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On Compactness and Convergence in Spaces of Measures

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1. Introduction

A famous result of Dieudonné ([2], Proposition 8, p. 37) and Grothendieck ([7], p. 150) states that for compact metric resp. locally compact spaces X convergence of a sequence of tight Borel measures on every open set entails its convergence on all Borel sets; in other words: The family $\mathscr{G}(X)$ of all open subsets of X is a convergence class for tight Borel measures.

It was rather natural that with recent developments within the area now called "Topology and Measure" this result became again of interest to various people (cf. [16, 15, 4, 5, 12, 11, 14]), where in [11, 14] there was even considered the case of measures taking their values in an abelian topological group; this will not be considered here. Rather we intend in the present survey to put emphasis onto a unified approach to several results on compactness and convergence of measures which will imply all the main theorems known up to now and various new results (cf. 3.7–3.13, 4.5, and 4.9). Among them there are the following generalizations of the Dieudonné-Grothendieck-Theorem [where (b), (e) and the second part of (c) are new]:

(a) $\mathscr{G}(X)$ is a convergence class for tight Borel measures in any Hausdorff space X ([4, 12]);

(b) $\mathscr{G}(X)$ is a convergence class for τ-smooth Borel measures in any regular space X ([1]);

(c) If $\mathscr{G}_0(X)$ denotes the exact open subsets of a topological space X, then $\mathscr{G}_0(X)$ is a convergence class for Baire measures ([11]); $\mathscr{G}_0(X)$ is a convergence class for τ-smooth Borel measures provided X is a completely regular space ([1]);

(d) $\mathscr{G}_0(X)$ is a convergence class for regular Borel measures in any normal space X ([11]);

(e) If $\mathscr{G}_r(X)$ denotes the regular open subsets of a topological space X, then $\mathscr{G}_r(X)$ is a convergence class for τ-smooth Borel measures provided X is a regular space ([1]);

(f) $\mathscr{G}_r(X)$ is a convergence class for regular Borel measures in any normal space X ([14]).

Although the main ideas for proving these theorems are due to Dieudonné and Grothendieck somewhat different additional methods have been applied for the proofs given in the papers just cited.

As already remarked it is the main aim of this paper to derive these and general compactness results as well (cf. our main Theorems 3.1–3.5 in Section 3) in a way which we think of as being the most unified one.

To make the paper as selfcontained as possible there is added an Appendix containing some known results (A1–A6) which are used within the text.

2. Basic Definitions and Auxiliary Results

Let X be an arbitrary non-empty set, $\mathscr{B}$ a σ-field of subsets of X and $M(X, \mathscr{B})$ [$M_+(X, \mathscr{B})$] the space of all countably additive realvalued [and nonnegative] set functions defined on $\mathscr{B}$; elements of $M(X, \mathscr{B})$ and $M_+(X, \mathscr{B})$ will be called measures and nonnegative measures, respectively. For $M \subset M(X, \mathscr{B})$ let $|M| := \{|\mu| : \mu \in M\}$ where $|\mu|$ denotes the total variation of $\mu \in M$ as defined in [8], Section 29. $M \subset M(X, \mathscr{B})$ is called bounded if M is bounded as a subset of the normed linear space $(M(X, \mathscr{B}), \|\cdot\|)$ with $\|\mu\| := |\mu|(X)$ for $\mu \in M(X, \mathscr{B})$.

Furthermore $B(X, \mathscr{B})$ denotes the space of all bounded $\mathscr{B}$-measurable realvalued functions defined on X.

$\mathbb{N}[\mathbb{R}]$ denotes the set of positive integers [real numbers], $\bar{A}$ the complement of a subset A of X, χ_A its indicator function, $A_1 \triangle A_2$ the symmetric difference of A_1 and A_2, and usually we shall write $A_1 \backslash A_2$ instead of $A_1 \cap \bar{A}_2$.

By a paving (in X) we will understand a non-empty family of subsets of X. For pavings $\mathscr{A}$ we shall use a terminology resembling that of P.A. Meyer, e.g. we will say that $\mathscr{A}$ is a $(\cup f)$- or a $(\cup c)$-paving if $\mathscr{A}$ is closed under finite or countable unions, respectively.

For a paving $\mathscr{A}$, $\bar{\mathscr{A}}$ denotes the paving of all $\bar{A}$ with $A \in \mathscr{A}$.

General Convention

If not specified otherwise we will tacitly assume that $\mathscr{B}$ is always a σ-field of subsets of X, and that $\mathscr{C}$, $\mathscr{F}$ and $\mathscr{H}$ are certain subpavings of $\mathscr{B}$. We assume also that sets denoted by letters B, C, F and H (with or without subscripts) are always elements of the pavings $\mathscr{B}$, $\mathscr{C}$, $\mathscr{F}$, and $\mathscr{H}$, respectively.

In our applications X will be a topological space and we shall write usually in that case $(X, \mathscr{G}(X))$ denoting with $\mathscr{G}(X)$ the class of all open subsets of X, and with $\mathscr{F}(X)$ [$\mathscr{K}(X)$] the class of all closed [compact] subsets of X. We will consider also the class $\mathscr{G}_r(X) := \{G \in \mathscr{G}(X) : (G^a)^0 = G\} = \{F^0 : F \in \mathscr{F}(X)\}$ of the so-called regular open sets, which is generally a proper subclass of $\mathscr{G}(X)$. Here A^a [A^0] denotes the closure [interior] of $A \subset X$.

Furthermore, $C(X)$ denotes the space of all continuous realvalued functions defined on X and $\mathscr{F}_0(X) := \{f^{-1}(0) : f \in C(X)\}$ is the class of all exact closed, $\mathscr{G}_0(X) := \overline{\mathscr{F}_0(X)}$ the class of all exact open subsets of X.

Finally, $\mathscr{B}_0(X)$ [$\mathscr{B}(X)$] denotes the Baire [Borel] σ-field in X, i.e. the smallest σ-field containing $\mathscr{G}_0(X)$ [$\mathscr{G}(X)$]; elements of $M(X, \mathscr{B}_0(X))$ [$M(X, \mathscr{B}(X))$] are called Baire [Borel] measures.

As to our general presentation it turns out that with respect to the pavings $\mathscr{C}$, $\mathscr{F}$, and $\mathscr{H}$ a certain separation property as well as certain regularity properties of the measures are essential. This will be formalized in the following definitions.

2.1. Definition. $\mathscr{C}$ is called $(\mathscr{H}, \mathscr{F})$-separating if for any H and F with $H \cap F = \emptyset$ there exists a pair C_1, C_2 of sets in $\mathscr{C}$ such that $C_1 \cap C_2 = \emptyset$ and $H \subset C_1$, $F \subset C_2$.

2.2. Definition. $\mu \in M(X, \mathscr{B})$ is called $\mathscr{F}$-regular if for any $A \in \mathscr{B}$ and any $\varepsilon > 0$ there exists F such that $F \subset A$ and $\sup\{|\mu(B)| : B \subset A \setminus F\} < \varepsilon$.

$M(X, \mathscr{B}, \mathscr{F})$ denotes the space of all $\mathscr{F}$-regular measures.

If X is a [Hausdorff] topological space then $\mu \in M(X, \mathscr{B}(X))$ is called regular [tight] if μ is $\mathscr{F}(X)$-regular [$\mathscr{K}(X)$-regular].

2.3. Definition. $\mu \in M(X, \mathscr{B})$ is called $(\mathscr{H}, \mathscr{C})$-regular if for any H and any $\varepsilon > 0$ there exists a pair C_1, C_2 of sets in $\mathscr{C}$ such that $H \subset C_1 \subset \bar{C}_2$ and $\sup\{|\mu(B)| : B \subset \bar{C}_2 \setminus H\} < \varepsilon$ [equivalently: For any $\bar{H} \in \bar{\mathscr{H}}$ and any $\varepsilon > 0$ there exists a pair C', C'' of sets in $\mathscr{C}$ such that $C' \subset \overline{C''} \subset \bar{H}$ and $\sup\{|\mu(B)| : B \subset \bar{H} \setminus C'\} < \varepsilon$].

2.4. Definition. (a) We shall say that $\mathscr{F}$ corresponds with $\mathscr{C}$ if $F \cap \bar{C} \in \mathscr{F}$ for any F and C.

(b) $\mathscr{C}$ is called $\mathscr{F}$-filtering if $\{C : F \subset C\}$ is filtering to the left (by inclusion) for all F.

2.5. Remarks. (a) Let $\mathscr{C}$ be $(\mathscr{H}, \mathscr{F})$-separating and $\mu \in M(X, \mathscr{B})$ be $\mathscr{F}$-regular; then μ is $(\mathscr{H}, \mathscr{C})$-regular;

(b) $(\mathscr{H}, \mathscr{C})$-regularity of $\mu \in M(X, \mathscr{B})$ implies its $(\mathscr{H}', \mathscr{C}')$-regularity for any $\mathscr{H}' \subset \mathscr{H}$ and $\mathscr{B} \supset \mathscr{C}' \supset \mathscr{C}$;

(c) If $\mathscr{F}$ is a $(\cup f)$-paving then $\mu_1, \mu_2 \in M(X, \mathscr{B}, \mathscr{F})$ implies $\mu_1 - \mu_2 \in M(X, \mathscr{B}, \mathscr{F})$;

(d) If $\mathscr{C}$ is a $(\cup f, \cap f)$-paving then $(\mathscr{H}, \mathscr{C})$-regularity of μ_i, $i = 1, 2$, implies $(\mathscr{H}, \mathscr{C})$-regularity of $\mu_1 - \mu_2$.

Let us illuminate the preceding definitions by some important examples concerning topological spaces $X = (X, \mathscr{G}(X))$. Remember that $\mu \in M(X, \mathscr{B}(X))$ is called τ-smooth iff for any paving $\mathscr{G}_0 \subset \mathscr{G}(X)$ filtering to the right ($\mathscr{G}_0 \uparrow$) and any $\varepsilon > 0$ there exists $G \in \mathscr{G}_0$ such that $\sup\{|\mu(B)| : \mathscr{B}(X) \ni B \subset \cup \mathscr{G}_0 \setminus G\} < \varepsilon$. A tight measure is regular and τ-smooth. By $M(X, r)$, $M(X, \tau)$, and $M(X, t)$ we denote the spaces of regular, τ-smooth and tight Borel measures, respectively. Before stating the examples let us remark that

(2.5.1) $\mu \in M(X, \mathscr{B}, \mathscr{F}) \Leftrightarrow |\mu| \in M_+(X, \mathscr{B}, \mathscr{F})$;

(2.5.2) $\mu \in M(X, \tau) \Leftrightarrow |\mu| \in M_+(X, \tau)$;

(2.5.3) $\mu \in M(X, \mathscr{B})$ is $(\mathscr{H}, \mathscr{C})$-regular $\Leftrightarrow |\mu|$ is $(\mathscr{H}, \mathscr{C})$-regular.

(This follows from the well known fact (cf. [3], III. 1.5) that

$$\sup\{|\mu(B)| : B \subset A\} \leq |\mu|(A) \leq 2 \cdot \sup\{|\mu(B)| : B \subset A\}$$

for any $A \in \mathscr{B}$.)

