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Unstable Bundles and Branched Structures on Riemann Surfaces

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1. Suppose M is a compact Riemann Surface of genus $g>1$. The Picard-Jacobi variety $J(M)$ of M can then be described as the group of analytically equivalent flat complex analytic line bundles over M. More explicitly we can identify the set of flat line bundles over M with the group $H^1(M,\mathbb{C}^*)$ $\simeq \mathrm{Hom}(\pi_1(M),\mathbb{C}^*)\simeq(\mathbb{C}^*)^{2g}$. We then consider two flat line bundles analytically equivalent if they have the same image under the homomorphism i in the exact sequence of groups

$$0\to\Gamma(M,\mathcal{O}^{1,0})\xrightarrow{\delta}H^1(M,\mathbb{C}^*)\xrightarrow{i}H^1(M,\mathcal{O}^*)$$

arising out of the sheaf exact sequence

$$0\to C^*\xrightarrow{i}\mathcal{O}^*\xrightarrow{d}\mathcal{O}^{1,0}\to 0$$

(where $\mathcal{O}^*$ is the multiplicative sheaf of germs of holomorphic nowhere vanishing functions; $\mathcal{O}^{1,0}$ is the sheaf of germs of holomorphic differentials of type $(1,0)$ and d is the mapping $f\to\frac{1}{2\pi i}\,d\log f$ of $\mathcal{O}^*$ onto $\mathcal{O}^{1,0}$). Geometrically, of course, analytic equivalence simply means that the flat line bundles determine the same analytic line bundle.

In terms of representations of the fundamental group if we represent M as the quotient space U/G of the unit disc modulo a group $G\approx\pi_1(M)$ of linear fractional transformations, then two representations $\alpha,\beta\in\mathrm{Hom}(G,\mathbb{C}^*)$ are analytically equivalent if and only if there exists a complex analytic mapping $P:U\to\mathbb{C}^*$ such that $P(gz)\,\alpha(g)=\beta(g)\,P(z)$ for all $g\in G$ and $z\in U$. In any case the equivalence classes of flat complex line bundles are simply cosets of the Lie subgroup $\delta\Gamma(M,\mathcal{O}^{1,0})\subset(\mathbb{C}^*)^{2g}$ and are all complex analytic submanifolds of $(\mathbb{C}^*)^{2g}$ analytically equivalent to $\mathbb{C}^g$. Thus the group of flat line bundles on M turns out to be a $\mathbb{C}^g$ – bundle over $J(M)$.

In Gunning [3, 4, 6] this construction is generalized to the set of flat complex vector bundles on M. In particular the cohomology set $H^1(M,\mathrm{SL}(2,\mathbb{C}))$, represented as the quotient space $\mathrm{Hom}(\pi_1(M),\mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C}))$ is studied. This set, though not having a natural group structure does have a natural complex structure. Again as in the case of the line bundles, two flat vector bundles are analytically equivalent when they determine the same complex analytic vector bundle and, in terms of representations of the fundamental group, we again call $\alpha,\beta\in\mathrm{Hom}(G,\mathrm{SL}(2,\mathbb{C}))$ representing classes $[\alpha]$ and $[\beta]$ in $H^1(M,\mathrm{SL}(2,\mathbb{C}))$ analytically equivalent if there exists a complex analytic mapping $P:U\to\mathrm{SL}(2,\mathbb{C})$ such that $P(gz)\,\alpha(g)=\beta(g)\,P(z)$ for all $g\in G$ and $z\in U$. If we restrict our attention to those representations not having scalar commutants we obtain a $6g-6$

dimensional complex manifold $S \subset H^1(M, \mathrm{SL}(2,\mathbb{C}))$ which analytic equivalence foliates into complex analytic submanifolds of dimension $3g-3$. However as shown explicitly in Gunning [3, 4] these submanifolds are not necessarily even topologically equivalent. A detailed description of the individual leaves in the above foliation is then a matter of some interest. In particular the leaf consisting of those flat bundles representing a maximally unstable analytic vector bundle corresponds to all possible projective structures subordinate to the complex structure of M. In the present paper we generalize this description and establish a correspondence between all leaves corresponding to irreducible bundles of any divisor order (not necessarily maximal) and the branched projective structure studied in [7, 8]. Our notation will be that of [1–3] and [7, 8] throughout.

2. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 \to \mathbf{Z}_2 \xrightarrow{e} & \mathrm{SL}(2,\mathbb{C}) & \xrightarrow{\mu_2} & \mathrm{PL}(1,\mathbb{C}) \to 0 \\ & \downarrow i & & \downarrow i \\ & \mathscr{S}\mathscr{L}(2,\mathbb{C}) & \xrightarrow{\hat{\mu}} & \mathscr{P}\mathscr{L}(1,\mathbb{C}) \end{array}$$

where we identify $\mathrm{SL}(2,\mathbb{C})$ [resp. $\mathrm{PL}(1,\mathbb{C})$] with the subsheaf of constant maps in $\mathscr{S}\mathscr{L}(2,\mathbb{C})$ [resp. $\mathscr{P}\mathscr{L}(1,\mathbb{C})$]. On the level of cohomology we then get

$$\begin{array}{ccccc} H^1(M,\mathbf{Z}_2) \xrightarrow{e} & H^1(M,\mathrm{SL}(2,\mathbb{C})) & \xrightarrow{\mu_2^*} & H^1(M,\mathrm{PL}(1,\mathbb{C})) \\ & \downarrow i_* & & \downarrow i_* \\ & H^1(M,\mathscr{S}\mathscr{L}(2,\mathbb{C})) & \xrightarrow{\hat{\mu}^*} & H^1(M,\mathscr{P}\mathscr{L}(1,\mathbb{C})). \end{array}$$

