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The Characters of Discrete Series as Orbital Integrals

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Suppose that G is a Lie group, which for the purpose of this introduction, we take to be a real form of a simply connected complex semisimple group. Suppose that square integrable representations for G exist and that f is a matrix coefficient of a square integrable representation belonging to the unitary equivalence class ω. Harish-Chandra has shown how to evaluate the integral of f with respect to the G-invariant measure on any regular semisimple conjugacy class. In fact suppose that h is a regular semisimple element of G. The Cartan subgroup T which centralizes h may be assumed to be stable with respect to a fixed Cartan involution θ. In other words, there is a θ-stable decomposition

$$T = T_I \, T_{\mathbb{R}},$$

where T_I is compact and $T_{\mathbb{R}}$ is a vector group. Then according to Harish-Chandra,

$$\int_{T_{\mathbb{R}}\backslash G} f(x^{-1} h x) \, dx = \varepsilon(T) \, \Theta_\omega(f) \, \Theta_\omega(h), \tag{1}$$

where Θ_ω is the character of ω and $\varepsilon(T)$ equals 1 if T is compact and is 0 otherwise. Implicit in this formula is the absolute convergence of the integral on the left. The vanishing statement (the case that T is noncompact) is sometimes known as the Selberg principle. The purpose of this paper is to establish a formula which generalizes (1).

If P is a parabolic subgroup of G, let

$$P = NAM$$

be the "Langlands decomposition". It is not P that we want to fix, but rather A, and its centralizer MA. In fact, let $\mathscr{P}(A)$ be the set of all parabolic subgroups for which A is the vector group in the above decomposition. This set is finite. For each

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$P=NAM$ in $\mathscr{P}(A)$ and $x \in G$, put

$$x=M(x)\exp(H_P(x))\,N(x)\,K(x),$$

for $M(x)\in M$, $H_P(x)\in \mathfrak{a}=\log A$, $N(x)\in N$ and $K(x)$ an element in K, the maximal compact subgroup of G fixed by θ. Fix a Euclidean measure on $\mathfrak{a}$. For any $x\in G$, let $v(x)$ be the volume in $\mathfrak{a}$ of the convex hull of

$$\{H_P(x): P\in\mathscr{P}(A)\}.$$

As a function of x, $v(x)$ is left MA-invariant. Choose T so that it contains A. Then

$$A\subset T_{\mathbb{R}}\subset T\subset MA.$$

Our formula is

$$\int_{T_{\mathbb{R}}\backslash G} f(x^{-1}hx)\,v(x)\,dx=\varepsilon(T,A)(-1)^p\,\Theta_\omega(f)\,\Theta_\omega(h), \tag{1*}$$

where p is the dimension of A and $\varepsilon(T,A)$ equals 1 if $A=T_{\mathbb{R}}$, and is 0 otherwise.

In the particular case that $A=T_{\mathbb{R}}$ we see that the character value of ω on any regular semisimple conjugacy class can be obtained as the weighted average of any matrix coefficient of ω over the conjugacy class. This is a surprising and striking coincidence. The matrix coefficients of the discrete series are of central importance in the harmonic analysis on G but there has never been anything resembling a general closed formula for them. On the other hand, in the spirit of Weyl's character formula, $\Theta_\omega(h)$ can be expressed as the quotient of two exponential polynomials.

The product of the left hand side of (1) by a suitable function of h yields Harish-Chandra's invariant integral, $F_f(h)$, which one studies not just for the function given above but for any f in the Schwartz space of G. We shall consider the product of the left side of (1*) by the same function of h, again for any f in the Schwartz space. The resulting distributions

$$f\to R_f(h),$$

turn out to have analogues of three fundamental properties of the invariant integrals. $F_f(h)$ satisfies a family of differential equations in h. The same is true for $R_f(h)$, although the equations here are more complicated. $F_f(h)$ satisfies boundary conditions at the hypersurface defined by any singular imaginary root of (G,T). So does $R_f(h)$. However, $R_f(h)$ also satisfies boundary conditions for each *real* root of (G,T), and it is these latter which most concern us. Finally, both $F_f(h)$ and $R_f(h)$ are rapidly decreasing in the $T_{\mathbb{R}}$-component of h. Our starting point is a remarkable combinatorial lemma of Langlands, which is similar to the result announced in [4(b), § 8]. We reproduce Langlands unpublished proof in § 2. Section 3 contains the main application of this lemma. There we derive a formula for $v(x)$ which is used in everything that follows. In § 4 we define the distributions and give some of their elementary properties. Sections 5–8 are essentially devoted to establishing the three main properties described above. We then prove formula (1*) in § 9 by comparing these properties with the known behaviour of the functions $\Theta_\omega(h)$.

The results of this paper are the generalizations to arbitrary rank of most of what is contained in [1(b)], although in that paper we did not recognize the appearance of the characters of discrete series. There is one formula for real rank one, though, ([1(b), Corollary 7.3]) which does not yet have a general analogue in higher rank. It would be desirable to define distributions for *any* $h_0 \in T$, and to relate them to the values of $R_f(h)$ as h approaches h_0.

The integral on the left side of (1*) actually came up in another context. It was observed by Langlands some time ago that integrals of this type would arise if one attempted to generalize the Selberg trace formula to arbitrary reductive groups over $\mathbb{Q}$. A general trace formula does not exist at this time. However, for groups of rank 1 we can see the integrals occurring in the trace formula in term (9.1) of [1(a)]. The role played by (1) in deducing the formula for the multiplicity of ω in the regular representation of G on $L^2(\Gamma \backslash G)$, where Γ is a discrete co-compact subgroup of G, is by now well known (see [4(c)]). It seems reasonable to expect that (1*) will play the same role in the more general situation where $\Gamma \backslash G$ is only assumed to have finite G-invariant volume.

Contents

§ 0. Notation

If G is any Lie group, its Lie algebra will be denoted by $\mathfrak{g}$, and the complexification of $\mathfrak{g}$ will be written as $\mathfrak{g}_{\mathbb{C}}$. We will generally let $\mathscr{G}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. We shall write G^0 for the connected component of 1 in G.

§ 1. Split Parabolic Subgroups

Let G be a Lie group. Suppose that K, θ and B are fixed and that (G, K, θ, B) satisfies Harish-Chandra's general assumptions ([2(h)], [2(i)]). For convenience we recall these assumptions.

For a start, G itself satisfies the following four conditions.

(i) $\mathfrak{g}$ is reductive.

(ii) If $G_{\mathbb{C}}$ is the connected complex adjoint group of $\mathfrak{g}_{\mathbb{C}}$, $\mathrm{Ad}(G) \subset G_{\mathbb{C}}$.

(iii) Let G_1 be the analytic subgroup of G corresponding to $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Then the center of G_1 is finite.

(iv) G^0 has finite index in G.

K is a maximal compact subgroup of G which meets every connected component of G and such that $K \cap G^0$ is a maximal compact subgroup of G^0. θ is an involution on $\mathfrak{g}$ for which $\mathfrak{k}$ is the $+1$ eigenspace. Finally, B is a real symmetric bilinear form on $\mathfrak{g}$ such that

(i) $B([X,Y],Z)+B(Y,[X,Z])=0, \qquad X,Y,Z\in\mathfrak{g},$

(ii) the quadratic form

$\|X\|^2=-B(X,\theta X), \qquad X\in\mathfrak{g},$

is positive definite on $\mathfrak{g}$, and

(iii) $B(\theta X,\theta Y)=B(X,Y), \qquad X,Y\in\mathfrak{g}.$

Suppose that $\mathfrak{t}$ is any θ-stable abelian subspace of $\mathfrak{g}$. Write

$$\mathfrak{t}=\mathfrak{t}_I\oplus\mathfrak{t}_{\mathbb{R}}$$

for the decomposition of $\mathfrak{t}$ into its $+1$ and -1 θ-eigenspaces. The restriction to $\mathfrak{t}$ of the form B is just denoted by $\langle\ ,\ \rangle$. It is nondegenerate. We extend $\langle\ ,\ \rangle$ to a symmetric form on $\mathfrak{t}_{\mathbb{C}}$ and use it to identify $\mathfrak{t}_{\mathbb{C}}$ with its complex dual space. This convention applies in particular to the case that $\mathfrak{t}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. In that case define T, the Cartan subgroup associated to $\mathfrak{t}$, to be the centralizer of $\mathfrak{t}$ in G. Then

$$T=T_I T_{\mathbb{R}},$$

where $T_I=T\cap K$, and $T_{\mathbb{R}}=\exp\mathfrak{t}_{\mathbb{R}}$. T normalizes each root space of $(\mathfrak{g},\mathfrak{t})$. It follows that each root α of $(\mathfrak{g},\mathfrak{t})$ gives rise to a quasi-character ξ_α of T.

Recall that a subgroup P of G is called parabolic if it is the normalizer in G of a subalgebra $\mathfrak{p}$ whose complexification contains a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{p}$ is the Lie algebra of P and $G=PK$. As usual, let N and $\mathfrak{n}$ denote the nilradicals of P and $\mathfrak{p}$ respectively, and put

$$\mathfrak{l}=\mathfrak{p}\cap\theta(\mathfrak{p}).$$

Let $\mathfrak{a}_1$ be the -1 θ-eigenspace of the center of $\mathfrak{l}$. Finally, let L be the centralizer of $\mathfrak{a}_1$ in G. Then $\mathfrak{l}$ is the Lie algebra of L and $P=NL$.

It is customary to call $\exp\mathfrak{a}_1$ the split component of P and to refer to $(P,\exp\mathfrak{a}_1)$ as a parabolic pair. However, for some applications one wants to consider pairs which arise, via extension of scalars, from parabolic pairs over a subfield of $\mathbb{R}$. The resultant objects over $\mathbb{R}$ have been axiomitized in the early pages of [4(a)]. Suppose that $\mathfrak{a}$ is a subspace of $\mathfrak{a}_1$ and $A=\exp\mathfrak{a}$. The action of $\mathfrak{a}$ on $\mathfrak{g}$ can be diagonalized over $\mathbb{R}$. Let Q be the set of roots of $(\mathfrak{g},\mathfrak{a})$, and let Q_P be the subset whose root spaces lie in $\mathfrak{n}$. For any $\gamma\in Q$ we denote the root space by $\mathfrak{g}_\gamma$, and also by $\mathfrak{n}_\gamma$ if γ happens to lie in Q_P. Let $\mathfrak{m}$ be the B-orthogonal complement of $\mathfrak{a}$ in $\mathfrak{l}$. A is said to be a *split component* of P if for any $Y\in\mathfrak{m}$ and $\gamma\in Q_P$,

$$\operatorname{tr}(\operatorname{ad}Y)_{\mathfrak{n}_\gamma}=0.$$

We shall call the pair (P,A) a *split parabolic subgroup* (of G).

Suppose that (P,A) is a split parabolic subgroup. Let M be the group of all $m\in L$ such that

$$\det(\operatorname{Ad}m)_{\mathfrak{n}_\gamma}=\pm1$$

for each γ in Q_P. Then $\mathfrak{m}$ is the Lie algebra of M. Moreover $M \cap A = \{1\}$ and $L = AM$, so that

$$P = MAN.$$

It is easily checked that no element in Q_P is zero and that Q is the disjoint union of Q_P, $\{0\}$, and $-Q_P$. In particular the Lie algebra $\mathfrak{p}$, and hence the groups P, N, and M, are uniquely determined by A together with the subset Q_P of Q. (See [4(a), pp. 2.2–2.5].) If $x \in G$ we write

$$x = M(x)(\exp H_P(x))\, N(x)\, K(x)$$

for $M(x) \in M$, $H_P(x) \in \mathfrak{a}$, $N(x) \in N$ and $K(x) \in K$. The vector $H_P(x)$ is uniquely determined. In the future we shall sometime index a split parabolic subgroup by a subscript or a superscript. In this case all the various objects associated to the parabolic subgroup (such as N, M, H_P etc.) will be indexed the same way, usually without further comment.

Suppose now that A is any vector subgroup of G such that θ is -1 on $\mathfrak{a}$. As above we write Q for the set of roots of $(\mathfrak{g}, \mathfrak{a})$. Put

$$\mathfrak{a}^0 = \{H \in \mathfrak{a} : \langle \gamma, H \rangle = 0 \text{ for every } \gamma \in Q\},$$

and let $\mathfrak{a}^1$ be the orthogonal complement of $\mathfrak{a}^0$ in $\mathfrak{a}$. Then

$$\mathfrak{a} = \mathfrak{a}^0 \oplus \mathfrak{a}^1,$$

and Q spans $\mathfrak{a}^1$. We shall always denote the dimension of $\mathfrak{a}^1$ by p. We shall say that A is a *special subgroup* of G if it is the split component of some parabolic subgroup P. Since

$$Q = Q_P \cup \{0\} \cup -Q_P$$

for any such P, A is special if and only if

$$\operatorname{tr}(\operatorname{ad} X)_{\mathfrak{g}_\gamma} = 0$$

for all $X \in \mathfrak{m}$ and $\gamma \in Q$. We shall write $\mathscr{P}(A)$ for the set of all parabolic subgroups with A as split component.