2.6. Example. (a) Let $(X, \mathscr{G}(X))$ be a topological space, $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{C} = \mathscr{G}_0(X)$, $\mathscr{F} = \mathscr{F}_0(X)$; then $\mathscr{C}$ is $(\mathscr{F}, \mathscr{F})$-separating;

(b) Let $(X, \mathscr{G}(X))$ be a Hausdorff space, $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{C} = \mathscr{G}(X)$, $\mathscr{F} = \mathscr{K}(X)$; then $\mathscr{C}$ is $(\mathscr{F}, \mathscr{F})$-separating; if $(X, \mathscr{G}(X))$ is a regular space, $\mathscr{C}$ is also $(\mathscr{H}, \mathscr{F})$-separating with $\mathscr{H} = \mathscr{F}(X)$;

(c) Let $(X, \mathscr{G}(X))$ be a normal space, $\mathscr{B}=\mathscr{B}(X)$, $\mathscr{C}=\mathscr{G}_0(X)$ or $=\mathscr{G}(X)$, $\mathscr{F}=\mathscr{F}(X)$; then $\mathscr{C}$ is $(\mathscr{H}, \mathscr{F})$-separating w.r.t. $\mathscr{H}=\mathscr{F}$ or $=\bar{\mathscr{C}}$.

2.7. Example. (a) Let $(X, \mathscr{G}(X))$ be a topological space and $\mu \in M(X, \mathscr{B}_0(X))$; then (i) $\mu \in M(X, \mathscr{B}_0(X), \mathscr{F}_0(X))$ and (ii) μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}_0(X)$, $\mathscr{H}=\mathscr{F}_0(X)$;

(b) Let $(X, \mathscr{G}(X))$ be a Hausdorff space and $\mu \in M(X, t)$; then μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}(X)$ and $\mathscr{H}=\mathscr{K}(X)$;

(c) Let $(X, \mathscr{G}(X))$ be a regular space and $\mu \in M(X, \tau)$; then (i) $\mu \in M(X, r)$ and (ii) μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}(X)$ and $\mathscr{H}=\mathscr{F}(X)$;

(d) Let $(X, \mathscr{G}(X))$ be a completely regular space and $\mu \in M(X, \tau)$; then μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}_0(X)$, $\mathscr{H}=\mathscr{F}(X)$ or $\overline{\mathscr{G}_0(X)}$;

(e) Let $(X, \mathscr{G}(X))$ be a normal space and $\mu \in M(X, r)$; then μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}_0(X)$ and $\mathscr{H}=\mathscr{F}(X)$ or $=\overline{\mathscr{G}_0(X)}$ and, under the same assumptions on $(X, \mathscr{G}(X))$ and μ, it is true that μ is $(\mathscr{H}, \mathscr{C})$-regular with respect to $\mathscr{C}=\mathscr{G}(X)$ and $\mathscr{H}=\mathscr{F}(X)$.

2.8. Example. (a) Let $(X, \mathscr{G}(X))$ be a topological space; then $\mathscr{F}=\mathscr{F}_0(X)$ corresponds with $\mathscr{C}=\mathscr{G}_0(X)$ as well as $\mathscr{F}=\mathscr{F}(X)$ with $\mathscr{C}=\mathscr{G}_0(X)$ or $=\mathscr{G}(X)$;

(b) Let $(X, \mathscr{G}(X))$ be a Hausdorff space; then $\mathscr{F}=\mathscr{K}(X)$ corresponds with $\mathscr{C}=\mathscr{G}(X)$.

Proof of 2.6–2.8. As to 2.6 (a)–(c) cf. the proofs of the Corollaries 4, 7, and 10 in [11]. 2.7 (a), (b), and (e) follow from 2.6 and 2.5 (a) (as to 2.7 (a) (i) cf. Proposition 15 in [11]).

Proof of 2.7 (c). According to (2.5.1)–(2.5.3) it suffices to prove the assertion for $\mu \in M_+(X, \tau)$. As $(X, \mathscr{G}(X))$ is a regular space, for any $G_0 \in \mathscr{G}(X)$ we have $G_0=\cup \mathscr{G}_0$ with $\mathscr{G}_0=\{G \in \mathscr{G}(X): G^a \subset G_0\}\uparrow$, hence for any $\varepsilon>0$ there exists $G_1 \in \mathscr{G}_0$ such that $\mu(G_0 \backslash G_1)<\varepsilon$. Put $G_2:=\bar{G}_1^a$; then $G_1 \subset \bar{G}_2=G_1^a \subset G_0$ and $\mu(G_0 \backslash G_1)<\varepsilon$, which proves (ii). Simultaneously the proof so far shows that $\mathscr{G}(X) \subset \mathscr{A}_0:=\{A \in \mathscr{B}(X): \forall \varepsilon>0 \exists F \in \mathscr{F}(X), G \in \mathscr{G}(X)$ s.t. $F \subset A \subset G$ and $\mu(G \backslash F)<\varepsilon\}$; but $\mathscr{A}_0$ is a σ-field and therefore $\mathscr{B}(X)=\mathscr{A}_0$ which implies (i). Proof of 2.7 (d): As before, w.l.o.g. $\mu \in M_+(X, \tau)$, and it suffices to prove that μ is $(\mathscr{H}, \mathscr{C})$-regular w.r.t. $\mathscr{C}=\mathscr{G}_0(X)$ and $\mathscr{H}=\mathscr{F}(X)$ [cf. 2.5 (b)]. For any $G_0 \in \mathscr{G}(X)$, we have $G_0=\cup \mathscr{G}_0$ with $\mathscr{G}_0=\{G \in \mathscr{G}_0(X): \exists G' \in \mathscr{G}_0(X)$ s.t. $G \subset \bar{G}' \subset G_0\}\uparrow$: It follows from the $(\cup f)$-closedness of $\mathscr{G}_0(X)$ and $\overline{\mathscr{G}_0(X)}$ that $\mathscr{G}_0$ is filtering to the right; also $\cup \mathscr{G}_0 \subset G_0$ by definition of $\mathscr{G}_0$. Hence it suffices to show that $G_0 \subset \cup \mathscr{G}_0$: As $(X, \mathscr{G}(X))$ is completely regular, for any $x \in G_0$ there exists $f_x \in C(X)$ s.t. $0 \leqq f_x \leqq 1$, $f_x(x)=1$ and $f_x|\bar{G}_0 \equiv 0$. Then $f_1:=(f_x-\frac{1}{4})^-$ and $f_2:=(f_x-\frac{1}{2})^+$ belong to $C(X)$, $0 \leqq f_i \leqq 1$, $\{f_i>0\} \in \mathscr{G}_0(X)$, $i=1, 2$, and $x \in \{f_x>\frac{1}{2}\}=\{f_2>0\} \subset \{f_x \geqq \frac{1}{4}\}=\{f_1=0\} \subset G_0$; hence $x \in \{f_2>0\} \in \mathscr{G}_0$ and therefore $x \in \cup \mathscr{G}_0$. Now, as $\mu \in M_+(X, \tau)$, for any $\varepsilon>0$ there exists $G_1 \in \mathscr{G}_0$ s.t. $\mu(G_0 \backslash G_1)<\varepsilon$ which implies, according to the definition of $\mathscr{G}_0$, that μ is $(\mathscr{H}, \mathscr{C})$-regular w.r.t. $\mathscr{C}=\mathscr{G}_0(X)$ and $\mathscr{H}=\mathscr{F}(X)$. Finally, the assertions in 2.8 are immediate.

2.9. Definition. (a) We shall say that $\mathscr{B}_0 \subset \mathscr{B}$ approximates $\mathscr{A}_0 \subset \mathscr{B}$ from below [above] with respect to $\mu \in M_+(X, \mathscr{B})$ if

$$\inf\{\mu(A \backslash B): \mathscr{B}_0 \ni B \subset A\}=0 [\inf\{\mu(B \backslash A): \mathscr{B}_0 \ni B \supset A\}=0]$$

for every $A \in \mathscr{A}_0$.

(b) We shall say that $\mathscr{B}_0 \subset \mathscr{B}$ approximates $\mathscr{A}_0 \subset \mathscr{B}$ from below [above] uniformly with respect to $M \subset M_+(X, \mathscr{B})$ if

$$\inf\left\{\sup_{\mu \in M} \mu(A \setminus B) : \mathscr{B}_0 \ni B \subset A\right\} = 0 \quad \left[\inf\left\{\sup_{\mu \in M} \mu(B \setminus A) : \mathscr{B}_0 \ni B \supset A\right\} = 0\right]$$

for every $A \in \mathscr{A}_0$.

Remark. If $\mu \in M(X, \mathscr{B})$ is $(\mathscr{H}, \mathscr{C})$-regular, then $\mathscr{C}$ approximates $\mathscr{H}$ from above with respect to $|\mu|$. For an analogous result for families of measures see Lemma 2.17 below.

2.10. Definition. (a) $M \subset M(X, \mathscr{B})$ is said to be dominated by $\lambda \in M_+(X, \mathscr{B})$ $(M \ll \lambda)$ if for all $\mu \in M$ $\lambda(A) = 0$ implies $\mu(A) = 0$.

(b) $M \subset M(X, \mathscr{B})$ is said to be uniformly dominated by $\lambda \in M_+(X, \mathscr{B})$ $(M \lll \lambda)$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\lambda(A) < \delta(\varepsilon)$ implies $\sup_{\mu \in M} |\mu(A)| < \varepsilon$.

2.11. Remark. (a) $M \ll \lambda [M \lll \lambda]$ iff $|M| \ll \lambda [|M| \lll \lambda]$.

(b) $M \ll \lambda$ iff for every $\mu \in M$ and for any $\varepsilon > 0$ there exists $\delta(\mu, \varepsilon) > 0$ such that $\lambda(A) < \delta(\mu, \varepsilon)$ implies $|\mu(A)| < \varepsilon$.

2.12. Definition. $M \subset M(X, \mathscr{B})$ is called equicontinuous if $\lim_{n \to \infty} \sup_{\mu \in M} |\mu(A_n)| = 0$ for each sequence $(A_n \in \mathscr{B})_{n \in \mathbb{N}}$ with $A_n \downarrow \emptyset$ as $n \to \infty$.

In the sequel M will always denote a subset of $M(X, \mathscr{B})$ and as before we maintain our General Convention.

2.13. Proposition. *Let* $M^{\Sigma} := \left\{ \nu := \sum_{m \in \mathbb{N}} 2^{-m} \frac{|\mu_m|}{1 + \|\mu_m\|} : \mu_m \in M, m \in \mathbb{N} \right\}$*; then*

(a) *If* $\mathscr{F}$ *is a* $(\cup f)$*-paving then* $M \subset M(X, \mathscr{B}, \mathscr{F})$ *implies* $M^{\Sigma} \subset M_+(X, \mathscr{B}, \mathscr{F})$*;*

(b) $M \subset M(X, \tau)$ *implies* $M^{\Sigma} \subset M_+(X, \tau)$*;*

(c) *If* $\mathscr{C}$ *is a* $(\cup f)$*-paving which is* $\mathscr{H}$*-filtering and if* μ *is* $(\mathscr{H}, \mathscr{C})$*-regular for all* $\mu \in M$*, then* ν *is* $(\mathscr{H}, \mathscr{C})$*-regular for all* $\nu \in M^{\Sigma}$.