Now it is clear that $\mu_2^{*-1}(\mu_2^*(T)) = \{v \otimes T \mid v \in H^1(M,\mathbf{Z}_2) \simeq (\mathbf{Z}_2)^{2g}\}$. Since tensoring by $H^1(M,\mathbf{Z}_2)$ preserves such things as irreducibility, indecomposability, divisor order etc. it is clear that we can extend such notions to $\mu_2^*(H^1(M,\mathrm{SL}(2,\mathbb{C}))$ in the obvious fashion. Furthermore passing from $H^1(M,\mathrm{SL}(2,\mathbb{C}))$ to $H^1(M,\mathscr{S}\mathscr{L}(2,\mathbb{C}))$ also preserves divisor order and therefore the notion of $\operatorname{div}\Phi$ can be extended to $i_*\mu_2^*(H^1(M,\mathrm{SL}(2,\mathbb{C}))) \subset H^1(M,\mathscr{P}\mathscr{L}(1,\mathbb{C}))$. Thus we can speak of stable and unstable projective bundles in $\hat{\mu}^*H^1(M,\mathscr{S}\mathscr{L}(2,\mathbb{C}))$ in a fashion analogous to the same notion in $H^1(M,\mathscr{S}\mathscr{L}(2,\mathbb{C}))$. In the sequel we shall be primarily interested in the map i_* above. Now $H^1(M,\mathbf{Z}_2)$ acts in a discrete fashion on $H^1(M,\mathrm{SL}(2,\mathbb{C}))$ and to understand what analytic equivalence does in $H^1(M,\mathrm{SL}(2,\mathbb{C}))$ it will suffice to analyze what happens in $\mu_2^*(H^1(M,\mathrm{SL}(2,\mathbb{C})))$. We call $\mu_2^*(H^1(M,\mathrm{SL}(2,\mathbb{C})))$ the set of associated flat projective bundles and $i_*\mu_2^*(H^1(M,\mathrm{SL}(2,\mathbb{C})))$ the set of assocociated analytic projective bundles. We recall that for any $\Phi \in i_*H^1(M,\mathrm{SL}(2,\mathbb{C}))$ we have that (1) $-g \leqq \operatorname{div}\Phi \leqq g-1$ and (2) $\operatorname{div}\Phi \neq 0 \Rightarrow \Phi$ is analytically indecomposable and if $\Phi = i_*(T)$ then T is irreducible. A corresponding result also holds in the projective case.

3. Before proceeding any further we review some of the facts about branched structures proved in [7, 8].

Let $\{U_\alpha, z_\alpha\}$ be a coordinate cover for a Riemann surface M. Suppose for each α, $W_\alpha : U_\alpha \to Y_\alpha$ is a meromorphic function on U_α to the open subset Y_α of the complex projective line $\boldsymbol{P}$. We note that this implies $W_\alpha : U_\alpha \to Y_\alpha$ is a locally branched covering map, and we let $O_w(p)$ denote the branching order of w at p. Suppose further that for each nonempty intersection $U_\alpha \cap U_\alpha$ there exists a meromorphic homeomorphism $\phi_{\alpha\beta}: W_\beta(U_\alpha \cap U_\beta) \to W_\alpha(U_\alpha \cap U_\beta)$ such that $W_\alpha = \phi_{\alpha\beta} \circ W_\beta$ on $U_\alpha \cap U_\beta$. Then we shall say $\{U_\alpha, W_\alpha, \phi_{\alpha\beta}\}$ is an analytic branched cover on M. An analytic branched structure on M is then simply an equivalence class of covers (where two branched covers are equivalent if their union is a branched cover). It is easily verifiable that branching orders are preserved by the above equivalence relation and we can speak of the branch points of a structure.

By adding the restriction that all the homeomorphisms $\phi_{\alpha\beta}$ belong to some pseudo-group G of meromorphic transformations of $\boldsymbol{P}$ we can also speak of analytic branched G-structures on M. The only such structures arise when $G = \mathrm{PL}(1, \mathbb{C})$ or when $G = GA(1, \mathbb{C})$ and we call such structures projective or affine branched structures on the Riemann surface M. We note that if M has such a branched G-structure $[\mathscr{U}]$, then we can always choose a representative branched G-cover $\{U_\alpha, W_\alpha\}$ of M such that each U_α has at most one branch point and no branch point p is in two or more distinct U_α's. Furthermore if U_α has no branch points then W_α is in fact a homeomorphism. We call such a cover a restricted cover on M and henceforth tacitly assume all our covers are restricted.

Now given a branched analytic cover $\{U_\alpha, W_\alpha\}$ on M we can canonically associate to it the positive divisor $\mathfrak{D}(\{U_\alpha, W_\alpha\}) = \sum_{p \in M} O_{W_\alpha}(p) \cdot p$ (for some α such that $p \in U$). If M is compact $\sum_{p \in M} O_{W_\alpha}(p) \cdot p$ is a finite sum and therefore recalling that branching orders are invariants of the structure represented by $\{U_\alpha, W_\alpha\}$ we have a map $\mathfrak{D}$: {branched structure on M} $\to$ {positive divisors on M}. Given a positive divisor $\mathfrak{D}$ on M we shall say a branched analytic structure is of type $\mathfrak{D}$ if and only if, for some representative $\{U_\alpha, W_\alpha\}$ of that structure, $\mathfrak{D}\{U_\alpha, W_\alpha\} = \mathfrak{D}$. We let $B(\{U_\alpha, W_\alpha\}) = \deg \mathfrak{D} = \sum_{p \in m} O_{W_\alpha}(p)$ and call this the branching order of the structure.

We let $\mathscr{P}(M) =$ {space of branched projective structures on M}, ${}_pV_{\mathfrak{D}} =$ {projective structures of type $\mathfrak{D}$ on M} for any positive divisor $\mathfrak{D}$, and for any positive integer B

$$ {}_pV(B) = \bigcup_{\{\mathfrak{D} \in M^{(B)}\}} {}_pV_{\mathfrak{D}} \subset \mathscr{P}(M) $$

(i.e. ${}_pV(B)$ is the space of all projective structures of total branching order B).

Now by Theorem 5 of [8] each ${}_pV(B)$ is a constructible set. In fact for each partition σ of the positive integer B there exists a complex analytic variety ${}_pV_B(\sigma)$ such that ${}_pV(B)$ is the disjoint union of the ${}_pV_B(\sigma)$ for all partitions σ of B.

Now suppose $\mathscr{U} = \{U_\alpha, W_\alpha, \phi_{\alpha\beta}\}$ represents a projective structure in $\mathscr{P}(M)$. We immediately note that the transition functions $\{\phi_{\alpha\beta}\}$ satisfy $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ and $\phi_{\alpha\beta}\phi_{\beta\alpha} = \phi_{\alpha\gamma}$ and therefore define a cocycle in $Z^1(\mathscr{U}, \mathrm{PL}(1, \mathbb{C}))$. Furthermore equivalent representative covers define equivalent cocycles and this equivalence is preserved under refinement. We therefore clearly have a well-defined map

$$ \mathscr{P}(M) \xrightarrow{j_2} H^1(M, \mathrm{PL}(1, \mathbb{C})) \xrightarrow{i_*} H^1(M, \mathscr{P}\mathscr{L}(1, \mathbb{C})). $$

By Theorem 6 of [8] j_2 is in fact holomorphic on each ${}_pV_B(\sigma) \subset \mathscr{P}(M)$.