Lemma 1.1. *Let* (P, A) *be a split parabolic subgroup. There is a uniquely determined subset* Φ_P *of* Q_P *for which any element in* Q_P *can be uniquely written as a nonnegative integral linear combination of elements in* Φ_P. Φ_P *forms a basis of* $\mathfrak{a}^1$. *If* γ^1 *and* γ^2 *are distinct elements in* Φ_P,

$$\langle \gamma^1, \gamma^2 \rangle \leqq 0.$$

For a proof this lemma see [4(a), Lemma 2.2]. The proof goes the same way as that of [2(c), Lemma 1]. □

For the rest of this section A will be a fixed special subgroup of G. L is just the centralizer of $\mathfrak{a}$ in G, so is defined independently of $P \in \mathscr{P}(A)$. The elements of $Q - \{0\}$ define hyperplanes which partition $\mathfrak{a}$ into a finite number of connected components called *chambers*. If $P \in \mathscr{P}(A)$, put

$$c_P(\mathfrak{a}) = \{H \in \mathfrak{a} : \langle \gamma, H \rangle > 0 \text{ for all } \gamma \in Q_P\}.$$

This is called the *positive chamber* of P. In view of Lemma 1.1. it is defined by p hyperplanes, corresponding to the elements of Φ_P. Distinct groups in $\mathscr{P}(A)$ give rise to distinct positive chambers. On the other hand, suppose that c is an arbitrary chamber in $\mathfrak{a}$. Define Q_P to be the set of roots in Q which are positive on c and let $\mathfrak{n}$ be the sum of the corresponding root spaces. Clearly Q is the disjoint union of Q_P, $-Q_P$ and $\{0\}$. It follows easily that $\mathfrak{l}+\mathfrak{n}$, evidently a subalgebra of $\mathfrak{g}$, is actually a parabolic subalgebra. Therefore, P, its normalizer in $\mathfrak{g}$, is a parabolic subgroup of G. A is a split component of P and $c_P(\mathfrak{a})=c$. We have shown that

$$P\to c_P(\mathfrak{a}),\qquad P\in\mathscr{P}(A),$$

is a bijection from $\mathscr{P}(A)$ on the set of chambers in $\mathfrak{a}$.

Suppose that $P\in\mathscr{P}(A)$. Let (P^*,A^*) be a split parabolic subgroup with $P\subset P^*$ and $A\supset A^*$. We shall say that (P,A) *dominates* (P^*,A^*) and write $(P,A)<(P^*,A^*)$ if there is a sequence

$$(P_1,A_1),\ldots,(P_k,A_k)$$

of split parabolic subgroups such that

$$P=P_1\subsetneqq P_2\subsetneqq\cdots\subsetneqq P_k=P^*,$$
$$A=A_1\supset A_2\supset\cdots\supset A_k=A^*,$$

and

$$\dim A_{i+1}-\dim A_i=1,\qquad 1\leqq i\leqq k.$$

Suppose that F is a subset of Φ_P. Define

$$\mathfrak{a}_F=\{H\in\mathfrak{a}:\langle\gamma,H\rangle=0\ \text{ for all }\ \gamma\in F\},$$

and let $\mathfrak{l}_F$ be the centralizer of $\mathfrak{a}_F$ in $\mathfrak{g}$. If Q_F is the set of roots in Q_P which do not vanish on $\mathfrak{a}_F$, put

$$\mathfrak{n}_F=\bigoplus\nolimits_{\gamma\in Q_F}\mathfrak{n}_\gamma.$$

Then $\mathfrak{l}_F+\mathfrak{n}_F$ is a parabolic subalgebra. Let P_F be its normalizer in G and put

$$A_F=\exp\mathfrak{a}_F.$$

Lemma 1.2. *The map*

$$F\to(P_F,A_F)$$

is a bijection from the collection of subsets of Φ_P onto the set of split parabolic subgroups which are dominated by (P,A).

For a proof see [4(a), Lemma 2.3]. □

Given (P,A), we write $P<P^*$ if $(P,A)<(P^*,A^*)$ for some A^*. A^* is uniquely determined, being equal to $L^*\cap A$. $(P\cap L^*,A)$ is a split parabolic subgroup of L^*. In fact given A and (P^*,A^*), the map

$$P\to P\cap L^*$$

is a bijection between the set of $P \in \mathscr{P}(A)$ which are contained in P^* and $P_{L^*}(A)$, the set of parabolic subgroups of L^* with A as split component.

Define two groups P and P' in $\mathscr{P}(A)$ to be *adjacent* if their chambers have a $p-1$ dimensional wall in common. P is adjacent to exactly p groups in $\mathscr{P}(A)$, one for each simple root of (P, A). Suppose that P and P' are arbitrary elements in $\mathscr{P}(A)$. A *path* of length n between P and P' is a set

$$P_1 = P, P_2, \ldots, P_n = P'$$

of elements in $\mathscr{P}(A)$ such that P_i and P_{i+1} are adjacent, $1 \leqq i \leqq n-1$. We write $d(P, P')$ for the length of the shortest path from P to P'.

In this paper we will generally not need to normalize Haar measures. An exception is the measure on $\mathfrak{a}$. Here we take the Euclidean measure defined by the norm $\|\ \|$. We do the same for any vector subspace of $\mathfrak{a}$, and in particular for $\mathfrak{a}^1$. Suppose that $P \in \mathscr{P}(A)$, and that

$$\Phi_P = \{\gamma^j : 1 \leqq j \leqq p\}.$$

Define

$$c(P) = |\det(\langle \gamma^j, \gamma^k \rangle)_{1 \leqq j, k \leqq p}|^{\frac{1}{2}}.$$

For any function $\varphi \in C_c^\infty(\mathfrak{a}^1)$,

$$\int_{\mathfrak{a}^1} \varphi(H)\, dH = c(P) \int \ldots \int \varphi(t_1 \gamma^1 + \cdots + t_p \gamma^p)\, dt_1 \ldots dt_p.$$

Lemma 1.3. *Fix i, $1 \leqq i \leqq p$, let $F = \{\gamma^i\}$, and define $(P^*, A^*) = (P_F, A_F)$, in the notation of Lemma 1.2. Then*

$$c(P) = c(P^*) \langle \gamma^i, \gamma^i \rangle^{\frac{1}{2}}.$$

Proof. If $j \neq i$, define γ_i^j to be the projection of γ^j onto $\mathfrak{a}^*$, the orthogonal complement of γ^i in $\mathfrak{a}$. By changing the t_i variable we see that for any $\varphi \in C_c^\infty(\mathfrak{a}^1)$,

$$\int \ldots \int \varphi(t_1 \gamma^1 + \cdots + t_p \gamma^p)\, dt_1 \ldots dt_p$$

equals

$$\int \ldots \int \varphi(t_1 \gamma_i^1 + \cdots + t_p \gamma_i^p)\, dt_1 \ldots dt_p.$$

But

$$\Phi_{P^*} = \{\gamma_i^j : j \neq i\}.$$

Therefore this last expression equals

$$c(P^*)^{-1} \int_{-\infty}^{\infty} \int_{(\mathfrak{a}^*)^1} \varphi(t_i \gamma^i + H^*)\, dH^*\, dt_i$$
$$= c(P^*)^{-1} \langle \gamma^i, \gamma^i \rangle^{-\frac{1}{2}} \int_{\mathfrak{a}^1} \varphi(H)\, dH.$$

On the other hand this equals

$$c(P)^{-1} \int_{\mathfrak{a}^1} \varphi(H)\, dH,$$

so the lemma follows. $\square$

Corollary 1.4. *If P and P' are in $\mathscr{P}(A)$,*

$$c(P)=c(P').$$

Proof. Suppose that P and P' are adjacent. Let γ^i be the root in Φ_P which is orthogonal to the common wall of P and P'. Then $-\gamma^i$ belongs to $\Phi_{P'}$. The other simple roots of P are different from those of P' but their projections onto the orthogonal complement of γ^i are not. It follows from the lemma that $c(P)=c(P')$. The general case is obtained by taking a path from P to P'. □

Since the factor $c(P)$ is independent of $P\in\mathscr{P}(A)$ we will in future simply denote it by c_A.

§ 2. Langlands' Combinatorial Lemma

Suppose that V is a finite dimensional Euclidean space with a basis

$$\Phi=\{\gamma^1,\dots,\gamma^p\}$$

such that

$$\langle\gamma^i,\gamma^j\rangle\leqq 0,$$

for $i\neq j$. Let $\{\mu^1,\dots,\mu^p\}$ be the corresponding dual basis. The following lemma is standard. For convenience we include the proof given in [4(a)].

Lemma 2.1. *For all i and j,*

$$\langle\mu^i,\mu^j\rangle\geqq 0.$$

Proof. The lemma is easy to establish if $p=2$. Suppose then that p is greater than 2, and assume inductively that the lemma is valid for spaces of dimension less than p. Fix i and j. Choose k not equal to i or j, and project

$$\{\gamma^l:\ l\neq k\}$$

onto the orthogonal complement of μ^k. This gives a basis $\{\delta^l: l\neq k\}$ of a $p-1$ dimensional Euclidean space. The dual basis is $\{\mu^l: l\neq k\}$. For $l\neq k$, $m\neq k$,

$$\langle\delta^l,\delta^m\rangle=\langle\delta^l,\gamma^m\rangle=\langle\gamma^l,\gamma^m\rangle-\frac{\langle\gamma^m,\gamma^k\rangle\langle\gamma^l,\gamma^k\rangle}{\langle\gamma^k,\gamma^k\rangle}.$$

If l and m are distinct this is no greater than 0. Applying the induction hypothesis we see that

$$\langle\mu^i,\mu^j\rangle\geqq 0.$$ □

Corollary 2.2. *Suppose that H is a point in V such that $\langle\gamma^i,H\rangle\geqq 0$ for all i. Then $\langle\mu^i,H\rangle\geqq 0$ for all i.*

Proof. We can write

$$H=\sum c_i\,\mu^i,$$

where each $c_i\geqq 0$. The corollary then follows from the lemma. □

If F is any subset of $\{1, \ldots, p\}$, let V_F denote the subspace of V spanned by the vectors $\{\gamma^i : i \in F\}$. If k belongs to F, set $\gamma_F^k = \gamma^k$. However, if k is not in F let γ_F^k be the projection of γ^k onto the orthogonal complement of V_F. Finally, let $\{\mu_F^k\}$ be the basis of V which is dual to $\{\gamma_F^k\}$.

Fix $F \subset \{1, \ldots, p\}$ and $j \notin F$. We can write

$$\gamma_F^j = \gamma^j + \sum_{i \in F} c_{ji} \gamma^i,$$

for real numbers c_{ji}. Now $\{\gamma^i : i \in F\}$ and $\{\mu_F^i : i \in F\}$ are dual bases of a subspace of V. Since $\langle \gamma^i, \gamma^j \rangle \leqq 0$ for each $i \in F$, we must also have $\langle \mu_F^i, \gamma^j \rangle \leqq 0$ for $i \in F$, by the above corollary. It follows that

$$c_{ji} \leqq 0, \qquad i \in F.$$

Consequently if $H \in V$, and $\langle \gamma^k, H \rangle \geqq 0$ for all k, then we must have $\langle \gamma_F^k, H \rangle \geqq 0$ for all k.

If $i \in F$ and $j \notin F$, then

$$\langle \gamma_F^i, \gamma_F^j \rangle = 0.$$

If both i and j are in F and $i \neq j$,

$$\langle \gamma_F^i, \gamma_F^j \rangle = \langle \gamma^i, \gamma^j \rangle \leqq 0.$$

If neither i nor j belongs to F,

$$\begin{aligned} \langle \gamma_F^i, \gamma_F^j \rangle &= \langle \gamma_F^i, \gamma^j \rangle + \sum_{k \in F} c_{jk} \langle \gamma_F^i, \gamma^k \rangle \\ &= \langle \gamma_F^i, \gamma^j \rangle \\ &= \langle \gamma^i, \gamma^j \rangle + \sum_{k \in F} c_{ik} \langle \gamma^k, \gamma^j \rangle, \end{aligned}$$

which is no greater than 0 if $i \neq j$. It follows that for arbitrary distinct indices i and j

$$\langle \gamma_F^i, \gamma_F^j \rangle \leqq 0.$$

This implies that for all i and j,

$$\langle \mu_F^i, \mu_F^j \rangle \geqq 0.$$

Any hyperplane of the form

$$\{X : \langle \gamma_F^i, X \rangle = 0\} \quad \text{or} \quad \{X : \langle \mu_F^i, X \rangle = 0\}, \qquad \text{for}$$

some i and F, will be called a special hyperplane. We shall say that a point $H \in V$ is regular if it does not lie on any special hyperplane. Fix a subset F of $\{1, \ldots, p\}$ and a regular point Λ. Define a function ψ_F^Λ on the set of regular points of V to be the product of the characteristic functions of

$$\{H : \langle \gamma_F^j, H \rangle > 0, \qquad \text{for all } j \textit{ not} \text{ in } F\}$$

and

$$\{H : \langle \mu_F^i, H \rangle \langle \gamma^i, \Lambda \rangle < 0, \qquad \text{for all } i \text{ in } F\}.$$

Let α_F^Λ denote the number of indices $i \in F$ such that $\langle \gamma^i, \Lambda \rangle < 0$. Langlands' combinatorial lemma is the following:

Lemma 2.3. *If* $\langle \gamma^i, \Lambda \rangle > 0$ *for all i then*

$$\sum_F (-1)^{\alpha_F^\Lambda} \psi_F^\Lambda(H) = 1$$

for all regular H. If $\langle \gamma^i, \Lambda \rangle < 0$ *for some i, then*

$$\sum_F (-1)^{\alpha_F^\Lambda} \psi_F^\Lambda(H) = 0$$

for all regular H.