Proof. Let us prove (c) and remark that (a) and (b) can be proved analogously. Suppose that μ is $(\mathscr{H}, \mathscr{C})$-regular for all $\mu \in M$ and consider $\nu = \sum_{m \in \mathbb{N}} 2^{-m} \frac{|\mu_m|}{1 + \|\mu_m\|}$, $\mu_m \in M$; then for any $H \in \mathscr{H}$ there exist sequences $(C_{1n})_{n \in \mathbb{N}}$ and $(C_{2n})_{n \in \mathbb{N}}$ of sets in $\mathscr{C}$ s.t. $H \subset C_{1n} \subset \overline{C_{2n}}$, $n \in \mathbb{N}$, and

$$|\mu_m|\left(\bigcap_n \overline{C}_{2n} \setminus H\right) = 0 \text{ for all } m \in \mathbb{N}, \text{ hence } \nu\left(\bigcap_n \overline{C}_{2n} \setminus H\right) = 0;$$

therefore for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ s.t. $\nu\left(\bigcap_{n \leq k} \overline{C}_{2n} \setminus H\right) < \varepsilon$. Since $\mathscr{C}$ is $\mathscr{H}$-filtering and $H \subset \bigcap_{n \leq k} C_{1n} \subset \bigcap_{n \leq k} \overline{C}_{2n} = \overline{\bigcup_{n \leq k} C_{2n}}$ with $\bigcup_{n \leq k} C_{2n} \in \mathscr{C}$, it follows that ν is $(\mathscr{H}, \mathscr{C})$-regular for all $\nu \in M^{\Sigma}$.

2.14. Proposition.(cf. [4], 2.6). (a) *If* $\mathscr{F}$ *is a* $(\cup f)$*-paving and if* $M \subset M(X, \mathscr{B}, \mathscr{F})$ $[M \subset M(X, \tau)]$ *is uniformly dominated by* $\lambda \in M_+(X, \mathscr{B})$*, then there exists* $\bar{\lambda} \in M_+(X, \mathscr{B}, \mathscr{F})$ $[\bar{\lambda} \in M_+(X, \tau)]$ *such that* $M \lll \bar{\lambda}$.

(b) *If* $\mathscr{C}$ *is a* $(\cup f)$*-paving which is* $\mathscr{H}$*-filtering and if* $M \lll \lambda$ *with* μ *being* $(\mathscr{H}, \mathscr{C})$*-regular for all* $\mu \in M$*, then there exists* $\bar{\lambda} \in M_+(X, \mathscr{B})$*,* $\bar{\lambda}$ *being* $(\mathscr{H}, \mathscr{C})$*-regular, such that* $M \lll \bar{\lambda}$.

Proof. Applying A2 with $M_k := \{\mu \in M : \|\mu\| \leqq k\}$, $k \in \mathbb{N}$, instead of M, and using A1 it follows that M_k is uniformly dominated by some $\lambda_k \in M_k^\Sigma \subset M^\Sigma$ and also that M is uniformly dominated by $\bar{\lambda} := \sum_{k \in \mathbb{N}} 2^{-k} \frac{\lambda_k}{1 + \|\lambda_k\|} \in (M^\Sigma)^\Sigma$; therefore the assertions follow from 2.13.

For the sequel the next concept proves to be essential; for one point sets M it reduces to the concept of "s-bounded" measures as introduced by Rickart in [13].

2.15. Definition. $M \subset M(X, \mathscr{B})$ is said to be uniformly s-bounded with respect to $\mathscr{A} \subset \mathscr{B}$ if $\sup_{\mu \in M} |\mu(A_n)| \to 0$ as $n \to \infty$ for each sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets in $\mathscr{A}$.

2.16. Proposition. *Let $M \subset M(X, \mathscr{B}, \mathscr{F})$ and $\mathscr{C}$ be a subpaving of $\mathscr{B}$. We consider the following assertions:*

(a) *$|M|$ is uniformly s-bounded w.r.t. $\mathscr{F}$;*

(b) *M is uniformly s-bounded w.r.t. $\mathscr{F}$;*

(c) *$|M|$ is uniformly s-bounded w.r.t. $\mathscr{C}$;*

(d) *M is uniformly s-bounded w.r.t. $\mathscr{C}$.*

Then (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d). *If, in addition, $\mathscr{C}$ is $\mathscr{F}$-filtering and μ is $(\mathscr{F}, \mathscr{C})$-regular for all $\mu \in M$, then (c)$\Leftrightarrow$(d).*

Proof. Let us prove that (d) implies (c) whenever $\mathscr{C}$ is $\mathscr{F}$-filtering and μ is $(\mathscr{F}, \mathscr{C})$-regular for all $\mu \in M$. The remaining assertions can be proved in an analogous manner without the additional assumptions. Suppose to the contrary that (c) does not hold; then there exist $\varepsilon > 0$, a sequence $(C_k)_{k \in \mathbb{N}}$ of pairwise disjoint sets in $\mathscr{C}$ and a sequence $(\mu_k \in M)_{k \in \mathbb{N}}$ such that $|\mu_k|(C_k) > \varepsilon$ for all $k \in \mathbb{N}$, hence by the $\mathscr{F}$-regularity of μ_k there exist $F_k \subset C_k$ with $|\mu_k(F_k)| > \varepsilon/2$, $k \in \mathbb{N}$. Now, since the μ_k are assumed to be $(\mathscr{F}, \mathscr{C})$-regular, it follows from the remark preceding 2.10 that $\mathscr{C}$ approximates $\mathscr{F}$ from above w.r.t. $|\mu_k|$, i.e. there exist $C'_k \supset F_k$ with

$$|\mu_k|(C'_k \setminus F_k) < |\mu_k(F_k)| - \varepsilon/2 .$$

Since $F_k \subset C_k \cap C'_k$ and $\mathscr{C}$ is $\mathscr{F}$-filtering there exists $C''_k \in \mathscr{C}$ such that $F_k \subset C''_k \subset C_k \cap C'_k$ and $|\mu_k(C''_k)| \geqq \mu_k(F_k)| - |\mu_k|(C'_k \setminus F_k) > \frac{\varepsilon}{2}$. This, however, contradicts (d), since the sets C''_k, $k \in \mathbb{N}$, are pairwise disjoint.

2.17. Lemma. *Let $M \subset M(X, \mathscr{B}, \mathscr{F})$ and assume that each $\mu \in M$ is $(\mathscr{F}, \mathscr{C})$-regular where $\mathscr{C}$ is $\mathscr{F}$-filtering. Then, if M is uniformly s-bounded with respect to $\mathscr{C}$, it follows that $\mathscr{C}$ approximates $\mathscr{F}$ from above uniformly with respect to $|M|$.*

Proof. According to (2.5.1), (2.5.3), and 2.16 we may assume w.l.o.g. that $\emptyset \neq M \subset M_+(X, \mathscr{B}, \mathscr{F})$. If $\mathscr{C}$ does not approximate $\mathscr{F}$ from above uniformly w.r.t. M, then there exist F and $\varepsilon > 0$ s.t. $\inf_C \left\{\sup_{\mu \in M} \mu(C \setminus F) : F \subset C\right\} > \varepsilon$; furthermore we claim that there exist sequences $(C_n)_{n \in \mathbb{N}}$ and $(C'_n)_{n \in \mathbb{N}}$ of sets in $\mathscr{C}$ and a sequence $(\mu_n \in M)_{n \in \mathbb{N}}$ s.t. for all $n \in \mathbb{N}$: (i) $C_{n+1} \subset C_n$, $C'_n \subset C_n \setminus C_{n+1}$, (ii) $F \subset C_n$ and (iii) $\mu_n(C'_n) > \varepsilon$. Since the C'_n, $n \in \mathbb{N}$, are pairwise disjoint, this contradicts the uniform s-boundedness of M w.r.t. $\mathscr{C}$.

To prove our claim, note first that by $(\mathscr{F}, \mathscr{C})$-regularity of $\mu \in M$, there exists C_1 with $F \subset C_1$; assume that $C_1, \ldots, C_k$ and, for $k > 1$, $C'_1, \ldots, C'_{k-1}$ and $\mu_1, \ldots, \mu_{k-1}$

have been already constructed so that (i)–(iii) are fulfilled. According to the choice of F there exists $\mu_k \in M$ s.t. $\mu_k(C_k \setminus F) > \varepsilon$. By the $(\mathscr{F}, \mathscr{C})$-regularity of μ_k one can find D_k, $D'_k \in \mathscr{C}$ with $F \subset D_k \subset \overline{D'_k}$ and $\mu_k(\overline{D'_k} \setminus F) < \mu_k(C_k \setminus F) - \varepsilon$, hence $\mu_k(C_k \cap D'_k) \geqq \mu_k(C_k \setminus F) - \mu_k(\overline{D'_k} \setminus F) > \varepsilon$. By the $\mathscr{F}$-regularity of μ_k we can find $F_k \subset C_k \cap D'_k$ with $\mu_k(F_k) > \varepsilon$. As $\mathscr{C}$ is $\mathscr{F}$-filtering one can find sets C'_k and C_{k+1} in $\mathscr{C}$ such that $F_k \subset C'_k \subset C_k \cap D'_k$, $F \subset C_{k+1} \subset C_k \cap D_k$ and thus $\mu_k(C'_k) > \varepsilon$. This proves our claim and hence the lemma.

2.18. Lemma. *Let $\mathscr{F}$ be a $(\cup f)$-paving and $\mathscr{C}$ be an $\mathscr{F}$-filtering paving. Let $M \subset M(X, \mathscr{B}, \mathscr{F})$ and assume that each $\mu \in M$ is both $(\mathscr{F}, \mathscr{C})$- and $(\overline{\mathscr{C}}, \mathscr{C})$-regular. Then, if M is uniformly s-bounded with respect to $\mathscr{C}$, it follows that $\mathscr{F}$ approximates $\mathscr{C}$ from below uniformly w.r.t. $|M|$.*

Proof. Again w.l.o.g. we may and do assume that $\emptyset \neq M \subset M_+(X, \mathscr{B}, \mathscr{F})$. If $\mathscr{F}$ does not approximate $\mathscr{C}$ from below uniformly w.r.t. M, then there exist $D \in \mathscr{C}$, $\varepsilon > 0$ s.t. $\inf_F \left\{ \sup_{\mu \in M} \mu(D \setminus F) : F \subset D \right\} > \varepsilon$; furthermore we claim that there exist sequences $(F_n)_{n \in \mathbb{N}}$, $(C_n, C'_n, C''_n)_{n \in \mathbb{N}}$ of sets in $\mathscr{F}$ and $\mathscr{C}$, respectively, and a sequence $(\mu_n \in M)_{n \in \mathbb{N}}$ s.t. for all $n \in \mathbb{N}$:

(i) $F_n \subset C_n \subset D \cap \bigcap_{i=1}^{n-1} C''_i$,

(ii) $\sup_{\mu \in M} \mu(C_n \setminus F_n) < \varepsilon \cdot 2^{-(n+1)}$

and

(iii) $C'_n \subset \overline{C''_n} \subset C_n$ and $\mu_n(C'_n) > \varepsilon/2$.

