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Absolutely P-Summing, P-Nuclear Operators and Their Conjugates

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Pietsch [16] has introduced the notion of absolutely p-summing operators between normed spaces. These operators provide a natural generalization of the Hilbert-Schmidt operators between two Hilbert spaces ([10, 16]). In addition, the theory of absolutely p-summing operators has application in the general theory of Banach spaces [9].

This paper is basically a detailed study of the absolutely p-summing operators and two new classes of operators, the strongly p-summing and p-nuclear operators. The introduction of these two classes was motivated by the following observations:

(i) *Absolutely P-Summing Operators and Tensor Products.* The concept of an absolutely 1-summing operator goes back to the early work of Grothendieck ([5], p. 155), where it is defined in terms of tensor products with the space ℓ_1. Grothendieck defined an operator T to be «semi-intégrale à droite» if the operator $I \otimes T$ mapping $\ell_1 \hat{\otimes}_\varepsilon E$ into $\ell_1 \hat{\otimes}_\pi F$ is continuous, where the ε and π topologies are the bi-equicontinuous and projective topologies as defined in Schaefer ([21], p. 92–96). One can show that an operator T is absolutely 1-summing if and only if it is semi-intégrale à droite. For the case $p > 1$, one can show that every operator T, such that $I \otimes T: \ell_p \otimes_\varepsilon E \to \ell_p \hat{\otimes}_\pi F$ is continuous, is also absolutely p-summing; however, the converse is not true. We have called the class of operators such that $I \otimes T$ is continuous the p-nuclear operators. It is important to note that the p-nuclear operators discussed here are not the same as the p-nuclear operators discussed in [11] and [12] (see Theorem 2.7.1).

(ii) *Conjugates of Absolutely P-Summing Operators.* The absolutely p-summing operators are not closed under conjugation. For example, Pietsch ([16], p. 338) has shown that the identity operator $I: \ell_1 \to \ell_2$ is absolutely 2-summing; however, the conjugate operator I' mapping ℓ_2 into ℓ_∞ is not absolutely 2-summing. The strongly p-summing operators

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($1 < p \leqq \infty$) are a characterization of the conjugates of absolutely q-summing operators $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

In Section IV we present a number of interesting applications and open questions related to the present work. In particular we present an operator characterization of inner product spaces (Theorem 4.2.2) and a tensor product characterization of nuclear operators in Hilbert space (Theorem 4.1.2).

Notation. Throughout the paper E and F will denote Banach spaces and E' and F' the continuous dual spaces. All linear operators are to be considered continuous. The space of continuous linear operators mapping E into F will be denoted by $\mathscr{L}(E,F)$; the space $\mathscr{B}(E,F)$ is the space of continuous bilinear forms on $E \times F$. The symbol $S(E)$ will denote the sequences with values in E and $F(E)$ will denote the sequences with all but a finite number of terms equal to zero.

I. Sequence Spaces and Tensor Products

1.1. Sequence Spaces. We shall begin by discussing various spaces of sequences with values in a Banach space E. These spaces will appear in the definitions of the operators studied in the following sections. A sequence $\{x_i\}$ with values in E is called weakly p-summable ($\ell_p(E)$) if for all $x' \in E'$, the sequence $\{x'(x_i)\} \in \ell_p$. The space $\ell_p(E)$ is a normed space; the norm is given by

$$\varepsilon_p(\{x_i\}) = \begin{cases} \sup\left\{\left(\sum_{i=1}^{\infty} |x'(x_i)|^p\right)^{1/p} : \|x'\| \leqq 1\right\}, & 1 \leqq p < \infty \\ \sup_i \{\sup\{|x'(x_i)| : \|x'\| \leqq 1\}\}, & p = \infty . \end{cases}$$

The following theorem, due to Grothendieck ([7], p. 86–88), provides a useful characterization of $\ell_p(E)$.

Theorem 1.1.1. *For $1 < p \leqq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, there is an isometric isomorphism between $\ell_p(E)$ and $\mathscr{L}(\ell_q, E)$. For $p = 1$, $\ell_1(E)$ is isometrically isomorphic with $\mathscr{L}(c_0, E)$. In both cases, a sequence $\{x_i\}$ in $\ell_p(E)$ is identified with the operator $T(\{c_i\}) = \sum_{i=1}^{\infty} c_i x_i$.*

A sequence $\{x_i\}$ is called absolutely p-summable ($\ell_p\{E\}$) if the sequence $\{\|x_i\|\} \in \ell_p$. The space $\ell_p\{E\}$ is a normed space; the norm is given by

$$\alpha_p(\{x_i\}) = \begin{cases} \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p}, & 1 \leqq p < \infty \\ \sup_i \|x_i\|, & p = \infty. \end{cases}$$

We have found it convenient to introduce the following additional sequence space. A sequence $\{x_i\}$ is called strongly p-summable ($\ell_p\langle E\rangle$) if for all sequences $\{x_i'\} \in \ell_q(E')$, $\frac{1}{p} + \frac{1}{q} = 1$, the series $\sum_{i=1}^{\infty} x_i'(x_i)$ converges.

Theorem 1.1.2. *The space $\ell_p\langle E\rangle$ is a normed space; the norm is given by*

$$\sigma_p(\{x_i\}) = \sup\left\{\left|\sum_{i=1}^{\infty} x_i'(x_i)\right| : \varepsilon_q(\{x_i'\}) \leqq 1\right\}.$$

Proof. Let $\{x_i\} \in \ell_p\langle E\rangle$; we first show $\sigma_p(\{x_i\})$ is finite. The sequence $\{x_i\}$ can be considered as the linear form F on $\ell_q(E')$ defined by $F(\{x_i'\}) = \sum_{i=1}^{\infty} x_i'(x_i)$. Define a sequence $\{F_n\}$ of linear forms on $\ell_q(E')$ by $F_n(\{x_i'\}) = \sum_{i=1}^{n} x_i'(x_i)$. It is easy to see that each F_n is continuous. Furthermore it follows directly from the definitions for F_n and F that $\{F_n\}$ converges to F at each point of $\ell_q(E')$. By Theorem 1.1.1 $\ell_q(E')$ is complete; therefore, applying the Banach-Steinhaus Theorem it follows that F is continuous and $\sigma_p(\{x_i\}) = \|F\| < \infty$. One can now easily verify that σ_p satisfies the properties of a norm. □

The relationships between the various sequence spaces are given in the following theorem:

Theorem 1.1.3 ([3], *p*. 15).

(i) *For* $1 \leqq p \leqq \infty$, $\ell_p\langle E\rangle \subseteqq \ell_p\{E\} \subseteqq \ell_p(E)$ *and* $\varepsilon_p \leqq \alpha_p \leqq \sigma_p$.

(ii) *For* $p = 1$, $\ell_1\langle E\rangle = \ell_1\{E\}$ *and* $\alpha_1 = \sigma_1$.

(iii) *For* $p = \infty$, $\ell_\infty\{E\} = \ell_\infty(E)$ *and* $\alpha_\infty = \varepsilon_\infty$.

1.2. Tensor Products. We shall have occasion to consider a number of norms in the tensor product $E \otimes F$. The projective or π-norm ([22], p. 434–445) is defined by $\|u\|_\pi = \inf\left\{\sum_i \|x_i\| \, \|y_i\|\right\}$, where the infimum is over all representations of $u = \sum_i x_i \otimes y_i$ in $E \otimes F$. The ε-norm is defined by

$$\|u\|_\varepsilon = \sup\left\{\left|\sum_i x'(x_i)\, y'(y_i)\right| : \|x'\| \leqq 1, \|y'\| \leqq 1\right\}.$$

We shall use the notations $E \otimes_\pi F$ and $E \otimes_\varepsilon F$ to denote the tensor product with the above norms. It is well-known ([22], p. 444) that the dual space for $E \otimes_\pi F$ is given by $\mathscr{B}(E, F)$; the dual space for $E \otimes_\varepsilon F$ is given by $J(E, F)$, the integral bilinear forms on $E \times F$ ([22], p. 500). Using the sequence spaces discussed in Section 1.1, we can define a number of other norms on $E \otimes F$. For $1 \leqq p \leqq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we define

$$w_p(u) = \inf(\varepsilon_p(\{x_i\})\, \varepsilon_q(\{y_i\})),$$
$$g_p(u) = \inf(\alpha_p(\{x_i\})\, \varepsilon_q(\{y_i\})),$$
$$d_p(u) = \inf(\varepsilon_p(\{x_i\})\, \alpha_q(\{y_i\})).$$

In each case the infimum is taken over all representations of $u = \sum_i x_i \otimes y_i$ in $E \otimes F$. The norms g_p and d_p have been considered by Saphar [20] and Chevet [2]. The tensor product with the norm g_p(d_p or w_p) will be denoted by $E \otimes_{g_p} F$ ($E \otimes_{d_p} F$ or $E \otimes_{w_p} F$). The completion of the tensor product with the norm $\tau = \varepsilon, \pi, w_p, d_p$ or g_p is denoted by $E \hat{\otimes}_\tau F$. In the following theorem we list a number of important properties of the above tensor norms.

Theorem 1.2.1. ([3], p. 20).

(i) *If* $u = x \otimes y$, *then* $g_p(u) = w_p(u) = d_p(u) = \|x\|\, \|y\|$.

