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On Moduli of Algebraic Varieties. I

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Introduction

It is natural in the theory of classification of projective, smooth varieties and compact complex manifolds to consider varieties and manifolds as fibre spaces.

Two types of natural fibrations for smooth, projective varieties and compact, complex manifolds are known. The first type is given by the pluricanonical mappings of a variety or manifold V; the second type is given by the Albanese mapping of V.

For surfaces the classification by Enriques [6] and Kodaira [15] is done mainly according to their natural fibrations.

The investigation of the classification problem and the mentioned natural fibrations for varieties and manifolds of dimension $\geqq 3$ has started only recently. Very interesting results have been obtained by Iitaka [12] and Ueno [28].

This is the idea:

To classify smooth, projective varieties and compact complex manifolds of a fixed dimension n one should consider their natural fibrations and distinguish them according to the nature of these fibrations. Then one can investigate and classify the fibrations on the lines of the papers of Kodaira [14] and Namikawa/Ueno [24].

However, there are smooth, projective varieties and compact, complex manifolds where the natural fibrations are "trivial".

The pluricanonical fibrations do not give information if the Kodaira dimension[1] of V is equal to the dimension of V or if the Kodaira dimension of V is 0 or $-\infty$. The Albanese fibration is "trivial" if the Albanese variety is zero or if the Albanese mapping is generically finite.

To classify projective varieties and complex manifolds, where one of these fibrations is "trivial", other methods than the theory of fibre spaces are needed. The present paper develops a method which gives for

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[1] Let K_V be the canonical line bundle of an irreducible, smooth projective variety V. If $\dim H^0(V, O(K_V^{\otimes m})) = N+1 \geqq 2$, m an integer $\geqq 1$, we have a rational map $\Phi_{mK}\colon V \to P^N$ of V into the projective space P^N. In this case the Kodaira dimension $\kappa(V)$ of V is defined by $\kappa(V) = \max_{m \in N_0} (\dim \phi_{mK}(V))$, where $N_0 = \{m \in \mathbb{N};\ \dim H^0(V, O(K_V^{\otimes m})) \geqq 2\}$.

If $\dim H^0(V, O(K_V^{\otimes m})) \leqq 1$ for all $m \in \mathbb{N}$ and $\dim H^0(V, O(K_V^{\otimes m}) = 1$ for one m, we define $\kappa(V) = 0$.

If $H^0(V, O(K_V^{\otimes m})) = 0$, $m \in \mathbb{N}$, we define $\kappa(V) = -\infty$.

certain unruled, smooth, polarized varieties $(V_0, \mathfrak{X}_0)$, where one of the canonical fibrations is trivial, a coarse moduli space for the global deformation functor of $(V_0, \mathfrak{X}_0)$. For instance, for a polarized K-3 surface $(V_0, \mathfrak{X}_0)$ an algebraic space is constructed which is a coarse moduli space for the global deformations of $(V_0, \mathfrak{X}_0)$.

We give a description of the method.

Let $(V_0, \mathfrak{X}_0)$ be a smooth, irreducible, projective variety of dimension n, defined over the complex numbers $\mathbb{C}$ and with a polarization $\mathfrak{X}_0$.

Let $(V/S, \mathfrak{X}/S)$ be a smooth, projective family of polarized varieties with a connected noetherian $\mathbb{C}$-scheme S as basis such that one of the geometric fibres of $(V/S, \mathfrak{X}/S)$ is isomorphic to $(V_0, \mathfrak{X}_0)$. The fibres of such a family $(V/S, \mathfrak{X}/S)$ are called *global deformations* of $(V_0, \mathfrak{X}_0)$.

Assume in the following that the families $(V/S, \mathfrak{X}/S)$ satisfy one of the three conditions:

1) The polarization $\mathfrak{X}/S$ is the canonical polarization.
2) The irregularity of the fibres of the families $(V/S, \mathfrak{X}/S)$ is zero.
3) $(V/S, \mathfrak{X}/S)$ is a polarized abelian variety.

Then consider more generally smooth families $(V/S, \mathfrak{X}/S)$ of polarized varieties such that the fibres are deformations of $(V_0, \mathfrak{X}_0)$. (Notice, $(V_0, \mathfrak{X}_0)$ does not have to be isomorphic to a fibre of $(V/S, \mathfrak{X}/S)$.) Call such a family a *family of deformations* of $(V_0, \mathfrak{X}_0)$ and let $\mathcal{M}(S)$ be the set of such families with basis S up to isomorphism. The collection $\mathcal{M}(S)$, S a noetherian $\mathbb{C}$-scheme, is then a contravariant functor from the category of noetherian $\mathbb{C}$-schemes S to the category of sets, where to a morphism $\alpha: T \to S$ a map $\mathcal{M}(\alpha): \mathcal{M}(S) \to \mathcal{M}(T)$ is defined by associating to a family $(V/S, \mathfrak{X}/S)$ the pullback family $(V \underset{S}{\times} T/T, \mathfrak{X} \underset{S}{\times} T/T)$. The functor $S \to \mathcal{M}(S)$ is called the *global deformation functor* of the polarized variety $(V_0, \mathfrak{X}_0)$.

We want $\mathcal{M}(S)$ to be a proper geometric object of algebraic geometry. Thus, we want to represent $\mathcal{M}(S)$ in the category of schemes or in the category of algebraic spaces (see [13] for this notion) or to find a coarse moduli space for the functor $\mathcal{M}(S)$ in one of these categories. Using projective methods we are able to show that the coarse moduli space for the functor $\mathcal{M}(S)$ in the category of algebraic spaces exists if $(V_0, \mathfrak{X}_0)$ belongs to one of the following types of polarized algebraic varieties[2].

a) $(V_0, \mathfrak{X}_0)$ is a (canonical polarized) smooth, projective curve of genus $g > 1$.

b) $(V_0, \mathfrak{X}_0)$ is a polarized abelian variety.

c) $(V_0, \mathfrak{X}_0)$ is a polarized K-3 surface.

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[2] In a forthcoming paper we will give a criterion for the representability of a functor as an algebraic space. This criterion will show that the deformation functor for the varieties with level n-structure in a)–d) is even representable in the category of algebraic $\mathbb{C}$-spaces. (n sufficiently big.)

d) $(V_0, \mathfrak{X}_0)$ is a smooth, projective variety, such that the canonical sheaf is very ample and contained in $\mathfrak{X}_0$. $(V_0, \mathfrak{X}_0)$ is then, in particular, canonically polarized.

Unfortunately, our method does not show the existence of a coarse moduli space for the global deformation functor of a canonical polarized variety, because we were not able to prove that the automorphism of these varieties operate faithfully on their integral cohomology.

The existence of the coarse moduli spaces in the cases a) and b) is well known. By [22] one knows even that in this cases the coarse moduli space is a quasi projective variety. But, our proof is more simple than the one given in [22]. In the remaining cases the existence of the coarse moduli space in the category of algebraic spaces seems to be unknown.

The method of the present paper which leads to the existence of such a coarse moduli space for $\mathcal{M}(S)$ is interesting in itself. We would like to briefly explain the idea of it.

We show first that the deformation functor $\mathcal{M}(S)$ of $(V_0, \mathfrak{X}_0)$ can locally be linearized if $\mathcal{M}$ satisfies one of the conditions 1) or 2)[3], i.e. if $(V/S, \mathfrak{X}/S)$ and $(V'/S, \mathfrak{X}'/S)$ are two families over the scheme S which belong to $\mathcal{M}(S)$. Then, after restricting $(V/S, \mathfrak{X}/S)$ and $(V'/S, \mathfrak{X}'/S)$ to open subsets of S, there exists a projective embedding of these restriction into a fixed projective space P^N, such that the families $(V/S, \mathfrak{X}/S)$, $(V'/S, \mathfrak{X}'/S)$ are locally isomorphic as polarized families if and only if these embeddings into P^N are projectively equivalent (compare p. 28). Furthermore, the fibres of the embeddings into P^N/S have a constant Hilbert polynomial $h(x)$. Denote by $H_{P^N}^{h(x)}$ the Hilbert scheme which parametrizes the flat families of subvarieties in P^N with Hilbert polynomial $h(x)$. One shows then that there exists a locally closed and connected subscheme H of $H_{P^N}^{h(x)}$ which parametrizes "locally" the families which belong to $\mathcal{M}(S)$. (Compare Prop. 2.14.)

The projective linear group $PGL(N)$ operates on H and one proves that the geometric quotient of H with respect to $PGL(N)$ in the category of algebraic spaces, if it exists, is a coarse moduli space of the functor $\mathcal{M}(S)$.

One of the difficulties of the paper is to show that the geometric quotient of H by $PGL(N)$ exists in the category of algebraic spaces.

For this purpose we prove in Chapter I that the geometric quotient of a connected algebraic group G, operating on a $\mathbb{C}$-scheme, exists in the category of algebraic spaces provided X is of finite type and reduced and G operates without fix points and with closed graph on X. The main difficulties in applying this result to H arise, because $PGL(N)$ operates on H in general with fix points[4]. As a matter of fact the non trivial auto-

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[3] The case of abelian varieties gets treated separately in Chapter III.

[4] There are also difficulties to overcome if H is not reduced. In this case one restricts the functor $\mathcal{M}$ to $\mathcal{M}_{\text{red}}$ as explained in Chapter II.

morphisms of a deformation $(V, \mathfrak{X})$ of $(V_0, \mathfrak{X}_0)$ lead to fix points of the operation of $PGL(N)$ on H.

In the cases we have considered this difficulty is overcome by showing that there exists a finite galois covering H' of H which is etale and on which $PGL(N)$ operates in a natural way (according to H) proper and fix point free. Λ' shall be the galois group of the covering $H' \to H$. Now, the geometric quotient M' of H' by $PGL(N)$ exists in the category of algebraic spaces (see Theorem 1.13). On the algebraic space M' the finite group operates and the geometric quotient M of M' by Λ' in the category of algebraic spaces also exists. (See Theorem 1.15.) This space M is a geometric quotient of H by $PGL(N)$ (in the category of algebraic spaces) and also a coarse moduli space for the functor $\mathcal{M}(S)$.

The construction of the covering H' involves new considerations. One has to show that the automorphisms of a deformation $(V, \mathfrak{X})$ of $(V_0, \mathfrak{X}_0)$ operate faithfully on the integral cohomology of V. This is the case if the variety $(V_0, \mathfrak{X}_0)$ belongs to one of the types a), c) or d) from p. 2.

Using this fact, one constructs H' first as a finite, unramified, topological covering of H which is galois with galois group Λ'. By the generalized Riemann existence theorem [2], XI, p. 12, the covering H' is then automatically a scheme on which $PGL(N)$ operates.

It is a interesting problem to find other types of polarized varieties $(V, \mathfrak{X})$ over $\mathbb{C}$ for which the automorphisms operate faithfully on the integral cohomology of V. In particular, one should decide for which algebraic varieties over $\mathbb{C}$ of general type[5] this is satisfied, our method leads for such varieties to a coarse moduli space for the corresponding global deformation functor. It is well known that a coarse moduli space for the deformation functor $\mathcal{M}(S)$ of $(V_0, \mathfrak{X}_0)$ in the category of analytic spaces exists if $(V_0, \mathfrak{X}_0)$ satisfies the statements 1) or 2) from p. 2. (We are not quite precise here, one has to modify the functor $\mathcal{M}(S)$ and allow any analytic space S as basis.) The reason that there is less trouble in the category of analytic spaces is that by the results of Holmann [10, 11], one can factor out in this category from the start the group $PGL(N)$. This leads, for instance, to coarse moduli spaces for canonical polarized algebraic varieties.

Finally, we would like to point out that the proof of the existence of the geometric quotient M, respectively M' of H, respectively H' by $PGL(N)$ (see Theorem 1.13) gives quite explicitly an etale neighbourhood of the points $P \in M$, resp., M'. The following is a method for constructing an etale neighborhood of a point $P \in M'$: Take a point $Q \in H'$ which is mapped to P under the canonical map $\varphi'\colon H' \to M'$. Let E_Q be the orbit of Q by $PGL(N)$ which is a closed subscheme of H'. Consider an affine open neighborhood W' of Q and an embedding $\lambda\colon W' \to A^n$ of W' into

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[5] An algebraic variety V is called of general type if $\kappa(V) = \dim V$.

the affine space $A^N/\mathbb{C}$. Take a linear subspace L of A^N of dimension $= N\text{-}\dim(E_Q)$ which passes through $\lambda(Q)$ and which intersects $\lambda(E_Q \cap W')$ transversally. Then an open subscheme of the intersection $L \cap \lambda(W')$ is an etale neighborhood of $P \in M'$. This shows that if one knows H' well, one knows also M' at least locally well. The description of an etale neighborhood of M is a little more complicated. We refer to the proof of Theorem 1.15.

This paper has been shortened at the suggestion of the referee.

I. Geometric Quotients in the Category of Algebraic Spaces

In this chapter all schemes and all algebraic spaces are *separated* and *k-spaces*, where k is an algebraically closed field. See [13] for the definition. The group G is an *algebraic group*, also defined over k. An irreducible, reduced k-scheme of finite type is called in the following a *k-variety*.

If X is an algebraic space and G a group we say that G acts on X if for all algebraic spaces Z the group $\mathrm{Hom}(Z, G)$ acts on $\mathrm{Hom}(Z, X)$ in a functorial way.

An algebraic space on which the group G acts will be called a *G-space*. *G-morphism* $f\colon X \to Y$ between G-spaces X, Y are defined in the usual way, i.e., for every algebraic space Z the map $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ induced by f is compatible with the actions of $\mathrm{Hom}(Z, G)$ on $\mathrm{Hom}(Z, X)$ respectively $\mathrm{Hom}(Z, Y)$.

1.1. *Definition.* X shall be a G-space. We say that a representable etale covering

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & V \arrow[r, "\pi"] & X \end{tikzcd}$$

of X (V is a k-scheme and R a subscheme of $V \underset{k}{\times} V$, see [13]) is a *G-stable* etale covering of the G-space X if there exist morphisms of schemes $\Phi'\colon V \times G \to V$ and $\Phi''\colon R \times G \to R$ such that

1) $\Phi'\colon V \times G \to V$ defines an action of the group G on the scheme V. (See [22].)

2) Φ'' is induced by Φ', i.e. Φ'' is the restriction of the morphism $\Phi' \times \Phi'\colon (V \times V) \times (G \times G) \cong (V \times G) \times (V \times G) \to V \times V$ to the subscheme $R \times \Delta_G$, where Δ_G is the diagonal of $G \times G$. The diagram

$$\begin{tikzcd} R \times G \arrow[r, shift left, "\pi_1 \times \mathrm{Id}"] \arrow[r, shift right, "\pi_2 \times \mathrm{Id}"'] \arrow[d, "\Phi''"] & V \times G \arrow[r, "\pi \times \mathrm{Id}"] \arrow[d, "\Phi'"] & X \times G \arrow[d, "\Phi"] \\ R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & V \arrow[r, "\pi"] & X \end{tikzcd}$$

is commutative and defines therefore an action of G on X.

