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Characterizations and Metrization of Proper Analytic Spaces

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1. Introduction

All topological spaces considered will be completely regular Hausdorff spaces. The word space will refer to such a topological space. A proper map from a space X into a space Y is a closed continuous map with compact point inverses. An analytic subset of a Polish (separable completely metrizable) space is one which is either empty or the image of the Baire 0-dimensional product space N^N, where $N=\{1, 2, \ldots\}$.

Definition. A proper analytic space is one which admits a proper map onto an analytic subset of a Polish space. A proper Borel space is one which admits a proper map onto a Borel subset of a Polish space.

Characterizations of proper analytic and proper Borel spaces are given in Theorems 1 and 2, respectively. Necessary and sufficient conditions for the metrizability of such spaces are given in Theorem 3, and an application to the metrization of compact convex subsets of Hausdorff locally convex real topological vector spaces in terms of the topological and Baire set structures of their sets of extreme points is given. In particular, as part of Theorem 4, we prove: If X is a compact convex set whose set of extreme points $\mathscr{E}(X)$ is a proper analytic space, then X is metrizable if and only if $\mathscr{E}(X)$ with its algebra of Baire subsets is a standard Borel space.

By the Baire sets of a space we mean the smallest family of sets containing the zero sets of continuous real-valued functions (that is, of the form $Z(f)=\{x: f(x)=0\}$) and closed under countable unions and complementation. By the Borel sets of a space we mean the smallest family containing the closed sets and closed under countable unions and complementation. For a metrizable space the Baire and Borel sets coincide.

A measurable space is a pair $(X, \mathscr{H})$ where X is a set and $\mathscr{H}$ is a family of subsets closed under countable unions and complementation. A standard Borel space is a measurable space which is measurably isomorphic to a measurable space $(X, \mathscr{B})$, where X is a Polish space and $\mathscr{B}$ is its algebra of Borel sets. The word standard derives from the fact that two Polish spaces are Borel isomorphic if and only if they have the same cardinality (which must be finite, countable, or that of the continuum). Note that Borel isomorphic means that there exists a bijective point map which takes Borel sets to Borel sets in both directions. The reader is referred to [2] for an exposition of the theory of standard Borel spaces and their applications, to [14] for the classical theory of analytic sets, and to [8] for a survey of the recent theory of non metrizable analytic sets.

Proper analytic spaces have been referred to as ZS-spaces by the author [12] and proper Borel spaces coincide with Frolík's bianalytic spaces ([5, 6, 8].)

2. Characterizations of Proper Analytic Spaces

Let $\mathscr{H}$ be a family of subsets of a set X. The Souslin-$\mathscr{H}$ subsets of X are the sets admitting a representation of the form

$$\bigcup_{\sigma\in N^N} \bigcup_{s<\sigma} H_s, \quad H_s \in \mathscr{H}$$

where $s<\sigma$ means that s is a finite restriction of the sequence of integers σ. The space of bounded continuous real-valued functions on a space X is denoted by $C^*(X)$, the family of zero sets of functions $f\in C^*(X)$ by $\mathscr{Z}(X)$, and the family of closed sets of X by $\mathscr{F}(X)$. The Stone-Čech compactification of a space X is denoted by βX. If

$$f\colon X \to Y$$

is a function and $A\subseteq X$ then $f|_A$ will denote the restriction of f to A.

A space X is defined to be analytic if it is a Souslin-$\mathscr{F}(\beta X)$ subset of βX. It is the case that a subset of a Polish space is analytic in the classical sense if and only if it is analytic in this sense ([3, 8]). The continuous image of an analytic space is analytic ([3, 8]), the countable product of analytic spaces is analytic ([5, 8]), and every analytic space is Lindelöf [8].

Theorem 1. *For any space X the following are equivalent:*

1) *X is proper analytic,*

2) *X is a Souslin-$\mathscr{Z}(\beta X)$ subset of βX,*

3) *X is homeomorphic to a Souslin-$\mathscr{Z}(K)$ subset of some compact Hausdorff space K,*

4) *X is homeomorphic to a closed subset of a product space $K\times A$ for some compact Hausdorff space K and an analytic subset A of some Polish space.*

The following properties of proper maps will be used:

1) If $\{f_\gamma: \gamma\in\Gamma\}$ is a class of proper maps (Γ denotes an index set) of spaces X_γ onto spaces Y_γ respectively, then the map

$$\Phi\colon \prod_{\gamma\in\Gamma} X_\gamma \to \prod_{\gamma\in\Gamma} Y_\gamma$$

defined by

$$\Phi(\{x_\gamma\colon \gamma\in\Gamma\}) = \{f_\gamma(x_\gamma)\colon \gamma\in\Gamma\}$$

is a proper map [16, p. 297].

