\cap\text{Fron }Q\neq\varnothing$, then $P\cap\mu(A)=\varnothing$; otherwise, $P\subset\mu(A)$, and P and Q would be disjoint relatively open subsets of the submanifold $\mu(A)\sim\bigcup\mathscr{G}$, contradicting $P\cap\text{Fron }Q\neq\varnothing$. It follows that $P\cap\mu(B)\neq\varnothing$ for some $B\subset\text{Fron }A$,

$$d_P\leqq\dim B<\dim A=d_Q\quad\text{and}\quad P\subset\text{Fron }Q$$

because $P = P \cap \mu(B) \subset P \cap \mu(\mathrm{Clos}\, A) = P \cap \mathrm{Clos}\, \mu(A) = P \cap \mathrm{Clos}\, [\mu(A) \sim \bigcup \mathscr{G}] = P \cap \mathrm{Clos}\, Q$.

Assuming now that $R \in \mathscr{V}$ is chosen so that $R \cap \mathrm{Fron}\, S \neq \varnothing$ (hence $R \subset \mathrm{Fron}\, S$) and d_R is maximal we infer that

$$\begin{aligned} &R \cap \mathrm{Clos}\, [(\mathrm{Fron}\, S) \sim R] \\ &\subset R \cap [(\bigcup \mathscr{G}) \cup \bigcup \mathscr{V} \cap \{P\colon R \cap \mathrm{Fron}\, P \neq \varnothing \text{ and } P \cap \mathrm{Fron}\, S \neq \varnothing\}] = \varnothing. \end{aligned}$$

Fix two points, $x \in R$ and $y \in S$, and let $p\colon \mathbf{R}^m \to \mathbf{R}^k$ be an orthogonal projection so that $\dim p[\mathrm{Tan}\,(R, x)] = k = \dim p[\mathrm{Tan}\,(S, y)]$. By 3.2(2)(3)

$$Z_S = S \cap \{z\colon \dim p\,[\mathrm{Tan}\,(S, y)] < k\},$$

is a proper analytic subset of S and has Hausdorff dimension less than k. By [3, 2.2 (7)] and the rank theorem we may choose a positive

$$r < \mathrm{dist}\,(x, \mathrm{Clos}\, [(\mathrm{Fron}\, S) \sim R])$$

so that $\dim\, [S \cap \mathrm{Fron}\, \mathbf{U}\,(x, r)] < k$ and $p | R \cap \mathbf{U}(x, r)$ is an analytic isomorphism. For any component V of

$$p[R \cap \mathbf{U}(x, r)] \sim p[Z_S] \sim p[S \cap \mathrm{Fron}\, \mathbf{U}(x, r)],$$

$p | S \cap \mathbf{U}(x, r) \cap p^{-1}(V)$ is covering map with infinitely many sheets because

$$R \cap p^{-1}(V) \subset \mathrm{Clos}\, [S \cap \mathbf{U}(x, r) \cap p^{-1}(V)].$$

However then $S \cap \mathbf{U}(x, r) \cap p^{-1}\{v\}$, for $v \in V$, is a relatively compact, infinite, zero dimensional semianalytic shadow, contradicting the theorem with $k = 0$.

Case 3. $R \in \mathscr{U}$ and $S \in \mathscr{V}$. Here $\dim R < k = \dim S$, and $\mathrm{Fron}\, S \subset \bigcup \mathscr{U}$ by Case 2. If $R \neq R' \in \mathscr{U}$, $R \cap \mathrm{Clos}\, R' \neq \varnothing$ and $R' \subset \mathrm{Fron}\, S$, then $\dim R < \dim R'$ and $R \subset \mathrm{Clos}\, R' \subset \mathrm{Fron}\, S$. Thus, replacing R if necessary, we may now assume

$$K = \bigcup \mathscr{U} \cap \{R'\colon R' \neq R \text{ and } R' \cap \mathrm{Clos}\, S \neq \varnothing\}$$

is compact. Choosing $A \in \mathscr{A}$ so that $S \cap \mu(A) \neq \varnothing$, and $\dim A = d_S$ (see Case 2) we infer that $A \in \mathscr{A} \sim \mathscr{B}$, S is a connected component of $\mu(A) \sim (R \cup K)$ and

$$S \cap \mu(\mathrm{Fron}\, A) \subset \bigcup \{S \cap \mu(B)\colon B \in \mathscr{A} \text{ and } \dim B < \dim A\} = \varnothing.$$

By the connectedness of R it suffices to show that $R \cap \mathrm{Clos}\, S$ is open relative to R. Fixing $x \in R \cap \mathrm{Clos}\, S$, we will find an open neighborhood U of x so that $U \cap R \subset \mathrm{Clos}\, S$ in the two possible cases.