3) The action of G on X, defined by Φ, gives the G-space X.

1.2. *Definition.* A G-space X is called *trivial* if for every algebraic space Z the group $\mathrm{Hom}(Z, G)$ acts trivially on $\mathrm{Hom}(Z, X)$.

1.3. *Remark.* If G is a connected algebraic group which operates trivially on an algebraic space X of finite type over k, then G operates trivially on every representable G-stable etale covering V of X. For the proof let V be any representable and G-stable etale covering of the G-space X. Then the stabilizer of every k-valued point P of V is a subgroup S_P of G of dimension $= \dim G$. This implies $S_P = G$, because G is connected and S_P is closed. Hence, G operates trivially on V.

Let X be a G-space and Y an algebraic space. A *G-invariant morphism* is a G-morphism $f: X \to Y$ where Y is G-trivial. (Note that in this case one has for the map of sheaves $O_Y \xrightarrow{f^*} O_X^G$, $O_X^G =$ sheaf of fixed elements of O_X under G.)

1.4. *Definition.* An algebraic space Y together with a G-invariant morphism $\varphi: X \to Y$ is called a *quotient* of X by G, if for every G-invariant morphism $f: X \to Z$ there exists a unique morphism $\tilde{f}: Y \to Z$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "\tilde{f}"] \\
Z &
\end{tikzcd}$$

is commutative.

1.5. *Definition.* Let G be a group operating on an algebraic space X. A *geometric quotient* of X by G in the category of algebraic spaces, is a pair (Y, φ) consisting of an algebraic space Y and a G-invariant morphism $\varphi: X \to Y$, satisfying

1) the map $\varphi: X \to Y$ is surjective (see [13], 107) and for any algebraic closed field k^* which contains k, the k^*-valued points of Y are precisely the orbits of the k^*-valued points of X.

2) The structure sheaf of O_Y is the sheaf O_X^G, consisting of the elements of O_X, which are kept fixed under G.

3) For every G-invariant map $f: X \to Z$ there exists a unique map $\tilde{f}: Y \to Z$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "\tilde{f}"] \\
Z &
\end{tikzcd}$$

is commutative, in other words, Y is a quotient of X by G and in particular uniquely determined.

1.6. **Theorem.** *Let X be a normal k-variety*[6]. *G shall be an irreducible, algebraic group, defined over k which operates freely and with closed graph on X. Assume also that, if $\rho_P\colon G \to E_P$ is the canonical map of G onto the orbit E_P of an arbitrary point $P \in X(\mathrm{Spec}(k))$, the differential map of the tangent spaces $d\rho_P\colon t(G)_l \to t(E_P)_P$ is surjective*[7]. *Then there exists a separated algebraic space Y of finite type over k which is a geometric quotient of X under the action of G in the category of algebraic spaces.*

The proof of this theorem will be done in several steps. We recall first some notations from the theory of G-varieties which can be found in [27].

Let X be a reduced k-scheme of finite type and G an algebraic group defined over k and operating on X in the sense of schemes, see [22]. Let Y be the quotient set of $X(\mathrm{Spec}(k))$ by G, endowed with the quotient topology and $\varphi\colon X(\mathrm{Spec}(k)) \to Y$ the canonical mapping of $X(\mathrm{Spec}(k))$ onto Y. It is clear that Y is a topological noetherian space. Y carries a canonical structure of ringed space, in fact, if $\varphi_*(O_X)$ is the direct image of the structure-sheaf O_X on Y then G operates on $\varphi_*(O_X)$ and we take $O_Y = \varphi_*(O_X)^G$ the subsheaf of $\varphi_*(O_X)$ left invariant by G. Denote this ringed space by (Y, O_Y). If the ringed space (Y, O_Y) is a k-scheme, this scheme, together with the map φ is a geometric quotient of X by G in the category of k-schemes in the sense of the following definition:

1.7. *Definition.* Let X be a reduced k-scheme and G an algebraic group operating on X. A *geometric quotient* of X by G *in the category of k-schemes* consists of a k-scheme Y and a morphism $\varphi\colon X \to Y$, satisfying

(1) For each closed point $y \in Y$, $\varphi^{-1}(y)$ is an orbit of G on X.

(2) For each invariant open subset $U \subset X$ there exists an open subset $V \subset Y$ such that $U = \varphi^{-1}(V)$.

(3) For each open set $V \subset Y$, $\varphi^*\colon \Gamma(V, O_Y) \to (\varphi^{-1}(V), O_X)$ is an isomorphism of $\Gamma(V, O_Y)$ onto the ring $\Gamma(\varphi^{-1}(V), O_X)^G$ of invariant functions on $\varphi^{-1}(V)$.

Remarks. a) One shows as in [23], prop. 5, that a geometric quotient in the sense of Definition 1.7 is also a quotient of X by G in the category of k-schemes.

b) The definition for the geometric quotient of X by G in the category of k-schemes, which we use, is weaker than the definition used by Mumford [22], p. 4. There are however cases when these definitions coincide. One such situation is described by the following proposition:

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[6] The irreducibility of X is actually not necessary.

[7] In characteristic 0 this is always true, compare [3], p. 180.

1.8. **Proposition.** *X is a k-scheme on which the group G operates. Assume that (Y, φ) is the geometric quotient of X by G in the category of k-schemes. Assume further that there exists a section $r: Y \to X$ over $\varphi: X \to Y$. Then (Y, φ) is a geometric quotient of X by G in the sense of* [22], p. 4.

Proof. It is easy to see that the existence of the section $r: Y \to X$ implies that the map $\varphi: X \to Y$ is universally submersive, i.e., for all base extensions $Y' \to Y$ by a k-scheme Y', if $X' = X \times_Y Y'$ and $\varphi': X' \to Y'$ are induced by φ then a subset $U' \subset Y'$ is open if and only if $\varphi^{-1}(U')$ is open in X'.

Criterion for geometric quotients. Let X and U be k-schemes of finite type and $\varphi: X \to U$ a surjective morphism such that G-orbits on X map to distinct points of U. (G-orbits are always G-orbits of k-valued points of X.) Then the k-valued points of U can be identified with the orbit set $Y = X/G$. Now, if there exists a section of X over U, i.e., a morphism $r: U \to X$ such that $\varphi \circ r: U \to U$ is the identity morphism, one shows that the ringed space (U, O_U) can be canonically identified with (Y, O_Y) by the map $\varphi \circ r$ and that (U, O_U) is therefore a geometric quotient of the G-scheme X. (Geometric quotient in the sense of Definition 1.7 and by Proposition 1.8 also in the sense of [22], p. 4.)

We would like to indicate the proof of this fact:

Clearly, the morphism $X \xrightarrow{\varphi} U$ gives a morphism of sheaves $\varphi^*: O_U \to O_X^G$. The section $r: U \to X$ defines a morphism $O_X \xrightarrow{r^*} O_U$ which has, restricted to O_X^G, a trivial kernel, because an element f in O_X^G, considered as a function on U, is the zero function if and only if f is zero in O_X and therefore zero in O_X^G. (Notice that O_X has no nilpotent elements, X is a k-scheme of finite type where k is an algebraically closed field.) Hence, $r^*: O_X^G \to O_U$ is injective. Now the equation $r^* \circ \varphi^* = \mathrm{Id}^*$ on O_U implies that r^* is an isomorphism form O_X^G onto O_U and this shows the statement.

Let X be a reduced k-scheme of finite type on which the algebraic group G is operating with finite stabilizers. Let the operation of G on X be defined by $\Phi: X \times G \to X$. $\Gamma \subset X \times X$ shall be the graph of Φ, i.e., $\Gamma = \{(x, \Phi(x, g)) = (x, g(x));\ x \in X,\ g \in G\}$.

We assume that Γ is closed in $X \times_k X$ (in the Zariski topology). Let U be a reduced k-scheme of finite type and $i: U \to X$ be a k-morphism from U to X. (In the following U is mainly a subscheme of X and $i: U \to X$ the embedding.) $q: U \to Y$ shall be the mapping of $U(\mathrm{Spec}(k))$ in Y, defined by $\varphi \circ i$.

Consider the fibre products $U \times_Y X = \{(u, x);\ (u, x) \in U(\mathrm{Spec}(k)) \times X(\mathrm{Spec}(k))$ and $q(u) = \varphi(x)\}$.

Claim. $U \underset{Y}{\times} X$ is a closed subset[8] (in the Zariski topology) of $U \underset{k}{\times} X$. Take $X \underset{k}{\times} X$ with the Zariski topology and let $Y \times Y$ be the quotient of $(X \underset{k}{\times} X)(\mathrm{Spec}(k))$ modulo $G \times G$ in the sense of topological spaces. Then the map $\varphi \times \varphi$: $(X \underset{k}{\times} X)(\mathrm{Spec}(k)) \to Y \times Y$ is continuous. Also, the map (i, Id): $U \times X \to X \times X$, Id = identity of X, is continuous. Hence, the composit map

$$U \times X \xrightarrow{(i,\mathrm{Id})} X \times X \xrightarrow{\varphi \times \varphi} Y \times Y,$$

which is the map (q, φ), is continuous. Now Γ is closed in $X \underset{k}{\times} X$ if and only if the diagonal Δ_Y of $Y \times Y$ is closed in $Y \times Y$ and since $U \underset{Y}{\times} X = (q, \varphi)^{-1}(\Delta_Y)$ the assertion follows.

As $U \underset{Y}{\times} X$ is closed in $U \times X$, it is in a canonical way a reduced k-scheme of finite type. *This k-scheme is in the following denoted by $U \underset{Y}{\times} X$.*

The group G acts in a natural way on $U \times X$ and on $U \underset{Y}{\times} X$ by the rule $g((u, x)) = (u, g(x))$, $x \in X(\mathrm{Spec}(k))$ and we have a canonical map $U \underset{Y}{\times} X \xrightarrow{f} X$ induced by the projection $U \times X \to X$ which is a G-morphism. We have also a morphism φ: $U \underset{Y}{\times} X \to U$ induced by the projection $U \times X \to U$. This morphism maps distinct G-orbits of $U \underset{Y}{\times} X$ to distinct points of U. Furthermore the map $U \xrightarrow{s} U \underset{Y}{\times} X$ defined by $s(u) = (u, i(u))$ is a section with respect to φ. This implies, see p. 8, that U is the geometric quotient of $U \underset{Y}{\times} X$ by G and also that $U \underset{Y}{\times} X$ is irreducible, provided U is irreducible. The last statement can be shown as follows:

Consider the map $U \times G \xrightarrow{\delta} U \underset{Y}{\times} X$ defined by $\delta((u, g)) = (u, g(u))$, where u and g are k-valued points of U, respectively G. This map is surjective for the k-valued points. Assume $U \underset{Y}{\times} X = V_1 \cup V_2$, where V_1, V_2 are closed subschemes of $U \underset{Y}{\times} X$ with $V_1 \nsubseteq V_2$ and $V_2 \nsubseteq V_1$. $\delta^{-1}(V_1), \delta^{-1}(V_2)$ are then closed subschemes of $U \times G$ and $\delta^{-1}(V_1) \cup \delta^{-1}(V_2) = U \times G$. This contradicts the irreducibility of $U \times G$ if U is irreducible. Q.E.D.

The stabilizers of the action of G on X are finite and hence, the fibres of the map δ: $U \times G \to U \underset{Y}{\times} X$ are finite. By the dimension theorem one concludes from this that

$$\dim(U \underset{Y}{\times} X) = \dim(U \times G) = \dim U + \dim G$$

and

$$\dim(U \underset{Y}{\times} X) = \dim(X) \quad \text{if} \quad \dim(U) = \dim(X) - \dim(G).$$

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[8] Only the k-valued points are considered.

We go now back to the situation of Theorem 1.6.

1.9. **Main Lemma.** *Let X be a normal k-variety and G an irreducible algebraic group which operates freely on X with closed graph Γ. Then there exists a representable etale covering* $R = X' \underset{X}{\times} X' \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X' \xrightarrow{\pi} X$ *which is G-stable, and there exist schemes $\overline{X'}$ and $\overline{R}$ of finite type over k which are geometric quotients of X', respectively R by G (in the sense of Definition* 1.7) *such that the induced maps* $\overline{R} \overset{\overline{\pi}_1}{\underset{\overline{\pi}_2}{\rightrightarrows}} \overline{X'}$ *define an etale equivalence equation on $\overline{X'}$. The diagram* $\overline{R} \overset{\overline{\pi}_1}{\underset{\overline{\pi}_2}{\rightrightarrows}} \overline{X'}$ *defines therefore an algebraic space Y and this space Y is a geometric quotient of X under the action of G in the category of algebraic spaces.*

Proof. We use the following result of Seshadri, see [27], Proposition 1.

1.10. **Proposition.** *Let X be a normal k-variety and G an algebraic group which operates with finite stabilizer on X. The orbit map ρ_p: $G \to E_p$ shall satisfy the assumption of Theorem* 1.6 *for all points $P \in X$. Given an orbit E of G on X there exists a G-stable open subset W of X containing E and an irreducible, normal, locally closed subvariety U of W of dimension* $\dim(X) - \dim(E)$, *which intersects E transversally in finitely many points, such that the morphisms* f: $U \underset{Y}{\times} W \to W$ *and* κ: $U \times G \to W$, *where κ is the composite of the maps δ and f via* $U \times G \xrightarrow{\delta} U \underset{Y}{\times} W \xrightarrow{f} W$, *have the following properties:*

1) *f is surjective and quasi finite, i.e.* $\forall x \in W, f^{-1}(x)$ *is not empty and finite.*

2) *The geometric quotient of* $U \underset{Y}{\times} W$ *under G exists (in the category of k-schemes) and is isomorphic to U.*

3) *The map κ is etale.*

4) *The field extension* $k(U \underset{Y}{\times} W)/k(W)$, $k(U \underset{Y}{\times} W)$*=function field of* $U \underset{Y}{\times} W$, *is finite separable and if G operates freely on X, δ is an isomorphism and therefore f etale.*

Using this proposition of Seshadri we can pick a finite open covering $\bigcup_{i=1}^{n} W_i$ of X and normal locally closed subvarieties U_i of W_i such that $U_i \underset{Y}{\times} W_i \to W_i$ are etale maps and satisfy the statements of Proposition 1.10. Then $X' = \coprod_{i=1}^{n} (U_i \underset{Y}{\times} W_i) \xrightarrow{\pi} X$ is a representable etal covering of X. If R denotes $X' \underset{X}{\times} X'$, the diagram $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X'$ defines an etale equivalence relation in the category of schemes with X as quotient. (Compare [13],

p. 93.) $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X' \xrightarrow{\pi} X$ is a G-stable etale covering of X, where the operation of G on R is as follows.