2) If $f\colon X\to Y$ and $g\colon Y\to W$ are continuous maps on the respective spaces and

$$g\circ f\colon X\to W$$

is a proper map, then both f and g are proper maps [13, p. 1583].

An immediate consequence of this result is

3) If $\{f_\gamma: \gamma\in\Gamma\}$ is a class of continuous maps from a space X into spaces Y_γ respectively, at least one of which is proper, then the map

$$\Phi\colon X\to \prod_{\gamma\in\Gamma} Y_\gamma$$

defined by

$$\Phi(x) = \{f_\gamma(x)\colon \gamma\in\Gamma\}$$

is a proper map.

Proof of Theorem 1. 1)⇒2) Let f be a proper map of X onto a metrizable analytic space A. Let $\tilde{A}$ denote a metrizable compactification of A and

$$\hat{f}\colon \beta X\to \tilde{A}$$

the Stone extension of f. Then, since f is proper,

$$\hat{f}^{-1}\circ\hat{f}[X] = X.$$

Since A is analytic in the metric space $\tilde{A}$, it has a representation of the form

$$A = \bigcup_{\sigma\in N^N} \bigcap_{s<\sigma} F_s, \qquad F_s\in\mathscr{F}(\tilde{A}).$$

For each s let $f_s\in C^*(\tilde{A})$ be such that $F_s = Z(f_s)$. Then

$$X = \bigcup_{\sigma\in N^N} \bigcap_{s<\sigma} Z(f_s\circ\hat{f})$$

and for each s $f_s\circ\hat{f}\in C^*(\beta X)$.

2)⇒3) Trivial.

3)⇒4) Suppose X is homeomorphic to a Souslin-$\mathscr{Z}(K)$ subset X' of a compact space K. Then X' is analytic ([3], [8]) and has a representation of the form

$$X' = \bigcup_{\sigma\in N^N} \bigcap_{s<\sigma} Z(f_s), \qquad f_s\in C^*(K).$$

Reindex the countable set $\{f_s: \sigma\in N^N, s<\sigma\}$ by $\{f_n: n=1,2,\ldots\}$ and define

$$\Phi\colon K\to R^N$$

by

$$\Phi(x)=(f_1(x), f_2(x), \ldots).$$

Then

$$\Phi^{-1}\circ\Phi[X']=X'$$

and, since Φ is continuous, $\Phi[X']$ is an analytic subset of R^N. The map

$$g\colon X'\to K\times\Phi[X']$$

definded by

$$g(x)=(x, \Phi(x))$$

is a homeomorphism of X' into $K\times\Phi[X']$.

Let

$$h\colon K\times\Phi[X']\to\Phi[K]\times\Phi[X']$$

be defined by

$$h(x, y)=(\Phi(x), y).$$

Since the set

$$\Delta=\{(y, y): y\,\Phi[X']\}$$

is closed in $\Phi[K]\times\Phi[K']$ and since $h^{-1}[\Delta]=g[X']$, we have that $g[X']$ is closed in $K\times\Phi[X']$. This completes the argument since X is homeomorphic to $g[X']$.

4)⇒1) Suppose X is homeomorphic to a closed subset X' of $K\times A$ where K is compact and A is a metrizable analytic space. Let $\tilde{A}$ denote a metrizable compactification of A and

$$A=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(g_s),\qquad g_s\in C^*(\tilde{A}),$$

be a representation of A in $\tilde{A}$. Then

$$K\times A=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}K\times Z(g_s)$$

is a Souslin-$\mathscr{Z}(K\times\tilde{A})$ representation of $K\times A$, since each

$$K\times Z(g_s)=Z(h_s)$$

where $h_s\in C^*(K\times\tilde{A})$ is defined by

$$h_s(x, y)=g_s(y).$$

Let $\overline{X'}$ denote the closure of X' in $K\times\tilde{A}$. Consider the functions h_s as being restricted to $\overline{X'}$ and reindex them by $\{h_n: n=1,2,\ldots\}$.