For future use we introduce some notation.

Suppose $k \in \mathbf{Z}$ such that $-g \leqq k \leqq g-1$. Then

$$S = \{T \in H^1(M, \mathrm{SL}(2, \mathbb{C})) \mid T \text{ has only scalar commutants}\} \tag{1}$$

$$S_k = \{T \in S \mid \operatorname{div} T = k\}\,; \quad S_{k^+} = \bigcup_{j \geqq k} S_j\,; \quad S_u = S_{0^+}\,. \tag{2}$$

$$S^* = i_*(S)\,; \quad S_k^* = i_*(S_k)\,; \quad S_{k^+}^* = i_*(S_{k^+}) \text{ and } i_k = i \mid S_k\,. \tag{3}$$

$$P = \mu_2^*(S)\,; \quad P_k = \mu_2^*(S_k)\,; \quad P_{k^+} = \mu_2^*(S_{k^+})\,; \quad P_u = \mu_2^*(S_u)\,. \tag{4}$$

$$P^* = i_*(P)\,; \quad P_k^* = i_*(P_k)\,; \quad P_{k^+}^* = i_*(P_{k^+})\,; \quad P_u^* = i_*(P_u)\,. \tag{5}$$

We note that $S = S_{-g^+}$ and S_u is the space of unstable flat vector bundles in M. By [3, 4] we have that $S^* = i_*(H^1(M, \mathrm{SL}(2, \mathbb{C})))$, that is any flat vector bundle is analytically equivalent to one having only scalar commutants.

Furthermore if $\Phi \in H^1(M, \mathrm{SL}(2, \mathbb{C})) - S$ then $\operatorname{div}\Phi = 0$ and Φ is reducible. Now as we mentioned previously $i_* : S \to S^*$ foliates S into a disjoint union of $3g-3$ dimensional complex submanifolds of S. We will be perticularly concerned with those submanifolds lying above S_u^*.

We have

Proposition 1. *Let k be an integer with $-g \leqq k \leqq g-1$. Then*

(1) *$T \in S_k$ and T is irreducible implies*

$$u_2^*(T) \in j_2({}_pV(2g-2-2k))\,.$$

(2) *$\Phi \in j_2({}_pV(2g-2-2k))$ and Φ is irreducible implies $\Phi = \mu_2^*(T)$ for some $T \in S_{k^+}$. If $k \geqq 0$ then $T \in S_k$.*

In particular for $k > 0$ there is a $1-1$ correspondence between P_k and ${}_pV(2g-2-2k)$.

Proof. (1) Let $T \in S_k$ and represent T by $(T_{\alpha\beta}) = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix}$ for some coordinate cover $\{U_\alpha\}$ of M. Since $\operatorname{div} T = k$ there exists a line bundle ξ with $c(\xi) = k$ such that $\xi^{-1} \otimes T$ has a holomorphic cross-section h. Let $\{h_\alpha\} = \begin{pmatrix} h_{1\alpha} \\ h_{2\alpha} \end{pmatrix} \in \Gamma(M, \mathcal{O}(\xi^{-1} \otimes T))$ represent such a cross-section and let $H_\alpha = \begin{pmatrix} h_{1\alpha} & (d/dz_\alpha)(h_{1\alpha}) \\ h_{2\alpha} & (d/dz_\alpha)(h_{2\alpha}) \end{pmatrix}$. Then we find that $\det H_\alpha = \xi_{\alpha\beta}^{-2} \kappa_{\alpha\beta} \det H_\beta$ (where κ denotes the canonical line bundle on M) so if $\det H_\alpha$ isn't identically zero we obtain $(\det H_\alpha) \in \Gamma(M, \theta(\xi^{-2}\kappa))$ and therefore by [1, Theorem 11], $\sum_{p \in M} v_p(\det H_\alpha) = c(\xi^{-2}\kappa) = 2g-2-2k$. However letting $w_\alpha = h_{1\alpha}/h_{2\alpha}$ in U_α we see that $w_\alpha = \dfrac{a_{\alpha\beta} w_\beta + b_{\alpha\beta}}{c_{\alpha\beta} w_\beta + d_{\alpha\beta}}$ in $U_{\alpha\beta}$ so that $\{w_\alpha\}$ represents a cross-section w of $\mu_2^*(T)$. Furthermore since $\operatorname{div}(\xi^{-1} \otimes T) = 0$, $h_{1\alpha}$ and $h_{2\alpha}$ have no common zeros. Then observing that $(d/dz_\alpha)\, w = w' = -(h_{2\alpha})^{-2} \det H_\alpha$ and $(1/w_\alpha)' = (h_{1\alpha})^{-2} \det H_\alpha$ we see that

$$B(w_\alpha) = \sum_{p \in m} 0_{w_\alpha}(p) = \sum_{p \in M} v_p(\det H_\alpha) = 2g-2-2k$$

and so by [7], $\mu_2^*(T) \in {}_pV(2g-2-2k)$ provided $\det H_\alpha$ doesn't vanish identically.

Now pick $\eta \in H^1(M, \mathbb{C}^*)$ so that $\eta\xi$ will have a holomorphic cross-section g_α if $k \geqq 0$ (meromorphic if $k<0$) with $\{g_\alpha\}$ not vanishing identically. Let $f_\alpha = g_\alpha h_\alpha$ in U_α. Then $f_\alpha \in \Gamma(M, \mathscr{M}^*(\eta \otimes T))$ [$\in \Gamma(M, \mathscr{O}(\eta \otimes T))$ if $k \geqq 0$]. Let $F_\alpha = \begin{pmatrix} f_{1\alpha} & f'_{1\alpha} \\ f_{2\alpha} & f'_{2\alpha} \end{pmatrix}$. Then an easy calculation shows that $\det F_\alpha = g_\alpha^2 \det H_\alpha$ and since g_α is not identically zero we must show $\det F_\alpha$ is not identically zero to conclude the same fact about $\det H_\alpha$. We argue by contradiction.