(ii) *The magnitudes of the norms are related in the following way:*

$$\begin{array}{ccccc} & & g_p & & \\ & \nearrow & & \searrow & \\ \varepsilon \rightarrow w_p & & & & \pi \\ & \searrow & & \nearrow & \\ & & d_p & & \end{array}$$

(iii) *If* $p = \infty$, $g_\infty(u) = w_\infty(u)$.

(iv) *If* $p = 1$, $d_1(u) = w_1(u)$.

(v) $\|u\|_\pi = g_1(u) = d_\infty(u)$.

In the next few theorems we develop the basic relationships between the norms on the various sequence spaces and the norms on the tensor product $\ell_p \otimes E$. It is well-known that $\ell_p \otimes E$ can be considered as a subspace of $S(E)$. In fact ([14], p. 98), the mapping

$$\phi\left(\sum_{i=1}^{n} \{c_{ij}\} \otimes x_i\right) = \left\{\sum_{i=1}^{n} c_{ij} x_i\right\}$$

is an algebraic isomorphism of $\ell_p \otimes E$ into $S(E)$. In most cases we shall consider $\ell_p \otimes E$ as a vector subspace of $S(E)$ without any specific reference to ϕ.

Theorem 1.2.2. ([7], p. 87). *Let* $1 \leqq p \leqq \infty$. *The space* $\ell_p(E)$ *induces the* ε*-norm on* $\ell_p \otimes E$.

Theorem 1.2.3. *Let* $1 \leqq p < \infty$.

(i) $\ell_p \otimes E \subsetneqq \ell_p\langle E\rangle$.

(ii) $\ell_p\langle E\rangle$ *induces the projective* (π) *norm on* $\ell_p \otimes E$.

Proof. (i) Let $\hat{\phi}$ be the mapping from $\ell_p \times E$ into $S(E)$ defined by $\hat{\phi}(\{c_j\}, x) = \{c_j x\}$. We show $\hat{\phi}(\{c_j\}, x) \in \ell_p\langle E\rangle$. If $\{x'_j\} \in \ell_q(E')\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, we have

$$\left|\sum_j x'_j(c_j x)\right| \leqq \sum_j |c_j x'_j(x)|$$
$$\leqq \begin{cases} \left(\sum_j |c_j|^p\right)^{1/p} \left(\sum_j |x'_j(x)|^q\right)^{1/q} & \text{for} \quad 1 < p < \infty \\ \sum_j |c_j| \sup_j \{|x'_j(x)|\} & \text{for} \quad p = 1 \end{cases}$$
$$\leqq \left(\sum_j |c_j|^p\right)^{1/p} \varepsilon_q(\{x'_j\}) \, \|x\| \, .$$

Since $\{c_j\} \in \ell_p$, the above inequality shows $\sum_j x'_j(c_j x)$ converges. Therefore $\{c_j x\} \in \ell_p\langle E\rangle$. Consequently, using the definition of the tensor product, we conclude that ϕ maps $\ell_p \otimes E$ into $\ell_p\langle E\rangle$.

(ii) It remains to show $\ell_p\langle E\rangle$ induces the π-norm on $\ell_p \otimes E$. Let $u = \sum_{i=1}^n \{c_{ij}\} \otimes x_i \in \ell_p \otimes E$. Since $\mathscr{B}(\ell_p, E) = \mathscr{L}(\ell_p, E')$, it follows from Theorem 1.1.1 that $\mathscr{B}(\ell_p, E) = \ell_q(E')$. Therefore

$$\left\|\sum_{i=1}^n \{c_{ij}\} \otimes x_i\right\|_\pi = \sup_{\substack{\|B\| \leqq 1 \\ B \in \mathscr{B}(\ell_p, E)}} \left|B\left(\sum_{i=1}^n \{c_{ij}\} \otimes x_i\right)\right|$$
$$= \sup_{\varepsilon_q(\{x'_j\}) \leqq 1} \left|\sum_{j=1}^\infty x'_j\left(\sum_{i=1}^n c_{ij} x_i\right)\right|$$
$$= \sigma_p\left(\sum_{i=1}^n \{c_{ij}\} \otimes x_i\right). \quad \square$$

Let $1 < p < \infty$. If we consider the norm induced on $\ell_p \otimes E$ as a normed subspace of $\ell_p\{E\}$ we obtain another norm on the tensor product. We shall use the notation $\ell_p \otimes_p E$ to denote the tensor product with this norm.

Theorem 1.2.4. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $d_q(u) \leqq \|u\|_p \leqq g_p(u)$.

Proof. We shall show $d_q(u) \leqq \|u\|_p$. The other inequality is proved in a similar way ([3], p. 25). Let $u = \sum_{i=1}^n \{c_{ij}\} \otimes x_i \in \ell_p \otimes E$ and define $\{c_{ij}\}^m = (c_{i1}, c_{i2}, \ldots, c_{im}, 0, 0, \ldots)$. Since $\lim_{m \to \infty} \{c_{ij}\}^m = \{c_{ij}\}$ in ℓ_p, it follows from Theorem 1.2.1(i) that

$$\lim_{m\to\infty} d_q\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i - \sum_{i=1}^{n} \{c_{ij}\} \otimes x_i\right)$$

$$\leqq \lim_{m\to\infty} \sum_{i=1}^{n} d_q((\{c_{ij}\}^m - \{c_{ij}\}) \otimes x_i)$$

$$= \lim_{m\to\infty} \sum_{i=1}^{n} \|\{c_{ij}\}^m - \{c_{ij}\}\| \, \|x_i\| = 0 .$$

This implies

$$\begin{aligned} d_q(u) &= d_q\left(\sum_{i=1}^{n} \{c_{ij}\} \otimes x_i\right) \\ &= \lim_{m\to\infty} d_q\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i\right). \end{aligned} \tag{1.2.1}$$

Let $\{e_j\}$ be the standard basis in ℓ_p. Using standard manipulations for tensor products we have

$$\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i = \sum_{j=1}^{m} e_j \otimes \left(\sum_{i=1}^{n} c_{ij} x_i\right).$$

Therefore from the definitions for the norm d_q, we have

$$d_q\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i\right) = d_q\left(\sum_{j=1}^{m} \left(e_j \otimes \left(\sum_{i=1}^{n} c_{ij} x_i\right)\right)\right)$$

$$\leqq \varepsilon_q(\{e_j\}) \left(\sum_{j=1}^{m} \left\|\sum_{i=1}^{n} c_{ij} x_i\right\|^p\right)^{1/p}$$

$$= \left(\sum_{j=1}^{m} \left\|\sum_{i=1}^{n} c_{ij} x_i\right\|^p\right)^{1/p}$$

$$= \alpha_p\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i\right).$$

From the above inequality and Eq. (1.2.1) we obtain

$$d_q(u) = \lim_{m\to\infty} d_q\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i\right)$$

$$\leqq \lim_{m\to\infty} \alpha_p\left(\sum_{i=1}^{n} \{c_{ij}\}^m \otimes x_i\right)$$

$$= \lim_{m\to\infty} \left(\sum_{j=1}^{m} \left\|\sum_{i=1}^{n} c_{ij} x_i\right\|^p\right)^{1/p}$$

$$= \alpha_p(u) = \|u\|_p .$$

Therefore $d_q(u) \leqq \|u\|_p$. □

It can be shown that all norms considered in the previous theorem are equivalent when E is an $\mathscr{L}_{p\lambda}$ space (Theorem 3.2.5). Furthermore, for $p=q=2$, the three norms are equal if and only if E is an inner product space (Theorem 4.2.2).

II. The Operators $\pi_p(E, F)$, $D_p(E, F)$, $N_p(E, F)$

2.1. Basic Definitions and Properties. Let $1 \leqq p < \infty$. An operator T is absolutely p-summing ($\pi_p(E, F)$) if there exists a constant $C \geqq 0$, such that for all finite sets $x_1, \ldots, x_n$, the inequality $\alpha_p(\{Tx_i\}) \leqq C\varepsilon_p(\{x_i\})$ is satisfied. The smallest number C, such that the above inequality is satisfied, is called the absolutely p-summing norm ($\pi_p(T)$) of T.

A detailed study of $\pi_p(E, F)$ is given by Pietsch [16]. We introduce two related classes of operators.

(i) Let $1 < p \leqq \infty$. An operator T is strongly p-summing ($D_p(E, F)$) if there exists $C \geqq 0$ such that $\sigma_p(\{Tx_i\}) \leqq C\alpha_p(\{x_i\})$.

(ii) Let $1 < p < \infty$. An operator T is p-nuclear ($N_p(E, F)$) if there exists $C \geqq 0$ such that $\sigma_p(\{Tx_i\}) \leqq C\varepsilon_p(\{x_i\})$. We shall denote the strongly p-summing norm by $D_p(T)$ and the p-nuclear norm by $N_p(T)$.

In the above definitions we have excluded the cases $D_1(E, F)$, $N_1(E, F)$, $\pi_\infty(E, F)$ and $N_\infty(E, F)$. If we formulate the analogous definitions we observe, using Theorem 1.1.3, that $\pi_\infty(E, F) = \mathscr{L}(E, F)$; $D_1(E, F) = \mathscr{L}(E, F)$; $N_1(E, F) = \pi_1(E, F)$; and $N_\infty(E, F) = D_\infty(E, F)$. Therefore, nothing is lost by excluding these cases. In the following, when we refer to $\pi_p(E, F)$, we shall assume $1 \leqq p < \infty$; when we refer to $D_p(E, F)$ we shall assume $1 < p \leqq \infty$; and when we refer to $N_p(E, F)$ we shall assume $1 < p < \infty$.