Define

$$\Phi\colon \overline{X'}\to R^N$$

by

$$\Phi(x)=(h_1(x), h_2(x), \ldots).$$

Then $\Phi|_{X'}$ is a proper map and $\Phi[X']$ is an analytic set of the metric space R^N (since a closed subset of an analytic space is analytic and the continuous image of an analytic space is analytic).

Remarks. 1) Frolík [11, Theorem 2] has shown that an analytic subspace A of a space X is a Souslin-$Z(X)$ set if and only if there exists a continuous map f of X onto a separable metric space such that

$$f[A] \cap f[X \setminus A] = \emptyset .$$

The equivalence of 1), 2) and 3) in Theorem 1 is an elaboration of this observation.

2) The implication 1)⇒2) in Theorem 1 implies that proper analytic spaces are in fact analytic.

3) The fourth part of Theorem 1 implies that the class of proper analytic spaces is the smallest class of spaces containing all compact spaces and all analytic subsets of Polish spaces which is closed under the operations of taking finite products and passing to closed subspaces.

4) The proper analytic spaces form a strictly smaller class of spaces than the analytic spaces. The integers plus one point from its Stone-Čech compactification is an analytic non proper analytic space.

Frolík has extensively studied the class of proper Borel spaces. He has proved

Theorem [8]. *For any space X the following are equivalent:*

1) *X is a proper Borel space,*

2) *X is a Baire subset of its Stone-Čech compactification,*

3) *X is homeomorphic to a Baire subset of some compact space,*

4) *Both X and* $(\beta X) \setminus X$ *are analytic spaces.*

The name bianalytic, used by Frolík, derives from part 4 of this theorem.

Theorem 2. *For any space X the following are equivalent:*

1) *X is a proper Borel space,*

2) *X has a disjoint representation of the form*

$$X = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z(f_s), \qquad f_s \in C^*(\beta X),$$

(*Disjoint means that for* $\sigma \neq \sigma'$ *we have* $\bigcap_{s<\sigma} Z(f_s) \cap \bigcap_{s<\sigma'} Z(f_s) = \emptyset$.)

3) *X is homeomorphic to a closed subset of a product space* $K \times B$ *for some compact Hausdorff space K and a Borel subset B of some Polish space.*

Proof. 1)⇒2) First note that a subset C of a Polish space has a disjoint representation

$$C=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(g_s),\qquad g_s\in C^*(Y),$$

if and only if it is a Borel subset [10, p. 210].

Let f be a proper map of X onto a Borel subset of a Polish space and let $\tilde{B}$ be a metrizable compactification of B. Let

$$\hat{f}\colon \beta X\to\tilde{B}$$

be the Stone extension of f and let

$$B=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(g_s),\qquad g_s\in C^*(\tilde{B})$$

be a disjoint representation of B in $\tilde{B}$. Then

$$X=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(g_s\circ\hat{f}),\qquad g_s\circ\hat{f}\in C^*(\beta X)$$

is a disjoint representation of X in βX.

2)⇒1) Suppose X has a disjoint representation

$$X=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(f_s),\qquad f_s\in C^*(\beta X).$$

Reindex $\{f_s\colon \sigma\in N^N, s<\sigma\}$ by $\{f_n\colon n=1,2,\ldots\}$ and define

$$\Phi\colon \beta X\to R^N$$

by

$$\Phi(x)=(f_1(x),f_2(x),\ldots).$$

Then $\Phi|_X$ is a proper map and $\Phi[X]$ is a Borel set in R^N since it has the dispoint representation

$$\Phi[X]=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(f_s\circ\Phi^{-1}),\qquad f_s\circ\Phi^{-1}\in C^*(\Phi[\beta X]).$$

1)⇒3) Let f be a proper map of X onto a Borel subset B of a Polish space. Then as in the proof of 3)⇒4) of Theorem 1 we obtain that X is homeomorphic to a closed subspace of $(\beta X)\times B$.