So suppose $\det F_\alpha$ is identically zero. Then $f_{1\alpha}$ and $f_{2\alpha}$ are linearly dependent and there exists constants $a_\alpha, c_\alpha \in \mathbb{C}^*$, such that $f_\alpha = \phi_\alpha \begin{pmatrix} a_\alpha \\ c_\beta \end{pmatrix}$ for some $\phi_\alpha \in \mathscr{M}^*_{U_\alpha}$. Furthermore since $f_\alpha = g_\alpha h_\alpha$ and g_α is not identically zero we get $h_\alpha = \theta_\alpha \begin{pmatrix} a_\alpha \\ c_\alpha \end{pmatrix}$ where $\theta_\alpha = \phi_\alpha / g_\alpha \in \mathscr{O}_{U_\alpha}$. In fact since $\operatorname{div}(\xi^{-1} \otimes T) = 0$ and $h_\alpha \in \Gamma(M, \mathscr{O}(\xi^{-1} \otimes T))$ we have [3, p. 78] that $\theta_\alpha \in \mathscr{O}^*_{U_\alpha}$. Thus by [1, Theorem 11]

$$\sum_{p\in M} v_p(\phi_\alpha) = \sum_{p\in M} (v_p(\theta_\alpha g_\alpha)) = \sum_{p\in M} v_p(g_\alpha) = c(\eta\xi) = k.$$

However since $f_\alpha \in \Gamma(M, \mathscr{M}^*(\eta \otimes T))$ we find that

$$\phi_\alpha \begin{pmatrix} a_\alpha \\ c_\alpha \end{pmatrix} = \eta_{\alpha\beta} T_{\alpha\beta} \phi_\beta \begin{pmatrix} a_\beta \\ c_\beta \end{pmatrix}$$

or

$$a_\alpha \phi_\alpha = \eta_{\alpha\beta} \phi_\beta (a_{\alpha\beta} a_\beta + b_{\alpha\beta} c_\beta)$$
$$c_\alpha \phi_\alpha = \eta_{\alpha\beta} \phi_\beta (c_{\alpha\beta} a_\beta + d_{\alpha\beta} c_\beta).$$

But if we now let $r_{\alpha\beta} = a_\alpha^{-1}(a_{\alpha\beta} a_\beta + b_{\alpha\beta} c_\beta) \eta_{\alpha\beta}$ we can easily verify that $r = (r_{\alpha\beta}) \in H^1(M, \mathbb{C}^*)$ and $(\phi_\alpha) \in \Gamma(M, \mathscr{M}^*(r))$ [$\in \Gamma(M, \mathscr{O}(r))$ if $k \geqq 0$]. But then since ϕ_α is not identically zero

$$\sum_{p\in M} v_p(\phi_\alpha) = c(r) = 0.$$

Thus if $k \neq 0$ assuming $\det F_\alpha \equiv 0$ leads to a contradiction.

So suppose $k = 0$.

But then $\phi_\alpha \in \Gamma(M, \mathscr{O}(r))$ implies $r \sim 1$, and $(\phi_\alpha) \in \mathbb{C}^*$. But then f_α is locally constant so $(f_\alpha) \in \Gamma(M, \mathscr{F}(\eta \otimes T))$. Thus by [3, Theorem 18], $\eta \otimes T$ is reducible. But η is a flat line bundle so T must be reducible again giving a contradiction.

(2) Now suppose $\Phi \in j_2({}_pV(2g-2-2k))$ and irreducible. Then by [8] we can find a holomorphic cross-section $w = \{w_\alpha\}$ of Φ such that $B(w) = \sum_{p\in M} 0_{w_\alpha}(p) = 2g-2-2k$. Furthermore there exists a flat vector bundle $T \in H^1(M, \mathrm{SL}(2, \mathbb{C}))$ such that $\mu_2^*(T) = \Phi$ and a flat line bundle $\eta \in H^1(M, \mathbb{C}^*)$ such that $\eta \otimes T$ has a holomorphic cross-section $h = (h_\alpha) = \begin{pmatrix} h_{1\alpha} \\ h_{2\alpha} \end{pmatrix}$ with $h_{1\alpha}/h_{2\alpha} = w_\alpha$. By the construction of h in [8] we see that $\sum_{p\in M} v_p(h_\alpha) = k$. Thus $\operatorname{div}(\eta \otimes T) \geqq k$ and since η is flat, $\operatorname{div} T \geqq k$. Furthermore since Φ is irreducible so is T and thus $T \in S_{k^+}$. Now if $k \geqq 0$ and $\operatorname{div} T = l > k$. Then by Part 1 we find $\mu_2^*(T) \in j_2({}_pV(2g-2-2l))$.

However by [8, Theorem 2] since $0<l,\ k\leqq g-1$ we have that

$$j_2({}_pV(2g-2-2l))\cap j_2({}_pV(2g-2-2k))=\emptyset\,.$$

This gives a contradiction so $\operatorname{div} T=k$.

Lastly by [8, Theorem 3] j_2 is injective on ${}_pV(2g-2-2k)$ for $0\leqq k\leqq g-1$. Furthermore taking $k>0$ guarantees that $j_2({}_pV(2g-2-2k))$ consists entirely of irreducible bundles. Then combining Parts (1) and (2) above gives a $1-1$ correspondence.

Remark. If $k=0$ we can still get a $1-1$ correspondence between the irreducible bundles in P_0 and ${}_pV(2g-2)-{}_aV(2g-2)$. We shall henceforth denote these sets by P_0' and ${}_pV'(2g-2)$ respectively (and similarly for S_0, S_0^*, P_0^*). The subset of P_0' of analytically indecompasable bundles will be denoted by P_0'' and the corresponding space of branched projective structure by ${}_pV''(2g-2)$. We note by [3, Theorem 30] that $S_0^*=S_0^{*\prime}$ and $P_0^*=P_0^{*\prime}$. In the case of $k=g-1$ we get the following result first noted by Gunning in [1, 5].

Corollary 1. *Suppose* $k=g-1$.

Then $P_{g-1}^*\subset H^1(M,\mathscr{P}\mathscr{L}(1,\mathbb{C}))$ *consists of a single element representable by* $\Lambda_{\alpha\beta}(z_\beta)=\begin{pmatrix}\lambda_{\alpha\beta}(z_\beta) & (d/dz_\beta)\,\lambda_{\alpha\beta}(z_\beta)\\ 0 & \lambda_{\alpha\beta}^{-1}(z_\beta)\end{pmatrix}$ *on* $U_\alpha\cap U_\beta$ *(for some representative cover* $\{U_\alpha\}$*, where* $\lambda=(\lambda_{\alpha\beta})\in H^1(M,\mathbb{C}^*)$ *is any line bundle such that* $\lambda^2=\kappa$*) and* P_{g-1} *corresponds precisely to the unbranched projective structures on* M, ${}_pV(0)\simeq\mathbb{C}^{3g-3}$.