Theorem 2.1.1 ([3], p. 44).

(i) *The spaces $N_p(E, F)$ and $D_p(E, F)$ are normed linear spaces.*

(ii) *If T belongs to $N_p(E, F)$ (or $D_p(E, F)$), then T is continuous and* $\|T\| \leqq N_p(T)$ ($\|T\| \leqq D_p(T)$).

(iii) *If T belongs to $N_p(E, F)$ (or $D_p(E, F)$) and S belongs to $\mathscr{L}(F, G)$, then ST belongs to $N_p(E, G)$ (or $D_p(E, G)$) and* $N_p(ST) \leqq \|S\|\, N_p(T)$ ($D_p(ST) \leqq \|S\|\, D_p(T)$).

(iv) *If T belongs to $\mathscr{L}(E, F)$ and S belongs to $N_p(F, G)$ (or $D_p(F, G)$), then ST belongs to $N_p(E, G)$ (or $D_p(E, G)$) and* $N_p(ST) \leqq N_p(S)\, \|T\|$ ($D_p(ST) \leqq D_p(S)\, \|T\|$).

(v) *If F is a Banach space, then $D_p(E, F)$ and $N_p(E, F)$ are Banach spaces.*

The definitions for the spaces $\pi_p(E, F)$, $D_p(E, F)$ and $N_p(E, F)$ have been expressed in terms of sequences with all but a finite number of terms equal to zero. One can, however, easily reformulate these definitions in terms of arbitrary sequences. Let $T \in \mathscr{L}(E, F)$. The operator T induces an operator $\hat{T}$ mapping $S(E)$ into $S(F)$ defined by $\hat{T}(\{x_i\}) = \{Tx_i\}$. One can easily show $T \in \pi_p(E, F)$ if and only if $\hat{T}: \ell_p(E) \to \ell_p\{F\}$ is continuous. Furthermore, in this case, $\|\hat{T}\| = \pi_p(T)$. Analogous statements are true for $N_p(E, F)$ and $D_p(E, F)$. Pietsch ([16], p. 349) has shown it is possible to give a weaker formulation of the above statement. Indeed, an operator T is in $\pi_p(E, F)$ if and only if $\hat{T}$ maps $\ell_p(E)$ into $\ell_p\{F\}$. We do not require that $\hat{T}$ be continuous. A slight modification of Pietsch's proof shows the analogous result holds for $D_p(E, F)$ and $N_p(E, F)$.

Theorem 2.1.2.
(i) *An operator T is in $N_p(E, F)$ if and only if $\hat{T}$ maps $\ell_p(E)$ into $\ell_p\langle F\rangle$.*
(ii) *An operator T is in $D_p(E, F)$ if and only if $\hat{T}$ maps $\ell_p\{E\}$ into $\ell_p\langle F\rangle$.*

Theorem 2.1.3. *Let $1 < p < \infty$. An operator $T \in N_p(E, F)$ if and only if $I \otimes T: \ell_p \otimes_\varepsilon E \to \ell_p \otimes_\pi F$ is continuous. In this case $N_p(T) = \|I \otimes T\|$.*

Proof. Let $T \in N_p(E, F)$. By Theorems 1.2.2 and 1.2.3 we can consider $\ell_p \otimes_\varepsilon E$ as a normed subspace of $\ell_p(E)$ and $\ell_p \otimes_\pi F$ as a normed subspace of $\ell_p\langle F\rangle$. Furthermore, one can easily show that $\hat{T}$ restricted to $\ell_p \otimes_\varepsilon E$ is equal to $I \otimes T$. Let $F(E)$ be the space of sequences in E with all but a finite number of terms equal to zero. It follows that $F(E) \subseteqq \ell_p \otimes E \subseteqq \ell_p(E)$. Since $\hat{T}: \ell_p(E) \to \ell_p\langle F\rangle$ is continuous and since $\hat{T}$ obtains its norm by taking the supremum over elements in $F(E)$, it follows that $I \otimes T$ is continuous and $N_p(T) = \|\hat{T}\| = \|I \otimes T\|$. The converse follows in a similar way. □

2.2. Relationships Between $\pi_p(E, F)$, $D_p(E, F)$ and $N_p(E, F)$. The following theorem follows directly from Theorems 1.1.3 and 2.1.2.

Theorem 2.2.1.
(i) *$N_p(E, F) \subseteqq D_p(E, F)$ and $N_p(T) \geqq D_p(T)$.*
(ii) *$N_p(E, F) \subseteqq \pi_p(E, F)$ and $N_p(T) \geqq \pi_p(T)$.*
(iii) *If T belongs to $\pi_p(E, F)$ and S belongs to $D_p(F, G)$, then the composition ST belongs to $N_p(E, G)$ and $N_p(ST) \leqq \pi_p(T)\, D_p(S)$.*

Theorem 2.2.2. *Let $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

(i) *Let $1 \leqq p < \infty$. An operator T belongs to $\pi_p(E, F)$ if and only if the adjoint operator T' belongs to $D_q(F', E')$. In this case $D_q(T') = \pi_p(T)$.*

(ii) *Let $1 < q \leqq \infty$. An operator T belongs to $D_q(E, F)$ if and only if the adjoint operator T' belongs to $\pi_p(F', E')$. In this case $D_q(T) = \pi_p(T')$.*

Proof. (i) Let $1 \leqq p < \infty$ and let T belong to $\pi_p(E, F)$; we must show T' belongs to $D_q(F', E')$. If $y'_1, \ldots, y'_n$ is a finite set in F' and if $\{x''_i\} \in \ell_p(E'')$

we have

$$\begin{aligned}\left|\sum_{i=1}^{n} x_i''(T'y_i')\right| &= \left|\sum_{i=1}^{n} T''x_i''(y_i')\right| \\ &\leqq \sum_{i=1}^{n} |T''x_i''(y_i')| \\ &\leqq \sum_{i=1}^{n} \|T''x_i''\|\ \|y_i'\| \\ &\leqq \alpha_p(\{T''x_i''\})\,\alpha_q(\{y_i'\}).\end{aligned} \tag{2.2.1}$$

However, since T is absolutely p-summing, it follows that T'' is absolutely p-summing and $\pi_p(T)=\pi_p(T'')$ ([16], p. 345; [13], p. 87). Therefore

$$\alpha_p(\{T''x_i''\}) \leqq \pi_p(T)\,\varepsilon_p(\{x_i''\}). \tag{2.2.2}$$

Substituting Eq. (2.2.2) in (2.2.1) we obtain

$$\left|\sum_{i=1}^{n} x_i''(T'y_i')\right| \leqq \pi_p(T)\,\varepsilon_p(\{x_i''\})\,\alpha_q(\{y_i'\}).$$

Taking the supremum over the unit ball in $\ell_p(E'')$, we obtain $\sigma_q(\{T'y_i\}) \leqq \pi_p(T)\,\alpha_q(\{y_i'\})$. Therefore $T' \in D_q(F', E')$ and

$$D_q(T') \leqq \pi_p(T). \tag{2.2.3}$$

Conversely, assume T' belongs to $D_q(F', E')$. Let $x_1, \ldots, x_n$ be a finite set in E and let $\{y_i'\}$ belong to $\ell_q\{F'\}$. It follows that

$$\begin{aligned}\left|\sum_{i=1}^{n} y_i'(Tx_i)\right| &= \left|\sum_{i=1}^{n} T'y_i'(x_i)\right| \\ &\leqq \sigma_q(\{T'y_i'\})\,\varepsilon_p(\{x_i\}) \\ &\leqq D_q(T')\,\alpha_q(\{y_i'\})\,\varepsilon_p(\{x_i\}).\end{aligned}$$

Since $\ell_q\{F'\}=(\ell_p\{F\})'$, we take the supremum over the unit ball in $\ell_q\{F'\}$ and obtain $\alpha_p(\{Tx_i\}) \leqq D_q(T')\,\varepsilon_p(\{x_i\})$. Therefore T is absolutely p-summing and

$$\pi_p(T) \leqq D_q(T'). \tag{2.2.4}$$

Combining Eqs. (2.2.3) and (2.2.4) we obtain $\pi_p(T)=D_q(T')$. Part (ii) is proved in a similar way. □

The author has not been able to find a proof of the above result which does not in some way require measure theoretic techniques. In the above proof we need the fact that the bidual of an absolutely p-summing operator is also absolutely p-summing. Pietsch ([16], p. 345) has proved this fact using measure theoretic techniques. Alternately we could have proved the previous theorem by using the following result, from the work of Grothendieck, which also requires measure theoretic techniques. Recall a Banach space E has the metric approximation property if for all finite sets $x_1, \ldots, x_n$ in E, there exists an operator T with finite dimensional range and $\|T\|=1$ such that $\|x_i - Tx_i\| < \varepsilon$ for $i=1,2,\ldots,n$. In the following theorem $J(E,F)=(E\otimes_\varepsilon F)'$ is the collection of integral bilinear forms on $E\times F$ ([22], p. 500).