Consider the set of k-valued points $R(k)$ of R, i.e.

$$R(k)=\{(x_1', x_2');\ x_i' \in X'(k) \text{ and } \pi(x_1)=\pi(x_2)\}.$$

If we let $X'=\coprod_i (U_i \underset{Y}{\times} W_i)$ we can write $x_1'=(u_1, w_1), x_2'=(u_2, w_2)$ and we get $(x_1', x_2') \in R(k)$ if and only if $w_1=w_2$ as elements of $X(k)$. This implies in particular that $u_2=g_0(u_1)$ with $g_0 \in G(k)$. The operation of G on R is then defined by the rule

$$g\big((u_1, w_1), (u_2, w_2)\big)=\big((u_1, g(w_1)), (u_2, g(w_2))\big)$$

for $g \in G(k)$.

We show now that the geometric quotient of R modulo this operation of G exists and that this quotient defines an etale equivalence relation on the variety $\coprod U_i$ which is exactly the one induced by the operation of G on X on the scheme $\coprod U_i$.

To make this more precise, we consider the equivalence relation on X which is induced by the action of G. This equivalence induces a relation on $\coprod U_i$ via the map $\coprod U_i \xrightarrow{\amalg \text{inj}} X$, where $\coprod$ inj is the direct sum of the injection maps of the schemes U_i into X. The graph of these equivalence relation on $\coprod U_i$ is given by the closed subscheme $\bar{R}=(\coprod U_i) \underset{Y}{\times} (\coprod U_i)$ of $(\coprod_i U_i) \times (\coprod_i U_i)$. Notice that one has a natural continuous map

$$(\coprod U_i) \times (\coprod U_i) \xrightarrow{\varphi \times \varphi} Y \times Y$$

induced by $\varphi\colon X \to Y$ and that $\bar{R}=(\varphi \times \varphi)^{-1} \Delta_Y$. The diagonal Δ_Y of $Y \times Y$ is closed in $Y \times Y$ and therefore $\bar{R}$ in $(\coprod U_i) \times (\coprod U_i)$. For the k-valued point of $\coprod_i U_i$ this equivalence relation can be described as follows:

Let u_i, u_j be elements of $(\coprod U_i)(k)$ then u_i is equivalent to u_j if and only if there exists an element $g \in G(k)$ such that $u_j=g(u_i)$ as k-valued points of X. The set of k-valued points of $\bar{R}$ is, consequently,

$$\begin{aligned}\bar{R}(k)&=\{(u_1, u_2);\ \exists\, g \in G(k),\ g(u_1)=u_2 \text{ in } X\}\\ &\subseteq (\coprod U_i) \times (\coprod U_i).\end{aligned}$$

Let $\bar{R}_{ij}=(U_i \times U_j) \cap \bar{R}$. Then $\bar{R}=\coprod_{i,j} \bar{R}_{ij}$ (disjoint sum). Clearly, $\bar{R}_{ij}=(U_i \times U_j) \cap \Gamma$, where Γ is the graph of the operation of G on X. This shows again that $\bar{R}_{ij}$ is a closed subscheme of $U_i \times U_j$ and also that $\bar{R}$ is a closed subscheme of $(\coprod_i U_i) \times (\coprod_i U_i)$. We will use this fact later.

Claim. *$\bar{R}$ is the geometric quotient of R modulo G in the category of schemes.*

For the proof let $\varphi'': R \to \bar{R}$ be the morphism defined by the projection $((u_1, w_1), (u_2, w_2)) \to (u_1, u_2)$. This map is surjective and sends distinct G-orbits of R into distinct points of R. We show that there is an injective morphism $j: \bar{R} \to R$ with

$$j: (u_1, u_2) \to ((u_1, u_2), (u_2, u_2))$$

which is then a section of R over $\bar{R}$ with respect to $\varphi'': R \to \bar{R}$.

To get the morphism $j: \bar{R} \to R$ it is enough to define the restriction of j to the various varieties $\bar{R}_{ij}$.

Let $p_i: \bar{R}_{ij} \to U_i$ be the morphism induced by the projection map $U_i \times U_j \to U_i$ and $p_j: \bar{R}_{ij} \to U_j$ the morphism induced by the projection $U_i \times U_j \to U_j$. Then $p_j(u_i, u_j) = u_j$ and $u_j = g(u_i)$ for $g \in G$ if $(u_i, u_j) \in \bar{R}_{ij}$. Hence, $p_j(u_i, u_j) = u_j \in (U_j \cap W_i) \subset W_i$ as $U_i \subset W_i$ and W_i is G-stable.

This shows that p_j can be considered as a map from $\bar{R}_{ij}$ into W_i. On the one hand $\bar{R}_{ij} \to U_i \times_Y W_i$ is to be the map $p_{ij} = (p_i, p_j)$. On the other hand we have a morphism $q_j: \bar{R}_{ij} \to U_j \times_Y W_j$, defined by the diagram

$$\begin{tikzcd}
\bar{R}_{ij} \arrow[r, "p_j"'] \arrow[rr, bend left, "q_j"] & U_j \arrow[r] & U_j \times_Y W_j \\
(u_i, u_j) \arrow[r] & u_j \arrow[r] & (u_j, u_j).
\end{tikzcd}$$

If we use the universal properties of the product $R = X' \times_X X'$, we obtain a morphism

$$U_i \times_Y U_j = \bar{R}_{ij} \xrightarrow{p_{ij} \times q_j} R$$

$$(u_i, u_j) \to (p_{ij}(u_i, u_j), q_j(u_i, u_j)) = ((u_i, u_j)(u_j, u_j))$$

which is the restriction of the map j to $\bar{R}_{ij}$.

The criterion described on p. 8 shows that $\bar{R} = \coprod \bar{R}_{ij}$ is the geometric quotient of R modulo G.

There is a commutative diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d] & X' \arrow[r, "\pi"] \arrow[d] & X \\
\bar{R} \arrow[r, shift left, "\bar{\pi}_1"] \arrow[r, shift right, "\bar{\pi}_2"'] & \coprod U_i .
\end{tikzcd}$$

$\bar{R} \overset{\bar{\pi}_1}{\underset{\bar{\pi}_2}{\rightrightarrows}} \coprod U_i$ is the quotient of the pair $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X'$ under the action of G.

If we can show that the maps $\bar{\pi}_1$ and $\bar{\pi}_2$ are etale maps, the diagram

$$\begin{tikzcd}\bar{R} \arrow[r, shift left, "\bar{\pi}_1"] \arrow[r, shift right, "\bar{\pi}_2"'] & \coprod U_i\end{tikzcd}$$

will define an algebraic space Y (see [13], p. 93).

To show that the $\bar{\pi}_i$ are etale we direct our attention to the canonical surjective G-morphism

$$\varepsilon\colon \bar{R}\times G \to R,$$

defined by $\varepsilon(\bar{r}, g)=g(j(\bar{r}))$, where j is the map $j\colon \bar{R}\to R$ from above.

ε is an isomorphism, because G operates freely on R. (Use [22], Proposition 0.9.)

We have (Proposition 1.10) an isomorphism $\coprod U_i\times G \xrightarrow{\delta} X'$ and the diagram

$$\begin{tikzcd}\bar{R}\times G \arrow[r, "\bar{\pi}_2\times \mathrm{Id}"'] \arrow[d, "\varepsilon"] & (\coprod U_i)\times G \arrow[d, "\delta"] \\ R \arrow[r, "\pi_2"'] \arrow[d] & X' \arrow[d] \\ \bar{R} \arrow[r, "\bar{\pi}_2"'] & \coprod U_i\end{tikzcd}$$

is commutative.

The map $(\bar{\pi}_2\times \mathrm{Id})$ is etale and the restriction to $\bar{R}=\bar{R}\times\{e\}$ is the map $\bar{\pi}_2$. Using [7] we can conclude that the map $\bar{\pi}_2$ is etale. In the same way one proves that $\bar{\pi}_1$ is etale. The same arguments together with the section $j'\colon (u_1, u_2)\to((u_1, u_1), (u_2, u_1))$ of R over $R\xrightarrow{\varphi''}\bar{R}$ instead of j suffice. Let Y be the separated algebraic space of finite type over k which is defined by the diagram

$$\begin{tikzcd}\bar{R} \arrow[r, shift left] \arrow[r, shift right] & \coprod U_i \arrow[r] & Y.\end{tikzcd}$$

(Y is separated, because $\bar{R}$ is a closed subscheme of $(\coprod U_i)\times(\coprod U_i)$.)

Claim. *Y is a geometric quotient of X modulo G.*

Statement 1 of Definition 1.5 is obviously satisfied by the algebraic space Y.

For the proof of statement 2) of Definition 1.5 let $\bar{V}\xrightarrow{f} Y$ be an etale map where $\bar{V}$ is a scheme. Using the Definition of the structure sheaf on Y (see [13], p. 104) it is enough to show that for every point $\bar{P}\in\bar{V}$ there exists an etale neighbourhood $\bar{h}\colon \bar{N}\to\bar{V}$ of $\bar{P}$, where $\bar{N}$ is a scheme, and a G-stable etale morphism $N\to X$, where N is also a scheme, on which G operates such that $\bar{N}$ is the geometric quotient of N modulo G in the sense of schemes.

The structure sheaf of $\bar{N}$ is then in particular the fixed sheaf O_N^G of O_N.

Notice that the etale map $\bar{V}\to Y$ can be described by a commutative diagram

$$\begin{tikzcd}
R_{\bar{V'}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\bar{f}''"] & \bar{V'} \arrow[r] \arrow[d, "\bar{f}'"] & \bar{V} \arrow[d, "\bar{f}"] \\
\bar{R} \arrow[r, shift left] \arrow[r, shift right] & \coprod U_i \arrow[r] & Y
\end{tikzcd}$$

where $\bar{V'}$ is an affine scheme and $\bar{V'}\to\bar{V}$ a representable etale covering of $\bar{V}$ and where $\bar{f}'$ is an etale map from the scheme $\bar{V'}$ to the scheme $U=\coprod U_i$. See [13], p. 101, for the details. Consider the commutative diagram

$$\begin{tikzcd}
 & & & X'\underset{U}{\times}\bar{V'} \arrow[dl, "f'"'] \arrow[dd, "\chi"] & & \\
R \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[ddr, "\varphi''"] & & X' \arrow[rr, "\pi"] \arrow[ddr, "\varphi'"'] & & X \arrow[ddr, "\varphi"'] & \\
 & R_{\bar{V'}} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[d, "\bar{f}''"'] & & \bar{V'} \arrow[rr] \arrow[d, "\bar{f}'"] & & \bar{V} \arrow[d, "\bar{f}"] \\
 & \bar{R} \arrow[rr, shift left, "\bar{\pi}_1"] \arrow[rr, shift right, "\bar{\pi}_2"'] & & U=\coprod U_i \arrow[rr] & & Y
\end{tikzcd}$$

where $X'\underset{U}{\times}\bar{V'}$ is the scheme product and $f'\colon X'\underset{U}{\times}\bar{V'}\to X'$ the projection map. This map is etale, because $\bar{f}'$ is etale. Therefore the map $\pi\circ f'$: $X'\times\bar{V'}\to X$ is also etale. It remains to show that the geometric quotient of $X'\underset{U}{\times}\bar{V'}$ modulo G is $\bar{V'}$.

First, the product $X'\underset{U}{\times}\bar{V'}$ is a subscheme of the scheme $X'\times\bar{V'}$, namely the inverse image of the diagonal under the map

$$X'\times\bar{V'}\xrightarrow{(\varphi',\bar{f}')}U\times U.$$

Second, the operation of G on $X'\underset{U}{\times}\bar{V'}$ is induced by the operation of G on X' and the map $X'\underset{U}{\times}\bar{V'}\xrightarrow{\chi}\bar{V'}$ maps different G-orbits to different points of $\bar{V'}$ and is surjective. Third, the map j: $\bar{V'}\to X'\underset{U}{\times}\bar{V'}$

$$v'\to((u,i(u)),v'),\quad\text{with}\quad u=\bar{f}'(v'),$$

$$i=\coprod\text{inj}\colon\ U\to X,$$

is a section of $X'\underset{U}{\times}\bar{V'}$ over $\bar{V'}$.

We can therefore apply the criterion from p. 8 and obtain the geometric quotient of $X'\underset{U}{\times}\bar{V'}$ modulo G which is isomorphic to $\bar{V'}$. This finishes the proof of statement 2) of Definition 1.5, if one takes $\overline{N}=\overline{V'}$ and $N=X'\times_U \overline{V'}$.

For the proof of statement 3) of Definition 1.5 we need the following lemma.

1.11. **Lemma.** *Let $R\rightrightarrows X'\to X$ be the representable G-stable etale covering from above and $R^* \overset{\pi_1^*}{\underset{\pi_2^*}{\rightrightarrows}} X^* \xrightarrow{\pi^*} X$ be any representable G-stable etale covering of X such that there exist etale maps $\alpha'\colon X^*\to X'$, $\alpha''\colon R^*\to R$ which are G-invariant and which induce the identity morphism on X via the commutative diagram*

$$\begin{tikzcd}
R^* \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] \arrow[d, "\alpha''"'] & X^* \arrow[r, "\pi^*"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"'] \\
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X' \arrow[r, "\pi"] & X.
\end{tikzcd} \tag{1}$$

Then the geometric quotients $\overline{X^*}$ and $\overline{R^*}$ of X^* and R^* modulo G exist. The maps π_1^*, π_2^* induce maps $\overline{\pi_1^*}$, $\overline{\pi_2^*}$

$$\begin{tikzcd}
\overline{R^*} \arrow[r, shift left, "\overline{\pi_1^*}"] \arrow[r, shift right, "\overline{\pi_2^*}"'] & \overline{X^*}
\end{tikzcd} \tag{2}$$

which define an etale equivalence relation on $\overline{X^*}$ and the algebraic space which is defined by the diagram (2) is isomorphic to Y.

Proof. Consider the subscheme $U=\coprod U_i$ of X' from above[9] and let $U^*=\alpha^{-1}(U)$ be the inverse image of U in X^*. The natural G-morphism $U^*\times G\xrightarrow{\delta^*}X^*$, defined by $(u^*,g)\xrightarrow{\delta^*}g(i^*(u))$ where $i^*\colon U^*\to X^*$ is the injection of U^* into X^*, is an isomorphism from $U^*\times G$ to X^*. This can be seen as follows. It is clear that δ^* is surjective and that the geometric fibres $\delta^{*-1}(P^*)$ consist of only one point for all $P^*\in X^*$, as G operates without fix points on X. Hence, δ^* is a radical morphism.

On the other hand we have the commutative diagram

$$\begin{tikzcd}
U^*\times G \arrow[d, "(\alpha'_U\times \mathrm{Id})"'] \arrow[dr, "\delta^*"] & \\
U\times G \arrow[d, "\delta"'] & X^* \arrow[dl, "\alpha'"] \\
X' &
\end{tikzcd}$$

where α'_U is the restriction of α' to U^*.

