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On the Total Absolute Curvature of Closed Curves in Manifolds of Negative Curvature

Dedicated to Professor Wilhelm Klingenberg on his 50th birthday

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On the total absolute curvature of smooth closed curves, the following theorems are well known.

Theorem A. ([2, 3]) *The total absolute curvature of a smooth closed curve c in a Euclidean space, is greater than, or equal to* 2π. *It is equal to* 2π *if and only if c is a convex plane curve.*

Theorem B. ([4]) *Let M be a complete simply connected Riemannian manifold with non positive sectional curvature. Then the total absolute curvature of a smooth closed curve c in M, is greater than, or equal to* 2π. *It is equal to* 2π, *if and only if c is the boundary of a 2-dimensional totally geodesic submanifold isometric with a convex domain of a Euclidean plane.*

In this paper we prove the following conjecture proposed by Professor N.H. Kuiper in a lecture in april 1973 at Kyushu university.

Theorem 1. *Let M be a complete simply connected Riemannian manifold with negative sectional curvature. Then the total absolute curvature of a smooth closed curve in M is greater than* 2π.

Corollary. *The total absolute curvature of a smooth closed curve in a hyperbolic space is greater than* 2π.

In order to prove this theorem we use the following three theorems

Theorem. (Synge, cf. [1]) *Let N be an immersed 2-dimensional submanifold of M and* σ *be a curve of N such that* σ *is a geodesic of M. Let P be the plane section of N tangent to* σ *at x,* $K(P)$ *be the sectional curvature of M with respect to P and* $G(x)$ *be the Gaussian curvature of N at x. Then* $G(x) \leqq K(P)$. *The equality* $G(x) = K(P)$ *holds if and only if the plane section field P is parallel along* σ *in M.*

Theorem. (Hadamard-Cartan, cf. [1]) *Let M be a complete simply connected Riemannian manifold with non positive sectional curvature. Then, for any point p of M, the exponential mapping* $\exp_p$ *is a diffeomorphism of* M_p *onto M, where* M_p *is the tangent space to M at p.*

Theorem. (Gauss-Bonnet) *Let M be a compact orientable 2-dimensional Riemannian manifold. Let D be a simply connected region on M bounded by a piecewise differentiable curve c consisting of m differentiable curves. Then we have*

$$\int_c k_g ds + \sum_{i=1}^{m} (\pi - \alpha_i) + \int_D G\, dA = 2\pi ,$$

where G is the Gaussian curvature of the surface M, k_g is the geodesic curvature of c, dA is the area element of M, s is the arc-length parameter of c and $\alpha_1, \ldots, \alpha_m$ are the inner angles at the points where c is not differentiable.

Proof of Theorem 1. Let $c: [0, l] \to M$ be a smooth closed curve with arc-length parameter. We can assume that the subarc $c|([l-\varepsilon, l]\,[0, \varepsilon])$ is not geodesic for a small positive number ε. We set $p = c(0)$. Since $\exp_p$ is the diffeomorphism of M_p onto M, we have a smooth closed curve $\tilde{c}: [0, l] \to M_p$ that is the lift of c by $\exp_p$, i.e., $\exp_p(\tilde{c}) = c$. Let $g(t)$ be the line segment from $0 = \tilde{c}(0)$ to $\tilde{c}(t)$ in M_p. Then we get the family of line segments $g(t), 0 \leqq t \leqq l$, which generates a surface $\tilde{S}$ of M_p with the boundary $\tilde{c}$, i.e., $\tilde{S}: [0, l] \times [0, 1] \ni (s, t) \to t\tilde{c}(s) \in M_p \cdot \tilde{S}$ is a piecewise immersed 2-dimensional surface in the sense of the following lemma.