Proof. The fact that P_{g-1}^* consists of one element can be found in Gunning [5, Appendix] while its explicit form is determined in [1, Theorem 21] or alternatively follows from Theorem 1 of this paper.

4. If $k\neq g-1$ then P_k^* is no longer a point and ${}_pV(2g-2-2k)$ no longer a simple vector space. The problem of determining the relation between fibers over P_k^* and subvarieties of ${}_pV(2g-2-2k)$ is thus much more complex. So suppose $0\leqq k\leqq g-1$. We can fiber both P_k^* and ${}_pV(2g-2-2k)$ over the Jacobi variety of M and this leads to a simplification of our problem. In particular let $J(M)$ be the Jacobi variety of M represented as $J(M)=\{\xi\in H^1(M,\mathcal{O}^*)\,|\,c(\xi)=0\}$ (see [1, Section 8]). If we pick any base point $m\in M$ and define $J_k(M)=\{\xi\in H^1(M,\mathcal{O}^*)\,|\,c(\xi)=k\}$ we obtain a complex analytic isomorphism of $J(M)=J_0(M)$ onto $J_k(M)$ by mapping ξ in J_0 to $\xi\zeta_m^k\in J_k$, (where ζ_m is a point bundle as defined in [1, Section 7] and we suppress the (M) where possible.) Fixing $m\in M$ we shall refer to J_k as the k^{th} order Jacobi variety of M. As in [2, Section 3] we can now define an analytic map $\psi_k: M^{(k)}\to J_k(M)$ (where $M^{(k)}=k^{\text{th}}$ fold symmetric product of M and we identify $M^{(k)}$ with the space of all positive divisors of degree k on M) by sending $\mathfrak{D}=p_1+\cdots+p_k\in M^{(k)}$ to $\xi_{\mathfrak{D}}=\zeta_{p_1}\cdots\zeta_{p_k}\in J_k(M)$. Now by [7] any projective structure $\{U_\alpha, w_\alpha\}$ in ${}_pV_k$ is associated to a unique divisor $\mathfrak{D}(\{U_\alpha, w_\alpha\})$ in $M^{(k)}$. We thus can define a map $\mathfrak{D}: {}_pV(k)\to M^{(k)}$ such that $\mathfrak{D}^{-1}(\mathfrak{d})={}_pV_{\mathfrak{d}}$ for $\mathfrak{d}\in M^{(k)}$ (where ${}_pV_{\mathfrak{d}}$ is the space of branched projective structures of type $\mathfrak{D}$ as defined in [7]. Letting $\psi=\psi_k\circ\mathfrak{D}$ we thus get a map

$$pV(k)\xrightarrow{\ \mathfrak{D}\ }M^{(k)}\xrightarrow{\ \psi_k\ }J_k,\qquad \psi=\psi_k\circ\mathfrak{D}: pV(k)\to J_k$$

giving the desired "fibering" of ${}_pV(k)$ over J_k.

Now by Lemma 15 of [3] if $0 \leqq k \leqq g-1$ and $\Phi \in S_k^*$ with Φ indecomposable then there exists a unique line bundle $\xi \subset \Phi$ with $c(\xi)=k$. We thus obtain a map $\hat{w}: S_k^* \to J_k$ and we let $\hat{\lambda} = \hat{w} \circ i : S_k \to S_k^* \to J_k$. To transfer these maps to P_k we simply note that if $\Phi \in P_k^*$ then ξ will only be known up to a factor in the finite group $H^1(M, \boldsymbol{Z}_2)$. Thus we get corresponding maps

$$\lambda = w \circ i : P_k \xrightarrow{i} P_k^* \xrightarrow{w} J_k/H^1(M, Z_2) = \tilde{J}_k .$$

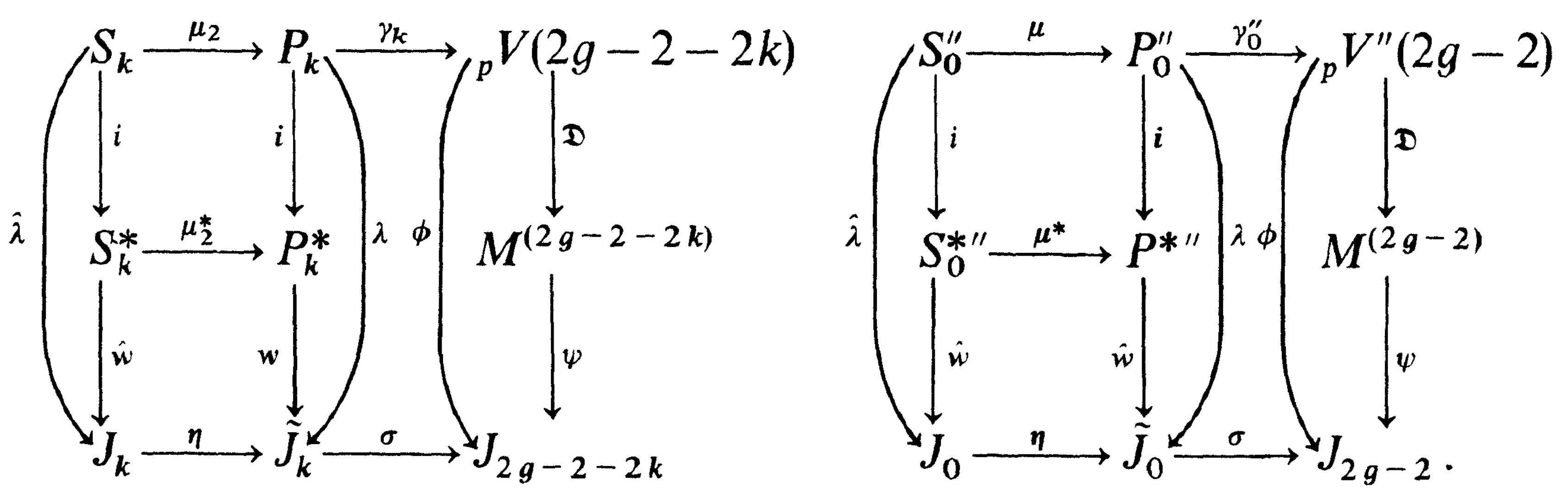
We let $\eta : J_k \to \tilde{J}_k$ be the natural surjection. We note that the map $[\xi] \in \tilde{J}_k \xrightarrow{\sigma} \kappa\xi^{-2} \in J_{2g-2-2k}$ is well defined (i.e. independent of choice of representative of $[\xi]$) and in fact establishes a biholomorphic equivalence between $\tilde{J}_k$ and $J_{2g-2-2k}$.