Proof. Suppose that $\langle \gamma^i, H \rangle$ is negative for all i. Then, as we have seen, each $\langle \gamma_F^i, H \rangle$, and therefore each $\langle \mu_F^i, H \rangle$ also, is negative. Hence $\psi_F^\Lambda(H)$ vanishes for all Λ unless F equals the set $F_1 = \{1, \dots, p\}$. Moreover, $\alpha_{F_1}^\Lambda = 0$, and $\psi_{F_1}^\Lambda(H)$ equals 1 or 0, depending on whether or not $\langle \gamma^i, \Lambda \rangle$ is positive for all i. This proves the lemma for H as above.

Now suppose that H and H' are two regular vectors. We have only to show that

$$\sum_F (-1)^{\alpha_F^\Lambda} \psi_F^\Lambda(H) = \sum_F (-1)^{\alpha_F^\Lambda} \psi_F^\Lambda(H').$$

H and H' can be joined by a polygonal path, no segment of which lies in a special hyperplane and no point of which lies on the intersection of 2 distinct special hyperplanes. We may assume that only one point of this path lies on a special hyperplane, say

$$\{X : \langle \alpha, X \rangle = 0\},$$

and that H and H' lie on opposite sides of this hyperplane. In other words $\langle \alpha, H \rangle$ and $\langle \alpha, H' \rangle$ are of opposite sign.

Suppose that F is a subset of $\{1, \dots, p\}$ and that no vector in either $\{\gamma_F^j : j \notin F\}$ or $\{\mu_F^i : i \in F\}$ is a multiple of α. Then

$$\langle \gamma_F^j, H \rangle = \langle \gamma_F^j, H' \rangle \quad \text{if } j \notin F,$$

and

$$\langle \mu_F^i, H \rangle = \langle \mu_F^i, H' \rangle \quad \text{if } i \in F.$$

It follows that $\psi_F^\Lambda(H) = \psi_F^\Lambda(H')$.

Let $\mathcal{S}_1$ be the collection of those F such that for some k not in F, γ_F^k is a multiple of α, and let $\mathcal{S}_2$ be the collection of those F such that for some k in F, μ_F^k is a multitude of α. In either case k is uniquely determined. $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint. Suppose $F_1 \in \mathcal{S}_1$. We might as well assume that $F_1 = \{1, \dots, k-1\}$, for some $k \leq p$, and that $\gamma_{F_1}^k$ is a multiple of α. Let $F_2 = \{1, \dots, k\}$. $\gamma_{F_1}^k$ and $\mu_{F_2}^k$ are both in the span of $\{\mu^k, \dots, \mu^p\}$ but both are orthogonal to $\{\mu^{k+1}, \dots, \mu^p\}$. Therefore, $\mu_{F_2}^k$ is a multiple of $\gamma_{F_1}^k$ and hence of α. We note for later use that since

$$\gamma_{F_1}^k = \gamma^k + \sum_{i<k} c_{ki} \gamma^i,$$

and

$$\mu_{F_2}^k = \mu^k + \sum_{j>k} d_{kj}\, \mu^j,$$

$\langle \mu_{F_1}^k, \mu_{F_2}^k \rangle$ equals 1, so that $\mu_{F_2}^k$ is actually a *positive* multiple of $\gamma_{F_1}^k$. At any rate, F_2 belongs to $\mathscr{S}_2$. Since the empty set does not belong to $\mathscr{S}_2$, this process can be reversed. In this way we set up a one to one correspondence between $\mathscr{S}_1$ and $\mathscr{S}_2$. Let F_1 and F_2 be two corresponding sets, which we assume for simplicity are as above. We have only to show that

$$(-1)^{\alpha_{F_1}^A} \psi_{F_1}^A(H) + (-1)^{\alpha_{F_2}^A} \psi_{F_2}^A(H) = (-1)^{\alpha_{F_1}^A} \psi_{F_1}^A(H') + (-1)^{\alpha_{F_2}^A} \psi_{F_2}^A(H').$$

Suppose that $j>k$. $\gamma_{F_2}^j$ is in the span of $\{\mu^{k+1}, \dots, \mu^p\}$ but is orthogonal to $\{\mu^l : l \geqq k+1, l \neq j\}$. $\gamma_{F_1}^j$ is in the span of $\{\mu^k, \dots, \mu^p\}$ but is orthogonal to $\{\mu^l : l \geqq k, l \neq j\}$. However,

$$\langle \gamma_{F_1}^j, \mu^j \rangle = \langle \gamma_{F_2}^j, \mu^j \rangle = 1.$$

Therefore, $\gamma_{F_2}^j - \gamma_{F_1}^j$ is in the span of $\{\mu^k, \dots, \mu^p\}$ but is orthogonal to $\{\mu^{k+1}, \dots, \mu^p\}$. Now $\mu_{F_1}^k$ also satisfies this property. It follows that

$$\gamma_{F_2}^j = \gamma_{F_1}^j + c_j\, \gamma_{F_1}^k, \qquad c_j \in \mathbb{R}.$$

If we take the inner product of this equation with $\gamma_{F_1}^k$, and note that

$$\langle \gamma_{F_2}^j, \gamma_{F_1}^k \rangle = \langle \gamma_{F_2}^j, \mu^k \rangle = 0,$$

we discover that $c_j \geqq 0$. Suppose that $i<k$. Then we can make the same argument to show that $\mu_{F_1}^i - \mu_{F_2}^i$ is in the span of $\{\mu^1, \dots, \mu^k\}$ but is orthogonal to $\{\mu^1, \dots, \mu^{k-1}\}$, and that

$$\mu_{F_1}^i = \mu_{F_2}^i + d_i\, \mu_{F_2}^k,$$

for $d_i \leqq 0$.

Now $\langle \gamma_{F_1}^k, H \rangle$ and $\langle \gamma_{F_1}^k, H' \rangle$ are of opposite signs. We can assume that $\langle \gamma_{F_1}^k, H' \rangle$ is negative. Then $\psi_{F_1}^A(H')$ vanishes for all A. We must show that

$$(-1)^{\alpha_{F_1}^A} \psi_{F_1}^A(H) = (-1)^{\alpha_{F_2}^A} \psi_{F_2}^A(H') - (-1)^{\alpha_{F_2}^A} \psi_{F_2}^A(H). \tag{2.1}$$

Suppose first of all that $\psi_{F_1}^A(H)$ vanishes for all A. Then there is a $j>k$ such that $\langle \gamma_{F_1}^j, H \rangle$ is negative. Since $\gamma_{F_1}^j$ is not a multiple of α, $\langle \gamma_{F_1}^j, H' \rangle$ is also negative. Therefore

$$\langle \gamma_{F_2}^j, H' \rangle = \langle \gamma_{F_1}^j, H' \rangle + c_j \langle \gamma_{F_1}^k, H' \rangle < 0.$$

This means that $\langle \gamma_{F_2}^j, H \rangle$ is also negative, so that $\psi_{F_2}^A(H)$ and $\psi_{F_2}^A(H')$ both vanish for all A. This proves (2.1) in the case under consideration.

The other case is that $\psi_{F_1}^A(H)$ does not vanish for some A. This means that $\langle \gamma_{F_1}^j, H \rangle > 0$ for each $j \geqq k$. But then, for any $j>k$,

$$\langle \gamma_{F_2}^j, H \rangle = \langle \gamma_{F_1}^j, H \rangle + c_j \langle \gamma_{F_1}^k, H \rangle$$

is positive. The same goes for $\langle \gamma^j_{F_2}, H'\rangle$. On the other hand, suppose $i<k$. If $\langle \mu^i_{F_2}, H\rangle$ is negative then

$$\langle \mu^i_{F_1}, H\rangle = \langle \mu^i_{F_2}, H\rangle + d_i \langle \mu^k_{F_2}, H\rangle$$

is also negative, since $\langle \mu^k_{F_2}, H\rangle$ is a positive multiple of $\langle \gamma^k_{F_1}, H\rangle$, which is positive. If $\langle \mu^i_{F_2}, H\rangle$ is positive then

$$\langle \mu^i_{F_1}, H'\rangle = \langle \mu^i_{F_2}, H'\rangle + d_i \langle \mu^k_{F_2}, H'\rangle$$

is positive, so that $\langle \mu^i_{F_1}, H\rangle$ is also positive. Therefore, the three numbers $\langle \mu^i_{F_1}, H\rangle$, $\langle \mu^i_{F_2}, H\rangle$, and $\langle \mu^i_{F_2}, H'\rangle$ all have the same sign. Thus, to relate the right side of (2.1) with the left side, we have only to consider the 2 opposite signs,

$$\operatorname{sign}(\langle \mu^k_{F_1}, H'\rangle \langle \gamma^k, \Lambda\rangle)$$

and

$$\operatorname{sign}(\langle \mu^k_{F_1}, H\rangle \langle \gamma^k, \Lambda\rangle).$$

Remembering that $\mu^k_{F_1}$ is a positive multiple of $\gamma^k_{F_1}$, we see that if $\langle \gamma^k, \Lambda\rangle$ is positive the first sign above is negative. In this case $\alpha^\Lambda_{F_2} = \alpha^\Lambda_{F_1}$. On the other hand, if $\langle \gamma^k, \Lambda\rangle$ is negative, $\alpha^\Lambda_{F_2} = \alpha^\Lambda_{F_1} + 1$, and the second sign above is negative. Either way, the right side of (2.1) equals the left hand side. The lemma is proved. □

When we apply the results of this section it will be a little simpler if we don't have to always index the elements of Φ. Therefore, in future F will denote only a subset of Φ. Modulo this technicality ψ^Λ_F and α^Λ_F will have the same meaning as above.

§ 3. The Volume of a Convex Hull

Fix a special subgroup A of G. Recall that $\mathfrak{a} = \mathfrak{a}^0 \oplus \mathfrak{a}^1$. In this section we shall not distinguish between functions on $\mathfrak{a}^1$ and $\mathfrak{a}^0$-invariant functions on $\mathfrak{a}$. Let $\hat{\Phi}_P$ be the basis of $\mathfrak{a}^1$ which is dual to Φ_P with respect to the bilinear form $\langle\,,\rangle$ on $\mathfrak{a}^1$. It follows from Lemma 1.1 that the space $\mathfrak{a}^1$, taken with the basis Φ_P, satisfies the assumptions of § 2. If Λ is any point in $\mathfrak{a}^1$, regular in the sense of § 2, and F is any subset of Φ_P, we can define the function $\psi^\Lambda_{P,F}$ and the integer $\alpha^\Lambda_{P,F}$. We have indexed them to denote their dependence on P.

Suppose that (P^*, A^*) is a split parabolic subgroup such that

$$(P^*, A^*) = (P_F, A_F)$$

for some subset F of Φ_P. If the bases Φ_P and $\hat{\Phi}_P$ are indexed as in § 2, we write χ_{P^*} for the characteristic function of the set

$$\{H \in \mathfrak{a} : \langle \gamma^j_F, H\rangle > 0 \quad \text{for all } \gamma^j \textit{ not} \text{ in } F\}$$

and we write φ^Λ_{P,P^*} for the characteristic function of

$$\{H \in \mathfrak{a} : \langle \mu^i_F, H\rangle \langle \gamma^i, \Lambda\rangle < 0 \quad \text{for all } \gamma^i \text{ in } F\}.$$

Then

$$\psi^\Lambda_{P,F}(H) = \chi_{P^*}(H)\, \varphi^\Lambda_{P,P^*}(H), \qquad H \in \mathfrak{a}.$$

Furthermore, we put

$$\beta^A_{P,P^*} = \alpha^A_{P,F}.$$

Finally, we write φ^A_P and β^A_P for $\varphi^A_{P,G}$ and $\beta^A_{P,G}$ respectively.