Lemma. *Let $(x^1, \ldots, x^n)$ be an orthonormal coordinate system of M. We set $\tilde{c}(s) = (\tilde{c}^1(s), \ldots, \tilde{c}^n(s))$. Then $\tilde{S}$ is not of maximal* rank $= 2$ *at (s, t) if and only if $t = 0$, or*

$$\frac{\dot{\tilde{c}}^1(s)}{\tilde{c}^1(s)} = \cdots = \frac{\dot{\tilde{c}}^n(s)}{\tilde{c}^n(s)}$$

hold, where $\dot{\tilde{c}}^i(s) = d\tilde{c}^i(s)/ds$. Hence the surface $\tilde{S}$ can be decomposed into two crescent shapes and fan shapes, as in Fig. 1, each of which is an immersed 2-dimensional submanifold except its boundary minus $\tilde{c}$, i.e.,

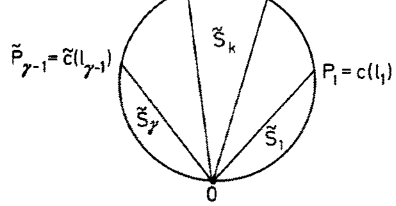

Fig. 1

$\tilde{S}=\tilde{S}_1\cup\cdots\cup\tilde{S}_k\cup\cdots\cup\tilde{S}_\gamma$, *where* $\tilde{S}_1$ *is the crescent shape* $O\tilde{P}_1$, $\tilde{S}_k$ *is the fan shape* $O\tilde{P}_{k-1}\tilde{P}_k$ *and* $\tilde{S}_\gamma$ *is the crescent shape* $O\tilde{P}_{\gamma-1}$.

Proof of lemma. Since $x^i(t\tilde{c}(s))=t\tilde{c}^i(s)$, $i=1,\dots,n$, we have

$$\frac{\partial}{\partial t}=\sum_{i=1}^{n}\frac{\partial x^i}{\partial t}\cdot\frac{\partial}{\partial x^i}=\sum_{i=1}^{n}\tilde{c}^i(s)\cdot\frac{\partial}{\partial x^i},$$

$$\frac{\partial}{\partial s}=\sum_{i=1}^{n}\frac{\partial x^i}{\partial s}\cdot\frac{\partial}{\partial x^i}=\sum_{i=1}^{n}t\dot{\tilde{c}}^i(s)\cdot\frac{\partial}{\partial x^i}.$$

Hence $\partial/\partial S$ and $\partial/\partial t$ are linearly independent at (s_0,t_0) if and only if $t_0=0$ or

$$\frac{\dot{\tilde{c}}^1(s_0)}{\tilde{c}^1(s_0)}=\cdots=\frac{\dot{\tilde{c}}^n(s_0)}{\tilde{c}^n(s_0)}$$

hold.

For the crescent shape $\tilde{S}_1$ we consider a sequence of the following figures ${}^2\tilde{P}_1^{(n)}\,{}^2\tilde{Q}_1^{(n)}\,{}^1\tilde{Q}_1^{(n)}$, $n=1,2,\dots$, in $\tilde{S}_1$, as in Fig. 2, which we call $\tilde{S}_1^{(n)}$, $n=1,2,\dots$, where ${}^2\tilde{P}_1^{(n)}=\tilde{c}({}^2l_1^{(n)})$, $0<{}^2l_1^{(n)}<l_1$, $\lim_{n\to\infty}{}^2l_1^{(n)}=l_1$, ${}^1\tilde{Q}_1^{(n)}$ is the point on $\tilde{c}$ that is at ε_n-distance from 0, ${}^2\tilde{Q}_1^{(n)}$ is the point on the line segment $O\,{}^2\tilde{P}_1^{(n)}$ that is at ε_n-distance from 0, and each point of the arc ${}^2\tilde{Q}_1^{(n)}\,{}^1\tilde{Q}_1^{(n)}$ is at ε_n-distance from 0 on $\tilde{S}_1$, $\lim_{n\to\infty}\varepsilon_n=0$. Then the surface $\tilde{S}_1^{(n)}, n=1,2,\dots$, is an immersed 2-dimensional submanifold of M_p. Now we set $S=\exp_p\tilde{S}$, $S_1=\exp_p\tilde{S}_1$, $S_k=\exp_p\tilde{S}_k$, $S_\gamma=\exp_p\tilde{S}_\gamma$, $S_1^{(n)}=\exp_p\tilde{S}_1^{(n)}$, ${}^2P_1^{(n)}=\exp_p{}^2\tilde{P}_1^{(n)}$, ${}^1Q_1^{(n)}=\exp_p{}^1Q_1^{(n)}$ and ${}^2Q_1^{(n)}=\exp_p{}^2\tilde{Q}_1^{(n)}$, $n=1,2,\dots$. We denote the inner angles at ${}^2P_1^{(n)}$, ${}^1Q_1^{(n)}$ and ${}^2Q_1^{(n)}$ in $S_1^{(n)}$ by $\beta_1^{(n)}$, ${}^1\delta_1^{(n)}$, and ${}^2\delta_1^{(n)}$ respectively and denote the inner angles at P and P_1 in S_1 by θ_1 and β_1 respectively. By Gauss lemma we have