We can now break P_k up as $P_k = \bigcup_{[\xi] \in \tilde{J}_k} \lambda^{-1}(\xi)$ and we break up ${}_pV(2g-2-2k)$ as ${}_pV(2g-2-2k) = \bigcup_{\{\zeta \in J_{2g-2-2k}\}} \phi^{-1}(\zeta)$. We shall denote $\lambda^{-1}(\xi)$ and $\phi^{-1}(\zeta)$ by $P_{[\xi]}$ and ${}_pV(\zeta)$ respectively and let $P_\xi^* = w^{-1}(\xi)$.

Theorem 1. *Suppose* $0 \leqq k \leqq g-1$.

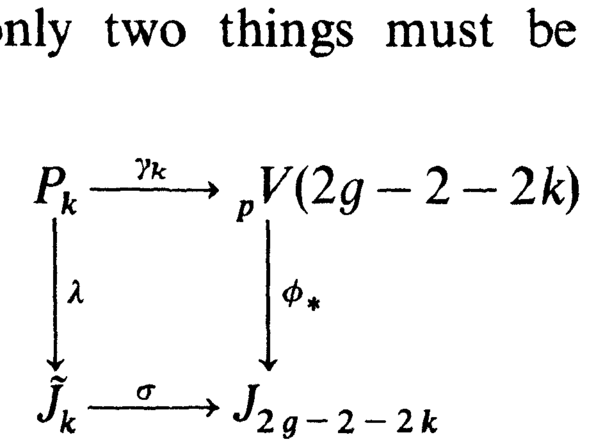
Let $\gamma_k = (j_2 | {}_pV(2g-2-2k))^{-1}$ *for* $k>0$ *and* $\gamma_0'' = (j_2 | {}_pV''(2g-2))^{-1}$. *Then* γ_k *and* γ_0'' *are bijections such that*

(1) *The following diagrams commute*



(2) $\gamma_k(P_{[\xi]}) = {}_pV(\kappa\xi^{-2})$ *for any* $\xi \in [\xi] \in w(P_k^*)$ *and* $\gamma_0''(P_{[\xi]}) = {}_pV(\kappa\xi^{-2})$ *for any* $\xi \in [\xi] \in w(P_0^{*\prime})$.

Proof. γ_k and γ_0'' are bijections by Proposition 1. Furthermore the proof for γ_0'' is identical to that of γ_k so we concern ourselves only with the latter case. Again by Proposition 1 only two things must be checked. Firstly the commutativity of the block



and secondly that ${}_pV(\kappa\xi^{-2}) \subset \gamma_k(P_{[\xi]})$.

So let $\tilde{T} = (\tilde{T}_{\alpha\beta}) \in P_k$ be represented by $T_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \in S_k$. We can then find $\tilde{\Phi} \in H^1(M, \mathscr{P}\mathscr{L}(1, \mathbb{C}))$ analytically equivalent to T and representable by $\Phi = (\Phi_{\alpha\beta}) = \begin{pmatrix} \xi_{\alpha\beta} & \eta_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix} \in H^1(M, \mathscr{S}\mathscr{L}(2, \mathbb{C}))$ with $\xi_{\alpha\beta} \in J_k(M)$ unique up to multiplication by elements of $H^1(M, Z_2)$. Then $\sigma\lambda(\tilde{T}) = \kappa\xi^{-2}$. Referring back to the proof of Proposition 1 we find that $\gamma_k(\tilde{T}) = \{w_\alpha\} \in {}_pV_{\mathfrak{D}(\det H_\alpha)}$ where $\begin{pmatrix} h_{1\alpha} \\ h_{2\alpha} \end{pmatrix} \in \Gamma(M, \mathcal{O}(\xi_{\alpha\beta}^{-1} \otimes T))$, $w_\alpha = h_{1\alpha}/h_{2\alpha}$ and $H_\alpha = \begin{pmatrix} h_{1\alpha} & h'_{1\alpha} \\ h_{2\alpha} & h'_{2\alpha} \end{pmatrix}$.

Then $\phi\gamma_k(\tilde{T}) = \psi(\mathfrak{D}(\det H_\alpha)) = \kappa\xi^{-2}$ since $\{\det H_\alpha \in \Gamma(M, \mathcal{O}(\kappa\xi^{-2}))\}$ again by the construction of the proof of Proposition 1.

For Part (2) suppose $\{w_\alpha\} \in {}_pV(\kappa\xi^{-2})$ for some $\xi \in J_k$. Then $\{w_\alpha\} \in {}_pV_{\mathfrak{D}}$ for some $\mathfrak{D} \in M^{(2g-2-2k)}$ with $\psi(\mathfrak{D}) = \kappa\xi^{-2}$. Now $j_2(\{w_\alpha\}) = \tilde{T} \in H^1(M, \mathrm{PL}(1, C))$ and by Proposition 1, $\tilde{T} \in P_k$. Then it suffices to show that if $T \in S_k$ represents $\tilde{T}$ then T is analytically equivalent to some $\Phi_{\alpha\beta} = \begin{pmatrix} \xi_{\alpha\beta} & \eta_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix} \in H(M, \mathcal{SL}(2, \mathbb{C}))$ with $(\xi_{\alpha\beta})$ a representative of ξ in J_k.

Writing $w_\alpha = h_\alpha(z_\alpha)$ we can assume that the h_α are holomorphic functions of z_α on U_α where $\{U_\alpha, z_\alpha\}$ is some representative analytic cover of M.

Let $T = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \in S_k$ and we note $w_\alpha = \dfrac{a_{\alpha\beta} w_\beta + b_{\alpha\beta}}{c_{\alpha\beta} w_\beta + d_{\alpha\beta}}$ for $p \in U_\alpha \cap U_\beta$.