3)⇒2) Suppose X is homeomorphic to a closed subset X' of $K\times B$ for a compact space K and Borel subset B of a Polish space. Let $\tilde{B}$ be a metrizable compactification of B and

$$B=\bigcup_{\sigma\in N^N}\bigcap_{s<\sigma}Z(f_s),\qquad f_s\in C^*(\tilde{B})$$

a disjoint representation of B in $\tilde{B}$.

Let $\overline{X'}$ denote the closure of X' in $K \times \tilde{B}$. Then

$$X' = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z(f_s|_{\overline{X'}}), \quad f_s|_{\overline{X'}} \in C^*(\overline{X'})$$

is a disjoint representation of X' in $\overline{X'}$.

Let

$$\hat{\imath}\colon \beta X' \to \overline{X'}$$

be the Stone extension of the identity map of X' onto itself. Then

$$X' = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z(g_s),$$

$$g_s = f_s|_{\overline{X'}} \circ \hat{\imath} \in C^*(\beta X'),$$

is a disjoint representation of X' in $\beta X'$.

Remark. The techniques used in proving the equivalence of 1) and 2) in Theorem 2 also prove:

If X is a proper Borel space and A is a Souslin-$Z(X)$ set, then A has a representation

$$A = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z(f_s), \quad f_s \in C^*(X)$$

with disjoint summands if and only if A is a Baire subset of X.

3. Metrization of Proper Analytic Spaces

We will need the following results:

A. (Frolík [9].) If X is a Lindelöf space and $\mathscr{A}$ is an algebra of bounded continuous real-valued functions on X which generate the topology of X, then every $f \in C^*(X)$ is the pointwise limit of some sequence in $\mathscr{A}$.

B. (Okuyama [18].) If X admits a proper map onto a metrizable space and the diagonal in $X \times X$ is a G_δ set, then X is metrizable.

A family of subsets $\mathscr{H}$ of a space X is said to be point countable if each point of X is contained in at most countable many members of $\mathscr{H}$.

C. (Nagata [17].) If X admits a proper map onto a metrizable space and has a point countable collection $\mathscr{U}$ of open sets such that for each pair $x, y \in X$, $x \neq y$, there is a $U \in \mathscr{U}$ such that $x \in U$ and $y \notin U$, then X is metrizable.

Theorem 3. *For a proper analytic space X the following are equivalent:*

1) *X is metrizable,*

2) *$X \times X$ is perfectly normal (that is, normal and $\mathscr{F}(X \times X) = \mathscr{Z}(X \times X)$),*

3) *The diagonal in* $X \times X$ *is a Souslin-*$\mathscr{Z}(X \times X)$ *set,*

4) X *is Baire isomorphic to a metrizable space,*

5) *The family of Souslin-*$\mathscr{Z}(X)$ *sets is countably generated; that is, there exists a sequence* $\{Z_n: n=1,2,\ldots\}$ *of zero sets such that the Souslin-*$\mathscr{Z}(X)$ *sets are the smallest family of sets containing this sequence and closed under the Souslin operation.*

Proof. Clearly 1)⇒2)⇒3) and 1)⇒4). 1)⇒5) since a metrizable analytic space has a countable base. It suffices to demonstrate that 4)⇒3)⇒1) and 5)⇒1).

4)⇒3) We first prove that the Baire sets in $X \times X$ coincide with the smallest family $\mathscr{H}$ of subsets closed under countable unions and countable intersections such that for each projection

$$\pi_n: X \times X \to X, \quad n=1,2$$

onto the first and second coordinates and for each Baire set B in X

$$\pi_n^{-1}[B] \in \mathscr{H}.$$

Since $X \times X$ is analytic, it is Lindelöf. Thus by result A above every $f \in C^*(X \times X)$ is the pointwise limit of a sequence of functions from

$$\mathscr{A} = \{g|_{X \times X}: g \in C^*(\beta X \times \beta X)\}.$$

Therefore the smallest family of functions containing $\mathscr{A}$ and closed under pointwise sequential convergence is the space of all real-valued Baire functions on $X \times X$.