Suppose that

$$\mathscr{Y} = \{Y_P : P \in \mathscr{P}(A)\}$$

is a set of points in $\mathfrak{a}$, indexed by $\mathscr{P}(A)$. We shall say that $\mathscr{Y}$ is an A-orthogonal set if for any adjacent pair P and P' in $\mathscr{P}(A)$, whose chambers share the wall defined by a uniquely determined simple root γ of (P, A), then

$$Y_P - Y_{P'} = r\gamma, \qquad r \in \mathbb{R}. \tag{3.1}$$

Let A^* be a special subgroup which is contained in A, and suppose that $P^* \in \mathscr{P}(A^*)$. If $\mathscr{Y}$ is A-orthogonal, the set

$$\mathscr{Y}_* = \{Y_P : P \in \mathscr{P}(A),\ P \subset P^*\},$$

indexed via the bijection described in § 1 by $\mathscr{P}_{L^*}(A)$, is an A-orthogonal set for the group L^*. The projection of

$$Y_P, \qquad P \in \mathscr{P}(A), \qquad P \subset P^*,$$

onto $\mathfrak{a}^*$ is independent of P. We denote it by Y_{P^*}. The set

$$\mathscr{Y}^* = \{Y_{P^*} : P^* \in \mathscr{P}(A^*)\}$$

is an A^*-orthogonal set.

Suppose that in (3.1) the number r is actually nonnegative for each adjacent pair P and P'. We shall label this stronger condition by saying that $\mathscr{Y}$ is a *positive* A-orthogonal set. Let P be a group in $\mathscr{P}(A)$, and let Λ be any point in $c_P(\mathfrak{a}) \cap \mathfrak{a}^1$. Then if P' is any other group in $\mathscr{P}(A)$, it is easily seen by induction on $d(P, P')$ that

$$\langle \Lambda, Y_P - Y_{P'} \rangle \geqq 0. \tag{3.2}$$

We shall say that a point $\Lambda \in \mathfrak{a}^1$ is *strongly regular* if for each $P \in \mathscr{P}(A)$, Λ is a regular point associated with $(\mathfrak{a}^1, \Phi_P)$ in the sense of § 2. Denote the set of strongly regular points in $\mathfrak{a}^1$ by $\mathfrak{a}^1_{sr}$. If $\mathscr{Y}$ is an A-orthogonal set, let $\mathfrak{a}_{sr}(\mathscr{Y})$ be the set of points $H \in \mathfrak{a}$ such that for each $P \in \mathscr{P}(A)$, the projection of

$$H + Y_P$$

onto $\mathfrak{a}^1$ is strongly regular.

Lemma 3.1. *Suppose that $\mathscr{Y}$ is an A-orthogonal set. Then for $\Lambda \in \mathfrak{a}^1_{sr}$ the function*

$$\psi(H, \mathscr{Y}) = \sum_{P \in \mathscr{P}(A)} (-1)^{\beta^A_P} \varphi^A_P(H - Y_P), \qquad H \in \mathfrak{a}_{sr}(\mathscr{Y}),$$

is independent of Λ.

Proof. By Lemma 2.2 the function is just the difference of 1 and

$$\sum_{P\in\mathscr{P}(A)}\sum_{F\subsetneq\Phi_P}(-1)^{\alpha_{P,F}^{A}}\psi_{P,F}^{A}(H-Y_P).$$

In this expression we change the sum over F to a sum over all parabolic subgroups dominated by (P, A). We obtain

$$\sum_{P\in\mathscr{P}(A)}\sum_{\substack{P^*>P\\ P^*\neq G}}(-1)^{\beta_{P,P^*}^{A}}\chi_{P^*}(H-Y_P)\,\varphi_{P,P^*}^{A}(H-Y_P)$$
$$=\sum_{P^*\neq G}\chi_{P^*}(H-Y_{P^*})\Big\{\sum_{\substack{P\in\mathscr{P}(A)\\ P<P^*}}(-1)^{\beta_{P,P^*}^{A}}\varphi_{P,P^*}^{A}(H-Y_P)\Big\}.$$

The expression in the brackets depends only on the projection of Λ onto the orthogonal complement of $\mathfrak{a}^*$ in $\mathfrak{a}$. It is simply the function

$$\psi_*(H,\mathscr{Y}_*)$$

associated with the group L^*. Our lemma therefore follows by induction on the dimension of $\mathfrak{a}^1$. □

If $\mathscr{Y}$ is an A-orthogonal set, there is a uniquely determined vector X in $\mathfrak{a}^0$ such that the set

$$\mathscr{Y}^1=\{Y_P-X: P\in\mathscr{P}(A)\}$$

lies in $\mathfrak{a}^1$. Denote the convex hull of $\mathscr{Y}^1$ by $C^1(\mathscr{Y})$, and put

$$C(\mathscr{Y})=C^1(\mathscr{Y})+\mathfrak{a}^0.$$

Lemma 3.2. *Suppose that $\mathscr{Y}=\{Y_P\}$ is a positive A-orthogonal set. Let H be a point in $\mathfrak{a}_{s,r}(\mathscr{Y})$. Then the following conditions on H are equivalent:*

(i) $\psi(H,\mathscr{Y})\neq 0$,

(ii) $H\in C(\mathscr{Y})$,

(iii) for each $P\in\mathscr{P}(A)$ and $\mu\in\hat{\Phi}_P$,

$$\langle\mu, H-Y_p\rangle<0,$$

and

(iv) $\psi(H,\mathscr{Y})=1$.

Proof. The characteristic function of $C(\mathscr{Y})$ and the function $\psi(H,\mathscr{Y})$ are both invariant under translation by $\mathfrak{a}^0$. We may therefore assume for the proof that $\mathfrak{a}^0=\{0\}$. We shall prove that (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

Suppose that (i) is true. Fix an arbitrary point Λ in $\mathfrak{a}_{s,r}$. In view of Lemma 3.1 we can find a $P\in\mathscr{P}(A)$ such that if γ is any element in Φ_P and μ is the corresponding dual basis element,

$$\langle\Lambda,\gamma\rangle\,\langle\mu, H-Y_P\rangle<0.$$

Summing over $\gamma\in\Phi_P$, we see that the number

$$\langle\Lambda, H-Y_P\rangle$$

is negative. If H were not in $C(\mathscr{Y})$ we could, by a fundamental property of convex sets, find a $\Lambda \in \mathfrak{a}$ such that

$$\langle \Lambda, H \rangle > \sup_{X \in C(\mathscr{Y})} \langle \Lambda, X \rangle.$$

We have just seen that this cannot happen if Λ is in $\mathfrak{a}_{sr}$, a dense subset of $\mathfrak{a}$. It follows that H belongs to $C(\mathscr{Y})$.

Next suppose that H is in $C(\mathscr{Y})$. Fix $P \in \mathscr{P}(A)$ and let Λ be any point in $c_P(\mathfrak{a})$. By the Krein-Millman theorem, the linear functional defined by Λ assumes a maximum on $C(\mathscr{Y})$ at some extreme point. Therefore, there is a $P' \in \mathscr{P}(A)$ such that

$$\langle \Lambda, Y_{P'} \rangle \geqq \langle \Lambda, H \rangle.$$

Combining this with (3.2) we find that

$$\langle \Lambda, H - Y_P \rangle \leqq 0.$$

But Λ was an arbitrary point of $c_P(\mathfrak{a})$. It follows that for each $\mu \in \hat{\Phi}_P$,

$$\langle \mu, H - Y_P \rangle \leqq 0.$$

Since H belongs to $\mathfrak{a}_{sr}(\mathscr{Y})$, this inequality is strict.

Suppose next that (iii) is valid. Fix $P \in \mathscr{P}(A)$ and $\Lambda \in c_P(\mathfrak{a})$. Then

$$(-1)^{\beta^{\Lambda}_{P'}} \varphi^{\Lambda}_{P'}(H - Y_{P'}), \qquad P' \in \mathscr{P}(A),$$

equals 1 if $P' = P$ and equals 0 otherwise. It follows from Lemma 3.1 that

$$\psi(H : \mathscr{Y}) = 1.$$

This is just condition (iv), which trivially implies condition (i). □

Corollary 3.3. *If $\mathscr{Y}$ is any A-orthogonal set, the support of the function*

$$H \to \psi(H, \mathscr{Y})$$

is contained in $C(\mathscr{Y})$.

The proof that (i) implied (ii) above only made use of the fact that $\mathscr{Y}$ was A-orthogonal. □

Let $\mathscr{Y}$ be a fixed A-orthogonal set. It is a consequence of Corollary 3.3 that

$$\int_{\mathfrak{a}^1} \psi(H, \mathscr{Y})\, e^{\langle \lambda, H \rangle}\, dH, \qquad \lambda \in \mathfrak{a}^1_{\mathbb{C}}, \tag{3.3}$$

is an entire function of λ. Remember that dH is supposed to be the Euclidean measure on $\mathfrak{a}^1$. We are going to evaluate (3.3) at $\lambda = 0$.

Let Λ be the real part of λ, and choose λ so that Λ is strongly regular. Fix $P \in \mathscr{P}(A)$. We claim that the integral

$$\int_{\mathfrak{a}^1} (-1)^{\beta^{\Lambda}_P} \varphi^{\Lambda}_P(H - Y_P)\, e^{\langle \lambda, H \rangle}\, dH \tag{3.4}$$

is absolutely convergent. If $\Phi_P = \{\gamma^1, \ldots, \gamma^p\}$ and

$$\hat{\Phi}_P = \{\mu^1, \ldots, \mu^p\}$$

is the corresponding dual basis, (3.4) is the integral of

$$(-1)^{\beta_P^A} e^{\langle\lambda, H\rangle}$$

over the set

$$S=\{H\in\mathfrak{a}^1:\langle\Lambda,\gamma^j\rangle\langle\mu^j, H-Y_P\rangle<0,\ 1\leqq j\leqq p\}.$$

In the integral, write

$$H=t_1\gamma^1+\cdots+t_p\gamma^p.$$

With this change of variables we gain the factor

$$c_A=|\det(\langle\gamma^j,\gamma^k\rangle)_{1\leqq j,k\leqq p}|^{\frac{1}{2}},$$

discussed in § 1. The integral becomes

$$c_A(-1)^{\beta_P^A}\prod_{j=1}^{p}\{\int_{S_j} e^{\langle\lambda,\gamma^j\rangle t_j}\,dt_j\},$$

where

$$S_j=\{t_j:\langle\Lambda,\gamma^j\rangle(t_j-\langle\mu^j, Y_P\rangle)<0\}.$$

The above integral over S_j is obviously absolutely convergent, and is easy to evaluate. It equals

$$\langle\lambda,\gamma^j\rangle^{-1} e^{\langle\lambda,\gamma^j\rangle\langle\mu^j, Y_P\rangle}\cdot\operatorname{sign}(\langle\Lambda,\gamma^j\rangle).$$

We have shown that the integral (3.4) is absolutely convergent and is equal to

$$c_A(-1)^{\beta_P^A}\prod_{j=1}^{p}(\langle\lambda,\gamma^j\rangle^{-1} e^{\langle\lambda,\gamma^j\rangle\langle\mu^j, Y_P\rangle}\cdot\operatorname{sign}(\langle\Lambda,\gamma^j\rangle))$$

$$=c_A\frac{e^{\langle\lambda, Y_P\rangle}}{\prod_{j=1}^{p}\langle\lambda,\gamma^j\rangle}.$$

Therefore the function (3.3) equals

$$c_A\sum_{P\in\mathscr{P}(A)}\frac{e^{\langle\lambda, Y_P\rangle}}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle}.$$

This function is entire in λ. To obtain its value at the origin we replace λ by

$$z\lambda,\quad z\in\mathbb{C},$$

and let z approach 0. The resulting expression,

$$c_A(p!)^{-1}\sum_{P\in\mathscr{P}(A)}\frac{\langle\lambda, Y_P\rangle^p}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle},$$

is independent of λ. We have proved the following:

Lemma 3.4. *Suppose that $\mathscr{Y}$ is an A-orthogonal set and that λ is a point in $\mathfrak{a}_{\mathbb{C}}^1$ whose real part is strongly regular. Then the integral over $H \in \mathfrak{a}^1$ of $\psi(H, \mathscr{Y})$ equals*

$$v(\mathscr{Y}) = c_A (p!)^{-1} \sum_{P \in \mathscr{P}(A)} \frac{\langle \lambda, Y_P \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle}.$$

In particular the right side of this formula is independent of λ. □

Corollary 3.5. *If $\mathscr{Y}$ is a positive A-orthogonal set, the expression $v(\mathscr{Y})$ equals the volume of the convex hull $C^1(\mathscr{Y})$.* □

The A-orthogonal sets which will concern us in this paper are given in the next lemma.