Theorem 2.2.3 ([5], p. 181). *Let E' or F' satisfy the metric approximation property. Then, the canonical mapping $\psi: E'\otimes_\pi F' \to J(E,F)$ is an isometry.*

We shall use this theorem in the proof of the following:

Theorem 2.2.4. *Let $1<p<\infty$. An operator $T\in N_p(E,F)$ if and only if $T'\in N_q(F',E')\left(\frac{1}{p}+\frac{1}{q}=1\right)$. In this case $N_p(T)=N_q(T')$.*

Proof. Let $T\in N_p(E,F)$. By Theorem 2.1.3, the operator $I\otimes T: \ell_p\otimes_\varepsilon E \to \ell_p\otimes_\pi F$ is continuous with $\|I\otimes T\| = N_p(T)$. Since $(\ell_p\otimes_\pi F)' = \ell_q(F')$, it follows that $(I\otimes T)'$ maps $\ell_q(F')$ into $J(\ell_p, E)$. However, by Theorem 1.2.2 $\ell_q\otimes_\varepsilon F'$ is a normed subspace of $\ell_q(F')$. Furthermore, since ℓ_p satisfies the metric approximation property, $\ell_q\otimes_\pi E'$ is a normed subspace of $J(\ell_p, E)$. Finally, $(I\otimes T)'$ restricted to $\ell_q\otimes_\varepsilon F'$ is equal to $I\otimes T'$. Therefore, $I\otimes T'$ is continuous and $N_q(T') = \|I\otimes T'\| \leqq \|(I\otimes T)'\| = \|I\otimes T\| = N_p(T)$. We conclude that $T'\in N_q(F',E')$ and

$$N_q(T') \leqq N_p(T)\,. \tag{2.2.5}$$

Conversely let T' belong to $N_q(F',E')$; we must show T belongs to $N_p(E,F)$. Let $x_1,\ldots,x_n$ be a finite set in E and let $\{y_i'\}$ belong to $\ell_q(F')$ with $\varepsilon_q(\{y_i'\}) \leqq 1$. We have

$$\begin{aligned}\left|\sum_{i=1}^n y_i'(Tx_i)\right| &= \left|\sum_{i=1}^n T'y_i'(x_i)\right| \\ &\leqq N_q(T') \sup_{\|x'''\|\leqq 1} \left(\sum_{i=1}^n |x'''(x_i)|^p\right)^{1/p} \varepsilon_q(\{y_i'\}) \\ &= N_q(T')\, \varepsilon_p(\{x_i\})\, \varepsilon_q(\{y_i'\})\,.\end{aligned}$$

Taking the supremum over all sequences $\{y_i'\}$ with $\varepsilon_q(\{y_i'\}) \leqq 1$, we have $\sigma_p(\{Tx_i\}) \leqq N_q(T')\, \varepsilon_p(\{x_i\})$. Therefore T belongs to $N_p(E,F)$ and

$$N_p(T) \leqq N_q(T')\,. \tag{2.2.6}$$

Combining Eqs. (2.2.5) and (2.2.6) we see $N_p(T) = N_q(T')$. □

The next corollary follows immediately from Pietsch's work. Pietsch ([16], pp. 343, 345) has shown that absolutely p-summing operators are weakly compact and completely continuous (an operator T is completely continuous if it takes weakly convergent sequences into strongly convergent sequences).

Corollary 2.2.5.

(i) *If $T \in D_q(E, F)$, then T is weakly compact and the conjugate T' is completely continuous.*

(ii) *If $T \in N_p(E, F)$, then T and T' are weakly compact and completely continuous.*

2.3. Integral Characterizations. One of the most useful and interesting results for absolutely p-summing operators is the integral characterization due to Pietsch ([16], p. 341). If an operator T is absolutely p-summing, then there exists a positive Radon measure μ, with $\|\mu\| = 1$, on the unit ball $S_{E'}$ in E' such that

$$\|Tx\| \leqq \pi_p(T)\Big(\int_{S_{E'}} |x'(x)|^p \, d\mu\Big)^{1/p}.$$

Conversely, if there exists a constant $C \geqq 0$ and a positive Radon measure μ, with $\|\mu\| = 1$, such that

$$\|Tx\| \leqq C\Big(\int_{S_{E'}} |x'(x)|^p \, d\mu\Big)^{1/p},$$

then T is absolutely p-summing and $\pi_p(T) \leqq C$. By using Theorem 2.2.2 we obtain

Theorem 2.3.1. *If $T \in D_p(E, F)$, then there exists a positive Radon measure μ, with $\|\mu\| = 1$, on the unit ball in $S_{F''}$ in F'' such that $\|Tx\|_\mu \leqq D_p(T)\, \|x\|$, where*

$$\|Tx\|_\mu = \sup\{|y'(Tx)| : y' \in F' \quad \textit{and} \quad \int_{S_{F''}} |y'(y'')|^q \, d\mu \leqq 1\}.$$

Conversely, if there exists a constant $C \geqq 0$ and a Radon measure μ, with $\|\mu\| = 1$, such that $\|Tx\|_\mu \leqq C\|x\|$, then $T \in D_p(E, F)$ and $D_p(T) \leqq C$.

In the next theorem we shall present an integral condition which is sufficient to guarantee that an operator T belong to $N_p(E, F)$. A straightforward calculation gives

Theorem 2.3.2. *Let T be a continuous operator mapping E into F. Suppose there exists a constant $C \geqq 0$ and positive Radon measures μ_1 and μ_2, with $\|\mu_1\| = 1$ and $\|\mu_2\| = 1$, on the unit balls in E' and F'', such that for all x in E and all y' in F', we have*

$$|y'(T(x))| \leqq C\Big(\int_{S_{E'}} |x'(x)|^p \, d\mu_1\Big)^{1/p}\Big(\int_{S_{F''}} |y'(y'')|^q \, d\mu_2\Big)^{1/q}.$$

Then T is p-nuclear and $N_p(T) \leqq C$.

Theorem 2.3.3.([3], *p.* 77). *Let T be an operator satisfying the conditions of the preceeding theorem. Then T is the product of an absolutely p-summing operator and a strongly p-summing operator.*

It is not known if the statement in Theorem 2.3.2 characterizes p-nuclear operators. However, if this were true every p-nuclear operator would be the product of an absolutely p-summing operator and a strongly p-summing operator.

2.4. Examples and Further Results. Pietsch ([16], p. 335) has shown if $p_1 \leqq p_2$, then $\pi_{p_1}(E,F) \subsetneqq \pi_{p_2}(E,F)$. From Theorem 2.2.2 we have

Theorem 2.4.1. *If $p_1 \leqq p_2$, then $D_{p_1}(E,F) \supseteqq D_{p_2}(E,F)$.*

We have not been able to prove or disprove a result similar to the above theorem for $N_p(E,F)$. Furthermore, we do not have examples of Banach spaces where $N_{p_1}(E,F) \neq N_{p_2}(E,F)$ for $p_1 \neq p_2$.

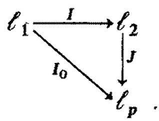
Theorem 2.4.2. *There exist Banach spaces such that*

(i) $\pi_p(E,F) \neq D_q(E,F)\left(\frac{1}{p}+\frac{1}{q}=1\right), 1 \leqq p < \infty$,

(ii) $N_p(E,F) \neq \pi_p(E,F), 1 < p < \infty$,

(iii) $N_p(E,F) \neq D_p(E,F), 1 < p < \infty$.

Proof. (i) Let $1 \leqq p \leqq 2$. Pietsch has shown that the canonical mapping $I:\ell_1 \to \ell_2$ is in $\pi_p(\ell_1,\ell_2)$ for $1 \leqq p < \infty$, but the conjugate $I':\ell_2 \to \ell_\infty$ is not in $\pi_p(\ell_2,\ell_\infty)$, $1 \leqq p \leqq 2$ ([16], p. 338; [13], p. 83). Therefore, by Theorem 2.2.2, I does not belong to $D_q(\ell_2,\ell_\infty)$ for $2 \leqq q \leqq \infty$. In a similar way, the operator I' belongs to $D_q(\ell_2,\ell_\infty)$ for $2 \leqq q \leqq \infty$, but it is not in $\pi_p(\ell_2,\ell_\infty)$ for $1 \leqq p \leqq 2$. Now let $2 \leqq p < \infty$. Consider the operator $I_0 = IJ$ defined by



In the above diagrams, I and J are the canonical mappings. Since J is continuous, we can conclude that I_0 is absolutely p-summing for all $p \geqq 2$. However, the conjugate mapping $I_0;\ell_q \to \ell_2$ is not absolutely p-summing. Indeed, if $x_i = (0,0,\ldots,0,1,0,\ldots)$ is the sequence with one in the i^{th} position and zero elsewhere, it follows that

$$\begin{aligned}\alpha_p(\{I_0'(x_i)\}) &= \left(\sum_{i=1}^{\infty} \|I_0(x_i)\|^p\right)^{1/p} \\ &= \lim_{n\to\infty} n^{1/p} = \infty,\end{aligned}$$

while

$$\varepsilon_p(\{x_i\}) = \sup_{\substack{\|x'\| \leqq 1 \\ x' \in \ell_p}} \left(\sum_{i=1}^{\infty} |x'(x_i)|^p \right)^{1/p} = 1 .$$

Consequently, I_0' is not absolutely p-summing; therefore, by Theorem 2.2.2, $I_0 \notin D_q(\ell_1, \ell_p)$, $\frac{1}{p} + \frac{1}{q} = 1$. In a similar way $I_0' \in D_q(\ell_q, \ell_\infty)$ but $I_0' \notin \pi_p(\ell_q, \ell_\infty)$.