If $R \cap [\mu(A) \sim \mu(\mathrm{Fron}\, A)] \neq \varnothing$, then $R \subset \mu(A)$, and as before we may choose a finite subset F of $A \cap \mu^{-1}\{x\}$ and an open neighborhood U of x in $M \sim K$ so that $U \cap R$ and $U \cap \mu(R_a)$, for $a \in F$, are connected submanifolds,

$$U \cap \mu(A) = U \cap \bigcup_{a \in F} \mu(R_a) \quad \text{and} \quad U \cap \mathrm{Fron}\, R = \varnothing = U \cap \bigcup_{a \in F} \mathrm{Fron}\, \mu(R_a).$$

Let Q be a connected component of $U \cap S$ whose closure contains x. For each $a \in F$, either $Q \subset \mu(R_a)$ or $\dim\, [Q \cap \mu(R_a)] < k$ because $Q \cap \mu(R_a)$ is analytic in Q.

Since $Q \subset \bigcup_{a \in F} \mu(R_a)$, we may find at least one $a \in F$ with $Q \subset \mu(R_a)$. Similarly $U \cap R \subset \mu(R_b)$ for some $b \in F$. Either $U \cap \mu(R_a) = U \cap \mu(R_b)$ or

$$\dim [U \cap \mu(R_a) \cap \mu(R_b)] < k$$

and, after shrinking U if necessary,

$$U \cap \mu(R_a) \cap \mu(R_b) = U \cap \delta(D) \quad \text{for some } D \in \mathscr{D},$$

hence $R \subset \delta(D)$. In any case $U \cap R \subset \mu(R_a)$. Since Q, being a k dimensional submanifold, is open relative to $\mu(R_a)$ and

$$U \cap \operatorname{Fron} Q \subset U \cap \operatorname{Fron} S \subset U \cap (R \cup K) \subset R,$$

Q is a connected component of $U \cap [\mu(R_a) \sim R]$. Thus by 2.4 with $M = U \cap \mu(R_a)$, $E = \varnothing$, and $\mathscr{F} = \{U \cap R\}$,

$$U \cap R \subset \operatorname{Clos} Q \subset \operatorname{Clos} S.$$

On the other hand, if $R \cap [\mu(A) \sim \mu(\operatorname{Fron} A)] = \varnothing$, then $R \cap \mu(\operatorname{Fron} A) \neq \varnothing$, and we may choose $B \in \mathscr{A}$ of smallest dimension so that $R \cap \mu(B) \neq \varnothing$ and $B \subset \operatorname{Fron} A$. Then $R \subset \mu(B)$, $S \cap \mu(B) = \varnothing$, and $R \cap \mu(\operatorname{Fron} B) = \varnothing$. For each $b \in B \cap \mu^{-1}\{x\}$, there is, by 4.1, a neighborhood W_b of b in $(M \times N) \sim \mu^{-1}(K)$ so that $B \cap W_b$ is contained in the closure of each component of $A \cap W_b$. Observing that $A \cap W_b \cap \mu^{-1}(S)$ is nonempty, open relative to $A \cap W_b$ by the rank theorem, and closed relative to $A \cap W_b$ because

$$\begin{aligned}(A \cap W_b) \cap \operatorname{Fron} \mu^{-1}(S) &\subset (A \cap W_b) \cap \mu^{-1}(\operatorname{Fron} S) \\ &\subset [A \cap \mu^{-1}(R)] \cup [W_b \cap \mu^{-1}(K)] = \varnothing,\end{aligned}$$

we infer that $B \cap W_b \subset \operatorname{Clos} \mu^{-1}(S)$. By the rank theorem and the compactness of $B \cap \mu^{-1}\{x\}$, there is a neighborhood U of x with

$$U \cap \mu(B) \subset \bigcup_{b \in B \cap \mu^{-1}\{x\}} \mu(B \cap W_b);$$

hence $U \cap R \subset U \cap \mu(B) \subset \operatorname{Clos} S$.