Since each $g \in C^*(\beta X \times \beta X)$ is the uniform limit of polynomials in

$$\{h \circ \pi_n^\beta\}, \quad n=1,2$$

where π_n^β is the projection on the n-th coordinate of $\beta X \times \beta X$ and $h \in C^*(\beta X)$ (Stone-Weirstrass theorem),

$$(g|_{X \times X})^{-1}[B] \in \mathscr{H}$$

for every Baire set B in the real line R. Now since every Baire function on $X \times X$ is obtained from these functions by iterating pointwise sequential limits, $\phi^{-1}[B] \in \mathscr{H}$ for every Baire function ϕ and every Baire set B in R. Therefore $\mathscr{H}$ contains the family of Baire sets of $X \times X$.

On the other hand, for each projection

$$\pi_n: X \times X \to X, \quad n=1,2$$

and each Baire set in X we have that $\pi^{-1}[B]$ is a Baire set in $X \times X$, since π_n is continuous. Therefore $\mathscr{H}$ coincides with the family of Baire sets of $X \times X$.

We have just proved that the Baire sets of $X \times X$ are completely determined by the Baire sets of X. Thus, if X is Baire isomorphic to a metrizable space M, then M must be separable and analytic [7, p. 1114, part C], and so the Baire sets in $M \times M$ are also completely determined by those in M. Thus $X \times X$ is Baire isomorphic to $M \times M$ and so the diagonal in $X \times X$ must be a Baire subset, since this is the case for the metrizable space $M \times M$. Therefore the diagonal in $X \times X$ is a Souslin-$Z(X \times X)$ set.

3)⇒1) We have for the diagonal $\Delta \subseteq X \times X$

$$\Delta = \bigcup_{\sigma \in N^N} \bigcap_{s<\sigma} Z(f_s), \qquad f_s \in C^*(X \times X).$$

Reindex $\{f_s\colon \sigma \in N^N, s<\sigma\}$ by $\{f_n\colon n=1,2,\ldots\}$ and define

$$\Phi\colon X \times X \to R^N$$

by

$$\Phi(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2), \ldots).$$

Then

$$\Phi^{-1} \circ \Phi[\Delta] = \Delta.$$

Since X is proper analytic, there is a proper map ϕ of X onto a metrizable analytic space A. The map

$$\phi \times \phi\colon X \times X \to A \times A$$

defined by

$$\phi \times \phi(x_1, x_2) = (\phi(x_1), \phi(x_2))$$

is proper. Therefore the map

$$\phi \times \phi \times \Phi\colon X \times X \to A \times A \times R^N$$

defined by

$$\phi \times \phi \times \Phi(x_1, x_2) = (\phi(x_1), \phi(x_2), \Phi(x_1, x_2))$$

is proper and

$$(\phi \times \phi \times \Phi)^{-1} \circ (\phi \times \phi \times \Phi)[\Delta] = \Delta.$$

Since Δ is closed in $X \times X$

$$(\phi \times \phi \times \Phi)[\Delta]$$

is closed in the metric space $(\phi \times \phi \times \Phi)[X]$ and is therefore a G_δ. So Δ is a G_δ in $X \times X$, which implies that X is metrizable by result B above.

5)⇒1) Suppose $\{Z_n\colon n=1,2,\ldots\}$ generates the Souslin-$Z(X)$ sets. If x, yX, $x \neq y$, then there is a Baire set B such that

$$x \in B, \qquad y \in X \setminus B.$$

We have

$$B = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z_s,$$

$$Z_s \in \{Z_n\colon\ n = 1, 2, \ldots\}.$$

Since $x \in B$, $x \in \bigcap_{s < \sigma_0} Z_s$ for some $\sigma_0 \in N^N$, and since $y \notin B$, $y \notin Z_{s'}$ for some $s' < \sigma_0$; that is,

$$y \in X \setminus Z_{s'}, \quad x \in Z_{s'}.$$

Thus $\{X \setminus Z_n\colon n = 1, 2, \ldots\}$ is a point-countable collection of open sets such that for all $x \neq y$ there is an m such that $x \in X \setminus Z_m$ and $y \in Z_m$. Therefore, since X is proper analytic, it is metrizable by result C above.

Remarks. 1) The hypothesis in Theorem 3 can not be weakened to include all analytic spaces. The integers plus one additional point from its Stone-Čech compactification is a non metrizable analytic space which satisfies 2) through 5) of Theorem 3.

2) A form of part 3) of Theorem 3 was announced in [11] and a form of part 4) was announced in [12, Theorem 1.12].