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[9] $U=\coprod U_i$ is a subscheme of X' by the map $U\to X'$, defined by $u\to(u,i(u))$, where $i=\coprod$ inj: $U\to X$.

The morphism $\alpha' \circ \delta^* = \delta \circ (\alpha'_U \times \mathrm{Id})$ is etale because, α'_U is etale and δ an isomorphism. Since α' is etale, δ^* is etale, see [7], p. 39. By [9], IV, 17.9.1 it follows that δ^* is an isomorphism. This fact implies, in particular, that the variety U^* is a geometric quotient of X modulo G in the scheme sense. The subvariety $U^* \underset{Y}{\times} X$ of $U^* \times X$ is defined on p. 9 with respect to the map $\pi^* i^*$: $U^* \to X$. G acts on $U^* \underset{Y}{\times} X$ by the rule $g((u^*, x)) = (u^*, g(x))$, for $g \in G(k)$, and the projection $U^* \underset{Y}{\times} X \xrightarrow{\varphi^*} U^*$ maps distinct orbits to distinct points of U. The morphism $U^* \xrightarrow{s^*} U^* \underset{Y}{\times} X$, defined by $u^* \xrightarrow{s^*} (u^*, \pi^* i^*(u^*))$ is a section of $U^* \underset{Y}{\times} X$ over U^*. Hence, (U^*, φ^*) is a geometric quotient of $U^* \underset{Y}{\times} X$ with respect to the action of G. This implies (use [22], Proposition 0.9) that the morphism $U^* \times G \xrightarrow{\rho^r} U^* \underset{Y}{\times} X$, defined by $\rho^*(u^*, g) = g(s^*(u^*))$ is an isomorphism. Then the morphism $X^* \xrightarrow{\rho^* \circ \delta^{*-1}} U^* \underset{Y}{\times} X$ is also an isomorphism.

We identify in the following X^* with $U^* \underset{Y}{\times} X$ via $\rho^* \circ \delta^{*-1}$. The map from $X^* = U^* \underset{Y}{\times} X$ to X which is induced by π^* coincides with the map from $U^* \underset{Y}{\times} X$ to X which is induced by the projection $U^* \times X \to X$.

After identifying X^* with $U^* \underset{Y}{\times} X$, R^* becomes a subscheme of $(U^* \underset{Y}{\times} X) \times (U^* \underset{Y}{\times} X)$ and its k-valued points are

$$R^*(k) = \{((u_1^*, x_1), (u_2^*, x_2));\ (u_i^*, x_i) \in U^* \underset{Y}{\times} X \text{ and } x_1 = x_2 \text{ in } X\}.$$

Let now $\overline{R^*}$ be the subvariety of $U^* \times U^*$ such that its k-valued points are

$$\overline{R^*}(k) = \{(u_1^*, u_2^*);\ u_i^* \in U^*(k) \text{ and there exists } g \in G(k) \text{ with } g(\pi^* i^*(u_1)) = \pi^* i^*(u_2)\}.$$

Then $\overline{R^*}$ defines an equivalence relation on U^* which is induced by the action of G on X.

One checks that the map $R^* \to \overline{R^*}$, defined by

$$((u_1^*, x_1), (u_2^*, x_2)) \to (u_1^*, u_2^*),$$

is a morphism which maps different orbits of R^* to different points of $\overline{R^*}$.

One checks also, as on p. 12, that the map

$$j^*\colon\ \overline{R^*} \to R^*,$$

defined by

$$j^*(u_1^*, u_2^*) = ((u_1^*, \pi^* i^*(u_2^*)), (u_2^*, \pi^* i^*(u_2^*)))$$

is a morphism which is a section of R^* over $\overline{R^*}$.

In addition the map $j^{*\prime}\colon \overline{R^*} \to R^*$, defined by

$$j^{*\prime}(u_1^*, u_2^*) = \big((u_1^*, \pi^* i^*(u_1^*)), (u_2^*, \pi^* i^*(u_1^*))\big)$$

is a morphism which is a section of R^* over $\overline{R^*}$.

In the same manner as on p. 12 ff. we now show that $\overline{R^*}$ is a geometric quotient of R^* modulo G in the sense of schemes.

From diagram (1) factoring out G we find

$$\begin{tikzcd}
\overline{R^*} \arrow[r, shift left, "\bar{\pi}_1^*"] \arrow[r, shift right, "\bar{\pi}_2^*"'] \arrow[d, "\bar{\alpha}''"] & U^* \arrow[d, "\bar{\alpha}'"] & \\
\bar{R} \arrow[r, shift left, "\bar{\pi}_1"] \arrow[r, shift right, "\bar{\pi}_2"'] & U \arrow[r, "\bar{\pi}"] & Y
\end{tikzcd}$$

where the maps $\bar{\alpha}''$ and $\bar{\alpha}'$ are etale and the diagram is commutative. (The proof is the same as on p. 13 for the map $\bar{\pi}_i$.) Furthermore, the maps $\bar{\pi}_1^*$ and $\bar{\pi}_2^*$ are etale and $\overline{R^*} \overset{\bar{\pi}_1^*}{\underset{\bar{\pi}_2^*}{\rightrightarrows}} U^*$ gives an etale equivalence relation on U^*. Let Y^* be the algebraic space, defined by it. We want to show that Y^* is equal to Y.

With respect to the etale maps $\bar{\pi}\circ\bar{\alpha}'\colon U^* \to Y$ the fibre product $U^* \underset{Y}{\times} U^*$ is isomorphic to the scheme $\overline{R^*}$. From this we see that $U^* \xrightarrow{\bar{\pi}\circ\bar{\alpha}'} Y$ is a representable etale covering of Y with $\overline{R^*}$ as equivalence relation and therefore that Y^* equals Y. See [13], p. 95. This proves Lemma 1.11.

Now to the proof of statement 3) of Definition 1.5. Let $f\colon X \to Z$ be a G-invariant morphism and let

$$R_{Z'} \rightrightarrows Z' \to Z$$

be a representable etale covering of Z, which we consider as a G-stable etale covering by taking the trivial action of G on Z' and $R_{Z'}$.

Consider the commutative diagram

$$\begin{tikzcd}
R_{\tilde{X}} \arrow[r, shift left, "\tilde{\pi}_1"] \arrow[r, shift right, "\tilde{\pi}_2"'] \arrow[d, "f''"] & \tilde{X} = X \underset{Z}{\times} Z' \arrow[r, "\tilde{\pi}"] \arrow[d, "f'"] & X \arrow[d, "f'"] \\
R_{Z'} \arrow[r, shift left] \arrow[r, shift right] & Z' \arrow[r] & Z.
\end{tikzcd}$$

The operation of G on X induces an operation of G on $\tilde{X}$ and on $R_{\tilde{X}}$ and the maps $\tilde{\pi}_i$, $\tilde{\pi}$ are etale. Let $R_{X'} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X' \xrightarrow{\pi} X$ be the G-stable etale covering of X from p. 11 and $\tilde{X} \underset{X}{\times} X'$ be the fibre product of $\tilde{X}$ and X'

over X. We have the diagram

$$\begin{tikzcd}
R_{\tilde{X}\times_X X'} \arrow[rr] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd, "\chi''"] & & R_{X'} \arrow[d, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] \\
& \tilde{X}\times_X X' \arrow[r] \arrow[d, "\chi'"] \arrow[dr, dashed] & X' \arrow[d, "\pi"] \\
R_{\tilde{X}} \arrow[r, shift left, "\tilde{\pi}_1"] \arrow[r, shift right, "\tilde{\pi}_2"'] \arrow[d, "f''"] & \tilde{X}=X\times_Z Z' \arrow[r, "\tilde{\pi}"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
R_{Z'} \arrow[r, shift left] \arrow[r, shift right] & Z' \arrow[r] & Z.
\end{tikzcd}$$

The group G operates on the schemes $\tilde{X}\times_X X'$ and $R_{\tilde{X}\times_X X'}$ in a way such that the induced action on X is the given one. Furthermore, by Lemma 1.11 the geometric quotients $\overline{\tilde{X}\times_X X'}$, respectively $\overline{R_{\tilde{X}\times_X X'}}$ of $\tilde{X}\times_X X'$, respectively, $R_{\tilde{X}\times_X X'}$ by the group G exist and we have an induced diagram

$$\overline{R_{\tilde{X}\times_X X'}} \rightrightarrows \overline{\tilde{X}\times X'}$$

which defines the algebraic space Y.

On the other hand, the maps $\tilde{X}\times_X X' \xrightarrow{f'\circ\kappa'} Z'$ and $R_{\tilde{X}\times_X X'} \xrightarrow{f''\circ\kappa''} R_{Z'}$ factor through $\overline{\tilde{X}\times_X X'}$ and $\overline{R_{\tilde{X}\times_X X'}}$. This gives the commutative diagram

$$\begin{tikzcd}
\overline{R_{\tilde{X}\times_X X'}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f''"] & \overline{\tilde{X}\times_X X'} \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "\bar{f}"] \\
R_{Z'} \arrow[r, shift left] \arrow[r, shift right] & Z' \arrow[r] & Z
\end{tikzcd}$$

and defines a unique morphism $\bar{f}$: $Y \to Z$ such that

$$\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "\bar{f}"] \\
Z &
\end{tikzcd}$$

is commutative. Lemma 1.9 and Theorem 1.6 are proved.

Theorem 1.6 shows that the geometric quotient in the category of algebraic spaces of a reduced and normal k-scheme on which a connected algebraic group operates freely and with closed graph always exists. However, for applications one needs the existence of a geometric quotient of a reduced, normal k-scheme on which a connected algebraic group operates with closed graph and with finite stabilizers.

Sometimes this can be established.

First we recall a theorem proved by Deligne (see [13], p. 183).

Theorem. *Let X be a separated algebraic k-space and G a finite group operating on X. Then the geometric quotient of X by G exists as an algebraic space.*

This theorem of Deligne can be used to prove the existence of a geometric quotient by a group action in the following more general situation:

1.12. **Theorem.** *Let X be an irreducible, reduced and normal k-scheme on which the connected algebraic group G operates. Assume that there exists a finite covering $X' \xrightarrow{f} X$ of X which is etale and galois with galois group A' and on which the group G operates without fix points and with closed graph. Assume further that the operation of G and A' commute and that the map f is a G-morphism. Then there exists an algebraic k-space of finite type which is a geometric quotient of X by G in the category of algebraic k-spaces.*

Proof. By Theorem 1.6 the geometric quotient Y' of X' modulo G exists and Y' is a separated algebraic space. The finite group A' operates then on Y'. To see this, we have to choose the representable etale covering of X' from p. 11 more carefully.

Consider the etale and galois covering $X' \xrightarrow{f} X$. By the proof of Proposition 1.10 in [27] one finds that there exist finitely many orbits $E_1, \dots, E_n$ on X, open G-stable subvarieties $W_1, \dots, W_n$ of X and locally closed subvarieties U_i of W_i, $i = 1, \dots, n$, such that

1) $E_i \subset W_i$, $i = 1, \dots, n$.

2) $\dim(U_i) = \dim(X) - \dim(G)$ and U_i intersects E_i transversally in finitely many points.

3) The G-morphisms $U_i \underset{Y}{\times} W_i$ satisfy statements 1) and 2) of Proposition 1.10 and $\bigcup_{i=1}^{n} W_i = X$.

U_i is the quotient of $U_i \underset{Y}{\times} W_i$ by G.

Let $W_i' = f^{-1}(W_i)$ and $U_i' = f^{-1}(U_i)$ be the inverse image of W_i and U_i. Then W_i' is G-stable, U_i' is a normal, locally closed subvariety of W_i' and we can assume that W_i' and U_i' satisfy Proposition 1.10. If necessary, one has to make U_i' and W_i' smaller, but one can always do this in such a way that A' operates on U_i' and W_i'.

It follows that the canonical maps $U_i' \times_{Y'} W_i' \to W_i'$ are etale maps (Y' is the quotient set of $X'(\operatorname{Spec}(k))$ under the operation of G), because G operates freely on X' (see Proposition 1.10). If we pick the W_i, $i = 1, \ldots, n$, in a proper way then the W_i' will cover X', and we obtain an etale covering

$$\tilde{X}' = \coprod_i (U_i' \times_Y W_i') \to X'.$$

Let $\tilde{R} = \tilde{X}' \times_{X'} \tilde{X}'$. We have then the diagram

$$\begin{tikzcd} \tilde{R} \arrow[r, shift left, "\tilde{\pi}_1"] \arrow[r, shift right, "\tilde{\pi}_2"'] & \tilde{X}' \arrow[r] & X', \end{tikzcd} \qquad (3)$$

where $\tilde{\pi}_1, \tilde{\pi}_2$ are etale maps and where X' is the quotient as an algebraic space. By Theorem 1.6 the group G can be factored out from the diagram (3) in the category of algebraic spaces. In doing so, we get an algebraic space Y' together with a representable etale covering

$$\begin{tikzcd} \bar{\tilde{R}} \arrow[r, shift left, "\bar{\pi}_1"] \arrow[r, shift right, "\bar{\pi}_2"'] & \bar{\tilde{X}}' = \coprod_i U_i' \arrow[r] & Y' \end{tikzcd}$$

and Y' is a geometric quotient of X' modulo G.

By construction the finite group, Λ' operates on the schemes U_i' and therefore also on $\overline{\coprod U_i'}$. But Λ' operates also on $\bar{\tilde{R}}$ as one checks easily.

One finds that the finite group Λ' operates on the separated algebraic space Y'. Let Y be the geometric quotient of Y', respectively Λ' which exists by the theorem due to Deligne.

Claim. *Y is a geometric quotient of X modulo G.*

Proof. We have the diagram of algebraic spaces

$$\begin{tikzcd} & X' \arrow[dl, "f"'] \arrow[dr, "\varphi'"] & \\ X \arrow[dr, "\varphi"'] & & Y' \arrow[dl] \\ & Y. & \end{tikzcd} \qquad (4)$$

By the universal property of X as a quotient of X' modulo Λ' we have a unique map $\varphi\colon X \to Y$ such that (4) is commutative. It is now easy to see that Y, together with the map $\varphi\colon X \to Y$ is a geometric quotient of X by G in the sense of Definition 1.5.

In applications of Theorem 1.12 it is often difficult to show that the scheme X on which G operates is normal.

One should try to drop this assumption. By transcendental methods this can be done if X is a reduced scheme of finite type over the complex numbers $\mathbb{C}$.