Since the C'_n, $n \in \mathbb{N}$, are pairwise disjoint, this contradicts the uniform s-boundedness of M w.r.t. $\mathscr{C}$.

To prove our claim, note first that the $\mathscr{F}$-regularity of $\mu \in M$ implies that there exist $F_1 \subset D$ and $\mu_1 \in M$ s.t. $\mu_1(F_1) > \varepsilon$. It follows from 2.17 that there exists $C^*_1 \in \mathscr{C}$ s.t. $F_1 \subset C^*_1$ and $\sup_{\mu \in M} \mu(C^*_1 \setminus F_1) < \varepsilon/4$. As $\mathscr{C}$ is $\mathscr{F}$-filtering choose $C_1 \in \mathscr{C}$ with $F_1 \subset C_1 \subset C^*_1 \cap D$; hence $\mu_1(C_1) > \varepsilon$ and $\sup_{\mu \in M} \mu(C_1 \setminus F_1) < \varepsilon/4$. Furthermore, by the $(\overline{\mathscr{C}}, \mathscr{C})$-regularity of μ_1 there exists a pair C'_1, C''_1 of sets in $\mathscr{C}$ s.t. $C'_1 \subset \overline{C''_1} \subset C_1$ and $\mu_1(C'_1) > \varepsilon/2$, which proves our claim for $n = 1$.

Now, assume that the quantities fulfilling (i)–(iii) have been already constructed up to $n = k$. Now $D \supset \bigcup_{i=1}^{k} F_i \in \mathscr{F}$, hence according to the choice of D there exists $\mu_{k+1} \in M$ s.t. $\mu_{k+1}\left(D \setminus \bigcup_{i=1}^{k} F_i\right) > \varepsilon$. Therefore, using the fact that (by the inductive assumption) $\sup_{\mu \in M} \mu\left(\left(\bigcup_{i=1}^{k} C_i\right) \setminus \left(\bigcup_{i=1}^{k} F_i\right)\right) < \varepsilon/2$ and $\bigcup_{i=1}^{k} \overline{C''_i} \subset \bigcup_{i=1}^{k} C_i \subset D$, one obtains that $\mu_{k+1}\left(D \setminus \bigcup_{i=1}^{k} \overline{C''_i}\right) > \varepsilon/2$, hence by the $\mathscr{F}$-regularity of μ_{k+1} we can find $F_{k+1} \subset D \cap \bigcap_{i=1}^{k} C''_i$ s.t. $\mu_{k+1}(F_{k+1}) > \varepsilon/2$. Now, it follows again from 2.17 that there exists $C^*_{k+1} \in \mathscr{C}$ s.t. $F_{k+1} \subset C^*_{k+1}$ and $\sup_{\mu \in M} \mu(C^*_{k+1} \backslash F_{k+1}) < \varepsilon \cdot 2^{-(k+2)}$. As $\mathscr{C}$ is $\mathscr{F}$-filtering choose $C_{k+1} \in \mathscr{C}$ with

$$F_{k+1} \subset C_{k+1} \subset C^*_{k+1} \cap D \cap \bigcap_{i=1}^{k} C''_i .$$

Then $\sup_{\mu \in M} \mu(C_{k+1} \backslash F_{k+1}) < \varepsilon \cdot 2^{-(k+2)}$, hence (i) and (ii) are fulfilled for $n = k+1$.

Finally, since $\mu_{k+1}(C_{k+1}) \geqq \mu_{k+1}(F_{k+1}) > \varepsilon/2$ and since μ_{k+1} is $(\bar{\mathscr{C}}, \mathscr{C})$-regular, there exists a pair C'_{k+1}, C''_{k+1} of sets in $\mathscr{C}$ s.t. $C'_{k+1} \subset \overline{C''_{k+1}} \subset C_{k+1}$ and $\mu_{k+1}(C'_{k+1}) > \varepsilon/2$, i.e. (iii) holds also true for $n = k+1$.

2.19. Lemma. *Let $M \subset M(X, \mathscr{B}, \mathscr{F})$ and assume that $\mathscr{F}$ corresponds with $\mathscr{C}$. If* (a) *$\mathscr{C}$ approximates $\mathscr{F}$ from above uniformly w.r.t. $|M|$ and if* (b) *$\mathscr{F}$ approximates $\{X\}$ from below uniformly w.r.t. $|M|$, then* (a) *and* (b) *together imply that $\mathscr{F}$ approximates $\mathscr{C}$ from below uniformly w.r.t. $|M|$.*

Proof. Let C and $\varepsilon > 0$ be given; according to (b) there exists F s.t. $\sup_{\mu \in M} |\mu| (\bar{F}) < \varepsilon/2$. Now, as $\mathscr{F}$ corresponds with $\mathscr{C}$, $F \cap \bar{C} \in \mathscr{F}$, and therefore (a) implies that there exists C_1 s.t. $F \cap \bar{C} \subset C_1$ and $\sup_{\mu \in M} |\mu| (C_1 \backslash (F \cap \bar{C})) < \varepsilon/2$. It follows that $F_1 := F \cap \bar{C}_1 \in \mathscr{F}$, $F_1 \subset C \cap F$ and $\sup_{\mu \in M} |\mu| ((C \cap F) \backslash F_1) < \varepsilon/2$, hence $F_1 \subset C$ and

$$\sup_{\mu \in M} |\mu| (C \backslash F_1) < \varepsilon .$$

2.20. Definition. (a) Let $\mathscr{T}_s$ denote the topology in $M(X, \mathscr{B})$ of set-wise convergence on $\mathscr{B}$, i.e. $\mathscr{T}_s$ is the weakest topology in $M(X, \mathscr{B})$ for which all mappings $\mu \to \mu(A)$, $A \in \mathscr{B}$, are continuous. In other words: If $\mu \in M(X, \mathscr{B})$ and if (μ_β) is a net in $M(X, \mathscr{B})$, then (μ_β) $\mathscr{T}_s$-converges to $\mu ((\mu_\beta) \underset{s}{\to} \mu)$ iff $\mu_\beta(A) \to \mu(A)$ for all $A \in \mathscr{B}$.

(b) $\mu_n \in M(X, \mathscr{B})$, $n \in \mathbb{N}$, is said to converge on $\mathscr{B}_0 \subset \mathscr{B}$ [to zero] if $(\mu_n(A))_{n \in \mathbb{N}}$ is convergent in $\mathbb{R}$ $\left[\lim_{n \to \infty} \mu_n(A) = 0\right]$ for all $A \in \mathscr{B}_0$.

(c) $\mathscr{B}_0 \subset \mathscr{B}$ is called a convergence class for $M \subset M(X, \mathscr{B})$ (w.r.t. $\mathscr{T}_s$) if any sequence $(\mu_n \in M)_{n \in \mathbb{N}}$ which converges on $\mathscr{B}_0$ does also converge on $\mathscr{B}$.

2.21. Proposition. *If $\mu_n \in M(X, \mathscr{B})$, $n \in \mathbb{N}$, converges on $\mathscr{B}$, then there exists $\mu \in M(X, \mathscr{B})$ such that $\mu_n \underset{s}{\to} \mu$; furthermore*

(a) *if $\mathscr{F}$ is a $(\cup f)$-paving then $\mu \in M(X, \mathscr{B}, \mathscr{F})$ provided $\mu_n \in M(X, \mathscr{B}, \mathscr{F})$ for all $n \in \mathbb{N}$;*

(b) *$\mu \in M(X, \tau)$ provided $\mu_n \in M(X, \tau)$ for all $n \in \mathbb{N}$;*

(c) *if $\mathscr{C}$ is a $(\cup f)$-paving which is $\mathscr{H}$-filtering then μ is $(\mathscr{H}, \mathscr{C})$-regular provided μ_n is $(\mathscr{H}, \mathscr{C})$-regular for all $n \in \mathbb{N}$.*

Proof. The first assertion is Nikodym's theorem (cf. A4). According to the Vitali-Hahn-Saks-Theorem (A3) $\{\mu_n : n \in \mathbb{N}\}$ is uniformly dominated by $\lambda \in M_+(X, \mathscr{B})$, where according to 2.14 we may assume w.l.o.g. that $\lambda \in M_+(X, \mathscr{B}, \mathscr{F})$ in case (a), $\lambda \in M_+(X, \tau)$ in case (b), and $\lambda (\mathscr{H}, \mathscr{C})$-regular in case (c), whence the assertion follows since λ does also dominate the limit measure μ.

From Lemma 10 in [10] (cf. A5) we obtain the following result.

2.22. Lemma. *Assume that $\mathscr{C}$ is a $(\cup c)$-paving and let $\mu_n \in M(X, \mathscr{B})$, $n \in \mathbb{N}$, be convergent on $\mathscr{C}$; then $\{\mu_n : n \in \mathbb{N}\}$ is uniformly s-bounded with respect to $\mathscr{C}$.*

Proof. Assume the contrary; then there exist $\varepsilon > 0$, a subsequence $(\mu_{n_k})_{k \in \mathbb{N}}$ of $(\mu_n)_{n \in \mathbb{N}}$ and a sequence $(C_k)_{k \in \mathbb{N}}$ of pairwise disjoint sets in $\mathscr{C}$ such that $\inf_{k \in \mathbb{N}} |\mu_{n_k}(C_k)| > \varepsilon$.

But this is in contradiction to Lemma 10 in [10] (cf. A5) with $b_{ik} := \mu_{n_i}(C_k)$ for i, $k \in \mathbb{N}$.

2.23. Proposition. *Let $\mu_m \in M(X, \mathscr{B})$, $m \in \mathbb{N}$, and $A_n \in \mathscr{B}$, $n \in \mathbb{N}$, be such that $A_n \uparrow A$ $[A_n \downarrow A]$ as $n \to \infty$; then the following holds true:*

(a) *If $\{\mu_m : m \in \mathbb{N}\} \lll \lambda$, then $\sup_{m \in \mathbb{N}} |\mu_m| (A \setminus A_n) \to 0$ $\left[\sup_{m \in \mathbb{N}} |\mu_m| (A_n \setminus A) \to 0\right]$ as $n \to \infty$;*

(b) *If $(\mu_m(A_n))_{m \in \mathbb{N}}$ is convergent in $\mathbb{R}$ (convergent to zero) for all $n \in \mathbb{N}$, and if $\sup_{m \in \mathbb{N}} |\mu_m(A \setminus A_n)| \to 0$ $\left[\sup_{m \in \mathbb{N}} |\mu_m(A_n \setminus A)| \to 0\right]$ as $n \to \infty$, then $(\mu_m(A))_{m \in \mathbb{N}}$ is convergent in $\mathbb{R}$ (convergent to zero).*

The simple proof of 2.23 is left to the reader.