Now $\mathfrak{D}\{w_\alpha\} = \Sigma\, v_p\{h'_\alpha\} = 2g - 2 - 2k$. Then for $p \in U_\alpha \cap U_\beta$ letting $H_{\alpha\beta} = h'_\beta/h'_\alpha$ we easily check that $(H_{\alpha\beta}) \in H^1(M, \mathcal{O}^*)$ with $c(H_{\alpha\beta}) = -\deg \mathfrak{D}$. Now $dw_\alpha/dw_\beta = (c_{\alpha\beta} w_\beta + d_{\alpha\beta})^{-2} = \dfrac{h'_\alpha}{h'_\beta} \cdot \dfrac{dz_\alpha}{dz_\beta} = H_{\alpha\beta}^{-1} \kappa_{\alpha\beta}^{-1}$. Letting $\xi_{\alpha\beta} = c_{\alpha\beta} w_\alpha + d_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ we have $(\xi_{\alpha\beta}) \in H^1(M, \mathcal{O}^*)$ and $\xi_{\alpha\beta}^2 = \kappa_{\alpha\beta} H_{\alpha\beta}$ so $c(\xi) = k$ and $\psi(\mathfrak{D}) = H^{-1} = \kappa\xi^{-2}$ as desired. Now let $\eta_{\alpha\beta} = \dfrac{1}{h'_\beta} \dfrac{d}{dz_\beta} \xi_{\alpha\beta} = c_{\alpha\beta}$.

Then in $U_\alpha \cap U_\beta \cap U_\gamma$ we find

$$\begin{aligned} \eta_{\alpha\gamma} = \frac{1}{h'_\gamma} \frac{d}{dz_\gamma} \xi_{\alpha\gamma} &= \frac{1}{h'_\gamma} \left[\left(\frac{d}{dz_\gamma} \xi_{\alpha\beta} \right) \cdot \xi_{\beta\gamma} + \xi_{\alpha\beta} \left(\frac{d}{dz_\gamma} \xi_{\beta\gamma} \right) \right] \\ &= \left(H_{\beta\gamma}^{-1} \frac{1}{h'_\beta} \right) \left(\kappa_{\beta\gamma}^{-1} \frac{d}{dz_\beta} \xi_{\alpha\beta} \right) \xi_{\beta\gamma} + \xi_{\alpha\beta} \eta_{\beta\gamma} \\ &= \xi_{\beta\gamma}^{-1} \eta_{\alpha\beta} + \xi_{\alpha\beta} \eta_{\beta\gamma} . \end{aligned}$$

Thus by [1, Section 4] $\xi_{\alpha\beta}^{-1} \eta_{\alpha\beta} \in H^1(M, \mathcal{O}(\xi^2))$ and so by [3, Theorem 13] we have

$$\Phi_{\alpha\beta} = \begin{pmatrix} \xi_{\alpha\beta} & \eta_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix} \quad \text{is in} \quad H^1(M, \mathcal{SL}(2, \mathbb{C})) .$$

Now

$$\begin{aligned} \begin{pmatrix} w_\alpha & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_{\alpha\beta} & \eta_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix} &= \begin{pmatrix} w_\alpha \xi_{\alpha\beta} & w_\alpha \eta_{\alpha\beta} + \xi_{\alpha\beta}^{-1} \\ \xi_{\alpha\beta} & \eta_{\alpha\beta} \end{pmatrix} \\ &= \begin{pmatrix} a_{\alpha\beta} w_\beta + b_{\alpha\beta} & c_{\alpha\beta} w_\alpha + (c_{\alpha\beta} w_\beta + d_{\alpha\beta})^{-1} \\ c_{\alpha\beta} w_\beta + d_{\alpha\beta} & c_{\alpha\beta} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} \begin{pmatrix} w_\beta & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} a_{\alpha\beta} w_\beta + b_{\alpha\beta} & a_{\alpha\beta} \\ c_{\alpha\beta} w_\beta + d_{\alpha\beta} & c_{\alpha\beta} \end{pmatrix} \\ &= \begin{pmatrix} a_{\alpha\beta} w_\beta + b_{\alpha\beta} & \dfrac{a_{\alpha\beta} d_{\alpha\beta} + a_{\alpha\beta} c_{\alpha\beta} w_\beta}{d_{\alpha\beta} + c_{\alpha\beta} w_\beta} \\ c_{\alpha\beta} w_\beta + d_{\alpha\beta} & c_{\alpha\beta} \end{pmatrix} \end{aligned}$$

This $\{\Phi_{\alpha\beta}\}$ is analytically equivalent to $\{T_{\alpha\beta}\}$ and we are done.

We note that we have explicitly determined the form of $i_*(S_k)$, namely if $T_{\alpha\beta} \in S_k$ is as above then $i_*(T_\alpha) \sim \begin{pmatrix} \xi_{\alpha\beta} & \frac{1}{h'_{\alpha\beta}} \frac{d}{dz_\beta} \xi_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix}$ using the notation of the theorem. In particular if $k = g-1$ then $H_{\alpha\beta}$ is equivalent to the trivial bundle, $\xi_{\alpha\beta}^2 = \kappa_{\alpha\beta}$ and $i_*(T) = \begin{pmatrix} \xi_{\alpha\beta} & \frac{d}{dz_\beta} \xi_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix}$ as stated in Corollary 1.

5. We consider the consequences of Theorem 1 in some cases of low genus.

Suppose $g(M) = 2$ so that $c(\kappa) = 2$. Thus $k = 0$ or $k = g-1 = 1$. If $k = 1$ then we have the case of Corollary 1 and P_1^* consists of one element while $P_1 \approx \mathbb{C}^3$. If $k = 0$ then $P_0'' \sim {}_pV''(2)$ and $P_{[\xi]}'' \sim {}_pV''(\kappa\xi^{-2})$. Now by the Riemann-Roch Theorem if $[\xi] \neq 1$ then $\dim H^1(M, \mathcal{O}(\xi^2)) = 1$ and therefore by [3, Theorem 14] $P_{[\xi]}^{*''}$ consists of at most one point for any $[\xi] \in \tilde{J}_0 - \{1\}$. Analogously for such $[\xi]$, $\psi^{-1}(\xi)$ also consists of at most one point. Following the notation in [2] we shall let $W_l^r = \{\xi \in J_l \mid \gamma(\xi) = \dim \Gamma(M, \mathcal{O}(\xi)) \geqq r\}$, and $G_l^r = \psi^{-1}(W_l^r)$. Now $G_l^1 = M^{(l)}$ for any l and by Theorem 10 of [2] if $1 \leqq r \leqq g$ we have $M^{(l)} - G_l^2$ is analytically isomorphic to $W_l^1 - W_l^2$. In our case $l = 2$, $W_2^2 = \{\kappa\}$ and G_2^2 is analytically isomorphic to $\boldsymbol{P}^1$. Then we find that both $P_0^{*''}$ and $M^{(2)}$ can be identified with the complex manifold obtained by blowing up $J_2 = W_2^1$ at the point κ and we can thus define a complex analytic isomorphism $\delta: P_0^{*''} \to M^{(2)}$ such that the following diagram commutes

$$\begin{array}{ccc} P_0'' - P_{[1]} & \xrightarrow{\gamma''} & {}_pV(2) - {}_pV \\ \downarrow i & & \downarrow \mathfrak{D} \\ P_0^{*''} - P_{[1]}^* & \xrightarrow{\delta} & M^{(2)} - G_2^2 \end{array}$$ with γ'' and δ isomorphisms.