1.13. **Theorem.** *Let X be a reduced $\mathbb{C}$-scheme of finite type which may be reducible ($\mathbb{C}$ = complex numbers) and G a connected algebraic group defined over $\mathbb{C}$ which acts on X without fix points and with closed graph. There exists an algebraic $\mathbb{C}$-space of finite type which is a geometric quotient of X by G.*

Proof. Let P be a point of X and E_P the orbit of P by G. X_0 shall be an affine open subset of X containing P. Let λ: $X_0 \to \mathbb{C}^N$ be a closed embedding which is fixed in the following. Take an irreducible, smooth subspace L of the $\mathbb{C}^N$ of dimension N-dim (E_P) with the following properties.

1) $L \cap \lambda(E_P \cap X_0)$ is non empty and consists of finitely many points. Furthermore, the intersection of L and $\lambda(E_P \cap X_0)$ is transversal.

2) If $\hat{X}_0$ is the maximal open subvariety of X_0 consisting of the normal points of X, then $\hat{X}_0 \cap L$ is a normal variety. (By [26] this is always possible.)

Let $U' = \lambda^{-1}(L \cap \lambda(X_0))$ and consider the morphism

$$U' \times G \xrightarrow{\kappa} X$$

defined by $(u', g) \to g(i(u'))$, where i: $U' \to X$ is the injection morphism.

Let $Q \in \lambda^{-1}(L \cap \lambda(E_P \cap X_0)$. Then $\kappa^{-1}(Q) = \{(u', g) \in U' \times G, g(i(u')) = Q\}$ is finite.

Claim. *For a point $Q' = (u', g) \in \kappa^{-1}(Q)$ the map κ is a local analytic isomorphism from $U' \times G$ to X.*

A proof of this claim is omitted here, for it is done along the lines of the proof of Hilfsatz 1 of Homann's paper [10].

The fact that κ is a local analytic isomorphism at a point $Q' \in \kappa^{-1}(Q)$ signifies that the completions of the local rings of $Q' \in U' \times G$ and $Q \in X$ are isomorphic. Hence the map κ is etale at every point $Q' \in \kappa^{-1}(Q)$.

Let V be the maximal open subvariety of $U' \times G$ on which κ is etale. Then $\kappa(V) = W$ is an open subvariety of X, because the map κ: $V \to X$ is open. (See [7].) Also W contains the orbit E_P and is G-stable. The image U of V by the projection of $U' \times G$ onto U' is an open subvariety of U' too, for the map $G \to \operatorname{Spec}(\mathbb{C})$ is universally open and therefore the map $U' \times G \to U'$ open.

Consider the product $U \times G$. Obviously one has $U \times G \supset V$. On the other hand if $(u, g) \in V$, $\{u\} \times G \subseteq V$, as $U' \times G \to X$ is a G-morphism. This shows that $U \times G \cong V$ and in addition that $U \subset W$.

Let $U \underset{Y}{\times} W$ be the variety defined on p. 9. Then one shows as before that the morphism δ: $U \times G \to U \underset{Y}{\times} W$, defined by $\delta(u, g) = g((u, i(u)))$, is an isomorphism ($i$ = embedding of U in W).

We have shown so far the following.

1.14. **Proposition.** *Let the assumption be as in Theorem* 1.13 *and let* E_P *be an orbit of the action of* G *on* X. *There exists an open neighbourhood* W *of* E *which is* G*-stable and a locally closed subvariety* U *of* W *such that the* G*-map* $U \underset{Y}{\times} W \cong U \times G \to W$ *is etale and surjective.*

Obviously, Proposition 1.14 is analog to the proposition of Seshadri from p. 10. Now the proof of Theorem 1.13 is the same as the proof of Theorem 1.6. One has only to use Proposition 1.14 instead of the proposition from p. 10. Minor changes in the proof of Theorem 1.12 lead to the following:

1.15. **Theorem.** *Let* X *be a reduced* $\mathbb{C}$*-scheme of finite type on which the connected algebraic group* G, *which is defined over* $\mathbb{C}$, *operates. Assume that there exists a finite covering* $X' \xrightarrow{f} X$ *of* X *which is etale and galois with galois group* Λ'. *Assume further that the group* G *operates with closed graph and without fix points on* X'.

Also assume that the operation of G *and* Λ' *commute and that* f *is a* G*-morphism. Then the geometric quotient of* X *modulo* G *exists and is an algebraic space of finite type over* $\mathbb{C}$.

Remark. Theorem 1.6 and Theorem 1.13 give a quite explicit description of the quotient Y of X modulo G. Roughly speaking, the following holds: Let E be an orbit of G on X and P the corresponding point on Y, then Y is locally at P isomorphic to a subvariety U of X which has a dimension equal to $\dim(X) - \dim(E)$ and which intersects E transversally. If G operates with fix points the local description of Y, as given in Theorem 1.12 and Theorem 1.15, is a little more delicate.

II. The Moduli Space of Polarized Varieties

In this chapter we describe a method which gives, under certain conditions, the existence of the coarse moduli space for the global deformation functor of a polarized variety.

All schemes and algebraic spaces are $\mathbb{C}$-spaces, where $\mathbb{C}$ is the complex number field[10].

An irreducible, smooth and projective $\mathbb{C}$-scheme is called a *smooth variety* over $\mathbb{C}$.

2.1. *Definition.* $(V, \mathfrak{X})$, where V is a smooth variety over $\mathbb{C}$, and $\mathfrak{X}$ is an algebraic equivalence class of divisors of V which contain a very

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[10] That we work over $\mathbb{C}$ is not essential in the first part of this chapter; we could take instead of $\mathbb{C}$ an arbitrary field k. It becomes essential when we later apply Theorem 1.15.

ample divisor is called a *polarized variety*[11]. A divisor $X \in \mathfrak{X}$ is called a *polar divisor* of $(V, \mathfrak{X})$.

2.2. *Definition.* A polarized variety $(V, \mathfrak{X})$ is called *canonically polarized* if $\mathfrak{X}$ contains a very ample multiple of a canonical divisor of V.

Automorphisms and *isomorphisms* of polarized varieties are defined as usual. A *projective embedding* of a polarized variety V is a projective embedding of the underlying variety V into a P^N/k, determined by a very ample linear system of V which consists of polar divisors.

Let $\mathrm{Pic}(V)$ denote the Picard-scheme of V. See [9]. $\mathrm{Pic}^0(V)$ shall denote the connected component of $\mathrm{Pic}(V)$ which contains the identity. Using the properties of $\mathrm{Pic}(V)$ there is another equivalent manner to define the notion of a polarization on V.

2.3. *Definition.* $(V, \mathfrak{X})$, where V is a smooth variety and $\mathfrak{X}$ a coset of $\mathrm{Pic}^0(X)$ in $\mathrm{Pic}(X)$ which contains a very ample invertible sheaf, is called a *polarized variety*.

We notice for the equivalence of Definitions 2.1 and 2.3 that on V the divisor classes with respect to linear equivalence, are functorially in a 1-1 correspondence with the classes of invertible sheaves on V. By the functorial properties of $\mathrm{Pic}(V)$ the k-rational points of $\mathrm{Pic}(V)$ parametrize the divisor classes of V. By Matsusaka [18] the k-rational points of $\mathrm{Pic}^0(V)$ parametrize exactly the divisor classes which are algebraically equivalent to zero. With these remarks the equivalence of the definitions is obvious.

Let $(V, \mathfrak{X})$ be a polarized variety and Z a polar divisor. $\chi(V, mZ) = h(m)$, where $\chi(V, mZ)$ is the Euler characteristic of the invertible sheaf determined by the divisor mZ, is then a polynomial in m which is independent of the choice of the polar divisor and therefore uniquely determined by $(V, \mathfrak{X})$. (Notice that if Z and Z' are algebraically equivalent divisors of V, then $\chi(V, mZ) = \chi(V, mZ')$ for all integers m. Compare [19], Prop. 3.1 and Prop. 3.2.) We shall call this polynomial the *Hilbert polynomial* of $(V, \mathfrak{X})$.

We need to consider families of polarized varieties.

2.4 *Definition.* A *smooth* and *projective* morphism $f: V \to S$ of schemes, where the base S is noetherian and the fibres are varieties is called a *projective family of varieties* with base S. Denote such a family by V/S.

$\mathrm{Pic}(V/S)$ shall be the relative Picard-scheme of the family V/S. $\mathrm{Pic}(V/S)$ is a group scheme over S, locally of finite type, such that among other things the following holds: For every S-scheme T with $\alpha: T \to S$ as map

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[11] This notion is actually what is called in the literature an inhomogeneous polarized variety. Compare [22] and the remark on p. 25.

for which $V \underset{S}{\times} T \to T$ has a section over T one has

$$\operatorname{Hom}(T, \operatorname{Pic}(V/S)) = \frac{\{\text{group of invertible sheaves of } V \underset{S}{\times} T\}}{\left\{\begin{array}{c}\text{subgroup of sheaves } f^*(L), L \text{ an} \\ \text{invertible sheaf on } T\end{array}\right\}}.$$

For the existence and for further properties of Pic(V/S) see Grothendieck [9].

$\operatorname{Pic}^0(V/S)$ shall denote the connected component of Pic(V/S) which contains the neutral element of Pic(V/S).

2.5. *Definition.* $(V/S, \mathfrak{X}/S)$ is called a *family of polarized varieties* if V/S is a projective family of varieties over S, S noetherian, and $\mathfrak{X}/S$ a coset of $\operatorname{Hom}(S, \operatorname{Pic}^0(V/S))$ in $\operatorname{Hom}(S, \operatorname{Pic}(V/S))$, containing an invertible sheaf which is very ample respectively S. An element $X \in \mathfrak{X}$ is called a *polar sheaf* of V/S.

$(V/S, \mathfrak{X}/S)$ is called *canoncially polarized* if $\mathfrak{X}/S$ contains a very ample multiple of the sheaf of regular differential n-forms of V/S, n = dimension of the fibres of V/S.

Let $(V/S, \mathfrak{X}/S)$ be a family of polarized (projective) varieties and let $X \in \mathfrak{X}/S$ be a polar sheaf. $\operatorname{Spec}(\Omega) \to S$ shall be a geometric point of S and $V_s/\Omega = V \underset{S}{\times} \operatorname{Spec}(\Omega)$ the fibre over s. X_s will signify the sheaf of V_s which is induced by X and $\mathfrak{X}_s$ the polarization of V_s which is determined by X_s. $(V_s, \mathfrak{X}_s)$ is then a polarized variety induced by $(V/S, \mathfrak{X}/S)$.

Let $h_s(x)$ denote the Hilbert polynomial of $(V_s, \mathfrak{X}_s)$ as defined above.

2.6. **Proposition.** *The polynomial $h_s(x)$ is independent of s.*

Proof. Follows from [8], III.

Let $(V_0, \mathfrak{X}_0)$ be a polarized variety defined over $\mathbb{C}$ and let $(V/S, \mathfrak{X}/S)$ be a family of polarized varieties (in the sense of Definition 2.5) such that $(V_0, \mathfrak{X}_0)$ is isomorphic to a geometric fibre of $(V/S, \mathfrak{X}/S)$. Assume that the base scheme S is a connected $\mathbb{C}$-scheme. Then, we define:

2.7. *Definition.* A geometric fibre $(V_s, \mathfrak{X}_s)$ of $(V/S, \mathfrak{X}/S)$ is called a *deformation* of $(V_0, \mathfrak{X}_0)$.

$(V_0, \mathfrak{X}_0)$ shall be fixed in the following. For a noetherian $\mathbb{C}$-scheme S we consider smooth, projective families $(V/S, \mathfrak{X}/S)$ with a polarization $\mathfrak{X}/S$ where the fibres of V/S are deformations of $(V_0, \mathfrak{X}_0)$. If $(V_0, \mathfrak{X}_0)$ is canonically polarized we consider only families $(V/S, \mathfrak{X}/S)$ of deformations of $(V_0, \mathfrak{X}_0)$ which are also canonically polarized.

We call such a family a *family of deformations* of $(V_0, \mathfrak{X}_0)$. (Notice that $(V/S, \mathfrak{X}/S)$ does not have to contain a fibre which is isomorphic to $(V_0, \mathfrak{X}_0)$.)

Let $\mathcal{M}(S)$ be the set of families $(V/S, \mathfrak{X}/S)$ of deformations of $(V_0, \mathfrak{X}_0)$ up to isomorphism. Clearly, the collection of sets $\mathcal{M}(S)$ form a contravariant functor from the category of noetherian $\mathbb{C}$-schemes to the category of sets, i.e., given a morphism f: $T \to S$, a map $\mathcal{M}(f)$: $\mathcal{M}(S) \to \mathcal{M}(T)$ is defined by associating to a polarized family $(V/S, \mathfrak{X}/S)$ the pullback $(V \underset{S}{\times} T/T, \mathfrak{X} \underset{S}{\times} T/T)$. This functor is called the *global deformation functor* of $(V_0, \mathfrak{X}_0)$.

Remark. As already stated there exists in the literature the notion of a homogeneous polarized variety. Compare [19]. This notion is more intrinsic than the notion of a polarized variety we are considering. For a homogeneous polarized variety $(V_0, \tilde{\mathfrak{X}}_0)$ one defines as above deformations, which are again homogeneous polarized varieties, and also the functor $\tilde{\mathcal{M}}$ of deformations of the homogeneous polarized variety $(V_0, \tilde{\mathfrak{X}}_0)$. If $(V_0, \mathfrak{X}_0)$ is a polarized variety in the sense of Definition 2.1 we associate to it a uniquely determined homogeneous polarized variety $(V_0, \tilde{\mathfrak{X}}_0)$. Let $\mathcal{M}$ respectively $\tilde{\mathcal{M}}$ be the deformation functors of $(V_0, \mathfrak{X}_0)$ respectively $(V_0, \tilde{\mathfrak{X}}_0)$ then the following can be shown:

If the deformations of $(V_0, \tilde{\mathfrak{X}}_0)$ satisfy statement $(*)$ in [19], p. 206, for sufficiently large integers m the deformation functor of the polarized variety $(V_0, m \cdot \mathfrak{X}_0)$ and the deformation functor of $(V_0, \tilde{\mathfrak{X}}_0)$ have the same sheafification with respect to the etale topology. For a proof of this fact the results in [19] and [8], IV, 17.16, have to be used. As $(*)$ is known to be true for canonically polarized varieties in characteristic 0, [20], the remark shows that for such varieties there is no essential difference between the functors $\mathcal{M}$ and $\hat{\mathcal{M}}$.

Back to the functor $\mathcal{M}$.

2.8. *Definition.* An algebraic space M of finite type over $\mathbb{C}$ and a morphism Φ from the functor $\mathcal{M}$ to the functor $h_M(S) = \operatorname{Hom}(S, M)$, represented by M in the category of algebraic spaces, is called a *coarse moduli space,* if

a) for every algebraically closed field k^* which contains $\mathbb{C}$ the map $\Phi(\operatorname{Spec}(k^*))$: $\mathcal{M}(\operatorname{Spec}(k^*)) \to h_M(\operatorname{Spec}(k^*))$ is an isomorphism.

b) Given an algebraic space N and a morphism φ from $\mathcal{M}$ to the representable functor h_N, there is a unique morphism χ: $h_M \to h_N$ such that $\psi = \chi \circ \Phi$.