2.24. Proposition. *Let $\mu_n \in M(X, \mathscr{B}, \mathscr{F})$, $n \in \mathbb{N}$, be convergent on $\mathscr{C}$ [to zero], where $\mathscr{C}$ is a $(\cup c)$-paving. If $\mathscr{C}$ approximates $\mathscr{F}$ from above uniformly with respect to $\{|\mu_n| : n \in \mathbb{N}\}$, then μ_n, $n \in \mathbb{N}$, is convergent on $\mathscr{B}$ [to zero].*

Proof. Let B and $\varepsilon > 0$ be given. By $\mathscr{F}$-regularity of μ_n there exists F_n such that $F_n \subset B$ and $|\mu_n| (B \setminus F_n) < \frac{\varepsilon}{6}$. Furthermore, for every $j \in \mathbb{N}$ we can find a set C_j satisfying $F_j \subset C_j$ and $|\mu_n| (C_j \setminus F_j) < \frac{\varepsilon}{3} \cdot 2^{-(j+1)}$ for all $n \in \mathbb{N}$. Putting $C := \bigcup_{j \in \mathbb{N}} C_j$ which pertains to $\mathscr{C}$ by assumption one obtains $B \triangle C \subset (B \setminus F_n) \cup \bigcup_{j \in \mathbb{N}} (C_j \setminus F_j)$ and therefore $|\mu_n| (B \triangle C) \leqq |\mu_n| (B \setminus F_n)$

$$+ \sum_{j \in \mathbb{N}} |\mu_n| (C_j \setminus F_j) < \frac{\varepsilon}{3} \text{ for all } n \in \mathbb{N}.$$

Since, by assumption, $(\mu_n(C))_{n \in \mathbb{N}}$ is convergent in $\mathbb{R}$, we get

$$\begin{aligned} |\mu_n(B) - \mu_m(B)| = {} & |\mu_n(B \setminus C) - \mu_m(B \setminus C) \\ & + \mu_n(B \cap C) - \mu_m(B \cap C)| \leqq |\mu_n|(B \triangle C) \\ & + |\mu_m|(B \triangle C) + |\mu_n(C) - \mu_m(C)| < \varepsilon \end{aligned}$$

for all sufficiently large m, $n \in \mathbb{N}$. Hence the sequence $(\mu_n(B))_{n \in \mathbb{N}}$ converges in $\mathbb{R}$. In an analogous manner one can prove that the sequence $(\mu_n)_{n \in \mathbb{N}}$ is convergent on $\mathscr{B}$ to zero, if it is convergent on $\mathscr{C}$ to zero.

2.25. Lemma. *Let $M \subset M(X, \mathscr{B})$ be such that $\mu_1 - \mu_2 \in M$ whenever $\mu_1, \mu_2 \in M$ and consider $\mathscr{H} \subset \mathscr{B}$ having the property that any sequence $\mu_n \in M$, $n \in \mathbb{N}$, which converges to zero on $\mathscr{H}$ does also converge to zero on $\mathscr{B}$; then $\mathscr{H}$ is a convergence class for M.*

Proof. Let $\mu_n \in M$, $n \in \mathbb{N}$, be s.t. $\lim_{n\to\infty} \mu_n(H)$ exists in $\mathbb{R}$ for all $H \in \mathscr{H}$. If $(\mu_n(B_0))_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$ for some B_0 then there exists $\varepsilon > 0$ and for any $n \in \mathbb{N}$ a natural number $m(n)$ s.t. $|\mu_n(B_0) - \mu_{n+m(n)}(B_0)| > \varepsilon$. $(+)$

Let $\nu_n := \mu_n - \mu_{n+m(n)}$; then $\nu_n \in M$ and $\lim_{n\to\infty} \nu_n(H) = 0$ for all $H \in \mathscr{H}$ and therefore $\lim_{n\to\infty} \nu_n(B) = 0$ for all $B \in \mathscr{B}$, which contradicts $(+)$.

2.26. Lemma. *Let* $(X, \mathscr{G}(X))$ *be a topological space and* $\mu_n \in M(X, \mathscr{B}(X))$, $n \in \mathbb{N}$, *be* $(\mathscr{F}(X), \mathscr{G}(X))$*-regular. Then the following holds true:*

If $(\mu_n)_{n\in\mathbb{N}}$ *converges on* $\mathscr{G}_r(X)$ *to zero, then* $\{\mu_n : n \in \mathbb{N}\}$ *is uniformly s-bounded with respect to* $\mathscr{G}(X)$.

Proof. Assume to the contrary that $\{\mu_n : n \in \mathbb{N}\}$ is not uniformly s-bounded w.r.t. $\mathscr{G}(X)$; then there exist $\varepsilon > 0$, a sequence of pairwise disjoint open sets G_k, $k \in \mathbb{N}$ and a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ s.t. $\inf_{k\in\mathbb{N}} |\mu_{n_k}(G_k)| > \varepsilon$. By (2.5.3) all $|\mu_{n_k}|$, $k \in \mathbb{N}$ are $(\mathscr{F}(X), \mathscr{G}(X))$-regular, hence there exist pairs G'_k, G''_k of open sets s.t. $\overline{G}_k \subset G'_k \subset \overline{G''_k}$ and

$$|\mu_{n_k}|\,(\overline{G''_k} \backslash \overline{G}_k) = |\mu_{n_k}|\,(G_k \backslash G''_k) < |\mu_{n_k}(G_k)| - \varepsilon\,, \qquad k \in \mathbb{N}\,.$$

Put $R_k := (\overline{G'_k})^0$, $k \in \mathbb{N}$; then $R_k \in \mathscr{G}_r(X)$, $\left(\bigcup_{m \neq k} R_m\right)^a \cap R_k^a \subset \left(\bigcup_{m \neq k} R_m\right)^a \cap \overline{G'_k} \subset \overline{G_k} \cap \overline{G'_k} = \emptyset$ and $|\mu_{n_k}(R_k)| \geqq |\mu_{n_k}(G_k)| - |\mu_{n_k}|\,(G_k \backslash G''_k) > \varepsilon$ for all $k \in \mathbb{N}$, which contradicts the assertion of Lemma 3.4 in [5] (see A6).

Let us conclude this section with the following lemma which is taken from [1] (Lemma 4.1.4).

2.27. Lemma. *Let* $(X, \mathscr{G}(X))$ *be a perfectly normal space and* $M \subset M(X, \mathscr{B}(X))$, *or let* $(X, \mathscr{G}(X))$ *be a completely regular space and* $M \subset M(X, \tau)$; *then in both cases the following holds true: If, for every uniformly bounded sequence of continuous functions* f_j, $j \in \mathbb{N}$, *converging to zero at every point* $x \in X$, *we have* $\lim_{j\to\infty} \int_X f_j d\mu = 0$ *uniformly with respect to* $\mu \in M$, *then* M *is uniformly s-bounded with respect to* $\mathscr{G}(X)$.

Proof. Assume to the contrary that M is not uniformly s-bounded w.r.t. $\mathscr{G}(X)$; then there exist a sequence of pairwise disjoint open sets G_n, $n \in \mathbb{N}$, $\varepsilon > 0$, and an infinite subset $\mathbb{N}_0$ of $\mathbb{N}$, s.t. $\inf_{n\in\mathbb{N}_0} |\mu_n(G_n)| > \varepsilon$ for some $\mu_n \in M$; w.l.o.g. we may take $\mathbb{N}_0 = \mathbb{N}$. Now, consider the first case, where $(X, \mathscr{G}(X))$ is supposed to be perfectly normal and $M \subset M(X, \mathscr{B}(X))$. Then, for every $n \in \mathbb{N}$, there exists a sequence of nonnegative continuous functions $f_{n,k}$, $k \in \mathbb{N}$, s.t. $f_{n,k} \uparrow \chi_{G_n}$ as $k \to \infty$. Hence for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ s.t.

$$|\mu_n(G_n)| < \left|\int_X f_{n,k_n} d\mu_n\right| + \varepsilon/2; \text{ put } f_n := f_{n,k_n},\ n \in \mathbb{N}.$$

If, as assumed in the second case, $(X, \mathscr{G}(X))$ is completely regular and $M \subset M(X, \tau)$, then it follows that $\chi_{G_n} = \sup \mathscr{H}_n$ with $\mathscr{H}_n := \{f \in C(X) : 0 \leqq f \leqq \chi_{G_n}\}$. Since μ_n is supposed to be τ-smooth, there exist $f_n \in \mathscr{H}_n$ s.t. $|\mu_n(G_n)| < \left|\int_X f_n d\mu_n\right| + \varepsilon/2$.

Therefore in both cases we did find a uniformly bounded sequence $(f_n \in C(X))_{n\in\mathbb{N}}$ converging to zero at every point $x\in X$, whence, by assumption $\sup_{\mu\in M}\left|\int_X f_n d\mu\right| < \varepsilon/2$ for n sufficiently large; but this will yield a contradiction to $\inf_{n\in\mathbb{N}} |\mu_n(G_n)| > \varepsilon$.

3. Compactness in Spaces of Regular or τ-Smooth Measures

This section is concerned with compactness criteria in the space $(M(X, \mathscr{B}, \mathscr{F}), \mathscr{T}_s)$ which will cover for topological basic spaces X all criteria known to us up to now, especially those given in [1] and [4] in the case of τ-smooth and tight measures, respectively. We shall make here consequent use of the concept of $(\mathscr{F}, \mathscr{C})$-regularity as defined in 2.3 which turns out to be essential also for the unified presentation of our convergence results in the next section.

Partially the proofs of the following theorem are taken from [1] and [4], respectively.

3.1. Theorem. *Let $\mathscr{F}$ be a $(\cup f)$-paving and $\mathscr{C}$ be a $(\cup c, \cap f)$-paving. Let M be a bounded subset of $M(X, \mathscr{B}, \mathscr{F})$ and suppose that μ is $(\mathscr{F}, \mathscr{C})$-regular for all $\mu\in M$. Then the following assertions are equivalent:*

(i) *M is conditionally compact in $(M(X, \mathscr{B}), \mathscr{T}_s)$;*

(ii) *M is conditionally sequentially compact in $(M(X, \mathscr{B}), \mathscr{T}_s)$;*

(iii) *M is uniformly dominated by some $\lambda\in M_+(X, \mathscr{B}, \mathscr{F})$ (and every $\nu\in M_+(X, \mathscr{B})$ dominating M dominates M uniformly);*

(iv) *For every uniformly bounded sequence $f_j\in B(X, \mathscr{B})$, $j\in\mathbb{N}$, and every $f\in B(X, \mathscr{B})$ with the property that for all $\mu\in M$ and all F $f_j\cdot\chi_F \to f\cdot\chi_F$ in $|\mu|$-measure, we have $\lim_{j\to\infty}\int_X f_j d\mu = \int_X f\,d\mu$ uniformly w.r.t. $\mu\in M$;*

(v) *M is uniformly s-bounded w.r.t. $\mathscr{F}$;*

(vi) *For every monotone decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{C}$ we have $\inf_{n\in\mathbb{N}}\sup_{\mu\in M}|\mu|\left(C_n\setminus\bigcap_{k\in\mathbb{N}} C_k\right) = 0$;*

(vii) (a) *$\mathscr{F}$ approximates $\mathscr{C}$ from below uniformly w.r.t. $|M|$ and* (b) *M is uniformly s-bounded w.r.t. $\mathscr{C}$.*