Case 4. $R \in \mathscr{U} \cup \mathscr{V}$ *and* $S \in \mathscr{W}$. Here, as in Case 3 we may assume

$$K = \bigcup (\mathscr{U} \cup \mathscr{V}) \cap \{R' \colon R' \neq R \text{ and } R' \cap \operatorname{Clos} S \neq \varnothing\}$$

is compact, infer that S is a component of $(M \sim K) \sim R$ and apply 2.4 with $M, E, \mathscr{F}$ replaced by the component of $M \sim K$ containing S, $\varnothing$, and $\{R\}$.

Step IV. $\mathscr{S}$ *is a finite family*. In fact $\mathscr{U}$ is finite because $\mathscr{T}$ is locally-finite and $\mu(\bigcup \mathscr{A})$ is compact. By 2.4 and Steps I and III, the finiteness of $\mathscr{W}$ will follow from that of $\mathscr{U}$ and $\mathscr{V}$.

To show that $\mathscr{V}$ is finite, fix a point $x_S \in S$ for each $S \in \mathscr{V}$, note that $\mathscr{V}$ is countable, and choose an orthogonal projection $p \colon \mathbf{R}^m \to \mathbf{R}^k$ so that $\dim p[\operatorname{Tan}(S, x_S)] = k$ for every $S \in \mathscr{V}$. Then the set Z_S, defined in the proof of Step III, has Hausdorff dimension less than k. Moreover $Z_S \in \mathscr{P}(M)$ because for any $A \in \mathscr{A} \sim \mathscr{B}$ with $S \subset \mu(A)$,

$$Z_S = S \cap \mu(A \cap \{x \colon \operatorname{rank}[(p \circ \mu)|A](x) < k\})$$

and $A \cap \{x\colon \operatorname{rank}[(p \circ \mu)|A](x) < k\}$ is a semianalytic subset of M by 3.2(3). Since, by Step III, Fron $S \in \mathscr{P}(M)$ and dim (Fron $S) < k$ whenever $S \in \mathscr{V}$,

$$Z = \bigcup_{S \in \mathscr{V}} p(Z_S \cup \operatorname{Fron} S)$$

is a compact at most $k-1$ dimensional member of $\mathscr{P}(\mathbf{R}^k)$. By induction and Lemma 2.4 (or by [3, 2.4]), $\mathbf{R}^k \sim Z$ has only finitely many components. Inasmuch as

$$p^{-1}\{y\} \cap \operatorname{Clos} \bigcup \mathscr{V} = \bigcup_{S \in \mathscr{V}} p^{-1}\{y\} \cap (S \sim Z_S)$$

is compact and discrete whenever $y \in R^k \sim Z$, $p|(\bigcup \mathscr{V}) \sim p^{-1}(Z)$ is a real analytic covering map with finite fibers. Therefore $(\bigcup \mathscr{V}) \sim p^{-1}(Z)$ has only finitely many components. Moreover every $S \in \mathscr{V}$ contains some component of $(\bigcup \mathscr{V}) \sim p^{-1}(Z)$ because, by the rank theorem,

$$S \cap p^{-1}(Z) = Z_S \cup [(S \sim Z_S) \cap p^{-1}(Z)]$$

has Hausdorff dimension less than k. Thus $\mathscr{V}$ is finite.

4.3. Corollary. *If M and N are real analytic manifolds, $\mu\colon M \times N \to M$ is the projection mapping, and G is a semianalytic shadow stratum in $M \times N$, then*

$$G \cap \{a\colon \operatorname{rank}(\mu|G)(a) < \sup\{\operatorname{rank}(\mu|G)(b)\colon b \in G\}\}$$

is contained in an at most (dim $G) - 1$ *dimensional semianalytic shadow.*

Proof. Letting $r = \sup\{\operatorname{rank}(\mu|G)(b)\colon b \in G\}$ and $Z = G \cap \{a\colon \operatorname{rank}(\mu|G)(a) < r\}$ and applying 4.2 with M and $\mathscr{C}$ replaced by $M \times N$ and $\{G\}$, we infer, for each $S \in \mathscr{S}$ with $S \subset G$ and $\dim S = \dim G$, that

$$Z \cap S = S \cap \mu_S(A_S \cap \{a\colon \operatorname{rank}[(\mu \circ \mu_S)|A_S] < r\} \in \mathscr{P}(M \times N)$$

because $D\mu_S(c)[\operatorname{Tan}(A_S, c)] = \operatorname{Tan}(S, \mu_S(c)) = \operatorname{Tan}(G, \mu_S(c))$ whenever $c \in A_S \cap \mu_S^{-1}(S)$. Since, by 3.2(2)(3), $Z \cap S$ has Hausdorff dimension at most $(\dim G) - 1$,

$$(\bigcup \mathscr{S} \cap \{T\colon \dim T < \dim G\}) \cup \bigcup \{Z \cap S\colon S \in \mathscr{S}, S \subset G, \dim S = \dim G\}$$

is an at most $(\dim G) - 1$ dimensional semianalytic shadow containing Z.