It is the aim of this paper to develop a method which implies the existence of a coarse moduli space for the deformation functor in certain cases. However, the proposed method will sometimes only work, if one changes the above situation as follows.

Consider the category C_{red} of reduced, noetherian $\mathbb{C}$-schemes and let $\mathcal{M}_{\text{red}}$ be the restriction of the deformation functor $\mathcal{M}$ of $(V_0, \mathfrak{X}_0)$ to this category.

2.9. *Definition.* A reduced, algebraic space $\overline{M}$ of finite type over $\mathbb{C}$

and a morphism Φ from $\mathcal{M}_{\mathrm{red}}$ to the functor $h_{\bar{M}}(S) = \mathrm{Hom}(S, \bar{M})$, $S \in C_{\mathrm{red}}$, is called a coarse moduli space for the functor $\mathcal{M}_{\mathrm{red}}$ if

a) for every algebraically closed field k^* which contains $\mathbb{C}$, the map $\Phi(\mathrm{Spec}(k^*))$: $\mathcal{M}(\mathrm{Spec}(k^*)) \to h_{\bar{M}}(\mathrm{Spec}(k^*))$ is an isomorphism.

b) Given a reduced algebraic space N and a morphism ψ from $\mathcal{M}_{\mathrm{red}}$ to the representable functor h_N, there is a unique morphism χ: $h_{\bar{M}} \to h_N$ such that $\psi = \chi \circ \Phi$.

Roughly speaking, the coarse moduli space for the functor $\mathcal{M}_{\mathrm{red}}$ is the reduced algebraic space which corresponds to the coarse moduli space M of the functor $\mathcal{M}$, provided M exists.

Let $\mathcal{M}(S)$ be again the global deformation functor of the polarized variety $(V_0, \mathfrak{X}_0)$. $h(x)$ shall be the Hilbert polynomial of $(V_0, \mathfrak{X}_0)$.

We assume in the following[12] that the families which belong to $\mathcal{M}(S)$ are canonically polarized or that the irregularity of $(V_0, \mathfrak{X}_0)$ together with the irregularities then of the fibres of the families which belong to $\mathcal{M}(S)$ are zero.

We *assume further* that the polarization of the families of $\mathcal{M}(S)$ is *sufficiently ample*, i.e., for every geometric fibre $(V_s, \mathfrak{X}_s)$ of a family $(V/S, \mathfrak{X}/S)$ of $\mathcal{M}(S)$ we have (1) X_s is very ample whenever $X_s \in \mathfrak{X}_s$; (2) $h^i(V_s, \mathcal{L}(n \cdot X_s)) = 0$ for all $X_s \in \mathfrak{X}_s$ and all $i > 0$, $n > 0$.

The following proposition shows that the last assumption made on $\mathcal{M}$ is not too serious and how one can change from a polarization to a sufficiently ample polarization.

2.10. **Proposition.** *There exists an integer $m > 0$ such that the deformation functor of the polarized variety $(V_0, m \cdot \mathfrak{X}_0)$ contains only families with a sufficiently ample polarization.*

Proof. By [19], Theorem 1, there exists an integer $c > 0$ which depends only on the polynomial $\chi(V_0, nX_0) = h(n)$, X_0 a divisor in $\mathfrak{X}_0$, such that for $t > c$ every deformation (V_s, Y_s) of $(V_0, t \cdot \mathfrak{X}_0)$ is sufficiently ample polarized. Pick a fixed integer $m > c$.

2.11. **Proposition.** *There exists a projective space P^N such that every family $V \xrightarrow{\omega} S$ which belongs to $\mathcal{M}(S)$ can be locally embedded into P^N, i.e., there exists a finite open covering $\{U_i\}$ of S such that the induced families $V \underset{S}{\times} U_i/U_i = V_i/U_i$ can be embedded into the projective space $P^N \times U_i$ in such a way that the fibres of V_i/U_i are embedded as polarized varieties with the induced polarization and do not lay in a hyperplane of the P^N. Let V_i/U_i be the embedding of V_i/U_i. Then, the Hilbert polynomials of the fibres of the families V_i/U_i are the same.*

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[12] We exclude the case of an abelian variety. There the considerations are different as explained in Chapter III.

Proof. Consider first the case when the families in $\mathcal{M}(S)$ are canonically polarized. Let $(V/S, \mathfrak{X}/S)$ be a family in $\mathcal{M}(S)$ and let $\Omega^n_{V/S}$ be the sheaf of relative differential n-forms $(n = \dim(V_0))$ of V/S. By the definition of canonically polarized families and by the assumption made on $\mathcal{M}$ there exists an integer $\alpha > 0$, which is the same for all families in $\mathcal{M}(S)$ and such that the sheaf $(\Omega^n_{V/S})^{\otimes\alpha}$, which belongs to $\mathfrak{X}/S$, is sufficiently ample relative to S. This implies (see [22], p. 19) that the sheaf $\omega_*((\Omega^n_{V/S})^{\otimes\alpha}) = \mathsf{E}$ is locally free of rank $(N+1)$. Let $\{U_i\}$ be an open covering of S such that E/U_i is a free sheaf for all i. Then $P(\mathsf{E}/U_i) = P^N \times U_i$ (notation as in [8], II) and the very ample sheaf $(\Omega^n_{V/S})^{\otimes\alpha}$ defines a closed immersion (over U_i) of the families $V_i \xrightarrow{\omega_i} U_i$ into the projective space $P^N \times U_i$. Denote this embedding by $\tilde{V}_i \to U_i$. It is then clear by the construction that the geometric fibres of the families $\tilde{V}_i/U_i$ do not lay in a hyperplane of the P^N.

Now, the geometric fibres $(V_s, \mathfrak{X}_s)$ of $(V/S, \mathfrak{X}/S)$ are deformations of $(V_0, \mathfrak{X}_0)$. Let $\tilde{V}_0/k$ be an embedding of $(V_0, \mathfrak{X}_0)$ into P^N/k determined by $(\Omega^n_{V_0})^{\otimes\alpha}$. Then $h(x)$ is the Hilbert polynomial of $\tilde{V}_0$. From [8], III, 7.9, we conclude once more that the Hilbert polynomials of the fibres of the embedding $\tilde{V}_i/U_i$ are equal to $h(x)$.

Next, we treat the case when the irregularity of the fibres is zero. Then, for every family $(V/S, \mathfrak{X}/S)$ of $\mathcal{M}(S)$, there is essentially only one invertible sheaf $\mathcal{L}$ of V in the set $\mathfrak{X}/S$ ($\mathcal{L}$ is up to tensoring with an invertible sheaf, coming from S, uniquely determined) and this sheaf is sufficiently ample. Let $\omega_*(\mathcal{L})$ be the direct image of the sheaf $\mathcal{L}$ on S. By [22], p. 19, $\omega_*(\mathcal{L})$ is again locally free of rank $N+1$, and we can pick an open covering $\{U_i\}$ of S with $\omega_* \mathcal{L}/U_i$ free so that the above arguments apply. Q.E.D.

In the following P^N is always the projective space from Proposition 2.11 and $h(x)$ the Hilbert polynomial of the geometric fibres of the embedded families which belong to $\mathcal{M}(S)$. $H^{h(x)}_{P^N}$ shall be the Hilbert scheme which parametrizes the proper and flat families of subvarieties of the P^N with $h(x)$ as Hilbert polynomial.

2.12. **Proposition.** *There exists a uniquely determined, connected subscheme H of the Hilbert scheme $H^{h(x)}_{P^N}$ such that, for every connected $\mathbb{C}$-scheme S and every morphism $f\colon S \to H^{h(x)}_{P}$, f factors through H if and only if the conditions* 1)–4) *are satisfied:*

1) *The family V/S in $P^N \times S$, induced by f via the universal mapping properties of $H^{h(x)}_{P^N}$ is a smooth family with connected fibres.*

2)[13] *If the families in $\mathcal{M}(S)$ are canonically polarized, the invertible sheaf on V/S, induced by the sheaf of hyperplane sections $O_{P^N}(1)$, is isomorphic to*

$$(\Omega^n_{V/S})^{\otimes\alpha} \otimes \omega^*(L)$$

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[13] If the polarization is not canonical, the assumption 2) is to be dropped.

for a suitable, invertible sheaf L on S. (ω: $V \to S$ is the morphism of the family V/S.)

3) *For every geometric point $s \in S$ the fibre V_s of V/S does not lie in a hyperplane.*

4) *Let $\Gamma_H \to H$ be the pullback of the universal family of the Hilbert scheme to H. Then every family V/S belonging to $\mathcal{M}$ is locally a pullback of $\Gamma_H \to H$. Furthermore, every deformation of $(V_0, \mathfrak{X}_0)$ in the sense of Definition 2.7 is isomorphic to a fibre of the family $\Gamma_H \to H$.*

Proof. Let $\Gamma/H^{h(x)}_{P^N}$ be the universal family in $P^N \times H^{h(x)}_{P^N}$ which belongs to $H^{h(x)}_{P^N}$. By the construction of $H^{h(x)}_{P^N}$ the polarized variety $(V_0, \mathfrak{X}_0)$ is isomorphic to a geometric fibre $(\Gamma_0, \mathfrak{Y}_0)$ of $\Gamma/H^{h(x)}_{P^N}$, where $\mathfrak{Y}_0$ denotes to polarization of Γ_0 determined by the hyperplane sections. Using that $(V_0, \mathfrak{X}_0)$ is sufficially ample polarized one concludes as in [22], Prop. 5.1, that there exists a subscheme U of $H^{h(x)}_{P^N}$ such that conditions 1)–3) are precisely realized on U. One checks that U contains a $\mathbb{C}$-valued point s such that the fibre of $\Gamma/H^{h(x)}_{P^N}$ over s is isomorphic to $(V_0, \mathfrak{X}_0)$. Then the connected component H of s in U satisfies the proposition.

Because of the universal mapping properties of the scheme $H^{h(x)}_{P^N}$, the group $PGL(N)$ operates on $H^{h(x)}_{P^N}$. This operation induces an operation of $PGL(N)$ on the scheme H, for the condition 1)–4) are invariant under $PGL(N)$.

Let $h_H = \mathrm{Hom}(-, H)$ be the Hom-functor of H. By Proposition 2.12 there is a natural morphism of functors

$$\pi\colon\ h_H(S) \to \mathcal{M}(S).$$

Let $\mathcal{PGL}(N)$ be the functor Hom $(S, PGL(N))$. Then, for every connected noetherian $\mathbb{C}$-scheme S, the action of $PGL(N)$ on H induces an action of the group $\mathcal{PGL}(N)(S)$ in the set $h_H(S)$ functorial in S. Denote this action by Φ. We have then

$$\pi \circ \Phi = \pi \circ p_2, \tag{5}$$

where p_2 is a projection morphism from $\mathcal{PGL}(N) \times h_H$ to h_H.

2.13. *Definition.* For all connected noetherian schemes S let $\mathcal{M}'(S)$ be the quotient of the set $h_H(S)$ by the action of the group $\mathcal{PGL}(N)(S)$. Let $\mathcal{M}'$ be the functor defined by this collection of sets and by the obvious maps between them. The Eq. (5) implies that π factors

$$\begin{tikzcd} h_H \arrow[r, "\pi'"] \arrow[rr, bend right, "\pi"'] & \mathcal{M}' \arrow[r, "I"] & \mathcal{M}. \end{tikzcd}$$

2.14. **Proposition.** *I is injective*[14] *and for an $\omega \in \mathcal{M}(S)$ there is a covering $\{U_j\}$ of S such that the restriction of ω to $\mathcal{M}(U_j)$ is in the image of I, for all j.*

---

[14] This means, roughly speaking, that two polarized families of deformation of $(V_0, \mathfrak{X}_0)$ are isomorphic if and only if their embeddings into P^N/S are projectively equivalent.

Proof. The first statement is proved exactly as in [22], p. 101. The second statement follows from Proposition 2.12.

The connection between the geometric quotient $\bar{H}$ of H by $PGL(N)$ and the coarse moduli space of the functor $\mathcal{M}$ is described in the following proposition.

2.15. **Proposition.** *The geometric quotient $\bar{H}$ of H by $PGL(N)$ is a coarse moduli space for the deformation functor (all in the category of algebraic spaces of $\mathbb{C}$).*

Proof. First, the morphisms ψ from $\mathcal{M}$ to a representable functor h_N, N an algebraic space over $\mathbb{C}$, and the set of morphisms f from the scheme H to the algebraic space N such that (6) $f \circ \Phi = f \circ p_2$ ($p_2: PGL(N) \times H \to H$ is the projection morphism, $\Phi =$ action of $PGL(N)$ on H) are canonically isomorphic. This is seen as follows. Consider the universal family Γ_H/H, given by the construction of H (see Proposition 2.12) and the element $\gamma \in \mathcal{M}(H)$, determined by Γ_H/H. Associate to the morphism $\psi: \mathcal{M} \to h_N$ the morphism $\psi(\gamma): H \to N$ which maps an S-valued point s of H to the S-valued point $\psi(\gamma)(s)$ of N where $\psi(\gamma)(s) = \psi((\mathcal{M}(s))(\gamma))$ via the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "s"'] & \mathcal{M}(S) \arrow[r, "\psi"] & h_N(S) \\
H \arrow[r] & \mathcal{M}(H). \arrow[u, "\mathcal{M}(s)"'] &
\end{tikzcd}$$

Obviously, $\psi(\gamma)$ satisfies then the Eq (6).

Conversely, if $f: h_H \to h_N$ is a morphism which satisfies (6) then f factors through the functor $\mathcal{M}'$, and we have an induced morphism

$$f': \mathcal{M}' \to h_N.$$

By Proposition 2.14 this morphism f' induces a unique morphism $\psi_f: \mathcal{M} \to h_N$. It is clear that

$$h_H(\mathrm{Spec}\,(k^*)) = \mathcal{M}(\mathrm{Spec}\,(k^*))$$

holds for all algebraically closed field k^* which contain $\mathbb{C}$. Q.E.D.

Next, let H_{red} be the reduced scheme which belongs to H. The action of $PGL(N)$ on H induces an action of $PGL(N)$ on H_{red}. If we consider H_{red} with this action, the arguments in the proof of Proposition 2.15 lead to the following:

2.16. **Proposition.** *If the geometric quotient $\overline{H_{\mathrm{red}}}$ of H_{red} by $PGL(N)$ exists in the category of algebraic spaces, it is a coarse moduli space in the sense of Definition 2.9 for the functor $\mathcal{M}_{\mathrm{red}}$ in this category.*

The method for the construction of the coarse moduli space for the deformation functors $\mathcal{M}$ or $\mathcal{M}_{\mathrm{red}}$ of a polarized variety $(V_0, \mathfrak{X}_0)$ follows:

We assume that $\mathcal{M}$ satisfies the assumption made on p. 26.