Lemma 3.6. *Fix $x \in G$. Then*

$$\mathscr{Y} = \{-H_P(x) : P \in \mathscr{P}(A)\}$$

is a positive A-orthogonal set.

Proof. Suppose that P and P' are adjacent groups in $\mathscr{P}(A)$. Let the common wall of $c_P(\mathfrak{a})$ and $c_{P'}(\mathfrak{a})$ be defined by $\gamma \in \Phi_P$. We must show that

$$-H_P(x) - (-H_{P'}(x)) \tag{3.5}$$

is non-negative multiple of γ.

Put

$$x = m\,a\,n'\,k, \qquad m \in M,\ a \in A,\ n' \in N',\ k \in K.$$

Then (3.5) equals $-H_P(n')$. Corresponding to the subset $\{\gamma\}$ of Φ_P we have the parabolic subgroup

$$P^* = M^* A^* N^*$$

of G, which contains both P and P'. Then

$$(P_*, A_*) = (P \cap M^*, A \cap M^*)$$

is a split parabolic subgroup of M^* of parabolic rank one. Put

$$n' = n^* n_*, \qquad n^* \in N \cap N' = N^*,\ n_* \in N' \cap M^*.$$

We have

$$-H_P(n) = -H_P(n_*) = -H_{P_*}(n_*).$$

This vector is certainly a multiple of γ. That it is actually a nonnegative multiple follows from [2(g), Lemma 85]. □

If $\mathscr{Y}$ and $\mathscr{Y}'$ are (positive) A-orthogonal sets then

$$\mathscr{Y} + \mathscr{Y}' = \{Y_P + Y'_P : P \in \mathscr{P}(A)\}$$

is a (positive) A-orthogonal set.

Corollary 3.7. *Suppose that $\mathcal{Y}$ is an A-orthogonal set. Then the function*

$$v(x,\mathcal{Y})=c_A(p!)^{-1}\sum_{P\in\mathcal{P}(A)}\frac{\langle\lambda, Y_P-H_P(x)\rangle^p}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle},\qquad x\in G,$$

is independent of $\lambda\in\mathfrak{a}_{\mathbb{C}}^1$. It is left-invariant under $L=MA$. □

§ 4. The Distributions

Let A be a special subgroup of G, and as always,

$$L=MA$$

is the centralizer of A. Fix a θ-stable Cartan subalgebra $\mathfrak{t}_M$ of $\mathfrak{m}$, and let T_M be the Cartan subgroup of M associated to $\mathfrak{t}_M$. Then

$$\mathfrak{t}=\mathfrak{t}_M\oplus\mathfrak{a}$$

is a Cartan subalgebra of $\mathfrak{g}$, and

$$T=T_M A$$

is the corresponding Cartan subgroup. We denote by T_{reg} the set of $h\in T$ such that $\xi_\alpha(h)\neq 1$ for every root α of $(\mathfrak{g},\mathfrak{t})$. Fix a Haar measure on G. On $T_{\mathbb{R}}=\exp\mathfrak{t}_{\mathbb{R}}$ we take the Haar measure which corresponds under the exponential map to the Euclidean measure on $\mathfrak{t}_{\mathbb{R}}$. These 2 measures determine a G-invariant measure on $T_{\mathbb{R}}\backslash G$ which we denote by dx. For any $h\in T_{\mathrm{reg}}$, any A-orthogonal set $\mathcal{Y}$ and each function $f\in C_c^\infty(G)$, define

$$\langle r(h:\mathcal{Y}),f\rangle=\int_{T_{\mathbb{R}}\backslash G} f(x^{-1}hx)\,v(x,\mathcal{Y})\,dx.$$

This integral is absolutely convergent and $r(h:\mathcal{Y})$ is easily seen to be a distribution. For any f, $\langle r(h:\mathcal{Y}),f\rangle$ is a smooth function of $h\in T_{\mathrm{reg}}$.

It is these distributions that we primarily want to understand. Their study involves an inductive argument, however, which forces us to enlarge the collection of distributions under consideration. Before doing this, we must agree to some notations for differential operators.

Let $\mathcal{G}$ and $\mathcal{A}$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}$ respectively. $\mathcal{G}$ can be identified with the algebra of left invariant differential operators on G. With this interpretation, an element $Y\in\mathcal{G}$ maps any $f\in C_c^\infty(G)$ to a new function whose value at $x\in G$ is denoted by $(Yf)(x)$ or $f(x;Y)$. On the other hand, from any element

$$X=X_1\ldots X_r,\qquad X_1,\ldots,X_r\in\mathfrak{g},$$

in $\mathcal{G}$ we obtain a right invariant differential operator D_X, defined by

$$(D_Xf)(x)=\frac{d}{dt_1}\cdots\frac{d}{dt_r}f(\exp t_1X_1\ldots\exp t_rX_r x)_{t_1=\cdots=t_r=0},$$

for each $f \in C^\infty(G)$. This last expression is also sometimes denoted by $f(X; x)$. The map

$$X \to D_X, \qquad X \in \mathscr{G},$$

is an anti-isomorphism from $\mathscr{G}$ onto the algebra of right invariant differential operators. When we want to specify that the differentiation D_X applies to the variable x, we write $D_X(x)$. Finally if

$$f(x, y_1, \dots, y_n)$$

is a function of several variables, we shall denote the value of the function

$$D_X(x) f(x, y_1, \dots, y_n)$$

at $x = x_0$ by

$$D_X(x|x_0) f(x, y_1, \dots, y_n).$$

If $\mathscr{U}$ is any vector subspace of $\mathscr{G}$, let $\mathscr{U}_A$ denote the set of elements U in $\mathscr{U}$ such that

$$\mathrm{Ad}(a) U = U$$

for any $a \in A$. The additional distributions will depend on an element $X \in \mathscr{G}_A$ as well as a point $h \in T_{\mathrm{reg}}$, and an A-orthogonal set $\mathscr{Y}$. We define them by

$$\langle r(h:\mathscr{Y}:X), f\rangle = \int_{T_{\mathbb{R}} \backslash G} f(x^{-1} h x) D_X v(x, \mathscr{Y})\, dx.$$

There is a formula for $D_X v(x, \mathscr{Y})$ which we must describe. Suppose that $P \in \mathscr{P}(A)$. Define

$$\mathfrak{p}^1 = \mathfrak{n} \oplus \mathfrak{m},$$

and

$$\mathfrak{v} = \theta(\mathfrak{n}).$$

$\mathfrak{p}^1$ and $\mathfrak{v}$ are subalgebras of $\mathfrak{g}$ and there is a vector space decomposition

$$\mathfrak{g} = \mathfrak{p}^1 \oplus \mathfrak{a} \oplus \mathfrak{v}.$$

Therefore, if $\mathscr{P}^1$ and $\mathscr{V}$ are the universal enveloping algebras of $\mathfrak{p}^1_{\mathbb{C}}$ and $\mathfrak{v}_{\mathbb{C}}$ respectively, $\mathscr{G}$ is linearly isomorphic to $\mathscr{P}^1 \oplus \mathscr{A} \oplus \mathscr{V}$. For each $X \in \mathscr{G}_A$, define $\mu_P(X)$ to be the unique element in $\mathscr{A}$ such that $X - \mu_P(X)$ belongs to

$$\mathfrak{p}^1 \mathscr{G} + \mathscr{G} \mathfrak{v}.$$

There is a decomposition

$$\mathscr{G} = \mathfrak{p}^1 \mathscr{G} \oplus \mathscr{A} \mathscr{V}$$

of $\mathscr{G}$ into subspaces which are normalized by the adjoint action of A. The component of $X - \mu_P(X)$ in $\mathscr{A}\mathscr{V}$ must actually lie in $\mathscr{A}$. By the definition of $\mu_P(X)$, this component must vanish. It follows that $X - \mu_P(X)$ belongs to $\mathfrak{p}^1 \mathscr{G}$. In particular, the map

$$X \to \mu_P(X), \qquad X \in \mathscr{G}_A,$$

is a homomorphism. From this discussion it is clear that for $x \in G$ and $X \in \mathscr{G}_A$,

$$D_X v(x, \mathscr{Y}) = c_A (p!)^{-1} \sum_{P \in \mathscr{P}(A)} D_{\mu_P(X)}(h|1) \frac{\langle \lambda, Y_P - \log h - H_P(x) \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle}. \tag{4.1}$$

Notice that for $m \in MA$,

$$D_X v(mx, \mathscr{Y}) = D_X v(x, \mathscr{Y}).$$

Another consequence of (4.1) is the commutativity formula

$$r(h:\mathscr{Y}:X_1 X_2) = r(h:\mathscr{Y}:X_2 X_1)$$

for two elements X_1 and X_2 in $\mathscr{G}_A$.

For any positive integer r let $\mathscr{G}_A(r)$ be the set of elements X in $\mathscr{G}_A$ such that for any $P \in \mathscr{P}(A)$, all the nonzero homogenous components of $\mu_P(X)$ have degree at least r. $\mathscr{G}_A(r)$ is an ideal in $\mathscr{G}_A$ and $\mathscr{G}_A(r)\,\mathscr{G}_A(r')$ is contained in $\mathscr{G}_A(r+r')$. If X is in $\mathscr{G}_A(p+1)$ then by (4.1)

$$D_X v(x, Y) = 0.$$

Suppose that A^* is another special subgroup of G which is contained in A. We claim that for any r, $\mathscr{G}_A(r)$ is contained in $\mathscr{G}_{A^*}(r)$. To see this, take any P^* in $\mathscr{P}(A^*)$. There is always a $P \in \mathscr{P}(A)$ which is contained in P^*, in which case the group

$$(P^*)^1 = M^* N^*$$

contains P^1. If X belongs to $\mathscr{G}_A(r)$, $\mu_{P^*}(X)$ is the projection of $\mu_P(X)$ onto $\mathscr{A}^*$. The claim follows.

There is a decomposition

$$\mathscr{G} = \mathbb{C}\, I \oplus \mathfrak{g}\mathscr{G},$$

where I is the identity. The projection of any $X \in \mathscr{G}$ onto $\mathbb{C}\, I$ yields a complex number, which we denote by $c_0(X)$.

Lemma 4.1. *For any* $X \in \mathscr{G}_A$,

$$X - c_0(X) I$$

belongs to $\mathscr{G}_A(1)$.

Proof. The element

$$Y = X - c_0(X) I$$

is in $\mathscr{G}_A$. But for any $P \in \mathscr{P}(A)$,

$$c_0(\mu_P(Y)) = c_0(Y) = 0,$$

which proves the lemma. $\square$

Suppose that $w \in K$. If $P \in \mathscr{P}(A)$, wPw^{-1} belongs to $\mathscr{P}(wAw^{-1})$. If

$$x = amnk, \qquad a \in A, m \in M, n \in N, k \in K,$$

write

$$wx = waw^{-1} \cdot wmw^{-1} \cdot wnw^{-1} \cdot wk$$

to see that

$$H_{wPw^{-1}}(wx) = \mathrm{Ad}(w) H_P(x).$$

If $\mathscr{Y}$ is an A-orthogonal set,

$$\begin{aligned} v(x, \mathscr{Y}) &= c_A (p!)^{-1} \sum_{P \in \mathscr{P}(A)} \frac{\langle \lambda, Y_P - \mathrm{Ad}(w^{-1}) H_{wPw^{-1}}(wx) \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle} \\ &= c_A (p!)^{-1} \sum_{P \in \mathscr{P}(wAw^{-1})} \frac{\langle \mathrm{Ad}(w) \lambda, \mathrm{Ad}(w) Y_{w^{-1}Pw} - H_P(wx) \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \mathrm{Ad}(w) \lambda, \gamma \rangle} = v(wx, w\mathscr{Y}), \end{aligned}$$

where

$$(wY)_P = \mathrm{Ad}(w)\, Y_{w^{-1}Pw}, \qquad P \in \mathscr{P}(wAw^{-1}).$$

$w\mathscr{Y}$ is a wAw^{-1}-orthogonal set which is positive whenever $\mathscr{Y}$ is. If we make a change of variables in the integral which defines $r(h:\mathscr{Y}:X)$ we obtain the formula

$$r(whw^{-1} : w\mathscr{Y} : \mathrm{Ad}(w)X) = r(h : \mathscr{Y} : X). \tag{4.2}$$

We know that if our distributions are anything like Harish-Chandra's invariant integrals, it is natural to multiply them by a certain function of h. We take the definition of this function from [2 (i), § 8]. If $\mathfrak{z}$ is the centralizer of $\mathfrak{t}_{\mathbb{R}}$ in $\mathfrak{g}$, let $Z(\mathfrak{t})$ be the centralizer of $\mathfrak{z}$ in K. Let R_I^+ be the set of positive roots of $(\mathfrak{z}, \mathfrak{t})$ relative to some order. Put

$$\Delta_+(h) = |\det(1 - \mathrm{Ad}(h^{-1}))_{\mathfrak{g}/\mathfrak{z}}|^{\frac{1}{2}}, \qquad h \in T,$$

and

$$\Delta_I(H) = \prod_{\beta \in R_I^+} \left(e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}\right), \qquad H \in \mathfrak{t}.$$