$$ {}^2\delta_1^{(n)}=\frac{\pi}{2}\quad\text{and}\quad\lim_{n\to\infty}{}^1\delta_1^{(n)}=\frac{\pi}{2}. \tag{1,1}$$

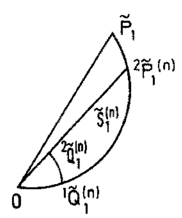

Fig. 2

By the construction we have

$$\lim_{n\to\infty} \beta_1^{(n)} = \beta_1 . \tag{1,2}$$

Since $\tilde{S}_1^{(n)}$ is an immersed 2-dimensional submanifold of M_p, $S_1^{(n)}$ is also immersed in M. So we can apply the Gauss-Bonnet formula for $S_1^{(n)}$. Then we have

$$\begin{aligned}\int_{\widehat{{}^1Q_1^{(n)}\,{}^2P_1^{(n)}}} k_g\,ds + \int_{\widehat{{}^2Q_1^{(n)}\,{}^1Q_1^{(n)}}} k_g\,ds + \int_{S_1^{(n)}} G\,dA \\ + (\pi - \beta_1^{(n)}) + \frac{\pi}{2} + (\pi - {}^1\delta_1^{(n)}) = 2\pi ,\end{aligned} \tag{1,3}$$

where k_g is the geodesic curvature of the respective curves on $S_1^{(n)}$, G is the Gaussian curvature and dA is the area element of $S_1^{(n)}$. And we have

$$\lim_{n\to\infty} \int_{\widehat{{}^2Q_1^{(n)}\,{}^1Q_1^{(n)}}} kg\,ds \geqq -\theta_1 . \tag{1,4}$$

For the fan shape $\tilde{S}^1$ we consider a sequence of the following figures ${}^1\tilde{P}_k^{(n)}\,{}^2\tilde{P}_k^{(n)}\,{}^2\tilde{Q}_k^{(n)}\,{}^1\tilde{Q}_k^{(n)}$, $n=1,2,\dots$, in $\tilde{S}_k$, as in Fig. 3, which we call $\tilde{S}_k^{(n)}$, $n=1,2,\dots$, where ${}^1\tilde{P}_k^{(n)} = \tilde{c}({}^1l_k^{(n)})$, ${}^2\tilde{P}_k^{(n)} = \tilde{c}({}^2l_k^{(n)})$, $l_{k-1} < {}^1l_k^{(n)} < {}^2l_k^{(n)} < l_k$, $\lim_{n\to\infty} {}^1l_k^{(n)} = l_{k-1}$, $\lim_{n\to\infty} {}^2l_k^{(n)} = l_k$, ${}^1\tilde{Q}_k^{(n)}$ is the point on the line segment $O\,{}^1\tilde{P}_k^{(n)}$ that is at ε_n-distance from O, ${}^2\tilde{Q}_k^{(n)}$ is the point on the line segment $O\,{}^2\tilde{P}_k^{(n)}$ that is at ε_n-distance from O and each point of the arc ${}^2\tilde{Q}_k^{(n)}\,{}^1\tilde{Q}_k^{(n)}$ is at ε_n-distance from O on $\tilde{S}_k$, $\lim_{n\to\infty} \varepsilon_n = 0$. Then the surface $\tilde{S}_k^{(n)}$, $n=1,2,\dots$, is an immersed 2-dimensional submanifold of M_p. Now we set $S_k^{(n)} = \exp_p \tilde{S}_k^{(n)}$, ${}^1P_k^{(n)} = \exp_p {}^1\tilde{P}_k^{(n)}$, ${}^2P_k^{(n)} = \exp_p {}^2\tilde{P}_k^{(n)}$, ${}^1Q_k^{(n)} = \exp_p {}^1\tilde{Q}_k^{(n)}$ and ${}^2\tilde{Q}_k^{(n)} = \exp_p {}^2\tilde{Q}_k^{(n)}$, $n=1,2,\dots$. We denote the inner angles at ${}^1P_k^{(n)}$, ${}^2P_k^{(n)}$, ${}^1Q_k^{(n)}$, and ${}^2Q_k^{(n)}$ in $S_k^{(n)}$ by $\alpha_k^{(n)}$, $\beta_k^{(n)}$, ${}^1\delta_k^{(n)}$ and ${}^2\delta_k^{(n)}$ respectively and denote the inner angles at P, P_{k-1} and P_k in S_k by θ_k, α_k and β_k respectively. By Gauss lemma we have