Let H be the scheme constructed in the proof of Proposition 2.12 with respect to $\mathcal{M}$ and H_{red} the corresponding reduced scheme. Then the group $PGL(N)$ operates on H and on H_{red} and the geometric quotients $\bar{H}$ of H modulo $PGL(N)$, respectively, $\overline{H_{\mathrm{red}}}$ of H_{red} by $PGL(N)$, if they exist as algebraic spaces, are coarse moduli spaces for the deformation functor $\mathcal{M}$, respectively, $\mathcal{M}_{\mathrm{red}}$. Compare Proposition 2.15 and Proposition 2.16. Of course, we would like to apply Theorem 1.13 or Theorem 1.15 to prove the existence of $\bar{H}$ and $\overline{H_{\mathrm{red}}}$. But there are difficulties.

First, one needs that the scheme H is reduced. So we have to consider H_{red} and the functor $\mathcal{M}_{\mathrm{red}}$ in general. Only if we can insure that H is reduced, we can employ H and $\mathcal{M}$.

Unfortunately, not much is known about the reducedness of the Hilbert scheme. Only in special cases (for curves and K-3 surfaces) is the scheme H known to be smooth and thus reduced. Secondly, one needs that the group $PGL(N)$ operates with closed graph on H_{red}. This is not too serious and is satisfied in many cases (see Chapter III). We show in this connection the following lemma.

Lemma. *If the fibres of the family* $\Gamma_H \to H$ *are unruled varieties, the action of* $PGL(N)$ *on* H *is proper, i.e., the map*

$$\Phi\colon H \times PGL(N) \longrightarrow H \times H$$
$$(x, g) \longrightarrow (x, g(x))$$

is proper. The induced action of $PGL(N)$ *on* H_{red} *is then also proper.*

Proof. By the valuation criterion for proper morphism ([8], Chapter II,7), we have to show the following:

Let R be a discrete valuation ring over $\mathbb{C}$ and g: Spec$(R) \to H \times PGL(N)$ a rational map. Let h: Spec$(R) \to H \times H$ be a morphism such that $h = \Phi \circ g$ as rational map. Then there exists a morphism g': Spec$(R) \to H \times PGL(N)$ which coincides with g on the general point of Spec(R).

Theorem 2 of Matsusaka and Mumford [21] states the following:

Let V and V' be polarized, smooth unruled varieties, defined over Spec(R), such that their specializations $\tilde{V}$ and $\tilde{V}'$ are also smooth, unruled polarized varieties. Then the specialization of any isomorphism ρ: $V \to V'$ is an isomorphism $\tilde{\rho}$: $\tilde{V} \to \tilde{V}'$.

This theorem can be applied to establish the valuation criterion for Φ.

Let h and g be as above.

Then h determines two varieties V and V' over Spec(R). The map g determines an isomorphism ρ: $V \times$ Spec$(K) \to V' \times$ Spec(K), $K =$ quotient field of R. g gives also a rational map $\operatorname{Spec}(R) \to PGL(N)$ and a morphism g' of $\operatorname{Spec}(R)$ into the projective closure $\overline{PGL(N)}$ of $PGL(N)$.

Let $\tilde{\rho}$ be the specialization of ρ. By the theorem of Matsusaka and Mumford, $\tilde{\rho}$ is an isomorphism of $\tilde{V}$and $\tilde{V}'$. Using Proposition 2.14 this implies that $\tilde{\rho}$ is induced by a projective transformation and that g' is a morphism of $\operatorname{Spec}(R)$ into $PGL(N)$. Hence, Φ satisfies the valuation criterion and is therefore proper.

Third, one needs that there exists a finite galois covering H' of H_{red} which is etale and on which $PGL(N)$ operates without fix points. Notice that $PGL(N)$ operates in general with fixed points on H_{red}. This follows from the definition of the action of $PGL(N)$ on H and H_{red} and the universal mapping properties of H which imply that for every geometric point s of H, respectively, H_{red}, the stabilizer of s by the action of $PGL(N)$ is isomorphic to the automorphism group of the polarized variety which is determined by s.

In certain cases a galois covering H'_{red} of H_{red} on which $PGL(N)$ operates without fix points can be constructed. Examples are curves, abelian varieties, polarized K-3 surfaces and canonical polarized varieties for which the canonical sheaf is very ample.

We consider for the construction of H'_{red} all $\mathbb{C}$-schemes of finite type and observe the universal family $\Gamma_H \to H$. By [16], Chapter II, III, the family $\Gamma_H \to H$ is locally a topological product with respect to the complex topology, i.e., for every $\mathbb{C}$-valued point s there exists a complex open neighbourhood U_s of s on H such that $f^{-1}(U_s)$ is homeomorphic to $\Gamma_s \times U_s$, where Γ_s is the fibre of Γ_H/H over s.

Let Γ_{s_0} be a fibre of $\Gamma_H \to H$ which is isomorphic to V_0 and let $H^i(\Gamma_{s_0}, \mathbb{Z})^*$ be the free part of the integral cohomology groups of Γ_{s_0}. $\Gamma_H \to H$ locally a topological product and H connected imply that for every fibre Γ_s of $\Gamma_H \to H$ the integral cohomology group $H^i(\Gamma_s, \mathbb{Z})^*$ is isomorphic to $H^i(\Gamma_{s_0}, \mathbb{Z})^*$ and hence to $H^i(V_0, \mathbb{Z})^*$, for $i = 1, \ldots, n = 2 \cdot \dim V$. Let $\sum_{i=1}^{n} H^i(\Gamma_s, \mathbb{Z})^*$ be the direct sum of the free part of the integral cohomology groups of Γ_s.

2.17. **Proposition.** *Assume that the automosphism group of the fibres* Γ_s *of* $\Gamma_H \to H$ *are finite for all* $\mathbb{C}$*-valued points* s *of* H, *and that the automorphism of* Γ_s *operate faithfully on* $\sum H^i(\Gamma_s, \mathbb{Z})^* (\cong \sum H^i(\Gamma_{s_0}, \mathbb{Z})^*)$. *Then, a finite galois covering* $H' \xrightarrow{f} H$ *exists which is etale and on which* $PGL(N)$ *operates without fix points, proper, and so that* f *is a* $PGL(N)$ *morphism.*

Proof. Let V/S be a family which is a pullback of the family $\Gamma_H \to H$ respectively a morphism $g: S \to H$. Then V/S is smooth and by [16] locally a topological product. Let V_s be a fibre of V/S over a $\mathbb{C}$-valued point s of S and $\sum H^i(V_s, \mathbb{Z})^*$ the direct sum of the free part of the integral cohomology groups $H^i(V_s, \mathbb{Z})$, considered as $\mathbb{Z}$-modul. Then $\sum H^i(V_s, \mathbb{Z})^*$ is independent of s and isomorphic to $\sum H^i(V_0, \mathbb{Z})^*$. Let $I(H^i(V_s, \mathbb{Z})^*) = I^i(V_s)$ be the set of minimal generator systems of the $\mathbb{Z}$-modul $H^i(V_s, \mathbb{Z})^*$ and $I(V_s) = \prod_i I^i(V_s)$ the direct product of these sets. Then the group $\Lambda = \prod \operatorname{Aut}(H^i(V_s, \mathbb{Z})^*)$ operates transitively on $I(V_s)$. Notice that $\operatorname{Aut}(H^i(V_s, \mathbb{Z})^*)$ is the group of invertible (n_i, n_i)-matrices with coefficients in $\mathbb{Z}$, where n_i is the rank of $H^i(V_s, \mathbb{Z})^*$ as $\mathbb{Z}$-module.

Consider the disjoint union $\tilde{S} = \bigcup_{s \in S(\mathbb{C})} I(V_s)$. Let γ: $\tilde{S} \to S$ be the map which sends an element $\tilde{s} \in I(V_s) \subset \tilde{S}$ to $s \in S(\operatorname{Spec}(\mathbb{C}))$. We know that the family V/S is locally a topological product. Using this fact one can introduce, by a standard argument, a natural topology on $\tilde{S}$ such that $\tilde{\gamma}$: $\tilde{S} \to S$ is continuous and $\tilde{S}$ with the map $\tilde{\gamma}$ is a topological covering of S. The group Λ operates, then, on $\tilde{S}$ as Decktransformation group. The assumption that the automorphism group $\operatorname{Aut}(V_s)$ of any fibre V_s of V/S is finite and acts faithfully on $\sum H^i(V_s, \mathbb{Z})^*$ implies that $\operatorname{Aut}(V_s)$ is isomorphic to a finite subgroup of $\Lambda = \prod \operatorname{Aut}(H^i(V_s, \mathbb{Z})^*)$. By the theorem of the appendix, the group Λ contains up to conjugency only finitely many subgroups of finite order. Let $\Lambda_1, \ldots, \Lambda_r$ be representatives of the conjugate classes of these finite subgroups. We can pick a sufficiently large integer n (independent of S) in such a way that the congruence subgroup $\Lambda^{(n)} = \prod (H^i(V_s, \mathbb{Z})^*)^{(n)}$ [15] of Λ does not contain any element of the groups Λ_i different from unity, for $i = 1, \ldots, r$.[16]

Let $\bar{\Lambda}^{(n)}$ be the quotient of Λ modulo $\Lambda^{(n)}$ and let $S^{(n)}$ be the finite unramified topological covering of S which is obtained from S by factoring out the group $\Lambda^{(n)}$. Then the covering $S^{(n)} \to S$ is galois with galois group $\bar{\Lambda}^{(n)}$. But $S^{(n)}$ is in a natural way also an analytic space and, by the generalized Riemann existence theorem [2], XI, p. 12, a $\mathbb{C}$-scheme such that the map $S^{(n)} \to S$ is an etale covering.

We have constructed so far for every family V/S, which is a pullback of Γ/H, a finite galois and etale covering $S^{(n)}$ of S with $\bar{\Lambda}^{(n)}$ as galois group. We denote this covering in the following by $P^{(n)}(V/S)$. One checks that the construction of $P^{(n)}(V/S)$ is for a fixed scheme S functorial in the families V/S in the following sense: Let V/S and V'/S be pullbacks of Γ/H such that the commutative diagram

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr] & & V' \arrow[dl] \\ & S & \end{tikzcd}$$

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[15] $\operatorname{Aut}(H^i(V_s, \mathbb{Z})^*)^{(n)}$ consists of all matrices M in $\operatorname{Aut}(H^i(V_s, \mathbb{Z})^*)$ which are congruent to the unity matrix I modulo n, i.e. $M - I$ is a matrix which has coefficients all divisible by n.

[16] The intersection of $\Lambda^{(n)}$ with any conjugate of one of the groups Λ_i, $i = 1, \ldots, r$, contains then also only the unity element, because $\Lambda^{(n)}$ is a normal subgroup of Λ.

gives an isomorphism from V/S onto V'/S. Then f induces an isomorphism $f^{(n)}\colon P^{(n)}(V'/S) \to P^{(n)}(V/S)$ and the diagram

$$\begin{tikzcd}
P^{(n)}(V'/S) \arrow[rr, "f^{(n)}"] \arrow[dr] & & P^{(n)}(V/S) \arrow[dl] \\
& S &
\end{tikzcd}$$

becomes commutative.

One checks also that the construction of $P^{(n)}(V/S)$ is compatible with taking pullbacks, i.e. if

$$\begin{tikzcd}
V \underset{S}{\times} T = V' \arrow[r] \arrow[d] & V \arrow[d] \\
T \arrow[r, "\alpha"] & S
\end{tikzcd}$$

is a pullback diagram, then $P^{(n)}(V'/T)$ is the pullback of $P^{(n)}(V/S)$ with respect to the morphism $T \xrightarrow{\alpha} S$.

We introduce in analogy to curves and abelian varieties the following notation.

2.18. *Definition.* Let V/S be a family which is a pullback of Γ_H/H and let $S^{(n)} = P^{(n)}(V/T)$ be the etale covering of S which is determined by V/S. A section of $S^{(n)}$ over S is called a *level n-structure* of the family V/S.

The construction described above gives in particular an etale covering $H^{(n)} = P^{(n)}(\Gamma_H/H)$ of H which has the following universal properties:

1) For every connected noetherian scheme S and morphism $S \to H$ the etale covering $P^{(n)}(\Gamma_H \underset{H}{\times} S/S) = S^{(n)}$ of S is isomorphic to the etale covering $H^{(n)} \underset{H}{\times} S$. (Pullback with respect to f.)

2) The S-valued point $h_H(n)(S)$ of $H^{(n)}$ correspond exactly to the families which belong to points in $h_H(S)$ with a level n-structure.

Proof of this Fact. Let $S \xrightarrow{\alpha} H^{(n)}$ be a morphism and $H^{(n)} \xrightarrow{\pi} H$ be the covering map. We get then a morphism $S \xrightarrow{\pi \circ \alpha} H$ and a family $V = \Gamma_H \underset{H}{\times} S$ over S. Because $S^{(n)} = S \underset{H}{\times} H^{(n)} = P^{(n)}(V/S)$, the map $S \xrightarrow[(\mathrm{Id}, \alpha)]{} S^{(n)} = S \underset{H}{\times} H^{(n)}$ is a section of $S^{(n)}$ over S and gives a level n-structure of V/S. Conversely, if $S \xrightarrow{\lambda} S^n$ is a section of $S^n = P^n(V/S)$ over S and V/S a pullback family of Γ_H/H with respect to a morphism $S \xrightarrow{\beta} H$, the map $\alpha\colon S \to H^{(n)}$ defined by the diagram

$$S \xrightarrow{\lambda} S^{(n)} = S \underset{H}{\times} H^{(n)} \to H^{(n)}$$

leads to an S-valued point of $H^{(n)}$ which returns the family V/S and the section $\lambda: S \to S^{(n)}$.

2.19. **Lemma.** *The group $PGL(N)$ operates on the scheme $H^{(n)}$ in a natural way where the operation is without fix points proper, and the covering map $H^{(n)} \to H$ is a $PGL(N)$-morphism.*

Proof. For every scheme S we have to define an operation of the group $PGL(N)(S)$ $(=S$-valued points of $PGL(N))$ on the S-valued points $h_{H^{(n)}}(S)$ of $H^{(n)}$ which is functorial in S. Let $s^{(n)}$ be a S-valued point of $H^{(n)}$. Then $s^{(n)}$ determines a unique S-valued point $s: S \to H$ of H and a section λ of $P^{(n)}(V/S)$ over S. (V/S is the pullback of Γ_H/H with respect to $s: S \to H$.) Let $\sigma \in PGL(N)(S)$. Then, σ maps the S-valued points s of H to an S-valued point s^σ of H. To s^σ corresponds a family V^σ/S which is the family gotten from V/S by applying the projective transformation σ to V. As the schemes $P^{(n)}(V/S)$ are functorial in V, we find that σ induces an isomorphism $\sigma^{(n)}$ from $P^{(n)}(V^\sigma/S)$ to $P^n(V/S)$ such that the diagram

$$\begin{tikzcd}
P^{(n)}(V^\sigma/S) \arrow[rr, "\sigma^{(n)}"] \arrow[dr] & & P^{(n)}(V/S) \arrow[dl] \\
& S &
\end{tikzcd}$$

is commutative. Let $\sigma^{(n)-1}$ be the inverse of $\sigma^{(n)}$. Then, for every section λ of $P^{(n)}(V/S)$ over S, we denote by $\lambda^\sigma = \sigma^{(n)-1}(\lambda)$ the section of $P(V^\sigma/S)$ over S which is the image of λ under the map $\sigma^{(n)-1}$.