So off the exceptional fiber the fibration $P_0'' \to P_0^{*''}$ corresponds precisely to the fibration ${}_pV''(2) \to M^{(2)}$; that is the fiber over a point $x \in P_0^{*''}$ s.t. $w(x) \neq [1]$ is analytically isomorphic to ${}_pV_{\mathfrak{D}}''$ for the unique $\mathfrak{D} \in M^{(2)}$ such that $\psi(\mathfrak{D}) = \kappa w^{-2}(x)$. In particular noting [4, 3 Theorem 30] we see that ${}_pV_{\mathfrak{D}}''$ is in fact a manifold for any $\mathfrak{D} \in M^{(2)} - G_2^2$.

Now suppose $g = 3$ so that $c(\kappa) = 4$ and $k = 0, 1, 2$. If $k = 2$ we again have the case of Corollary 1. Suppose $k = 1$. Then $P_1 \overset{\gamma_1}{\simeq} {}_pV(2)$ and $P_{[\xi]}$ corresponds to ${}_pV(\kappa\xi^{-1})$ for $[\xi] \in J_1$. Again $M^{(2)} - G_2^2$ is biholomorphic to $W_2^1 - W_2^2$ and W_2^2 consists of a finite set of points if M is hyperelliptic and is empty otherwise. If M is not hyperelliptic we thus find that $M^{(2)}$ is biholomorphic to $W_2^1 \subset J_2$. Similarly if $[\xi] \in J_1$ is such that $\kappa\xi^{-2} \in W_2^1$ then by [3, Theorem 14] $P_{[\xi]}^*$ consists of precisely one point and this can also be identified with W_2^1 and thus we can define an analytic isomorphism $\delta_1: \to M^{(2)}$ such that the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\gamma_1} & {}_pV(2) \\ \downarrow i & & \downarrow \mathfrak{D} \\ P_1^* & \xrightarrow{\delta} & M^{(2)} \end{array}$$

commutes with γ_1 and δ_1 both being isomorphisms.

We thus again find that the fibration $P_1 \to P_1^*$ corresponds precisely to that of ${}_pV(2) \to M'^{(2)}$ so that the fiber over each $x \in P_1^*$, which is a complex analytic manifold of Dimension 6 by [3, Theorem 30] is isomorphic to ${}_pV_{\mathfrak{D}}$ for $\mathfrak{D} = \delta(x) \in M^{(2)}$. In particular we find that each such ${}_pV_{\mathfrak{D}}$ is a manifold.

Now if M is hyperelliptic then $M^{(2)}$ and P_1^* will be identifiable with the minimal non-singular resolution of W_2^1 obtained by blowing it up at the finite number of points in its singular locus W_2^2. It will thus still be possible to define an analytic isomorphism of P_1^* onto $M^{(2)}$ however the commutativity properties enjoyed over non-hyperelliptic surfaces will now only be true for flat bundles $T \in P_1$ such that $\delta_0 \circ i(T) \notin G_2^2$.

Lastly suppose $k=0$, so that $P_0'' \overset{\gamma}{\sim} {}_pV''(4)$. Here $\dim \Gamma(M, \mathcal{O}(\kappa\xi^{-2})) \geqq 2$ for all $[\xi] \in \tilde{J}_0$ so we cannot improve upon the results of our theorem.

In general we can make the following statement

Corollary 2. *Suppose* $0 \leqq k \leqq g-1$ *and* $m = 2g-2-2k$. *Then*

(1) *If* $m \leqq g$ *then* G_m^2 *is a proper analytic subvariety of* $M^{(m)}$ *and there exists a bijection* $\delta: P_k^* - w^{-1}(\sigma^{-1}(W_m^2)) \to M^{(m)} - G_m^2 \overset{\Psi}{\simeq} W_m^1 - W_m^2$ *such that if* $\hat{P}_k = P_k^* - w^{-1}(\sigma^{-1}(w_m^2))$ *and;* $\hat{M}^m = M^{(m)} - G_m^2$; $\hat{P}_k = i^{-1}(\hat{P}_k^*)$ *and* $\mathfrak{D}^{-1}(M^{(m)}) = {}_p\hat{V}(m)$ *then the diagram*

$$\begin{array}{ccc} \hat{P}_k & \xrightarrow{\gamma} & {}_p\hat{V}(m) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathfrak{D}} \\ \hat{P}_k^* & \xrightarrow{\delta} & \hat{S}^{(m)} \end{array}$$

commutes with γ, δ *being bijections.*

(2) *There exists a proper analytic subset* A *of the Teichmuller Space* T^g *of Riemann Surfaces of genus* g *such that* $M \in T^g - A$ *implies that if* $m < \left[\frac{g+3}{2}\right]$ *the above map* δ *can be extended to all of* P_k^* *in such a way that commutativity is preserved.*

Proof. Follows immediately Proposition 1 and [11].

Thus we know that "in general" if $2g-2-2k$ is small enough the fibration $P_k^* \to P_k^*$ will coincide with the fibration ${}_pV(2g-2-2k) \to M^{(2g-2-2k)}$ thus generalizing Gunning's results in [6]. The problem of extending in a natural manner map δ of Corollary 2 from $\hat{P}_k^* \to \hat{S}^{(m)}$ to a map $P_k^* \to S^{(m)}$ is treated in [9] however it is not clear whether commutativity still occurs in the new diagram thus obtained. It would thus appear that the bijection between ${}_pV_{[\xi]}$ and ${}_pV(\kappa\xi^{-2})$ of Theorem 1 is the best result obtainable in the general case. Corresponding results for the case of affine bundles will appear in [10].

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