$${}^1\delta_k^{(n)} = {}^2\delta_k^{(n)} = \frac{\pi}{2} . \tag{k,1}$$

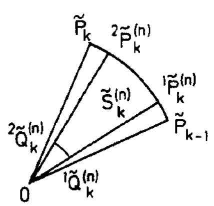

Fig. 3

By the construction we have

$$\lim_{n\to\infty} \alpha_k^{(n)} = \alpha_k, \lim_{n\to\infty} \beta_k^{(n)} = \beta_k \quad \text{and} \quad \alpha_k + \beta_{k-1} = 0. \tag{k,2}$$

Since $\tilde{S}_k^{(n)}$ is an immersed 2-dimensional submanifold of M_p, $S_k^{(n)}$ is also immersed in M. So we can apply the Gauss-Bonnet formula for $S_k^{(n)}$. Then we have

$$\int_{\widehat{{}^1P_k^{(n)}\,{}^2Q_k^{(n)}}} k_g\,ds + \int_{\widehat{{}^2Q_k^{(n)}\,{}^1Q_k^{(n)}}} k_g\,ds + \int_{S_k^{(n)}} G\cdot dA + (\pi - \alpha_k^{(n)}) + (\pi - \beta_k^{(n)}) + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi. \tag{k,3}$$

And we have

$$\lim_{n\to\infty} \int_{\widehat{{}^2Q_k^{(n)}\,{}^1Q_k^{(n)}}} k_g\,ds \geqq -\theta_k. \tag{k,4}$$

For the crescent shape $\tilde{S}_\gamma$ we consider a sequence of the similar figures ${}^1\tilde{P}_\gamma^{(n)}\,{}^2\tilde{Q}_\gamma^{(n)}\,{}^1\tilde{Q}_\gamma^{(n)}$, $n=1,2,\ldots$ in $\tilde{S}_\gamma$ as in $\tilde{S}_1^{(n)}$, as in Fig. 4, which we call $\tilde{S}_\gamma^{(n)}$, $n=1,2,\ldots$. Then the surface $\tilde{S}_\gamma^{(n)}$, $n=1,2,\ldots$, is an immersed 2-dimensional submanifold of M_p. Now we set $S_\gamma^{(n)} = \exp_p \tilde{S}_\gamma^{(n)}$, ${}^1P_\gamma^{(n)} = \exp_p {}^1\tilde{P}_\gamma^{(n)}$, ${}^1Q_\gamma^{(n)} = \exp_p {}^1\tilde{Q}_\gamma^{(n)}$ and ${}^2Q_\gamma^{(n)} = \exp_p {}^2\tilde{Q}_\gamma^{(n)}$, $n=1,2,\ldots$. We denote the inner angles at ${}^1P_\gamma^{(n)}$, ${}^1Q_\gamma^{(n)}$ and ${}^2Q_\gamma^{(n)}$ by $\alpha_\gamma^{(n)}$, ${}^1\delta_\gamma^{(n)}$ and ${}^2\delta_\gamma^{(n)}$ respectively and denote the inner angles at P and $P_{\gamma-1}$ in S_γ by θ_γ and α_γ respectively. By Gaussian lemma we have

$${}^1\delta_\gamma^{(n)} = \frac{\pi}{2} \quad \text{and} \quad \lim_{n\to\infty} {}^2\delta_\gamma^{(n)} = \frac{\pi}{2}. \tag{γ,1}$$