Finally, for $\zeta \in Z(\mathfrak{t})$ and $H \in \mathfrak{t}$, define

$$\tilde{\Delta}(\zeta, H) = \Delta_I(H)\, \Delta_+(\zeta \exp H).$$

Let $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ be the set of H in $\mathfrak{t}$ such that

$$\zeta \exp H$$

is in T_{reg}. Then if $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$ and X and $\mathscr{Y}$ are as above, we define

$$\langle R(\zeta, H : \mathscr{Y} : X), f \rangle = \tilde{\Delta}(\zeta, H) \langle r(\zeta, H : \mathscr{Y} : X), f \rangle, \qquad f \in C_c^\infty(G).$$

We shall often write

$$r_f(h:\mathcal{Y})=\langle r(h:\mathcal{Y}),f\rangle,$$

and

$$R_f(\zeta,H:\mathcal{Y}:X)=\langle R(\zeta,H:\mathcal{Y}:X),f\rangle,$$

and when we want to emphasize the dependence of $R_f(\zeta,H:\mathcal{Y}:X)$ on T and A, we will write

$$R_f^{T,A}(\zeta,H:\mathcal{Y}:X)=R_f(\zeta,H:\mathcal{Y}:X).$$

If the function $\tilde{\Delta}(\zeta,H)$ seems unfamiliar it is because we have not assumed that G is acceptable. Suppose that R^+ is the set of positive roots of $(\mathfrak{g},\mathfrak{t})$ relative to some order, which we always assume is taken so that R_I^+ is contained in R^+. For G to be acceptable the function

$$H\to e^{\rho(H)}=\exp\left(\tfrac{1}{2}\sum_{\alpha\in R^+}\alpha(H)\right),\qquad H\in\mathfrak{t},$$

must lift to a function ξ_ρ on T. For H in $\mathfrak{t}_{\text{reg}}(\zeta)$, let $n_{\mathbb{R}}(H)$ be the number of positive *real* roots β in R^+ such that $\beta(H)<0$. Define

$$\varepsilon_{\mathbb{R}}(H)=(-1)^{n_{\mathbb{R}}(H)}$$

and

$$\Delta(\zeta,H)=\varepsilon_{\mathbb{R}}(H)\,\tilde{\Delta}(\zeta,H).$$

Then if G is acceptable $\Delta(\zeta,H)$ equals the usual function

$$\Delta(\zeta\exp H)=\xi_\rho(\zeta\exp H)\prod_{\alpha\in R^+}(1-\xi_\alpha(\zeta\exp H)^{-1}).$$

In any case, $\Delta(\zeta,H)$ is analytic in H.

There are still some other related distributions which we need to define. Suppose that β is a fixed real root of $(\mathfrak{g},\mathfrak{t})$. Define

$$H'_\beta=\frac{2\beta}{\langle\beta,\beta\rangle}.$$

Let X'_β be a fixed root vector for β such that

$$[X'_\beta,-\theta X'_\beta]=H'_\beta,$$

and put

$$Y'_\beta=-\theta X'_\beta.$$

Define

$$\mathfrak{t}_0=\{H\in\mathfrak{t}:\beta(H)=0\}$$

and

$$\mathfrak{t}^*=\mathfrak{t}_0\oplus\mathbb{R}(X'_\beta-Y'_\beta).$$

$\mathfrak{t}^*$ is a Cartan subalgebra, and we denote the corresponding Cartan subgroup by T^*. Notice that $Z(\mathfrak{t}^*)$ is contained in $Z(\mathfrak{t})$. Suppose that ζ is in $Z(\mathfrak{t})$. From the theory of the split three dimensional real Lie algebra, we know that

$$\xi_\beta(\zeta) = \xi_{-\beta}(\zeta) = \pm 1.$$

This number equals 1 if and only if ζ is in $Z(\mathfrak{t}^*)$, which is the case if and only if $\mathrm{Ad}(\zeta)$ commutes with X'_β and Y'_β.

Define

$$\mathfrak{a}^* = \mathfrak{t}_0 \cap \mathfrak{a},$$

and put $A^* = \exp \mathfrak{a}^*$. Let $n_\beta(A)$ be the cosine of the angle between β and the linear subspace $\mathfrak{a}$ of $\mathfrak{t}_{\mathbb{R}}$. If H is any point in $\mathfrak{t}$ such that $\beta(H) \neq 0$, put

$$\tau(H) = \tau_\beta(H) = n_\beta(A) \, \|H'_\beta\| \log \left| e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}} \right|.$$

Suppose that $\zeta \in Z(\mathfrak{t}^*)$. Then for X and $\mathscr{Y}$ as above, $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$ and $f \in C_c^\infty(G)$, we define

$$\langle S^\beta(\zeta, H : \mathscr{Y} : X), f \rangle = S_f^\beta(\zeta, H : \mathscr{Y} : X)$$

to be

$$R_f(\zeta, H : \mathscr{Y} : X) + \tau(H) \, R_f^{T, A^*}(\zeta, H : \mathscr{Y}^* : X).$$

Here $\mathscr{Y}^*$ is the A^*-orthogonal set defined in § 3. Note that if $n_\beta(A)$ is not 0 there is a unique reduced root γ_β of $(\mathfrak{g}, \mathfrak{a})$ such that the restriction of β to $\mathfrak{a}$ is a positive integral multiple of γ_β. This gives rise to an injection

$$j_\beta : \mathscr{P}(A^*) \to \mathscr{P}(A),$$

whose image is the set of $P \in \mathscr{P}(A)$ for which γ_β is a simple root. The vector in $\mathscr{Y}^*$ associated to any $P^* \in \mathscr{P}(A^*)$ is the projection of $Y_{j_\beta(P^*)}$ onto A^*.

The role played by these new distributions will become clear in § 6.

Lemma 4.2. *For $X, \mathscr{Y}$ as above and $x \in G$,*

$$n_\beta(A) \, \|H'_\beta\| \, D_X \, v^*(x, \mathscr{Y}^*) = -D_{Y'_\beta X'_\beta X} \, v(x, \mathscr{Y}).$$

Proof. If $n_\beta(A) = 0$ the lemma is obvious, so we assume $n_\beta(A) \neq 0$. $D_{Y'_\beta X'_\beta X} \, v(x, \mathscr{Y})$ equals

$$c_A (p\,!)^{-1} \sum_{P \in \mathscr{P}(A)} D_{\mu_P(Y'_\beta X'_\beta X)}(h|1) \frac{\langle \lambda, Y_P - \log h - H_P(x) \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle}.$$

For any P,

$$\mu_P(Y'_\beta X'_\beta X) = \mu_P(Y'_\beta X'_\beta) \, \mu_P(X) = -\varepsilon_\beta(P) \, H'_\beta \, \mu_P(X),$$

where $\varepsilon_\beta(P)$ equals 1 or 0 according to whether γ_β is a root of (P, A) or not. Therefore $D_{Y'_\beta X'_\beta X} \, v(x, \mathscr{Y})$ equals

$$-c_A((p-1)\,!)^{-1} \langle \lambda, H'_\beta \rangle \cdot \sum_P^{+} D_{\mu_P(X)}(h|1) \frac{\langle \lambda, Y_P - \log h - H_P(x) \rangle^{p-1}}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle}, \tag{4.3}$$

where the summation is extended over all $P \in \mathscr{P}(A)$ for which γ_β is a root of (P, A).

Put

$$\lambda = z\gamma_\beta + \lambda^*, \qquad z \in \mathbb{C},$$

where λ^* is a point in $\mathfrak{a}^*$ such that $\langle \lambda^*, \gamma\rangle \neq 0$ for any root γ of $(\mathfrak{g}, \mathfrak{a})$ which has nontrivial restriction to $\mathfrak{a}^*$. We fix λ^* and let z approach 0. If γ_β is a root of (P, A), and $\gamma \in \Phi_P$, then $\langle \lambda^*, \gamma\rangle$ can be zero only if γ equals the reduced root γ_β. In particular, γ_β must belong to Φ_P, which is the same as saying that $P = j_\beta(P^*)$ for some $P^* \in \mathscr{P}(A^*)$. We find that (4.3) equals

$$-c_A((p-1)!)^{-1} \frac{\langle \gamma_\beta, H'_\beta\rangle}{\langle \gamma_\beta, \gamma_\beta\rangle} D_X(x) \sum_{P^* \in \mathscr{P}(A^*)} \frac{\langle \lambda, Y_{j_\beta(P^*)} - H_{P^*}(x)\rangle^{p-1}}{\prod_{\gamma \in \Phi_{P^*}} \langle \lambda, \gamma\rangle}.$$

From Lemma 1.3 and the definition of $n_\beta(A)$ we obtain

$$c_A \frac{\langle \gamma_\beta, H'_\beta\rangle}{\langle \gamma_\beta, \gamma_\beta\rangle} = c_A\, n_\beta(A) \frac{\langle H'_\beta, H'_\beta\rangle^{\frac{1}{2}}}{\langle \gamma_\beta, \gamma_\beta\rangle^{\frac{1}{2}}} = c_{A^*}\, n_\beta(A)\, \|H'_\beta\|.$$

It follows that (4.3) equals

$$-n_\beta(A)\, \|H'_\beta\|\, D_X\, v^*(x, \mathscr{Y}^*),$$

which is what we were required to prove. $\square$

Corollary 4.3. *For X, $\mathscr{Y}$ and ζ as above, $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$, and $f \in C_c^\infty(G)$,*

$$\begin{aligned} -n_\beta(A)\, \|H'_\beta\|\, R_f^{T, A^*}(\zeta, H : \mathscr{Y}^* : X) &= R_f^{T, A}(\zeta, H : \mathscr{Y} : Y'_\beta X'_\beta X) \\ &= S_f^\beta(\zeta, H : \mathscr{Y} : Y'_\beta X'_\beta X). \end{aligned}$$

Proof. The first equation follows from the lemma. The second is a consequence of the fact that $R_f^{T, A^*}(\zeta, H : \mathscr{Y}^* : Y'_\beta X'_\beta X)$ equals 0. $\square$

The next four sections will be devoted to a more detailed study of the distributions $R(\zeta, H : \mathscr{Y} : X)$ and $S^\beta(\zeta, H : \mathscr{Y} : X)$. In these sections A, $\mathfrak{t}$, T, $\mathscr{Y}$ and X will be fixed and are to have the meaning ascribed above. The element ζ is also to be fixed, and unless stated otherwise it belongs only to $Z(\mathfrak{t})$.

§ 5. The Differential Equations

As in the real rank one case ([1(b), § 5]) our distributions satisfy a linear nonhomogenous differential equation. However, unlike the real rank one case it is not sufficient for us to consider only the Casimir element. Rather, we must derive a differential equation for each element of $\mathscr{Z}$, the center of $\mathscr{G}$.

Let $\mathscr{T}$ be the universal enveloping algebra of $\mathfrak{t}_{\mathbb{C}}$. For any $X \in \mathfrak{g}_{\mathbb{C}}$ and $Y \in \mathscr{G}$ define

$$R_X(Y) = YX,$$
$$L_X(Y) = XY.$$

It is easily verified [2(a), Lemma 15] that for every $h \in T$ there is a unique linear mapping

$$\Gamma_h : \mathscr{G} \otimes \mathscr{T} \to \mathscr{G}$$

such that

(i) $\Gamma_h(1\otimes u)=u, \quad u\in\mathscr{T}$,

and

(ii) $\Gamma_h(X_1 \dots X_r\otimes u)=(L_{\mathrm{Ad}(h^{-1})X_1}-R_{X_1})\dots(L_{\mathrm{Ad}(h^{-1})X_r}-R_{X_r})\,u$,

for $X_1,\dots,X_r\in\mathfrak{g}$.

Let $\mathfrak{s}$ be the direct sum over all roots of $(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ of the corresponding root spaces. $\mathfrak{s}$ is a subspace of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathscr{S}$ be the image of the symmetric algebra on $\mathfrak{s}$ under the canonical map from the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$ to $\mathscr{G}$. $\mathscr{S}$ is a vector subspace of $\mathscr{G}$. Denote by $\mathscr{S}'$ the set of $X\in\mathscr{S}$ such that $c_0(X)=0$.