Proof. (i)⇔(ii)⇔(iii): A simple proof for these equivalences is given in Theorem 2.6, Remark 1.8, and Corollary 2.7 of [4] (cf. A1, A2, and 2.14). (iii)⇒(iv): Let $f_j\in B(X, \mathscr{B})$, $j\in\mathbb{N}$, be uniformly bounded, $C := \max\left\{1, \sup_{j\in\mathbb{N}}\sup_{x\in X}|f_j(x)|\right\} < \infty$,

and consider $f\in B(X, \mathscr{B})$ s.t. $f_j\chi_F\to f\chi_F$ in $|\mu|$-measure for all F and $\mu\in M$; as μ is $\mathscr{F}$-regular, this implies $|\mu|(\{|f|>C\}) = 0$ for all $\mu\in M$, and therefore f is integrable w.r.t. $|\mu|$. Now, assume (iv) to be wrong; then there exists a uniformly bounded sequence $f_j\in B(X, \mathscr{B})$, $j\in\mathbb{N}$, so that there exists $\varepsilon>0$, an infinite subset $\mathbb{N}_0\subset\mathbb{N}$ and $\mu_n\in M$, $n\in\mathbb{N}_0$, with $\inf_{n\in\mathbb{N}_0}\left|\int_X (f_n - f)d\mu_n\right| > \varepsilon$. W.l.o.g. we may assume $\mathbb{N}_0 = \mathbb{N}$. Since $\{|\mu_n| : n\in\mathbb{N}\} \lll \lambda\in M_+(X, \mathscr{B}, \mathscr{F})$, there exists F_0 s.t. $\sup_{n\in\mathbb{N}}|\mu_n|(\overline{F_0}) < \varepsilon/4C$. Hence,

for all $n\in\mathbb{N}$, $\left|\int_X (f_n - f)d\mu_n - \int_{F_0}(f_n - f)d\mu_n\right| \leq \int_{\overline{F_0}}|f_n - f|d|\mu_n| < \varepsilon/2$ and therefore

$(*)$ $\inf_{n\in\mathbb{N}}\left|\int_{F_0}(f_n - f)d\mu_n\right| > \varepsilon/2.$

On the other hand, if we define $\lambda_0:=\sum_{m\in\mathbb{N}} 2^{-m}\frac{|\mu_m|}{1+\|\mu_m\|}$, it follows that $f_n\cdot\chi_{F_0}\to f\cdot\chi_{F_0}$ in λ_0-measure, and $\{|\mu_m|: m\in\mathbb{N}\}\lll\lambda_0$ [cf. A1 and 2.11 (a)], hence there exists $\delta>0$ s.t. $\lambda_0(A)<\delta$ implies $\sup_{m\in\mathbb{N}}|\mu_m|(A)<\varepsilon/8C$. Now, as $f_n\cdot\chi_{F_0}\to f\cdot\chi_{F_0}$ in λ_0-measure, we obtain, given $\delta_0:=\varepsilon/\left(4\sup_{\mu\in M}\|\mu\|+1\right)$, that $\lambda_0(F_0\cap\{|f_n-f|>\delta_0\})<\delta$ for all $n\geqq n_0$, and therefore $\sup_{m\in\mathbb{N}}|\mu_m|(F_0\cap\{|f_n-f|>\delta_0\})<\varepsilon/8C$ for all $n\geqq n_0$. It follows that for all $n\geqq n_0$

$$\left|\int_{F_0}(f_n-f)d\mu_n\right|\leqq\int_{F_0\cap\{|f_n-f|>\delta_0\}}|f_n-f|d|\mu_n| + \int_{F_0\cap\{|f_n-f|\leqq\delta_0\}}|f_n-f|d|\mu_n|<\varepsilon/4+\delta_0\sup_{\mu\in M}\|\mu\|\leqq\varepsilon/2,$$

which contradicts (∗).

(iii) ⇒(v)–(vii): Obvious.

(iv)⇒(v): Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathscr{F}$; then $f_n:=\chi_{F_n}$, $n\in\mathbb{N}$, converges pointwise to zero, hence for all F and $\mu\in M$ we have $f_n\chi_F\to 0$ in $|\mu|$-measure, whence, by (iv), $\lim_{n\to\infty}\mu(F_n)=0$ uniformly w.r.t. $\mu\in M$.

For the rest of the proof we may and do assume w.l.o.g. that $M\subset M_+(X,\mathscr{B},\mathscr{F})$ (cf. (2.5.1), (2.5.3), 2.11 (a), and 2.16).

(v)⇒(vi): Assume (vi) to be wrong; then there exist $\varepsilon>0$, a monotone decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{C}$ and a sequence $(\mu_n\in M)_{n\in\mathbb{N}}$ s.t. $\inf_{n\in\mathbb{N}}\mu_n\left(C_n\setminus\bigcap_{k\in\mathbb{N}}C_k\right)>\varepsilon$. From this it follows that there exists a strictly increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ s.t. $\inf_{k\in\mathbb{N}}\mu_{n_k}(C_{n_k}\setminus C_{n_{k+1}})>\varepsilon$. By the $\mathscr{F}$-regularity of the μ_{n_k} there exist F_k with $F_k\subset C_{n_k}\setminus C_{n_{k+1}}$ and $\mu_{n_k}(F_k)>\varepsilon$, $k\in\mathbb{N}$; hence we arrive at a sequence of pairwise disjoint F_k, $k\in\mathbb{N}$, with $\sup_{\mu\in M}\mu(F_k)>\varepsilon$ for all $k\in\mathbb{N}$ which contradicts (v).

(vi)⇒(iii): According to Theorem 2.6, Remark 1.8 and Corollary 2.7 of [4] (cf. A1, A2, and 2.14) it suffices to show that for any sequence $(\mu_n\in M)_{n\in\mathbb{N}}$ it is true that $\{\mu_n: n\in\mathbb{N}\}$ is uniformly dominated by

$$\lambda:=\sum_{n\in\mathbb{N}}2^{-n}\frac{\mu_n}{1+\|\mu_n\|}.$$

Suppose the contrary; then there exist some sequence $(\mu_n\in M)_{n\in\mathbb{N}}$ and an $\varepsilon>0$ s.t. for every $\delta>0$ there exist $A\in\mathscr{B}$ and $n\in\mathbb{N}$ with $\lambda(A)<\delta$ and $\mu_n(A)>\varepsilon$, whence, by the $\mathscr{F}$-regularity of μ_n one can find $F\subset A$ s.t. $\lambda(F)<\delta$ and $\mu_n(F)>\varepsilon$. Now, by 2.13 (c), λ is $(\mathscr{F},\mathscr{C})$-regular, and therefore (cf. Remark preceding 2.10) there exists C s.t. $\lambda(C)<\delta$ and $\mu_n(C)>\varepsilon$. In that way, for $\delta=2^{-k}$, one arrives at C_k and $n_k\in\mathbb{N}$ s.t. $\lambda(C_k)<2^{-k}$ and $\mu_{n_k}(C_k)>\varepsilon$. Now, put $C_n':=\bigcup_{k\geqq n}C_k$, $n\in\mathbb{N}$; then $(C_n')_{n\in\mathbb{N}}$ is a monotone decreasing sequence of sets in $\mathscr{C}$ with $\lambda\left(\bigcap_{n\in\mathbb{N}}C_n'\right)=0$, since $\lambda(C_n')\leqq\sum_{k\geqq n}2^{-k}\to 0$ as $n\to\infty$; furthermore, $\mu_{n_k}\left(C_k'\setminus\bigcap_{n\in\mathbb{N}}C_n'\right)=\mu_{n_k}(C_k')\geqq\mu_{n_k}(C_k)>\varepsilon$ for all $k\in\mathbb{N}$; but this contradicts (vi).

(vii)⇒(vi): Suppose (vi) to be wrong; then (cf. the proof of "(v)⇒(vi)") there exist $\varepsilon>0$, a monotone decreasing sequence $(D_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{C}$, and a sequence $(\mu_n\in M)_{n\in\mathbb{N}}$ s.t. $\mu_n(D_n\setminus D_{n+1})>\varepsilon$ for all $n\in\mathbb{N}$; furthermore, we claim that there exist sequences $(C_n)_{n\in\mathbb{N}}$ and $(C'_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{C}$ such that for all $n\in\mathbb{N}$:

(i) $C_{n+1}\subset C_n$ and $C'_n\subset C_n\setminus C_{n+1}$, (ii) $C_n\subset D_n$ and $\sup\limits_{m\in\mathbb{N}}\mu_m(D_n\setminus C_n)<\varepsilon/2$ and (iii) $\mu_n(C'_n)>\varepsilon/2$. Since the C'_n, $n\in\mathbb{N}$, are pairwise disjoint, this contradicts the uniform s-boundedness of M w.r.t. $\mathscr{C}$.

To prove our claim, put $C_1:=D_1$ and suppose that $C_1,\dots,C_k$ and, for $k>1$, $C'_1,\dots,C'_{k-1}$ have been already constructed so that (i)–(iii) are fulfilled; then, since $C_k\subset D_k$ and $D_{k+1}\subset D_k$, $\sup\limits_{m\in\mathbb{N}}\mu_m(D_{k+1}\setminus(C_k\cap D_{k+1}))<\varepsilon/2$, where $C_k\cap D_{k+1}\in\mathscr{C}$, and hence by (vii) (a) there exists $F_k\subset C_k\cap D_{k+1}$ s.t. $\sup\limits_{m\in\mathbb{N}}\mu_m((C_k\cap D_{k+1})\setminus F_k)<\varepsilon/2-\sup\limits_{m\in\mathbb{N}}\mu_m(D_{k+1}\setminus(C_k\cap D_{k+1}))$. Therefore we obtain $(*)$ $\sup\limits_{m\in\mathbb{N}}\mu_m(D_{k+1}\setminus F_k)<\varepsilon/2$, since

$$\sup_{m\in\mathbb{N}}\mu_m(D_{k+1}\setminus F_k)\leqq\sup_{m\in\mathbb{N}}\mu_m(D_{k+1}\setminus(D_{k+1}\cap C_k))+\sup_{m\in\mathbb{N}}\mu_m((D_{k+1}\cap C_k)\setminus F_k)<\varepsilon/2\,.$$

Furthermore, as $F_k\subset D_{k+1}$,

$$\mu_k(C_k\setminus F_k)\geqq\mu_k(D_k\setminus D_{k+1})-\sup_{m\in\mathbb{N}}\mu_m(D_k\setminus C_k)>\varepsilon-\varepsilon/2=\varepsilon/2\,.$$

Since μ_k is $(\mathscr{F},\mathscr{C})$-regular, there exists a pair C''_k, C^*_k of sets in $\mathscr{C}$ s.t. $C''_k\subset\overline{C^*_k}\subset\overline{F_k}$ and $\mu_k(\bar F_k\setminus C''_k)<\mu_k(C_k\setminus F_k)-\varepsilon/2$, and therefore $\mu_k(C_k\cap C''_k)\geqq\mu_k(C_k\cap\bar F_k)-\mu_k(\bar F_k\setminus C''_k)>\varepsilon/2$. Now, $C'_k:=C_k\cap C''_k$ and $C_{k+1}:=D_{k+1}\cap C_k\cap C^*_k$ are sets in $\mathscr{C}$, and (iii) holds true for $n=k$. On the other hand, we obtain (i) from the fact that $C'_k\subset C_k$, $C'_k\subset C''_k\subset\overline{C^*_k}$ and $C_{k+1}\subset C_k\cap C^*_k$; finally, (ii) (with $n=k+1$) follows from $(*)$, since

$$F_k\subset C_{k+1}\subset D_{k+1}\,.$$

This completes the proof of 3.1.