4.4. Corollary. *For any analytic mapping $g\colon M \to N$, any locally finite families $\mathscr{C}$ os semianalytic shadows in M and $\mathscr{D}$ of semianalytic shadows in N and any open semianalytic shadow $L \subset M$ such that $g|\operatorname{Clos} L$ is proper, there exists a one-one stratification $(\mathscr{S}, \mathscr{T})$ of $f = g|L$ into semianalytic shadows so that $\mathscr{S}$ is compatible with $\mathscr{C}$ and $\mathscr{T}$ is compatible with $\mathscr{D}$.*

Proof. Combine 4.0, 4.2, 4.3, and 2.2.

5. Semianalytic Shadow Chains

A k dimensional locally flat chain T ([1, 4.1.24]) in an analytic manifold M is called a k dimensional *semianalytic shadow chain* if there exist a k dimensional semianalytic shadow A in M and a $k-1$ dimensional semianalytic shadow B in M with

$$\operatorname{spt} T \subset A \quad \text{and} \quad \operatorname{spt} \partial T \subset B.$$

Then, by [4.4] and the reasoning of [1, 4.2.28],

$$T=\sum_{B\in\mathscr{B}} m_B(\mathscr{H}^k \llcorner B)\wedge\xi_B$$

for some locally finite disjointed family $\mathscr{B}$ of k dimensional orientable, semianalytic shadow strata with orienting k vectorfields ξ_B and integer multiplicities m_B for $B\in\mathscr{B}$. Moreover by 4.0, 4.2, 4.3, and 2.5, the reasoning of [3, 2.9, § 4–§ 5] carries over to give an analogous slicing and intersection theory for semianalytic shadow chains.

References

1. Federer, H.: Geometric measure theory. Berlin-Heidelberg-New York: Springer 1969
2. Giesecke, B.: Simpliziale Zerlegung abzählbarer analytischer Räume. Math. Z. **83**, 177–213 (1964)
3. Hardt, R.: Slicing and intersection theory for chains associated with real analytic varieties. Acta Math. **129**, 57–136 (1972)
4. Lojasiewicz, S.: Sur le problème de la division. Rozprawy Mathematyczne, No. 22, Warsaw, 1961
5. Lojasiewicz, S.: Triangulation of semi-analytic sets. Annali Sc. Norm. Sup. Pisa, 5.3, **18**, 449–474 (1964)
6. Lojasiewicz, S.: Ensembles semianalytiques. Cours Faculté des Sciences d'Orsay, I.H.E.S. Bures-sur-Yvette, 1965
7. Lojasiewicz, S.: Une propriété topologique des sousensembles analytique reels. Editions du Centre National de la Recherche Scientifique [15, Quai Anatole-France, Paris] vol. 117, Les Equations aux Derivees Partielles pp. 87–89 (1962)
8. Mather, J.: Stratifications and mappings. Harvard University notes, 1971
9. Narashimhan, R.: Introduction to the theory of analytic spaces. New York: Springer 1966
10. Seidenberg, A.: A new decision method for elementary algebra. Ann. of Math. **60**, 365–374 (1954)
11. Thom, R.: Ensembles morphismes stratifiés. Bull. Amer. Math. Soc. **75**, 240–284 (1969)
12. Hironaka, H.: Subanalytic sets. Number Theory, Algebraic Geometry, and Commutative Algebra, in honor of Y. Akizuki, pp. 453–493. Tokyo: Kinokuniya Publications 1973
13. Hironaka, H.: Subanalytic sets. Lectures notes of Istituto matematico "Leonida Tonelli", Pisa, 1973
14. Herrera, M.: Integration on a semianalytic set. Bull. Soc. Math. France **94**, 141–180 (1966)
15. Herrera, M.: DeRham theorems on semianalytic sets. Bull. Amer. Math. Soc. **73**, 414–418 (1967)
16. Bloom, T., Herrera, M.: DeRham cohomology of an analytic space. Inventions math. **7**, 275–296 (1969)
17. Hardt, R.: Homology theory for real semianalytic sets. To appear in Annali Sc. Norm. Sup. Pisa
18. Hardt, R.: Topological properties of subanalytic sets. Preprint

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(Received September 12, 1973)