Suppose mow that h is regular. In [2(a), Lemma 22] Harish-Chandra proves that Γ_h maps $\mathscr{S}\otimes\mathscr{T}$ bijectively onto $\mathscr{G}$. In particular, for each $z\in\mathscr{Z}$ there is a unique element $\beta_h(z)$ in $\mathscr{T}$ such that

$$z-\beta_h(z)$$

belongs to $\Gamma_h(\mathscr{S}'\otimes\mathscr{T})$. Therefore there are elements

$$\{X_i: 1\leqq i\leqq r\}$$

in $\mathscr{S}'$ which commute with T, linearly independent elements

$$\{u_i: 1\leqq i\leqq r\}$$

in $\mathscr{T}$ and analytic functions

$$\{a_i: 1\leqq i\leqq r\}$$

on T_{reg} such that for any $h\in T_{\mathrm{reg}}$,

$$z-\beta_h(z)=\sum_{i=1}^{r} a_i(h)\,\Gamma_h(X_i\otimes u_i).$$

Fix $f\in C_c^\infty(G)$. For $y, y_1, y_2\in G$, let us write

$$f(y_1:y:y_2)=f(y_1\,y\,y_2).$$

We shall denote the function

$$f(x^{-1}yx)=f(x^{-1}:y:x), \qquad x,y\in G,$$

by $F(x:y)$. If z is, as above, in the center of $\mathscr{G}$,

$$f(x^{-1}hx;z)=f(x^{-1}:h:z;x)=F(x:h;z)$$

for each $x\in G$. Suppose that $X\in\mathfrak{g}$ and $Y\in\mathscr{G}$. Then

$$\begin{aligned}
&F(x:h;(L_{\mathrm{Ad}(h^{-1})X}-R_X)\,Y)\\
&\quad=f(x^{-1}:h;(L_{\mathrm{Ad}(h^{-1})X}-R_X)\,Y:x)\\
&\quad=f(x^{-1}:X;\,h;\,Y:x)-f(x^{-1}:h;\,YX:x)\\
&\quad=-\frac{d}{dt}f((\exp tX\cdot x)^{-1}:h;\,Y:x)\Big|_{t=0}-\frac{d}{dt}f(x^{-1}:h;\,Y:\exp tX\cdot x)\Big|_{t=0}\\
&\quad=-\frac{d}{dt}F(\exp tX\cdot x:h;\,Y)\Big|_{t=0}.
\end{aligned}$$

From this formula it follows easily that for any h in T_{reg},

$$F(x:h;\Gamma_h(X_i\otimes u_i))=D_{X_i'}(x)\,D_{u_i}(h)\,F(x:h).$$

Here X_i' is the image of X_i under the anti-automorphism of $\mathscr{G}$ defined by

$$Y_1\dots Y_r\to(-1)^r\,Y_r\dots Y_1,\qquad Y_1,\dots,Y_r\in\mathfrak{g}_{\mathbb{C}}.$$

Notice that for any pair of functions g_1 and g_2 in $C_c^\infty(G)$,

$$\int_G (D_{X'_i}\,g_1(x))\,g_2(x)\,dx=\int_G g_1(x)\,D_{X_i}\,g_2(x)\,dx.$$

It follows that

$$\int_{T_{\mathbb{R}}\backslash G} F(x:h;\Gamma_h(X_i\otimes u_i))\,D_X\,v(x,\mathscr{Y})\,dx$$

equals

$$D_{u_i}(h)\int_{T_{\mathbb{R}}\backslash G} f(x^{-1}h\,x)\,D_{XX_i}\,v(x,\mathscr{Y})\,dx.$$

Now we shall write

$$h=\zeta\exp H,\qquad \zeta\in Z(\mathfrak{t}),\qquad H\in\mathfrak{t}_{\mathrm{reg}}(\zeta).$$

Let $S(\mathfrak{t}_{\mathbb{C}})$ be the symmetric algebra on $\mathfrak{t}_{\mathbb{C}}$. It is canonically isomorphic to $\mathscr{Y}$. In particular any element $u\in S(\mathfrak{t}_{\mathbb{C}})$ defines a differential operator with constant coefficients on $\mathfrak{t}$, which sends any function $\varphi\in C^\infty(\mathfrak{t})$ to a new function which we denote by

$$\partial(u)\,\varphi(H)=\varphi(H;\partial(u)),\qquad H\in\mathfrak{t}.$$

Let $I(\mathfrak{t}_{\mathbb{C}})$ be the set of elements in $S(\mathfrak{t}_{\mathbb{C}})$ which are invariant under the Weyl group of $(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$, and let γ denote the isomorphism from $\mathscr{Z}$ onto $I(\mathfrak{t}_{\mathbb{C}})$. Harish-Chandra has proved [2(a), Theorem 2] that for $\varphi\in C^\infty(\mathfrak{t}_{\mathrm{reg}}(\zeta))$, $z\in\mathscr{Z}$ and h as above,

$$\tilde{\Delta}(\zeta,H)\,\beta_h(z)\,(\tilde{\Delta}(\zeta,H)^{-1}\,\varphi(H))=\partial(\gamma(z))\,\varphi(H).$$

Finally, for each i, let $\partial_H^i(z)$ be the operator which sends $\varphi\in C^\infty(\mathfrak{t}_{\mathrm{reg}}(\zeta))$ to

$$\tilde{\Delta}(\zeta,H)\,a_i(\zeta\exp H)\,D_{u_i}(\tilde{\Delta}(\zeta,H)^{-1}\,\varphi(H)).$$

Each $\partial_H^i(z)$ is a differential operator on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$. It is a consequence of [2(a), Lemma 23] that for every z and i there is a k such that the coefficients of the differential operator

$$\Delta(\zeta,H)^k\,\partial_H^i(z)$$

extend to analytic functions on $\mathfrak{t}$.

We have virtually established

Lemma 5.1. *For any $z\in\mathscr{Z}$ we can find elements*

$$\{X_i:1\leqq i\leqq r\}$$

in $G_A(1)$ *and differential operators*

$$\{\partial^i_H(z) : 1 \leqq i \leqq r\}$$

on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ *such that for any* $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$ *and* $f \in C_c^\infty(G)$,

$$R_{zf}(\zeta, H : \mathscr{Y} : X) - R_f(\zeta, H ; \partial(\gamma(z)) : \mathscr{Y} : X)$$

equals

$$\sum_{i=1}^{r} R_f(\zeta, H ; \partial^i_H(z) : \mathscr{Y} : X X_i).$$

Proof. The equality of the two given expressions has been established by the discussion above. The assertion that each X_i belongs to $\mathscr{G}_A(1)$ follows from Lemma 4.1. □

Suppose that β is a real root of $(\mathfrak{g}, \mathfrak{t})$. Adopt the notation of the previous section. Fix $\zeta \in Z(\mathfrak{t}^*)$. We can easily transform the assertion of the lemma into a differential equation for $S^\beta_f(\zeta, H : \mathscr{Y} : X)$. Define

$$\tilde{\tau}(H) = -\tfrac{1}{2} \log |e^{\frac{\beta(H)}{2}} - e^{-\frac{\beta(H)}{2}}| = -n_\beta(A)^{-1} \, \|H'_\beta\| \, \tau(H),$$

if $\beta(H) \neq 0$. From Corollary 4.3 we have the formula

$$S^\beta_f(\zeta, H : \mathscr{Y} : X) = R_f(\zeta, H : \mathscr{Y} : X) + \tilde{\tau}(H) \, R_f(\zeta, H : \mathscr{Y} : Y'_\beta X'_\beta X).$$

If D is any differential operator on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ we write

$$[D, \tilde{\tau}(H)]$$

for the commutator of D and the operator given by multiplication by $\tilde{\tau}(H)$. Applying the differential equation we have just proved to the two right hand terms in the above formula for S^β_f, we find that for any $z \in \mathscr{Z}$, $S^\beta_f(\zeta, H ; \partial(\gamma(z)) : \mathscr{Y} : X)$ is the sum of

$$S^\beta_{zf}(\zeta, H : \mathscr{Y} : X) - \sum_{i=1}^{r} S^\beta_f(\zeta, H ; \partial^i_H(z) : \mathscr{Y} : Y'_\beta X'_\beta X)$$

and

$$R_f(\zeta, H ; [\partial(\gamma(z)), \tilde{\tau}(H)] : \mathscr{Y} : Y'_\beta X'_\beta X)$$
$$+ \sum_{i=1}^{r} R_f(\zeta, H ; [\partial^i_H(z), \tilde{\tau}(H)] : \mathscr{Y} : Y'_\beta X'_\beta X X_i).$$

The element $Y'_\beta X'_\beta$ is in $\mathscr{G}_A(1)$. It follows from (4.1) and Corollary 4.3 that for any $\tilde{X} \in \mathscr{G}_A$,

$$R_f(\zeta, H : \mathscr{Y} : Y'_\beta X'_\beta \tilde{X}) = R_f(\zeta, H : \mathscr{Y} : \tilde{X} Y'_\beta X'_\beta) = S^\beta_f(\zeta, H : \mathscr{Y} : \tilde{X} Y'_\beta X'_\beta).$$

We have shown that there exist elements

$$\tilde{X}_j, \quad 1 \leqq j \leqq \tilde{r},$$

in $\mathscr{G}_A(1)$ and differential operators

$$\tilde{\partial}^j_H(z), \quad 1 \leqq j \leqq \tilde{r},$$

on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ such that

$$S_f^\beta(\zeta, H:\partial(\gamma(z)):\mathscr{Y}:X)=S_{zf}^\beta(\zeta, H:\mathscr{Y}:X)-\sum_{j=1}^{\tilde{r}} S_f^\beta(\zeta, H;\tilde{\partial}_H^j(z):\mathscr{Y}:X\tilde{X}_j). \tag{5.1}$$

The logarithm which is in the formula for $\tilde{\tau}(H)$ disappears from the coefficient functions of the differential operators $[\partial(\gamma(z)), \tilde{\tau}(H)]$ and

$$[\partial_H^i(z), \tilde{\tau}(H)], \quad 1 \leqq i \leqq r.$$

It follows that for each j and z there is an integer k such that the coefficients of the differential operator

$$\Delta(\zeta, H)^k\, \tilde{\partial}_H^j(z)$$

extend to analytic functions on $\mathfrak{t}$.

The above differential equations can be written down explicitly if z is the Casimir element in $\mathscr{G}$. The calculation proceeds as in the case of $\mathbb{R}$-rank one. The analogue of Corollary 4.3, for β an arbitrary root of $(\mathfrak{g}, \mathfrak{t})$, allows one to express the non-homogenous components of the differential equation in a particularly simple form. Here, however, we need this more explicit formula only in a special case, and we may as well just quote the result from [1].

Suppose that, as above, β is a real root of $(\mathfrak{g}, \mathfrak{t})$. Let $\mathfrak{g}_\beta$ be the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}$. Let G_β be the analytic subgroup of G corresponding to $\mathfrak{g}_\beta$. Finally let $\mathscr{Z}_\beta$ be the center of the universal enveloping algebra of $\mathfrak{g}_{\beta,\mathbb{C}}$, and let s_β denote the reflection in $\mathfrak{t}_{\mathbb{C}}$ about the hyperplane $\mathfrak{t}_{0,\mathbb{C}}$. Then we have the isomorphism γ_β from $\mathscr{Z}_\beta$ onto the set of elements in $S(\mathfrak{t}_{\mathbb{C}})$ which are invariant under s_β. Suppose that H is a point in $\mathfrak{t}$ such that $\beta(H) \neq 0$. For any $\varphi \in C_c^\infty(G)$ we write, in the usual notation for the invariant integral,

$$F_\varphi(H)=|e^{\frac{\beta(H)}{2}}-e^{-\frac{\beta(H)}{2}}| \int_{T_{\mathbb{R}}\backslash G_\beta} f(x^{-1}\exp H\, x)\,dx.$$

We also write

$$R_\varphi(H)=R_\varphi(1, H:0:1)$$

and

$$S_\varphi(H)=S_\varphi^\beta(1, H:0:1)$$

for the values at φ of the distributions R and S^β associated with the group G_β and for which $\zeta=1, \mathscr{Y}=0$ and $X=1$.

The element

$$\omega=\tfrac{1}{8}(H_\beta')^2+\tfrac{1}{4}X_\beta' Y_\beta'+\tfrac{1}{4}Y_\beta' X_\beta'+\tfrac{1}{8}$$

is in $\mathscr{Z}_\beta$. Its image in $S(\mathfrak{t}_{\mathbb{C}})$ under γ_β is $\frac{1}{8}(H_\beta')^2$.

Lemma 5.2. *For $\varphi \in C_c^\infty(G)$ we have*

$$S_\varphi(H;\partial(\gamma_\beta(\omega)))=S_{\omega\varphi}(H)+\tfrac{1}{4}\coth\beta(H)\cdot F_\varphi(H;\partial(H_\beta')).$$

Proof. First of all we quote the differential equation satisfied by $R_\varphi(H)$ from [1(b)]. G_β is not semisimple and need not be a matrix group but this does not affect the outcome. $R_\varphi(H)$ is actually a multiple of the distribution defined in [1(b)]. In fact if u belongs to the subgroup $\exp(\mathbb{R}\, Y'_\beta)$ of G_β, $v(u,0)$ equals

$$-\frac{\beta(H_\beta(u))}{\langle\beta,\beta\rangle^{\frac{1}{2}}}=\|H'_\beta\|\cdot\tfrac{1}{2}\,\beta(H_\beta(u)).$$

We must therefore multiply the differential equation of [1, Theorem 5.1] by $\|H'_\beta\|$. The result is

$$\tfrac{1}{8}\,R_\varphi(H;\partial(H'_\beta)^2)=R_{\omega\varphi}(H)+\tfrac{1}{2}\|H'_\beta\|\,(e^{\frac{\beta(H)}{2}}-e^{-\frac{\beta(H)}{2}})^{-2}\,F_\varphi(H).$$

(The reader might feel more comfortable deriving this equation directly.)