We define

$$\sigma: (s, \lambda) \to (s^\sigma, \lambda^\sigma).$$

In this way we obtain an operation of $PGL(N)(S)$ on the S-valued points of $H^{(n)}$. (First we realize an operation of $PGL(N)(S)$ on the set of families of polarized varieties $\Gamma_H \times S/S$ with a level n-structure and, therefore, an operation of $PGL(N)(S)$ on $H^{(n)}(S)$.)

This operation is functorial in S and defines therefore an operation of $PGL(N)$ on the scheme $H^{(n)}$.

It is easy to see that the operations of $PGL(N)$ and $\bar{A}^{(n)}$ commute and that the covering map $H^{(n)} \to H$ is a $PGL(N)$ morphism.

That $PGL(N)(S)$ operates on $H^{(n)}$ without fix points can be seen as follows.

Let $s^{(n)}$ be a $\mathbb{C}$-valued point of $H^{(n)}$ and let $(V/\mathrm{Spec}(\mathbb{C}), \lambda)$ be the variety over $\mathrm{Spec}(\mathbb{C})$ with level n-structure λ which is determined by $s^{(n)}$. If $\sigma \in PGL(\mathrm{Spec}(\mathbb{C}))$ has $s^{(n)}$ as fix points, then σ induces an automorphism of $V/\mathrm{Spec}(\mathbb{C})$ and maps λ to a level n-structure λ^σ of $V/\mathbb{C}$. By assumption $\lambda = \lambda^\sigma$. Therefore, $\sigma = \mathrm{Id}$, as the automorphism of $V/\mathbb{C}$ operates faithfully on the integer cohomology and by the construction of $H^{(n)}$.

It remains to show that the action of $PGL(N)$ on $H^{(n)}$ is proper. For this purpose we consider the commutative diagram

$$\begin{tikzcd}
H^{(n)} \times PGL(N) \arrow[r, "\Phi^{(n)}"] \arrow[d, "p"] & H^{(n)} \times H^{(n)} \arrow[d, "q"] \\
H \times PGL(N) \arrow[r, "\Phi"] & H \times H
\end{tikzcd}$$

where $\Phi^{(n)}$ and Φ are the maps to the graph. We know that $q \circ \Phi^{(n)}$ is proper, because p and Φ are proper maps. Therefore $\Phi^{(n)}$ is proper, because q is separable.

If one takes $H' = H^{(n)}$ for a qualified integer n, the Proposition 2.17 is satisfied.

Taking things together, we have proved the following.

2.20. **Theorem.** *Let $\mathcal{M}$ be the deformation functor of an unruled polarized variety $(V_0, \mathfrak{X}_0)$. The assumption made on p. 26 concerning $\mathcal{M}$ shall be satisfied. Assume that every deformation $(V, \mathfrak{X})$ of $(V_0, \mathfrak{X}_0)$ is unruled and that the automorphism group of $(V, \mathfrak{X})$ operates faithfully on the integral cohomology of V. Then the coarse moduli space for the functors $\mathcal{M}$, respectively, $\mathcal{M}_{\mathrm{red}}$ (take $\mathcal{M}_{\mathrm{red}}$ if H is not reduced) in the sense of Definition 2.9 exists.*

III. Applications

In this chapter we show the existence of the coarse moduli space for the global deformation functor in the sense of Definition 2.8 or 2.9 for curves, polarized abelian varieties, polarized K-3 surfaces and canonical polarized varieties for which the canonical bundle is very ample.

Curves. Let $(V_0, \mathfrak{X}_0)$ be a polarized, irreducible, smooth and projective curve of genus $g > 1$ defined over $\mathbb{C}$. Assume that $\mathfrak{X}_0$ is a divisor class respectively algebraic equivalence which contains $3K_V$, K_V a canonical divisor of V_0. Let $\mathcal{M}$ be the deformation functor of the canonical polarized curve $(V_0, \mathfrak{X}_0)$ in the sense of Chapter II. If V/S is a family in $\mathcal{M}(S)$ and $\Omega_{V/S}$ the sheaf of relative differential 1-forms of V/S, one knows that $\Omega_{V/S}^{\otimes 3}$ is very ample. Hence, the sheaf $\Omega_{V/S}^{\otimes 3}$ gives a local embedding of V/S into a projective space P^N called the 3-canonical embedding, so that Proposition 2.11 is satisfied. Let $H/\mathbb{C}$ be the scheme which parametrizes the (local) 3-canonical embeddings of the families in $\mathcal{M}(S)$ and which satisfies the requirements of Proposition 2.12.

One knows by [22] and [1] that the scheme H is smooth and connected. Furthermore, it is shown in [4], XVII that the automorphism group of every curve V of genus $g > 1$ operates faithfully on $H^1(V, \mathbb{Z})$.

Using Theorem 2.20 we conclude the existence of the coarse moduli space over $\mathbb{C}$ for the deformation functor of the curve $(V_0, \mathfrak{X}_0)$. This space is normal.

Remarks. a) It follows from [22] that, for every curve $V/\mathbb{C}$ of genus g, there exists a $\mathbb{C}$-valued point s on H such that the fibre of the universal curve Γ_H/H over s is isomorphic to $V/\mathbb{C}$. This implies that the algebraic space which is the coarse moduli space for the deformation functor of $(V_0, \mathfrak{X}_0)$ is also the coarse moduli space for curves of genus g over $\mathbb{C}$ in the sense of [22].

b) Of course one knows more about the moduli space of curves of genus g. It is proved in [22] that the coarse moduli space for curves of genus g is a normal, reduced scheme.

Polarized Abelian Varieties. Let $(V_0, \mathfrak{X}_0)$ be a polarized abelian variety of dimension g, defined over the complex numbers. Let H be the Hilbert scheme for polarized abelian varieties of dimension g, constructed in [22], p. 131 ff.

Let s_0 be a point of H which belongs to $(V_0, \mathfrak{X}_0)$ and let H_0 be the connected component of H which contains s_0. Then $PGL(N)$ operates on H_0 and the geometric quotient is the coarse moduli space of the deformation functor of $(V_0, \mathfrak{X}_0)$. By [22], p. 132 ff. there exists a finite etale galois covering H_0' of H_0 on which $PGL(N)$ operates without fix points. (H_0' parametrizes the polarized abelian varieties with n-partition point structure.) Applying Theorem 2.20, we get a coarse moduli space for the deformation functor of $(V_0, \mathfrak{X}_0)$. Again Mumford's results in [22] are stronger: They show that the coarse moduli space for polarized abelian varieties of dimension g exists and is a scheme.

Polarized K-3 Surfaces[17]. Let $(V_0, \mathfrak{X}_0)$ be a polarized K-3 surface defined over the complex numbers $\mathbb{C}$. $\mathfrak{X}_0$ shall contain a sufficiently ample sheaf $\mathscr{L}$. Let $V_0 \hookrightarrow P^N$ be the projective embedding defined by $\mathscr{L}$ and $\mathscr{M}$ be the deformation functor of $(V_0, \mathfrak{X}_0)$ in the sense of Chapter II. Then for every family $(V/S, \mathfrak{X}/S) \in \mathscr{M}(S)$, the sheaves in $\mathfrak{X}/S$ are sufficiently ample. Using one of these sheaves we can embed V/S locally into the P^N in such a way that the fibres do not lie in a hyperplane.

Let H be the scheme constructed in Proposition 2.12 and corresponding to the deformation functor $\mathscr{M}$ of the K-3 surface $(V_0, \mathfrak{X}_0)$.

Claim. *The family $\Gamma_H \to H$ is a family of K-3 surfaces, i.e., every fibre of $\Gamma_H \to H$ is a K-3 surface.*

Proof. By [9] the irregularity of the fibres of $\Gamma_H \to H$ is zero. By [19], p. 233, a specialization of a smooth, projective and ruled variety is ruled.

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[17] An irreducible, smooth and projective surface V is a K-3 surface if the irregularity of V is zero and if the canonical divisors of V are linearly equivalent to the zero divisor.

Hence, a general fibre of $\Gamma_H \to H$ which has a K-3 surface as specialization, is not a ruled or a rational surface. Using upper semi continuity of the cohomology, we get that such a general fibre of $\Gamma_H \to H$ has Kodaira dimension $\leqq 0$ and is therefore a K-3 surface or an Enriques surface by the classification table of surfaces [15]. The fundamental group of the fibres of $\Gamma_H \to H$ are by [7] the same, hence trivial. This implies that every such general fibre of $\Gamma_H \to H$ is a K-3 surface.

Let now V be a general fibre of $\Gamma_H \to H$ which is a K-3 surface and let V' be a specialization of V over a discrete valuation ring. By upper semi continuity we have

$$\dim H^0(V', K_{V'}) \geqq \dim H^0(V, K_V) = 1,$$

and

$$\dim H^0(V', -K_{V'}) \geqq \dim H^0(V, -K_V) = 1.$$

Also we have a map $H^0(V', K_{V'}) \times H^0(V', -K_{V'}) \to H^0(V', O_{V'})$, defined by $(f, g) \to f \otimes g$. Let $0 \neq f \in H^0(V', K_{V'})$ and $0 \neq g \in H^0(V', -K_{V'})$ then $f \otimes g$ is a global section in $H^0(V, O_V)$ which is not identically zero and hence a constant function $\neq 0$. This implies that the canonical class of V' is principal and that V' is a K-3 surface. From the proved facts one concludes that every fibre of $\Gamma_H \to H$ is a K-3 surface. Q.E.D.

By [25], Proposition 1, the scheme H is smooth. By [17] the automorphism group of a polarized K-3 surface is finite. Using [25], Proposition 2, we get from this fact that the automorphisms of such a surface act faithfully on $H^2(X, \mathbb{Z})$. We can apply the construction developed in Chapter II to obtain an algebraic space which is a coarse moduli space for the deformation functor $\mathscr{M}$ of $(V_0, \mathfrak{X}_0)$.

Canonically Polarized Varieties where the Canonical Sheaf is Very Ample. Let $(V_0, \mathfrak{X}_0)$ be such a polarized variety which is smooth, projective and defined over the complex numbers.

Let $(V, \mathfrak{X})$ be any deformation of $(V_0, \mathfrak{X}_0)$ in the sense of Definition 2.7. It is well known that the automorphism group $\mathrm{Aut}(V, \mathfrak{X})$ of $(V, \mathfrak{X})$ is finite[18] and operates faithfully on the global sections $H^0(V, \Omega^n)$ of the sheaf of holomorphic differential n-form, because this sheaf is by assumption very ample and determines $\mathfrak{X}$. Using Hodge theory we get that $H^n(V, \mathbb{C})$ contains in a natural way the $\mathbb{C}$-module $H^0(V, \Omega^n)$. This implies that $\mathrm{Aut}(V, \mathfrak{X})$ operates faithfully on $H^n(V, \mathbb{C})$ and, therefore, also on $H^n(V, \mathbb{Z})^*$ as $H^n(V, \mathbb{C}) = H^n(V, \mathbb{Z}) \otimes \mathbb{C}$. Applying the method of Chapter II, the existence of the coarse moduli space for the deformation functor $\mathscr{M}_{\mathrm{red}}$ of $(V_0, \mathfrak{X}_0)$ is realized as an algebraic space. (We have to take $\mathscr{M}_{\mathrm{red}}$ because it is not known if the scheme H from Proposition 2.12, which belongs to $\mathscr{M}$, is reduced.)

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[18] Use [17].

Appendix

Theorem. *The group $GL(n, \mathbb{Z})$ contains up to conjugation only finitely many finite subgroups.*

W. D. Geyer has communicated to me a proof of this theorem which we will indicate.

1. Having given up to conjugation a finite subgroup G of $GL(n, \mathbb{Z})$ means that a faithfull integral n-dimensional representation of G has been given or $\mathbb{Z}^n$ has been given as a faithfull G-modul.

2. As G is finite, every element $g \in G$ has a bounded order $\leqq N$. If $g^m = 1$ and m is minimal, then g (as an operator on $\mathbb{Z}^n$) satisfies a polynomial equation $f(g)=0$ of degree n^2. Because the m-th cyclotomic polynomial $\Phi_m(x)$ divides, in this case, f, degree $(f)=n^2 \geqq \varphi(m)$. But, $\lim_{m \to \infty} \varphi(m) = \infty$. Therefore, for only finitely many m, $\varphi(m) \leqq n^2$.

2a. Remark. If G is only known to be periodic, it follows from 2), together with [5] 36.1, that G is finite.

3. Assumption. G is abelian and finite. Consequently, the exponent of G divides $N!$. Look to the subgroup $G \cap SL(n, \mathbb{Z})$ which is of index $\leqq 2$ in G. After adjoining the $N!$-roots of unity to Q, one can diagonalize $G \cap SL(n, \mathbb{Z})$ and $G \cap SL(n, \mathbb{Z})$ becomes a subgroup of $(\mathbb{Z}/N!\,\mathbb{Z})^n$. This demonstrates: Up to isomorphism there are only finitely many finite abelian subgroups of $SL(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$.

4. G shall be any finite subgroup of $GL(n, \mathbb{Z})$. By [5] (36.13), there exists a number $F(n)$ (independent of G) such that G has an abelian normal subgroup of index $\leqq F(n)$. This implies together with 3): There exist up to isomorphism only finitely many finite subgroups of $GL(n, \mathbb{Z})$.

5. To conclude the theorem it remains to prove that a finite group has up to isomorphism only finitely many integral representations of dimension n.

This is a special case of [5] (79.12).

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Note added in proof. The assumptions in Theorem 1.13 can be weekened considerably. We are able to show the following:

Let X be a separated algebraic $\mathbb{C}$-space locally of finite type and G an algebraic group over $\mathbb{C}$. If G acts properly on X and with finite stabilizers, there exists a separated algebraic $\mathbb{C}$-space locally of finite type over $\mathbb{C}$ which is a geometric quotient of X by G.

A proof will appear in Compositio Mathematica.

The stronger quotient theorem yields over the complex numbers in connection with the considerations in Chapter II of this paper to coarse moduli spaces for canonical polarized, smooth varieties and also for polarized, smooth varieties with irregularity 0.

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