4. Applications to Compact Convex Sets

MacGibbon [15] has proved that a compact convex subset X of a Hausdorff locally convex real topological vector space is metrizable if its set of extreme points $\mathscr{E}(X)$ is a Souslin-$\mathscr{Z}(X)$ subset. In brief she considers the map $f\colon X \times X \to X$ defined by $f(x, y) = \frac{1}{2}(x + y)$. The diagonal in $\mathscr{E}(X) \times \mathscr{E}(X)$ is equal to $f^{-1}[\mathscr{E}(X)]$ and is thus a Souslin-$\mathscr{Z}(\mathscr{E}(X) \times \mathscr{E}(X))$ set. Since $\mathscr{E}(X)$ is Souslin-$\mathscr{Z}(X)$, it is proper analytic and thus by Theorem 3.3) above $\mathscr{E}(X)$ is metrizable. Since a metrizable proper analytic space is the continuous image of N^N, X must be metrizable by the main theorem of [3] which states that X is metrizable if $\mathscr{E}(X)$ is the continuous image of N^N.

In the same vein we have

Theorem 4. *If X is a compact convex set whose set of extreme points is a proper analytic space, then the following are equivalent:*

1) *X is metrizable,*

2) *$\mathscr{E}(X)$ with its own algebra of Baire subsets is a standard Borel space,*

3) *$\mathscr{E}(X)$ with its own algebra of Baire subsets is Baire isomorphic to a metrizable analytic space (equivalently, to an analytic subset of R),*

4) *The Boolean algebra of Baire subsets of $\mathscr{E}(X)$ is free on a countable number of generators or the cardinality of $\mathscr{E}(X)$ is at most countable,*

5) *Every point of $\mathscr{E}(X)$ is a Baire subset of $\mathscr{E}(X)$ and there exists a map of the algebra of Baire sets of $\mathscr{E}(X)$ onto the algebra of Baire sets of some*

metrizable analytic space A with the property that

$$B_1 \subseteq B_2 \quad \text{if and only if } f(B_1) \subseteq f(B_2).$$

Proof. 1)⇒2) The set of extreme points of a metrizable compact convex set is always a G_δ subset [1, p. 34] and is thus completely metrizable.

2)⇒3) Trivial.

3)⇒1) This follows directly from part 4) of Theorem 3.

2)⇒4) An uncountable standard Borel space is isomorphic to the Cantor set 2^N with its algebra of Borel sets and this latter algebra is free on a countable number of generators [20, p. 107].

4)⇒2) Let $\mathscr{B}$ denote the algebra of Baire subsets of the space $\mathscr{E}(X)$. Suppose $\mathscr{B}$ is free on a countable set of generators. Then there exists a Boolean algebra isomorphism f of the algebra of Borel sets of the Cantor set 2^N onto $\mathscr{B}$. The map f is induced by a point map ϕ from $\mathscr{E}(X)$ onto 2^N; that is,

$$\phi^{-1}[B] = f(B)$$

for every Borel set B of 2^N [19, p. 13]. Thus $(\mathscr{E}(X), \mathscr{B})$ is isomorphic as a measurable space to 2^N with its algebra of Borel sets. Therefore $(\mathscr{E}(X), \mathscr{B})$ is a standard Borel space.

3)⇒5) Trivial.

5)⇒3) Suppose f is the map in part 5). Then from [21, p. 137] there exists a point map ϕ of $\mathscr{E}(X)$ onto A such that

$$\phi[B] = f(B)$$

for every Baire subset B of $\mathscr{E}(X)$. The map ϕ is a Baire isomorphism.

Remarks. If $\mathscr{E}(X)$ is a Souslin-$\mathscr{Z}(X)$ subset, then it is proper analytic by Theorem 1. The converse is false since any compact space may be represented as the extreme points of a compact convex set.

The problem remains open as to whether or not the hypothesis that $\mathscr{E}(X)$ be proper analytic may be dropped from Theorem 4. If the cardinality of $\mathscr{E}(X)$ is at most countable, then the hypothesis may be dropped since $\mathscr{E}(X)$ will be the continuous image of the discrete space N and so of N^N. Thus the problem may be phrased as:

If X is a compact convex set and $\mathscr{E}(X)$ is Baire isomorphic to 2^N, then is X necessarily metrizable?

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