(ii) and (iii). It is well-known that every completely continuous operator mapping a reflexive space into a Banach space is compact. Therefore, using Corollary 2.2.5, we can conclude that a p-nuclear operator is compact whenever its domain or range is reflexive. Using this fact, it follows that all the operators considered in part (i) cannot be p-nuclear since they are not compact. Parts (ii) and (iii) follow from this observation. □

The above examples show that strongly p-summing and absolutely p-summing operators are not necessarily compact. Since $I: \ell_1 \to \ell_2$ is absolutely p-summing $(1 \leqq p < \infty)$ and since the conjugate $I': \ell_2 \to \ell_\infty$ is strongly p-summing $(1 < p \leqq \infty)$, it follows that $I'I: \ell_1 \to \ell_\infty$ is a p-nuclear operator $(1 < p < \infty)$ which is not compact.

2.5. Nuclear, P-Nuclear and Integral Operators. In the next few theorems we investigate the relationships between the nuclear, p-nuclear and integral operators. Recall an operator T is nuclear if it can be represented in the form $Tx = \sum_{i=1}^{\infty} a_i(x)\, y_i$, where $a_i \in E'$, $y_i \in F$ and $\sum_{i=1}^{\infty} \|a_i\| \; \|y_i\| < \infty$. The nuclear or trace norm is defined by $\partial(T) = \inf\left\{ \sum_{i=1}^{\infty} \|a_i\| \; \|y_i\| \right\}$, where the infimum is over all such representations of T. Recall an operator T can be considered as a linear form on $E \otimes F'$ according to the formula $T\left(\sum_{i=1}^{n} x_i \otimes y_i' \right) = \sum_{i=1}^{n} y_i'(Tx_i)$. An operator T is integral if $T \in (E \otimes_\varepsilon F')'$. We shall use the symbol $J(T)$ to denote the norm of T in $(E \otimes_\varepsilon F')'$.

Lemma 2.5.1. *$N_p(E,F) = \mathscr{L}(E,F) \cap (E \otimes_{w_p} F')'$ and $N_p(T) = \|T\|_{w_p'}$ (the symbol $\|T\|_{w_p'}$ denotes the norm of T in $(E \otimes_{w_p} F')'$).*

Proof. Let $T \in N_p(E,F)$ and let $u \in E \otimes_{w_p} F'$. If $u = \sum_{i=1} x_i \otimes y_i'$ is an arbitrary representation for u in $E \otimes_{w_p} F'$ we have

$$\left| T\left(\sum_{i=1}^{n} x_i \otimes y_i' \right) \right| = \left| \sum_{i=1}^{n} y_i'(Tx_i) \right|$$
$$\leqq N_p(T)\, \varepsilon_p(\{x_i\})\, \varepsilon_q(\{y_i'\}) .$$

By taking the infimum over all representations for u in $E \otimes_{w_p} F'$ we have $|T(u)| \leqq N_p(T)\, w_p(u)$. Therefore, $T \in (E \otimes_{w_p} F')'$ and $\|T\|_{w_p'} \leqq N_p(T)$. The converse is proved in a similar way. □

Theorem 2.5.2. *Every integral operator is p-nuclear and* $N_p(T) \leqq J(T)$.

Proof. Let T be an integral operator. Then, by definition, $T \in (E \otimes_\varepsilon F')'$. However, using Holder's inequality one can easily show for $u \in E \otimes F'$, $\|u\|_\varepsilon \leqq w_p(u)$. From this observation it follows that $T \in (E \otimes_{w_p} F')'$. Therefore, by the above lemma, T is p-nuclear and $N_p(T) \leqq J(T)$. □

Since every nuclear operator is integral we have

Theorem 2.5.3. *Every nuclear operator is p-nuclear and* $N_p(T) \leqq \partial(T)$.

Unfortunately we do not have an example of a p-nuclear operator which is not integral. In Section III we give sufficient conditions for every p-nuclear operator to be integral. The identity mapping $I: \ell_1 \to \ell_\infty$ is a p-nuclear operator $(1 < p < \infty)$ which is not nuclear.

2.6. $\mathscr{H}'$ Forms and 2-Nuclear Operators. In this section we show the 2-nuclear operators coincide with the $\mathscr{H}'$ forms introduced by Grothendieck ([6], p. 41). The following theorem is stated in [6] and [1] and proved in [1].

Theorem 2.6.1. *Let E and F be Banach spaces and let $B(x, y)$ be a continuous bilinear form on $E \times F$. Then there exists a unique tensor norm $\mathscr{H}$ in $E \otimes F$ with the following property; the bilinear form $B(x, y)$ belongs to $(E \otimes_{\mathscr{H}} F)'$; with $\|B\|_{\mathscr{H}'} \leqq 1$ if and only if $B(x, y) = (\phi(x), \psi(y))$, where ϕ is a linear mapping of E into a Hilbert space H, with $\|\phi\| \leqq 1$, and ψ is a linear mapping of F into H' with $\|\psi\| \leqq 1$.*

It can be shown ([1], p. 173–174) for $u \in E \otimes F$ that $\|u\|_{\mathscr{H}} \leqq 1$ if and only if there is a representation for $u = \sum_{i=1}^{n} x_i \otimes y_i$ with $\sum_{i=1}^{n} |x'(x_i)|^2 \leqq \|x'\|^2$ and $\sum_{i=1}^{n} |y'(y_i)|^2 \leqq \|y'\|^2$. From this fact we have

Theorem 2.6.2. *Let $u \in E \otimes F'$. Then $\|u\|_{\mathscr{H}} = w_2(u)$.*

A bilinear form $B(x, y')$ on $E \times F'$ is called an $\mathscr{H}'$ form if $B(x, y') \in (E \otimes_{\mathscr{H}} F')'$.

Theorem 2.6.3. *A linear operator T mapping E into F is an $\mathscr{H}'$ form if and only if $T \in N_2(E, F)$.*

2.7. The P-Nuclear Operators of Pietsch and Persson. In [12], Pietsch and Persson have introduced a class of operators called p-nuclear operators which differs from the p-nuclear operators discussed here. A linear operator T is called p-nuclear $(\eta_p(E, F))$ (in the terminology of Pietsch and Persson) if it has a representation $T(x) = \sum_{i=1}^{\infty} a_i(x)\, y_i$, where

$a_i \in E'$, $y_i \in F$ and

$$\left(\sum_{i=1}^{\infty} \|a_i\|^p\right)^{1/p} < \infty,$$

$$\sup_{\|y'\| \leqq 1} \left(\sum_{i=1}^{\infty} |y'(y_i)|^q\right)^{1/q} < \infty.$$

Theorem 2.7.1. *Let* $1 < p < \infty$.

(i) $N_p(\ell_1, \ell_\infty)$ *is not contained in* $\eta_p(\ell_1, \ell_\infty)$.

(ii) *Let* H_1 *and* H_2 *be Hilbert spaces. Then,* $\eta_p(H_1, H_2)$ *is not contained in* $N_p(H_1, H_2)$.

Proof. (i) The identity operator $I: \ell_1 \to \ell_\infty$ is in $N_p(\ell_1, \ell_\infty)$ for $1 < p < \infty$, but it is not compact. However, each operator in $\eta_p(\ell_1, \ell_\infty)$ is compact ([12], p. 24). Therefore, $N_p(\ell_1, \ell_\infty)$ is not contained in $\eta_p(\ell_1, \ell_\infty)$.

(ii) For H_1 and H_2 Hilbert spaces, $\eta_p(H_1, H_2)$ coincides with the Hilbert-Schmidt operators ([12], p. 57). In Section III we show $N_p(H_1, H_2)$ coincides with the nuclear operators. Therefore $\eta_p(H_1, H_2)$ is not contained in $N_p(H_1, H_2)$. □

For H_1 and H_2 Hilbert spaces, the p-nuclear operators discussed here coincide with the nuclear operators. For this reason, it seems appropriate to use the term p-nuclear for the operators discussed here.

III. Operators in $\mathscr{L}_{p\lambda}$ Spaces

In this section we investigate the relationships between the various classes of operators discussed in Section II when either the domain or range is an $\mathscr{L}_{p\lambda}$ space.

3.1. $\mathscr{L}_{p\lambda}$ *Spaces.* Let E and F be Banach spaces. The distance $d(E, F)$ between E and F is defined by $d(E, F) = \inf\{\|T\| \, \|T^{-1}\|\}$, where the infimum is taken over all invertible operators in $\mathscr{L}(E, F)$. The following definition is due to Lindenstrauss and Pełczyński [9].

Definition 3.1.1. Let $1 \leqq p \leqq \infty$ and $1 \leqq \lambda < \infty$. A Banach space is called an $\mathscr{L}_{p\lambda}$ space if for all finite dimensional subspaces $M \subseteqq E$ there exists a finite dimensional subspace N containing M such that $d(N, \ell_p^n) \leqq \lambda$, where $n = \dim(N)$.

It can be shown ([9], p. 283) that every $L_p(\mu)$ space is an $\mathscr{L}_{p\lambda}$ space for all $\lambda > 1$ and every space of type $C(X)$, where X is a compact Hausdorff space, is an $\mathscr{L}_{\infty\lambda}$ space for all $\lambda > 1$. Furthermore every $\mathscr{L}_{p\lambda}$ space is isomorphic to a subspace of an $L_p(\mu)$ space ([9], p. 284).