By the construction we have

$$\lim_{n\to\infty} \alpha_\gamma^{(n)} = \alpha_\gamma. \tag{γ,2}$$

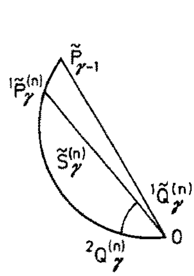

Fig. 4

Since $\tilde{S}_\gamma^{(n)}$ is an immersed 2-dimensional submanifold of M_p, $S_\gamma^{(n)}$ is also immersed in M. So we can apply the Gauss-Bonnet formula for $S_\gamma^{(n)}$. Then we have

$$\int_{\widehat{{}^1P_\gamma^{(n)}\,{}^2Q_\gamma^{(n)}}} k_g\,ds + \int_{\widehat{{}^2Q_\gamma^{(n)}\,{}^1Q_\gamma^{(n)}}} k_g\,ds + \int_{S_\gamma^{(n)}} G\cdot dA + (\pi-\alpha_\gamma^{(n)}) + \frac{\pi}{2} + (\pi - {}^2\delta_\gamma^{(n)}) = 2\pi\,. \tag{γ,3}$$

And we have

$$\lim_{n\to\infty} \int_{\widehat{{}^2Q_\gamma^{(n)}\,{}^1Q_\gamma^{(n)}}} k_g\,ds \geqq -\theta_\gamma\,, \tag{γ,4}$$

and

$$\theta_1 + \sum_k \theta_k + \theta_\gamma \geqq 2\pi\,. \tag{θ}$$

By Synge's lemma $S_1^{(n)}$, $S_k^{(n)}$, and $S_\gamma^{(n)}$ have strictly negative Gaussian curvature everywhere. Let ∇ be the covariant differentiation of M and ∇' be the respective covariant differentiation with respect to the induced metric on $S_1^{(n)}$, $S_k^{(n)}$, and $S_\gamma^{(n)}$ for the sake of simplicity. Let X be the tangent vector of c, i.e., $X = c_*(d/ds)$ and we set $D = d/ds$.

The absolute curvature ϱ of c and the absolute geodesic curvature ϱ_g of c on S_1^n, S_k^n, and S_γ^n, can be expressed as follows:

$$\varrho = |\nabla_D X| \quad \text{and} \quad \varrho_g = |k_g| = |\nabla_D^1 X|\,,$$

where $|X|$ is the length of X. On the other hand we have $\nabla_D' X = (\nabla_D X)^T$, where $(\nabla_D X)^T$ is the tangential component of $\nabla_D X$. So we have

$$\varrho(s) \geqq \varrho_g(s) \qquad s \in [0, l]\,.$$

By (1,1) ~ (1,4), (k,1) ~ (k,4), (γ,1) ~ (γ,4) and (θ) we have

$$\begin{aligned}
\int_0^l \varrho\,ds &\geqq \int_0^l \varrho_g\,ds \geqq \int_0^l k_g\,ds \\
&= \lim_{n\to\infty} \int_{\widehat{{}^1Q_1^{(n)}\,{}^2P_1^{(n)}}} k_g\,ds + \sum_k \lim_{n\to\infty} \int_{\widehat{{}^1P_k^{(n)}\,{}^2P_k^{(n)}}} k_g\,ds + \lim_{n\to\infty} \int_{\widehat{{}^1P_\gamma^{(n)}\,{}^2Q_\gamma^{(n)}}} k_g\,ds \\
&= \left(\theta_1 + \sum_k \theta_k + \theta_\gamma\right) - \lim_{n\to\infty}\left(\int_{S_1^{(n)}} G\cdot dA + \sum_k \int_{S_k^{(n)}} G\cdot dA + \int_{S_\gamma^{(n)}} G\cdot dA\right) \\
&> 2\pi\,.
\end{aligned}$$

Q.E.D.

We can get a generalization of Theorem 1.

Theorem 2. *Let M be a complete Riemannian manifold with negative sectional curvature. Then the total absolute curvature of a smooth closed curve contractible to a point is greater than 2π.*

In fact, if we consider the Riemannian universal covering manifold $\tilde{M}$ of M and lift the closed curve contractible to a point in M into $\tilde{M}$ by the covering mapping, then Theorem 2 follows immediately from Theorem 1.

Remark. We can get an alternative proof of Theorems A and B by analogous arguments.

Note added in proof. Recently F. Brickell and C. C. Hsiung proved the similar theorem by a different method.

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