3.2. Theorem. (cf. [1], 4.1.5). *Let the assumptions of* 3.1 *be fulfilled and suppose, in addition, that* $\mathscr{F}$ *is a* $(\cup f,\cap f)$*-paving. Then each of the assertions*

(i)–(vii) *in* 3.1 *is equivalent to the following assertion*

(viii) (a) $\mathscr{F}$ *approximates* $\mathscr{C}$ *from below uniformly w.r.t.* $|M|$;

(b) *For every monotone decreasing sequence* $(F_n)_{n\in\mathbb{N}}$ *of sets in* $\mathscr{F}$ *we have* $\inf\limits_{n\in\mathbb{N}}\sup\limits_{\mu\in M}|\mu|\left(F_n\setminus\bigcap\limits_{k\in\mathbb{N}}F_k\right)=0.$

Proof. (viii)⇒(vi): Suppose (vi) to be wrong; then there exist $\varepsilon>0$, a monotone decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{C}$, and a sequence

$$(\mu_n\in M)_{n\in\mathbb{N}}\text{ s.t. }(*)\inf_{n\in\mathbb{N}}|\mu_n|\left(C_n\setminus\bigcap_{k\in\mathbb{N}}C_k\right)>\varepsilon.$$

By (viii) (a) there exist F_n, $n\in\mathbb{N}$, s.t. $F_n\subset C_n$ and $\sup\limits_{\mu\in M}|\mu|(C_n\setminus F_n)<\varepsilon\cdot 2^{-n-1}$. Let $F'_n:=\bigcap\limits_{k=1}^{n}F_k$, $n\in\mathbb{N}$; then $(F'_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence of sets in $\mathscr{F}$, hence, by (viii) (b), $\sup_{\mu\in M}|\mu|\left(F'_n\setminus\bigcap_{k\in\mathbb{N}}F'_k\right)\to 0$ as $n\to\infty$. Furthermore,

$$\sup_{\mu\in M}|\mu|(C_n\setminus F'_n)\leqq\sup_{\mu\in M}|\mu|\left(\bigcup_{l=1}^{n}(C_l\setminus F_l)\right)\leqq\sum_{l=1}^{n}\sup_{\mu\in M}|\mu|(C_l\setminus F_l)<\varepsilon/2$$

for all $n\in\mathbb{N}$, and therefore

$$\sup_{\mu\in M}|\mu|\left(C_n\setminus\bigcap_{k\in\mathbb{N}}C_k\right)\leqq\sup_{\mu\in M}|\mu|\left(C_n\setminus\bigcap_{k\in\mathbb{N}}F'_k\right)\leqq\sup_{\mu\in M}|\mu|(C_n\setminus F'_n)+\sup_{\mu\in M}|\mu|\left(F'_n\setminus\bigcap_{k\in\mathbb{N}}F'_k\right)<\varepsilon$$

for n sufficiently large, which contradicts (∗).

3.3. Theorem. *Let the assumptions of* 3.1 *be fulfilled and suppose, in addition, that* μ *is* $(\bar{\mathscr{C}},\mathscr{C})$*-regular for all* $\mu\in M$. *Then each of the assertions* (i)–(vii) *in* 3.1 *is equivalent to the following assertion* (ix) *M is uniformly s-bounded w.r.t. $\mathscr{C}$.*

Proof. (ix)=(vii) (b), and (vii) (a) follows from 2.18.

3.4. Theorem. (cf. [1], 4.1.5). *Let the assumptions of* 3.1 *be fulfilled and suppose, in addition, that $\mathscr{F}$ corresponds with $\mathscr{C}$. Then each of the assertions* (i)–(vii) *in* 3.1 *is equivalent to the following assertion*

(x) (a) *$\mathscr{F}$ approximates $\{X\}$ from below uniformly w.r.t. $|M|$;*

(b) *M is uniformly s-bounded w.r.t. $\mathscr{C}$.*

Proof. (x)⇒(vii): Follows from 2.17 and 2.19. Since (vii) is equivalent to (iii), there exists $\lambda\in M_+(X,\mathscr{B},\mathscr{F})$ s.t. $|M|\lll\lambda$, which implies (x).

3.5. Theorem. (cf. [1], 4.1.5). *Let the assumptions of* 3.1 *be fulfilled and suppose, in addition, that $\mathscr{F}$ is a $(\cup f,\cap f)$-paving and that $\mathscr{F}$ corresponds with $\mathscr{C}$. Then each of the assertions* (i)–(vii) *in* 3.1 *is equivalent to the following assertion*

(xi) (a) *$\mathscr{C}$ approximates $\mathscr{F}$ from above uniformly w.r.t. $|M|$;*

(b) *$\mathscr{F}$ approximates $\{X\}$ from below uniformly w.r.t. $|M|$;*

(c) *For every monotone decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{F}$ we have*

$$\inf_{n\in\mathbb{N}}\sup_{\mu\in M}|\mu|\left(F_n\setminus\bigcap_{k\in\mathbb{N}}F_k\right)=0\,.$$

Proof. Note, that under the present assumptions, (i)–(vii) in 3.1 is equivalent to (viii) in 3.2. Now, the implication (xi)⇒(viii) follows from 2.19. On the other hand, since (viii) is equivalent to (iii), there exists $\lambda\in M_+(X,\mathscr{B},\mathscr{F})$ s.t. $|M|\lll\lambda$, where w.l.o.g. λ may be supposed to be $(\mathscr{F},\mathscr{C})$-regular by 2.14 (b), hence $\mathscr{C}$ approximates $\mathscr{F}$ from above w.r.t. λ(cf. Remark preceding 2.10). From this it follows that (xi) is implied by (viii).

3.6. Remark. The assumption in 3.1 of M being a bounded subset of $M(X,\mathscr{B},\mathscr{F})$ was only needed to prove the equivalence of (i)–(iv). The equivalence of (iii) with each of the other assertions proved in connection with 3.1–3.5 holds true without the boundedness assumption on M. We remark also, that, for bounded M, the compactness criteria 3.1–3.5 characterize as well the *so*-called (cf. [3], IV. 9) weakly conditionally (sequentially) compact subsets of $M(X,\mathscr{B},\mathscr{F})$; this follows from [4], 2.14 and 2.16.

Let us conclude this section by applying the results obtained so far to situations when X is supposed to be a topological space. In that case, using also 2.7 and 2.8, we obtain immediately the following corollaries, where we remark that in any case considered below $\mathscr{F}$ is a $(\cup f)$-paving and $\mathscr{C}$ is a $(\cup c, \cap f)$-paving (cf. [6], 1.14 for the $(\cup c)$-closedness of $\mathscr{G}_0(X)$).

3.7. Corollary (cf. [1], 4.1.7). *Let $(X, \mathscr{G}(X))$ be a topological space and let M be a bounded subset of $M(X, \mathscr{B}_0(X))$. Then, taking $\mathscr{B} = \mathscr{B}_0(X)$, $\mathscr{F} = \mathscr{F}_0(X)$ and $\mathscr{C} = \mathscr{G}_0(X)$, the assertions* (i)–(xi) *in* 3.1–3.5 *are all equivalent.*

3.8. Corollary (cf. [4], 3.7 and [1], 4.1.10). *Let $(X, \mathscr{G}(X))$ be a Hausdorff space and let M be a bounded subset of $M(X, t)$. Then, taking $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{F} = \mathscr{K}(X)$ and $\mathscr{C} = \mathscr{G}(X)$, the assertions* (i)–(viii), (x) *and* (xi) *in* 3.1–3.5 *are all equivalent and also equivalent to any of the following assertions*

(xii) *$\mathscr{K}(X)$ approximates $\mathscr{B}(X)$ from below uniformly w.r.t. $|M|$;*

(xiii) *$\mathscr{K}(X)$ approximates $\mathscr{G}(X)$ from below uniformly w.r.t. $|M|$;*

(xiv) (a) *$\mathscr{K}(X)$ approximates $\{X\}$ from below uniformly w.r.t. $|M|$;*
(b) *$\mathscr{G}(X)$ approximates $\mathscr{K}(X)$ from above uniformly w.r.t. $|M|$.*

Proof. (iii)⇒(xii), since $M \lll \lambda \in M_+(X, t)$ [cf. 2.14 (a)]. (xiii)⇒(vii): It suffices to show that M is uniformly s-bounded w.r.t. $\mathscr{G}(X)$. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint open sets; then, for $G := \bigcup_{n \in \mathbb{N}} G_n \in \mathscr{G}(X)$ and any $\varepsilon > 0$, there exists $K \in \mathscr{K}(X)$ s.t. $K \subset G$ and $\sup_{\mu \in M} |\mu|(G \setminus K) \leqq \varepsilon$. Since K is compact, there exists $m \in \mathbb{N}$ s.t. $\bigcup_{n \geqq m} G_n \subset G \setminus K$, and therefore $\sup_{\mu \in M} |\mu|(G_n) \leqq \sup_{\mu \in M} |\mu|(G \setminus K) \leqq \varepsilon$ for all $n \geqq m$.
From this and 2.17 it follows that (xiii) implies (xiv); on the other hand, (xiii) follows from (xiv) by 2.19.

3.9. Corollary (cf. [1], 4.1.12). *Let $(X, \mathscr{G}(X))$ be a regular space and let M be a bounded subset of $M(X, \tau)$. Then, taking $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{F} = \mathscr{F}(X)$ and $\mathscr{C} = \mathscr{G}(X)$, the assertions* (i)–(xi) *in* 3.1–3.5 *are all equivalent.*

3.10. Remark. 3.9 yields a remarkable generalization of 3.11 in [4] as well as the following corollary will generalize 3.12 in [4].

3.11. Corollary (cf. [1], 4.1.13 and [4], 3.12). *Let $(X, \mathscr{G}(X))$ be a completely regular space and let M be a bounded subset of $M(X, \tau)$. Then, taking $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{F} = \mathscr{F}(X)$ and $\mathscr{C} = \mathscr{G}_0(X)$ or $= \mathscr{G}(X)$, the assertions* (i)–(xi) *in* 3.1–3.5 *are all equivalent and also equivalent to the following assertion.*

(xv) *For every uniformly bounded sequence of continuous functions f_n, $n \in \mathbb{N}$, converging to zero at every point $x \in X$, we have $\lim_{n \to \infty} \int_X f_n d\mu = 0$ uniformly w.r.t. $\mu \in M$.*

Proof. (iv)⇒(xv): Obvious. (xv)⇒(ix): Follows from 2.27.