3.2. Absolutely Q-Summing Operators and Strongly P-Summing Operators in $\mathscr{L}_{p\lambda}$ *Spaces.* In the next few theorems we shall discuss the relationship between the absolutely q-summing operators and the strongly p-summing operators when either the domain or range is an $\mathscr{L}_{p\lambda}$ space. In the following lemma we use the well-known fact that an operator $T: E \to F$ with finite dimensional range can be considered a number of $E' \otimes F$.

Lemma 3.2.1. *Let T have finite dimensional range.*

(i) *For* $1 \leqq p < \infty$, $T \in \pi_p(E, F)$ *and* $\pi_p(T) \leqq g_p(T)$.

(ii) *For* $1 < p \leqq \infty$, $T \in D_p(E, F)$ *and* $D_p(T) \leqq d_q(T)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The essential details for $\pi_2(E, F)$ are given by Pietsch ([17], p. 239). The other cases are similar. □

In the next lemma, ℓ_p^n is n-dimensional Euclidean space (real or complex) with $\|(x_1, \ldots, x_n)\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

Lemma 3.2.2. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $1 < p \leqq \infty$.

(i) *Let T be an operator from ℓ_p^n into E. Then, T belongs to $\pi_q(\ell_p^n, E)$ and $D_p(\ell_p^n, E)$ with $D_p(T) \leqq \pi_q(T)$.*

(ii) *Let T be an operator from E into ℓ_q^n. Then, T belongs to $D_p(E, \ell_q^n)$ and $\pi_q(E, \ell_q^n)$ and $\pi_q(T) \leqq D_p(T)$.*

Proof. (i) Since T has finite dimensional range, we can conclude from the previous lemma that T belongs to $\pi_q(\ell_p^n, E)$ and $D_p(\ell_p^n, E)$. Let $\{e_i\}_{i=1}^n$ be the standard basis for ℓ_p^n. Since T is absolutely q-summing, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} \|Te_i\|^q\right)^{1/q} &\leqq \pi_q(T) \sup_{\substack{\|x'\| \leqq 1 \\ x' \in \ell_q^n}} \left(\sum_{i=1}^{n} |x'(e_i)|^q\right)^{1/q} \\ &\leqq \pi_q(T).\end{aligned}$$

If $x_1, \ldots, x_m$ is a finite set in ℓ_p^n, we can represent each x_j as $x_j = \sum_{i=1}^{n} a_{ij} e_i$. Consequently, if $\{y_j\} \in \ell_q(E')$, we have

$$\begin{aligned}\left|\sum_{j=1}^{m} y_j'(Tx_j)\right| &\leqq \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij} y_j'(Te_i)| \\ &\leqq \begin{cases} \left(\sum_{i,j} |a_{ij}|^p\right)^{1/p} \left(\sum_{i,j} |y_j'(Te_i)|^q\right)^{1/q}, & 1 < p < \infty \\ \sup_{i,j} |a_{ij}| \sum_{i,j} |y_j'(Te_i)|, & p = \infty \end{cases} \\ &\leqq \alpha_p(\{x_j\}) \left\{\sum_i \left(\|Te_i\|^q \left(\sum_j \left|\frac{y_j'(Te_i)}{Te_i}\right|^q\right)\right)\right\}^{1/q} \\ &\leqq \alpha_p(\{x_j\}) \left(\sum_i \|Te_i\|^q\right)^{1/q} \varepsilon_q(\{y_j'\}).\end{aligned}$$

Therefore, using Eq. (3.2.1), we have

$$\left|\sum_j y'_j(Tx_j)\right| \leqq \pi_q(T)\,\varepsilon_q(\{y'_j\})\,\alpha_p(\{x_j\}).$$

Taking the supremum over the unit ball in $\ell_q(F')$, we have $\sigma_p(\{Tx_j\}) \leqq \pi_q(T)\,\alpha_p(\{x_j\})$. From this equation we conclude $D_p(T) \leqq \pi_q(T)$.

(ii) Apply part (i) and Theorem 2.2.2. □

Theorem 3.2.3. *Let* $1 < p \leqq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *Let* E *be an* $\mathscr{L}_{p\lambda}$ *space. Then* $\pi_q(E, F) \subsetneqq D_p(E, F)$ *and* $D_p(T) \leqq \lambda\pi_q(T)$.

(ii) *Let* F *be an* $\mathscr{L}_{q\lambda}$ *space. Then* $D_p(E, F) \subsetneqq \pi_q(E, F)$ *and* $\pi_q(T) \leqq \lambda D_p(T)$.

Proof. (i) Fix $\delta > 0$ and let $x_1, \ldots, x_n$ belong to E. Since E is an $\mathscr{L}_{p\lambda}$ space, there exists a finite dimensional subspace $M \subsetneqq E$ which contains the linear subspace spanned by $x_1, \ldots, x_n$, and an invertible operator $S: \ell_p^m \to M$ $(\dim(M) = m)$, such that $\|S\|\,\|S^{-1}\| \leqq \delta + \lambda$. Consider the following diagram:

$$\begin{array}{ccccc} & & E & \xrightarrow{T} & F \\ & & \uparrow{\scriptstyle I_1} & & \uparrow{\scriptstyle \bar{T}} \\ \operatorname{sp}\{x_1, \ldots, x_n\} & \xrightarrow[I_2]{} & M & \xleftarrow[S]{} & \ell_p^m \end{array}.$$

In this diagram, the operators I_1 and I_2 are the canonical inclusion mappings and the operator $\bar{T}$ is defined by the equation $\bar{T} = T I_1 S$. Since $T \in \pi_q(E, F)$, it follows that $\bar{T} \in \pi_q(\ell_p^m, F)$ and $\pi_q(\bar{T}) \leqq \|S\|\,\|I_1\|\,\pi_q(T) \leqq \|S\|\,\pi_q(T)$. Therefore, using the previous lemma, we see that $\bar{T} \in D_p(\ell_p^m, F)$ and $D_p(\bar{T}) \leqq \pi_q(\bar{T}) \leqq \|S\|\,\pi_q(T)$. If we let $y_j = S^{-1}x_j$ in ℓ_p^m, we have

$$\begin{aligned}\sigma_p(\{Tx_j\}) &= \sigma_p(\{\bar{T}y_j\})\\ &\leqq D_p(\bar{T})\,\alpha_p(\{y_j\})\\ &\leqq \|S\|\,\pi_q(T)\,\alpha_p(\{y_j\}).\end{aligned}$$

Since $x_j = Sy_j$, we obtain

$$\begin{aligned}\alpha_p(\{Tx_j\}) &\leqq \|S\|\,\|S^{-1}\|\,\pi_q(T)\,\alpha_p(\{x_j\})\\ &\leqq (\lambda + \delta)\,\pi_q(T)\,\alpha_p(\{x_j\}).\end{aligned}$$

Therefore, T belongs to $D_p(E, F)$ and $D_p(T) \leqq (\lambda + \delta)\,\pi_q(T)$. Since this expressions holds for all $\delta > 0$, it follows that $D_p(T) \leqq \pi_q(T)\,\lambda$.

Part (ii) is proved in a similar way ([3], p. 96–97). □

In a similar way one can prove

Theorem 3.2.4. *Let* $1 < p < \infty$ *and let* F *be an* $\mathscr{L}_{p\lambda}$ *space. Then the identity mapping*

$$I \otimes I : E \otimes_{d_q} F \to E \otimes_{g_p} F$$

is continuous with $\|I \otimes I\| \leqq \lambda$.

Theorem 3.2.5. *Let $1<p<\infty$ and let E be an $\mathscr{L}_{p\lambda}$ space. Then, the norms $g_p(u)$, $d_q(u)$ and $\|u\|_p$ are equivalent in $\ell_p \otimes E$.*

3.3. P-Nuclear and Integral Operators in $\mathscr{L}_{p\lambda}$. Spaces. In the next few theorems we shall investigate the relationship between the p-nuclear and integral operators when the spaces considered are of type $\mathscr{L}_{p\lambda}$. We shall need the following lemma.