3.12. Corollary (cf. [1], 4.1.8). *Let $(X, \mathscr{G}(X))$ be a normal space and let M be a bounded subset of $M(X, r)$. Then, taking $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{F} = \mathscr{F}(X)$ and $\mathscr{C} = \mathscr{G}_0(X)$ or $= \mathscr{G}(X)$, the assertions* (i)–(xi) *in* 3.1–3.5 *are all equivalent.*

3.13. Corollary (cf. [1], 4.1.9). *Let $(X, \mathscr{G}(X))$ be a perfectly normal space and let M be a bounded subset of $M(X, \mathscr{B}(X))$. Then, taking $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{F} = \mathscr{F}(X)$ and $\mathscr{C} = \mathscr{G}(X)$, the assertions* (i)–(xi) *in* 3.1–3.5 *and* (xv) *in* 3.11 *are all equivalent.*

Proof. $M(X, \mathscr{B}(X)) = M(X, r)$, since $(X, \mathscr{G}(X))$ is a perfectly normal space, hence the assertion follows from 3.12 and the same argument as in the proof of 3.11.

4. Convergence-Theorems

Besides the compactness results of the preceding section our general tools presented in Section 2 enable us also to derive in a rather straightforward way the results (a)–(f) mentioned in the introduction. To this extent we will consider first the question whether the $(\cup c)$-paving $\mathscr{C}$ can serve as a convergence class for sequences $\mu_n \in M(X, \mathscr{B}, \mathscr{F})$, $n \in \mathbb{N}$, and secondly we will study the same question w.r.t. $\mathscr{G}_r(X)$ [being aware that $\mathscr{G}_r(X)$ is not even $(\cup f)$-closed].

A. The paving $\mathscr{C}$ as Convergence Class

Suppose that there is given an arbitrary non-empty set X, a σ-field $\mathscr{B}$ in X, an arbitrary subpaving $\mathscr{F}$ of $\mathscr{B}$ and a $\mathscr{F}$-filtering $(\cup c)$-paving $\mathscr{C} \subset \mathscr{B}$. Then the following theorem which is, for the present case of realvalued measures, a generalization of Theorem 1 in [11] [cf. 2.5 (a)] holds true.

4.1. Theorem. *Let $\mu_n \in M(X, \mathscr{B}, \mathscr{F})$, $n \in \mathbb{N}$ and suppose that all μ_n are $(\mathscr{F}, \mathscr{C})$-regular. Then, whenever $(\mu_n)_{n \in \mathbb{N}}$ is convergent on $\mathscr{C}$, there exists $\mu \in M(X, \mathscr{B})$ being $(\mathscr{F}, \mathscr{C})$-regular, too, such that $\mu_n \underset{s}{\rightarrow} \mu$ $\left[\text{i.e. } \lim_{n \to \infty} \mu_n(B) = \mu(B) \text{ for all } B \in \mathscr{B}\right]$. If $\mathscr{F}$ is a $(\cup f)$-paving, then μ is also $\mathscr{F}$-regular.*

Proof. According to 2.21 it suffices to show that $(\mu_n)_{n \in \mathbb{N}}$ is convergent on $\mathscr{B}$. Since $(\mu_n)_{n \in \mathbb{N}}$ is convergent on $\mathscr{C}$ 2.22 implies that $\{\mu_n : n \in \mathbb{N}\}$ is uniformly s-bounded w.r.t. $\mathscr{C}$. Now the assertion follows from 2.24 according to 2.17.

Now, using 4.1 and 2.7, we are in the position of obtaining immediately the results (a)–(d) mentioned in the introduction.

4.2. Corollary (cf. [11], Corollary 4). *Let $(X, \mathscr{G}(X))$ be a topological space; then $\mathscr{G}_0(X)$ is a convergence class for Baire measures.*

4.3. Corollary (cf. [12] and [4], Theorem 5.2). *Let $(X, \mathscr{G}(X))$ be a Hausdorff space; then $\mathscr{G}(X)$ is a convergence class for tight Borel measures.*

4.4. Corollary (cf. [1], 4.2.13). *Let $(X, \mathscr{G}(X))$ be a regular space; then $\mathscr{G}(X)$ is a convergence class for τ-smooth Borel measures.*

4.5. Corollary (cf. [1], 4.2.14). *Let $(X, \mathscr{G}(X))$ be a completely regular space; then $\mathscr{G}_0(X)$ is a convergence class for τ-smooth Borel measures.*

4.6. Corollary (cf. [11], Corollary 7). *Let $(X, \mathscr{G}(X))$ be a normal space; then $\mathscr{G}_0(X)$ is a convergence class for regular Borel measures.*

4.7. Remark. As it was shown in [11], Example 8, 4.5 does not hold for regular instead of τ-smooth Borel measures.

B. The Paving $\mathscr{G}_r(X)$ as Convergence Class

Suppose that $(X, \mathscr{G}(X))$ is a topological space, take $\mathscr{B} = \mathscr{B}(X)$, $\mathscr{C} = \mathscr{G}(X)$ and let $\mathscr{F}$ be a $(\cup f)$-paving contained in $\mathscr{B}(X)$. Then the following theorem holds true.

4.8. Theorem. *Let $\mu_n \in M(X, \mathscr{B}, \mathscr{F})$, $n \in \mathbb{N}$, and suppose that μ_n is $(\mathscr{H}, \mathscr{C})$-regular w.r.t. $\mathscr{H} = \mathscr{F}$ and $= \mathscr{C}$. Then, whenever $(\mu_n)_{n \in \mathbb{N}}$ converges on $\mathscr{G}_r(X)$, there exists $\mu \in M(X, \mathscr{B}, \mathscr{F})$, being also $(\mathscr{H}, \mathscr{C})$-regular, such that $\mu_n \underset{s}{\rightarrow} \mu$.*

Proof. According to 2.21 it suffices to prove that $\mathscr{G}_r(X)$ is a convergence class for $M := \{\mu \in M(X, \mathscr{B}, \mathscr{F}) : \mu(\mathscr{H}, \mathscr{C})\text{-regular w.r.t. } \mathscr{H} = \mathscr{F} \text{ and } = \bar{\mathscr{C}}\}$. To this extent [observing that by 2.5 (c) and (d) $\mu_1 - \mu_2 \in M$ whenever $\mu_1, \mu_2 \in M$] it suffices by 2.25 to show that the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges on $\mathscr{B}$ to zero whenever it converges on $\mathscr{G}_r(X)$ to zero. But if $(\mu_n)_{n \in \mathbb{N}}$ is convergent on $\mathscr{G}_r(X)$ to zero, then, by 2.26, $\{\mu_n : n \in \mathbb{N}\}$ is uniformly s-bounded w.r.t. $\mathscr{G}(X)$, and therefore 3.3 (cf. 3.6) yields that $\{\mu_n : n \in \mathbb{N}\} \lll \lambda \in M_+(X, \mathscr{B}, \mathscr{F})$. Since $|\mu_n|$, $n \in \mathbb{N}$, are $(\bar{\mathscr{C}}, \mathscr{C})$-regular (cf. 2.5.3), for every $G \in \mathscr{G}(X)$ there exist sequences $(C_m)_{m \in \mathbb{N}}$ and $(C'_m)_{m \in \mathbb{N}}$ of sets in $\mathscr{C} = \mathscr{G}(X)$ s.t. $C_m \subset \overline{C'_m} \subset G$ and $\sup_{n \in \mathbb{N}} |\mu_n| \left(G \setminus \bigcup_{m \in \mathbb{N}} C_m \right) = 0$. Now, consider $R_m := (\overline{C'_m})^0$; then $R_m \in \mathscr{G}_r(X)$, $C_m \subset R_m \subset G$ and therefore $\sup_{n \in \mathbb{N}} |\mu_n| \left(G \setminus \bigcup_{m \in \mathbb{N}} R_m \right) = 0$. Since $\mathscr{G}_r(X)$ is $(\cap f)$-closed, it follows that $\lim_{n \to \infty} \mu_n(A) = 0$ whenever A is a finite union of sets in $\mathscr{G}_r(X)$; hence by 2.23 it follows that

$$\lim_{n \to \infty} \mu_n \left(\bigcup_{m \in \mathbb{N}} R_m \right) = 0$$

and therefore $\lim_{n \to \infty} \mu_n(G) = 0$. Now the assertion follows from 2.24 according to 2.17.

Next, using 4.8 and 2.7, we are again in the position of obtaining also the remaining results (e) and (f) mentioned in the introduction.

4.9. Corollary (cf. [1], 4.2.18). *Let $(X, \mathscr{G}(X))$ be a regular space; then $\mathscr{G}_r(X)$ is a convergence class for τ-smooth Borel measures.*

4.10. Corollary(cf. [14], Theorem 1). *Let $(X, \mathscr{G}(X))$ be a normal space; then $\mathscr{G}_r(X)$ is a convergence class for regular Borel measures.*

4.11. Remark. 4.9 generalizes Theorem 3.1 of [5].

Appendix

A1 (cf. [4], Remark 1.8 and Corollary 2.7). *$M \subset M(X, \mathscr{B})$ is uniformly dominated by some $\lambda \in M_+(X, \mathscr{B})$ iff M is equicontinuous; in that case any $\nu \in M_+(X, \mathscr{B})$ dominating M dominates M uniformly.*

A2 (cf. [4], Theorem 2.6). *Let M be a bounded subset of $M(X, \mathscr{B})$. Then the following assertions are equivalent:*

(i) *M is conditionally compact in $(M(X, \mathscr{B}), \mathscr{T}_s)$;*

(ii) *M is conditionally sequentially compact in $(M(X, \mathscr{B}), \mathscr{T}_s)$;*

(iii) *M is uniformly dominated by some $\lambda \in M^\Sigma$ and every $\nu \in M_+(X, \mathscr{B})$ dominating M dominates M uniformly;*

(iv) *M is equicontinuous;*

(v) *Every countable subset of M is equicontinuous.*

A3 (Vitali-Hahn-Saks). *Let $\mu_n \in M(X, \mathscr{B})$, $n \in \mathbb{N}$, be convergent on $\mathscr{B}$; then $\{\mu_n : n \in \mathbb{N}\}$ is uniformly dominated by some $\lambda \in M_+(X, \mathscr{B})$.*

A4 (Nikodym). *Let* $\mu_n \in M(X, \mathscr{B})$, $n \in \mathbb{N}$, *be convergent on* $\mathscr{B}$; *then*

$$\mu(A) := \lim_{n \to \infty} \mu_n(A),$$

$A \in \mathscr{B}$, *defines a measure.*

A5 (cf. [10], Lemma 10). *Let* b_{ik} *be real numbers for* $i, k \in \mathbb{N}$ *such that*

$$\left(\sum_{k \in M} b_{ik}\right)_{i \in \mathbb{N}}$$

is convergent in $\mathbb{R}$ *for all subsets* M *of* $\mathbb{N}$. *Then* $\lim_{i \to \infty} b_{ii} = 0$.

A6 (cf. [5], Lemma 3.4). *Let* $\mu_n \in M(X, \mathscr{B}(X))$, $n \in \mathbb{N}$, *be convergent on* $\mathscr{G}_r(X)$ *to zero, and let* $(R_n)_{n \in \mathbb{N}}$ *be a sequence of sets in* $\mathscr{G}_r(X)$ *with* $\left(\bigcup_{m \neq n} R_m\right)^a \cap R_n^a = \emptyset$ *for all* $n \in \mathbb{N}$; *then* $\lim_{n \to \infty} \mu_n(R_n) = 0$.

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