Lemma 3.3.1. *Let $1<p<\infty$. Let E be a Banach space and F a finite dimensional Banach space. Furthermore, suppose there exists an isomorphism $S: F \to \ell_p^n$ ($\dim F = n$) such that $\|S\| \, \|S^{-1}\| \leqq C$. Then for $u \in E \otimes F$, we have $\|u\|_\varepsilon \leqq w_p(u) \leqq \|u\|_\varepsilon C$.*

Proof. Using Hölder's inequality, one can easily show $\|u\|_\varepsilon \leqq w_p(u)$. We shall show $w_p(u) \leqq C\|u\|_\varepsilon$. Let $u = \sum_{i=1}^{m} x_i \otimes y_i$ belong to $E \otimes F$ and let S be an isomorphism from F to ℓ_p^n with $\|S\| \, \|S^{-1}\| \leqq C$. If $\{e_j\}$ is the standard basis for ℓ_p^n, we can represent $S(y_i) = \sum_{j=1}^{n} a_{ij} e_j$, for $i = 1, 2, \ldots, m$. If $I: E \to E$ is the identity mapping, we have

$$\begin{aligned}(I \otimes S)(u) &= \sum_{i=1}^{m} x_i \otimes S\, y_i \\ &= \sum_{i=1}^{m} \left(x_i \otimes \left(\sum_{j=1}^{n} a_{ij} e_j\right)\right) \\ &= \sum_{j=1}^{n} u_j \otimes e_j,\end{aligned}$$

where $u_j = \sum_{i=1}^{m} a_{ij} x_i$. Now, since $\|I \otimes S^{-1}\| \leqq \|I\| \, \|S^{-1}\|$ and since $u = (I \otimes S^{-1})(I \otimes S)(u)$, it follows that

$$\begin{aligned}w_p(u) &\leqq w_p((I \otimes S^{-1})(I \otimes S)(u)) \\ &\leqq \|S^{-1}\| \, \|I\| \, w_p((I \otimes S)(u)).\end{aligned}$$

Therefore, by definition of the w_p norm,

$$\begin{aligned}w_p(u) &\leqq \|S^{-1}\| \, \varepsilon_p(\{u_j\}) \, \varepsilon_q(\{e_j\}) \\ &\leqq \|S^{-1}\| \sup_{\substack{\|x'\| \leqq 1 \\ x' \in E'}} \left(\sup_{\substack{\|y'\| \leqq 1 \\ y' \in \ell_q^n}} \left(\sum_{j=1}^{n} |x'(u_j)|^p\right)^{1/p} \left(\sum_{j=1}^{n} |y'(e_j)|^q\right)^{1/q} \right).\end{aligned} \tag{3.3.1}$$

We observe that

$$\left(\sum_{j=1}^{n} |x'(u_j)|^p\right)^{1/p} = \sup_{\substack{\|y'\| \leqq 1 \\ y' \in \ell_q^n}} \left| \sum_{j=1}^{n} x'(u_j)\, y'(e_j) \right|. \tag{3.3.2}$$

Combining Eqs. (3.3.1) and (3.3.2) we obtain

$$\begin{aligned} w_p(u) &\leqq \|S^{-1}\| \sup_{\substack{\|x'\|\leqq 1 \\ \|y'\|\leqq 1}} \left|\sum_{j=1}^{n} x'(u_j)\, y'(e_j)\right| \\ &= \|S^{-1}\| \, \|(I\otimes S)\, u\|_\varepsilon \\ &\leqq \|S^{-1}\| \, \|S\| \, \|u\|_\varepsilon \leqq C\|u\|_\varepsilon . \end{aligned}$$

Therefore, $w_p(u) \leqq C\|u\|_\varepsilon$. □

Theorem 3.3.2. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose either E is an* $\mathscr{L}_{q\lambda}$ *space or F is an* $\mathscr{L}_{p\lambda}$ *space. Then, for* $u \in E\otimes F$, *we have* $\|u\|_\varepsilon \leqq w_p(u) \leqq \|u\|_\varepsilon \lambda$.

Proof. The inequality $\|u\|_\varepsilon \leqq w_p(u)$ follows from Hölder's inequality; we prove $w_p(u) \leqq \|u\|_\varepsilon \lambda$. Let F be an $\mathscr{L}_{p\lambda}$ space and let $\delta > 0$. If M is a finite dimensional subspace of E and N is a finite dimensional subspace of F, the symbol $w_p(u)_{M\otimes N}$ will denote the w_p norm on $M\otimes N$. From the definition of the w_p norm it follows that

$$w_p(u) = \inf w_p(u)_{M\otimes N}$$

where the infimum is taken over all finite dimensional subspaces $M \subsetneqq E$ and $N \subsetneqq F$ such that $u \in M\otimes N$. Since F is of type $\mathscr{L}_{p\lambda}$, for each finite dimensional subspace N, there exist finite dimensional $N' \supsetneqq N$ such that $d(N', \ell_p^{n'}) \leqq \lambda$, where $n' = \dim(N')$. Furthermore, with each subspace N', there is an isomorphism $S_{N'}: N' \to \ell_p^{n'}$ such that $\|S_{N'}\| \, \|S_{N'}^{-1}\| \leqq \lambda + \delta$. Therefore, by the previous lemma,

$$w_p(u)_{M\otimes N'} \leqq \|u\|_\varepsilon(\lambda+\delta). \tag{3.3.2}$$

From the definition for w_p, we have

$$w_p(u) = \inf_{u\in M\otimes N} w_p(u)_{M\otimes N} \leqq \inf_{u\in M\otimes N'} w_p(u)_{M\otimes N'} .$$

However, since $N \subsetneqq N'$, we have

$$w_p(u)_{M\otimes N} \geqq w_p(u)_{M\otimes N'} .$$

Therefore, we can conclude that

$$w_p(u) = \inf_{M\otimes N'} w_p(u)_{M\otimes N'} .$$

Using Eq. (3.3.2), we have

$$\begin{aligned} w_p(u) &= \inf_{M\otimes N'} w_p(u)_{M\otimes N'} \\ &\leqq \|u\|_\varepsilon(\lambda+\delta). \end{aligned}$$

Since δ was arbitrary, we see $w_p(u) \leqq \|u\|_\varepsilon \lambda$. If E is of type $\mathscr{L}_{q\lambda}$, the proof follows in a similar way. □

Using Lemma 2.5.1 and the previous theorem we have

Theorem 3.3.3. *Suppose either E is an $\mathscr{L}_{q\lambda}$ space or F' is an $\mathscr{L}_{p\lambda}$ space. Then, the integral and p-nuclear operators coincide and* $\frac{1}{\lambda} J(T) \leqq N_p(T) \leqq J(T)$.

Since every Hilbert space is an $\mathscr{L}_{2\lambda}$ space for all $\lambda > 1$ we have

Theorem 3.3.4. *Let E or F be Hilbert space.*

(i) *For $u \in E \otimes F$, $\|u\|_\varepsilon = w_2(u)$.*

(ii) *The integral and 2-nuclear operators coincide and $J(T) = N_2(T)$.*

IV. Applications and Open Questions

In this final section we present a number of interesting applications and open questions related to the present work.

4.1. Characterization of Nuclear and Hilbert-Schmidt Operators. Let E and F be Hilbert spaces. An operator T mapping E into F is a Hilbert-Schmidt operator if for each orthonormal basis $\{e_\alpha\}$ in E, $\sum_\alpha \|Te_\alpha\|^2 < \infty$. Pełczyński [10] has shown T is Hilbert-Schmidt if and only if $T \in \pi_p(E, F)$ for $1 \leqq p < \infty$. By taking conjugates we have

Theorem 4.1.1. *An operator T is Hilbert-Schmidt if and only if $T \in D_p(E, F)$ for $1 < p \leqq \infty$.*

An operator T is nuclear if and only if $T(x) = \sum_{n=1}^{\infty} \lambda_n (x \cdot e_n) f_n$. In this representation $\{e_n\}$ is an orthonormal set in E, $\{f_n\}$ is an orthonormal set in F, $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Modifying Pełczyński's methods we obtain

Theorem 4.1.2. *Let $1 < p < \infty$. If E and F are Hilbert spaces, an operator T is p-nuclear if and only if it is nuclear.*

Proof. From Theorem 2.5.3, we know every nuclear operator is p-nuclear; we show the opposite inclusion. Let $T \in N_p(E, F)$. Since E and F are Hilbert spaces, T is compact; therefore, there exist orthonormal sets $\{e_n\}$ in E and $\{f_n\}$ in F such that $T(x) = \sum_{n=1}^{\infty} \lambda_n (x \cdot e_n) f_n$, where $\lambda_n \geqq 0$ and $\lim_{n \to \infty} \lambda_n = 0$. We must show $\sum_{n=1}^{\infty} \lambda_n < \infty$. Let $R_n(t)$ be the n^{th} Rademacher function defined by

$$R_n(t)=\begin{cases}(-1)^k, & \frac{k}{2^n}<t<\frac{k+1}{2^n}\\ 0, & t=\frac{k}{2^n}, \quad t=1\end{cases}$$

where $k=0, 1, 2, \ldots, 2^n-1$ and n is a positive integer. For a fixed positive integer m, define

$$x_k=\sum_{n=1}^{m} R_n\left(\frac{2k+1}{2^{m+1}}\right)e_n,$$

and

$$y'_k=\sum_{n=1}^{m} R_n\left(\frac{2k+1}{2^{m+1}}\right)f_n,$$

where $k=0, 1, \ldots, 2^m-1$. Pełczyński ([10], p. 358, Eq. 6) has shown for $x'\in E'$

$$\left(\sum_{k=0}^{2^m-1}|x'(x_k)|^p\right)^{1/p}\leqq 2^{m/p}B_p\|x'\|,$$

where B_p is a constant depending only on p. Therefore

$$\varepsilon_p(\{x_k\})\leqq 2^{m/p}B_p. \tag{4.1.1}$$

In a similar way

$$\varepsilon_q(\{y'_k\})\leqq 2^{m/q}B_q. \tag{4.1.2}$$

Now by direct computation we have

$$\begin{aligned}\left|\sum_{k=0}^{2^m-1}(y'_k\cdot Tx_k)\right| &= \left|\sum_{k=0}^{2^m-1}\left(\sum_{n=1}^{m} R_n\left(\frac{2k+1}{2^{m+1}}\right)f_n\cdot\sum_{n=1}^{m} R_n\left(\frac{2k+1}{2^{m+1}}\right)\lambda_n f_n\right)\right|\\ &= \left|\sum_{k=0}^{2^m-1}\left(\sum_{n=1}^{m} R_n^2\left(\frac{2k+1}{2^{m+1}}\right)\lambda_n\right)\right|\\ &= \sum_{k=0}^{2^m-1}\left(\sum_{n=1}^{m}\lambda_n\right)=2^m\left(\sum_{n=1}^{m}\lambda_n\right).\end{aligned}$$

From Eq. (4.1.1) and (4.1.2) and the fact that T is p-nuclear we obtain

$$\begin{aligned}2^m\left(\sum_{n=1}^{m}\lambda_n\right) &= \left|\sum_{k=0}^{2^m-1}(y'_k\cdot Tx_k)\right|\\ &\leqq \varepsilon_q(\{y'_k\})\,\varepsilon_p(\{x_k\})\,N_p(T)\\ &\leqq 2^{m/q}B_q 2^{m/p}B_pN_p(T).\end{aligned}$$

Therefore, for each m

$$\sum_{n=1}^{m}\lambda_n\leqq B_qB_pN_p(T).$$

Since m is arbitrary we can conclude $\sum_{n=1}^{\infty} \lambda_n < \infty$ which shows T is nuclear. □

It was mentioned earlier that the definition of an absolutely 1-summing operator can be traced back to the early work of Grothendieck. In Grothendieck's original formulation an operator T is called *semi-intégrale à droite* if the mapping

$$I \otimes T : \ell_1 \otimes_\varepsilon E \to \ell_1 \otimes_\pi F \tag{4.1.3}$$

is continuous. It is well-known that $T \in \pi_1(E, F)$ if and only if T is semi-intégrale à droite. Therefore, for E and F Hilbert spaces, Eq. (4.1.3) characterizes the Hilbert-Schmidt operators. For the case $p > 1$, we apply Theorems 2.1.3 and 4.1.2 and obtain the following result.

Theorem 4.1.3. *Let E and F be Hilbert spaces. A mapping T is nuclear if and only if the mapping $I \otimes T : \ell_p \otimes_\varepsilon E \to \ell_p \otimes_\pi F$ is continuous for $1 < p < \infty$.*

4.2. Characterization of Inner Product Spaces. A normed linear space E is an inner product space if there is an inner product defined in E such that $\|x\|^2 = (x \cdot x)$. In [4] the author has proven the following result.

Theorem 4.2.1. *Let E be a normed space. Then E is an inner product space if and only if for all Banach spaces F and for all absolutely 2-summing operators T mapping E into F, the conjugate operator T' is absolutely 2-summing and $\pi_2(T') \leqq \pi_2(T)$.*

Kwapień [8], has given a similar characterization of spaces isomorphic to inner product spaces. Using the operators discussed here, Theorem 4.2.1 can be restated as follows.

Theorem 4.2.2. *Let F be a normed linear space. The following statements are equivalent:*

(i) *E is an inner product space.*

(ii) *For all Banach space F, $\pi_2(E, F) \subsetneq D_2(E, F)$ and $D_2(T) \leqq \pi_2(T)$.*

(iii) *For all Banach spaces F, $D_2(F, E) \subsetneq \pi_2(F, E)$ and $\pi_2(T) \leqq D_2(T)$.*

(iv) *Let $u \in \ell_2 \otimes E$. Then $g_2(u) = d_2(u) = \|u\|_2$ ($\|u\|_2$ is the norm induced on $\ell_2 \otimes E$ as a subspace of $\ell_2\{E\}$.)*

Problem. Let $u \in E \otimes F$. According to Theorem 3.3.4, $\|u\|_\varepsilon = w_2(u)$ when either E or F is Hilbert space. It is interesting to consider the converse question. Suppose E and F are Banach spaces and for $u \in E \otimes F$, $\|u\|_\varepsilon = w_2(u)$. Does this imply either E or F is Hilbert space?

4.3. Extension Theorems for Absolutely P-Summing Operators. Saphar ([18], p. 134) has proven the following: Let E and F be Banach spaces and let M be a closed subspace of E. Then, the canonical mapping

$M \otimes_{d_2} F$ into $E \otimes_{d_2} F$ is an isometry. This result can be restated in terms of absolutely 2-summing operators as follows:

Theorem 4.3.1. ([3], p. 63). *Let $T \in \pi_2(M, F)$. Then, there exists an extension T_e of T which is an absolutely 2-summing operator mapping E into F'' with $\pi_2(T) = \pi_2(T_e)$.*

Using the results from Sections II and III we can show the analogous result is not true for absolutely 1-summing operators. Indeed, if such a result were true every absolutely 1-summing operator T mapping a Banach space E into a reflexive Banach space F would have a conjugate T' which is absolutely 1-summing. The examples discussed in Theorem 2.4.2 show this is certainly not the case. We shall need the following theorem which is also of independent interest.

Theorem 4.3.2. *Let T be an absolutely 1-summing operator mapping $C(X)$ (X compact, Hausdorff) into F. Then T' is absolutely 1-summing and $\pi_1(T') \leqq \pi_1(T)$.*

Proof. The space $C(X)$ is an $\mathscr{L}_{\infty\lambda}$ space for all $\lambda > 1$. The result follows from Theorems 2.2.2 and 3.2.3. □

To show Theorem 4.3.1 is not valid for absolutely 1-summing operators we argue as follows. Let E be a Banach space, F a reflexive Banach space and let $T \in \pi_1(E, F)$. Let us assume an extension theorem were valid. Let T_e be an extension of T which maps $C(S_{E'})$ into F and let I be the canonical embedding of E into $C(S_{E'})$. By Theorem 4.3.2, T_e' is absolutely 1-summing and $\pi_1(T_e') \leqq \pi_1(T_e)$. Therefore, since $T' = I' T_e'$, it follows immediately that T' is absolutely 1-summing and $\pi_1(T') \leqq \pi_1(T)$. However, if $E = \ell_1$ and $F = \ell_2$, the canonical embedding $I: \ell_1 \to \ell_2$ is absolutely 1-summing, but its conjugate I' is not absolutely 1-summing. Therefore, an extension T_e does not exist.

Problem. Does there exist an extension theorem for absolutely p-summing operators $1 < p < 2$, $2 < p < \infty$?

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References

1. Amemiya, I., Shiga, K.: On tensor products of Banach spaces. Kodai Math. Seminar Reports **9**, 161—176 (1957).
2. Chevet, S.: Sur certains produits tensoriels topologiques d'espaces de Banach. C. R. Acad. Sc. Paris, t. **266**, 1968.
3. Cohen, J.: Absolutely p-summing, p-nuclear operators and their conjugates. Dissertation, Univ. of Md., College Park, Md., Jan. 1970.

4. Cohen, J.: A Characterization of inner product spaces using absolutely 2-summing operators. Studia Math. **38**, 271—276 (1970).
5. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Memoires Amer. Math. Soc. **16** (1955).
6. Grothendieck, A.: Résumé de la théorie métrique des produits tensoriels topologiques. Bol. Soc. Mat. São Paulo **8**, 1—79 (1956).
7. Grothendieck, A.: Sur certaines classes de suites dans les espaces de Banach et le théorème de Dvoretzky-Rogers. Boletim Soc. Mat. São Paulo **8**, 81—110 (1956).
8. Kwapień, S.: A linear topological characterization of inner-product spaces. Studia Math. **38**, 271—278 (1970).
9. Lindenstrauss, J., Pełczyński, A.: Absolutely summing operators between $\mathscr{L}_p$-spaces. Studia Math. **29**, 275—326 (1968).
10. Pełczyński, A.: A characterization of Hilbert-Schmidt operators. Studia Math. **28**, 355—360 (1967).
11. Persson, A.: On some properties of p-integral and p-nuclear operators. Studia Math. **33**, 213—222 (1969).
12. Persson, A., Pietsch, A.: P-nukleare und p-integrale Abbildungen in Banachräumen. Studia Math. **33**, 19—62 (1969).
13. Pietsch, A.: Absolut summierende Abbildungen in lokalkonvexen Räumen. Math. Nachr. **27**, 77—103 (1963).
14. Pietsch, A.: Nukleare Lokalkonvexe Räume. Berlin: Akademie-Verlag 1965.
15. Pietsch, A.: Quasinukleare Abbildungen in normierten Räumen. Studia Math. **28**, 79—90 (1967).
16. Pietsch, A.: Absolut p-summierende Abbildungen in normierten Räumen. Studia Math. **28**, 333—353 (1967).
17. Pietsch, A.: Hilbert-Schmidt Abbildungen in Banach Räumen. Math. Nachr. **37**, 237—245 (1968).
18. Saphar, P.: Applications à puissance nucléaire et applications Hilbert-Schmidt dans les espaces de Banach. Ann. Ecole Sup. **84**, 113—152 (1966).
19. Saphar, P.: Produits tensoriels topologiques et classes d'applications linéaires. C.R. Acad. Sc. Paris **266**, 526—528 (1968).
20. Saphar, P.: Comparison de Normes des produits tensoriels d'espaces de Banach. Applications., C. R. Acad. Sc. Paris, **266** (1968).
21. Schaefer, H. H.: Topological vector spaces. New York: Macmillan 1966.
22. Treves, F.: Topological vector spaces, distributions and kernels. New York: Academic Press 1967.

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