Now

$$S_\varphi(H)=R_\varphi(H)+\|H'_\beta\|\,\log|e^{\frac{\beta(H)}{2}}-e^{-\frac{\beta(H)}{2}}|\,F_\varphi(H).$$

Since

$$F_\varphi(H;\partial(H'_\beta)^2)=F_{\omega\varphi}(H),$$

we have only to look at the commutator

$$\tfrac{1}{8}\,[\partial(H'_\beta)^2,\log|e^{\frac{\beta(H)}{2}}-e^{-\frac{\beta(H)}{2}}|].$$

This is just

$$\tfrac{1}{4}\coth\beta(H)\cdot\partial(H'_\beta)-\tfrac{1}{2}(e^{\frac{\beta(H)}{2}}-e^{-\frac{\beta(H)}{2}})^{-2}.$$

The lemma follows. □

Corollary 5.3. *Suppose that $z\in\mathscr{Z}_\beta$ and $\varphi\in C_c^\infty(G_\beta)$. Then modulo a function which extends to a smooth function of $H\in\mathfrak{t}$, $S_\varphi(H;\partial(\gamma_\beta(z))$ equals $S_{z\varphi}(H)$.*

Proof. $\mathscr{Z}_\beta$ is generated, over the universal enveloping algebra of the center of $\mathfrak{g}_{\beta,\mathbb{C}}$, by ω. The corollary follows by induction provided that we can prove the result for $z=\omega$. It is known that $F_\varphi(H)$ extends to a smooth function of $H\in\mathfrak{t}$. Since

$$F_\varphi(s_\beta H;\partial(H'_\beta))=-F_\varphi(H;\partial(H'_\beta)),$$

we have

$$F_\varphi(H_0)=0$$

if $\beta(H_0)=0$. Therefore

$$H\to\tfrac{1}{4}\coth\beta(H)\cdot F_\varphi(H)$$

extends to a smooth function on $\mathfrak{t}$. The corollary now follows from the lemma. □

§ 6. Boundary Values

Let β be a real root of $(\mathfrak{g},\mathfrak{t})$. In this section we shall investigate the behaviour of $S_f^\beta(\zeta,H:\mathscr{Y}:X)$ as H approaches the hyperplane $\mathfrak{t}_0$. Adopt the notation of § 4 and

§ 5. In addition, define $K_\beta = K \cap G_\beta$, $N_\beta = \exp(\mathbb{R} X'_\beta)$ and $U_\beta = \exp(\mathbb{R} Y'_\beta)$. Then

$$G_\beta = T_{\mathbb{R}} N_\beta K_\beta = T_{\mathbb{R}} U_\beta K_\beta.$$

We have already normalized the Haar measure on $T_{\mathbb{R}}$, and we have also fixed some Haar measure on G. In this chapter we shall need to use Haar measures on G_β, N_β, U_β and K_β, as well as a G-invariant measure on $G_\beta \diagdown G$. At this point we assume only that they are normalized to satisfy the obvious compatibility conditions. Put

$$\Lambda = \exp\left\{-(-1)^{\frac{1}{2}} \frac{\pi}{4} \operatorname{ad}(X'_\beta + Y'_\beta)\right\}.$$

Λ is an automorphism of $\mathfrak{g}_{\beta,\mathbb{C}}$. We have $\Lambda(\mathfrak{t}_{\mathbb{C}}) = \mathfrak{t}^*_{\mathbb{C}}$ and

$$\Lambda H'_\beta = i(X'_\beta - Y'_\beta).$$

Let us decree that an imaginary root α of $(\mathfrak{g}, \mathfrak{t}^*)$ with nontrivial restriction to $\mathbb{C}(X'_\beta - Y'_\beta)$ is positive if and only if $\alpha(i(X'_\beta - Y'_\beta))$ is positive. This condition, together with our fixed order on the imaginary roots of $(\mathfrak{g}, \mathfrak{t})$, serves to order the imaginary roots of $(\mathfrak{g}, \mathfrak{t}^*)$.

In this section we assume that ζ actually lies in $Z(\mathfrak{t}^*)$. Then we can consider the distributions $R^{T^*, A^*}(\zeta, H^* : \mathscr{Y}^* : X)$. Moreover, any element

$$\zeta \exp H_0, \qquad H_0 \in \mathfrak{t}_0,$$

commutes with G_β. Define $\mathfrak{t}_{0,\mathrm{reg}}(\zeta)$ to be the set of points H_0 on $\mathfrak{t}_0$ such that

$$\xi_\gamma(\zeta \exp H_0) \neq 1$$

for any root γ of $(\mathfrak{g}, \mathfrak{t})$ not equal to β or $-\beta$. It is an open dense subset of $\mathfrak{t}_0$. If S is any function on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$, and $H_0 \in \mathfrak{t}_{0,\mathrm{reg}}(\zeta)$, we shall write

$$S(H_0)^{\pm} = \lim_{t \to \pm 0} S(H_0 + t H'_\beta).$$

Fix an open subset Ω_0 of $\mathfrak{t}_{0,\mathrm{reg}}(\zeta)$ which is relatively compact in $\mathfrak{t}_{0,\mathrm{reg}}(\zeta)$.

Theorem 6.1. *Suppose that $\zeta \in Z(\mathfrak{t}^*)$, $u \in S(\mathfrak{t}_{\mathbb{C}})$ and $f \in C_c^\infty(G)$. Then the limits $S_f^\beta(\zeta, H_0; \partial(u) : \mathscr{Y} : X)^+$ and $S_f^\beta(\zeta, H_0; \partial(u) : \mathscr{Y} : X)^-$ both exist, uniformly for $H_0 \in \Omega_0$. The first limit minus the second one equals*

$$n_\beta(A) \lim_{\theta \to 0} R_f^{T^*, A^*}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta); \partial(\Lambda(s_\beta u - u)) : \mathscr{Y}^* : X).$$

Proof. We shall put

$$H = H_0 + t H'_\beta, \qquad H_0 \in \Omega_0,\ t \in (-\varepsilon, \varepsilon),$$

for some fixed positive number ε. Let $\bar{\Omega}$ be the set of H so obtained, and let Ω be the set of $H \in \bar{\Omega}$ for which $t \neq 0$. We take ε to be so small that for any H in the closure of $\bar{\Omega}$ and any root γ of $(\mathfrak{g}, \mathfrak{t})$, $\gamma \neq \pm\beta$,

$$\xi_\gamma(\zeta \exp H) \neq 1.$$

The function

$$\tilde{\Delta}_0(\zeta, H) = \tilde{\Delta}(\zeta, H)|e^t - e^{-t}|^{-1}$$

is a smooth function of $H \in \bar{\Omega}$. Define

$$\varphi_y(x) = f(y^{-1} \zeta x y), \qquad x \in G_\beta, \ y \in G.$$

Then for $H \in \Omega$, $S_f^\beta(\zeta, H : \mathscr{Y} : X)$ equals the product of $\tilde{\Delta}_0(\zeta, H)$ with

$$\int_{G_\beta \backslash G} |e^t - e^{-t}| \int_{T_{\mathbb{R}} \backslash G_\beta} \varphi_y(x^{-1} \exp H x)(D_X v(x y, \mathscr{Y}) + \tau(H) D_X v^*(y, \mathscr{Y}^*)) \, dx \, dy.$$

We have used the fact that the function

$$D_X v^*(y, \mathscr{Y}^*), \qquad y \in G,$$

is left invariant under G_β. It follows from the definition of Ω and the fact that f is compactly supported, that the integral over $G_\beta \backslash G$ may be taken over a relatively compact subset Γ of G which is independent of H. We rewrite the above integral as

$$\begin{aligned} &\int_\Gamma |e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1} u^{-1} \exp H u k) . \\ &(D_X v(u k y, \mathscr{Y}) + \tau(H) D_X v^*(y, \mathscr{Y}^*)) \, du \, dk \, dy. \end{aligned} \tag{6.1}$$

If $n_\beta(A) = 0$, G_β is contained in M. As a result, (6.1) reduces to the integral over $y \in \Gamma$ of the product of $D_X v(y, \mathscr{Y})$ with

$$|e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1} u^{-1} \exp H u k) \, du \, dk.$$

This last expression is a smooth function of $H \in \bar{\Omega}$, so there is nothing further to prove.

Therefore we may assume that $n_\beta(A) \neq 0$, and that the restriction of β to $\mathfrak{a}$ is a positive multiple of the reduced root γ_β of $(\mathfrak{g}, \mathfrak{a})$. In the formula for

$$D_X v(u k y, \mathscr{Y}), \qquad u \in U_\beta, \ k \in K_\beta, \ y \in G,$$

namely,

$$c_A(p!)^{-1} \sum_{P \in \mathscr{P}(A)} D_{\mu_P(X)}(h|1) \frac{\langle \lambda, Y_P - \log h - H_P(u k y) \rangle^p}{\prod_{\gamma \in \Phi_P} \langle \lambda, \gamma \rangle},$$

we consider separately summations $\sum_P^+$ and $\sum_P^-$ over those $P \in \mathscr{P}(A)$ for which γ_β is, and respectively is not, a root of (P, A). If γ_β is not a root of (P, A),

$$H_P(u k y) = H_P(k y).$$

If γ_β is a root of (P, A), write

$$u = \exp H_\beta(u) \cdot N(u) K(u),$$

where $N(u)\in N_\beta$, $K(u)\in K_\beta$, and $H_\beta(u)$ is a multiple of H'_β. Then

$$H_P(u\,k\,y)=H_\beta(u)+H_P(K(u)\,k\,y).$$

$D_X\,v(u\,k\,y,\mathcal{Y})$ becomes the sum of

$$c_A(p!)^{-1}\,D_X(\eta|k\,y)\sum_P^{-}\frac{\langle\lambda,Y_P-H_P(\eta)\rangle^p}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle} \tag{6.2}$$

and

$$c_A(p!)^{-1}\,D_X(\eta|K(u)\,k\,y)\cdot\sum_P^{+}\frac{\langle\lambda,Y_P-H_\beta(u)-H_P(\eta)\rangle^p}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle}.$$

This last expression we rewrite as the sum of

$$c_A(p!)^{-1}\,D_X(\eta|K(u)\,k\,y)\cdot\sum_P^{+}\frac{\langle\lambda,Y_P-H_P(\eta)\rangle^p}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle}, \tag{6.3}$$

$$c_A((p-1)!)^{-1}\,D_X(\eta|K(u)\,k\,y)\cdot\sum_P^{+}\langle\lambda,H_\beta(u)\rangle\frac{\langle\lambda,Y_P-H_P(\eta)\rangle^{p-1}}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle} \tag{6.4}$$

and

$$\langle\lambda,H_\beta(u)\rangle^2\cdot\sum_P^{+}\frac{Q_P(\lambda)}{\prod_{\gamma\in\Phi_P}\langle\lambda,\gamma\rangle}. \tag{6.5}$$

In (6.5), $Q_P(\lambda)$ is a polynomial in λ, which of course depends on $K(u)\,k\,y$.

Put

$$\lambda=z\,\gamma_\beta+\lambda^*,\qquad z\in\mathbb{C},$$

where λ^* is a point in $\mathfrak{a}^*$ such that $\langle\lambda^*,\gamma\rangle\neq 0$ for any root γ of $(\mathfrak{g},\mathfrak{a})$ which has nontrivial restriction to $\mathfrak{a}^*$. For u as above

$$\langle\lambda,H_\beta(u)\rangle=z\,\langle\gamma_\beta,H_\beta(u)\rangle.$$

We fix λ^* and let z approach 0. If γ_β is a root of (P,A), and $\gamma\in\Phi_P$, then $\langle\lambda^*,\gamma\rangle$ can be zero only if γ equals the reduced root γ_β. In particular γ_β must be in Φ_P, which is the same as saying that $P=j_\beta(P^*)$, $P^*\in\mathcal{P}(A^*)$. At any rate, there is, for fixed P, at most one γ in Φ_P such that $\langle\lambda^*,\gamma\rangle=0$. It follows that the limit of (6.5) as z approaches 0 is 0, while the limit of (6.4) is

$$-c_A((p-1)!)^{-1}\frac{\langle\gamma_\beta,H_\beta(u)\rangle}{\langle\gamma_\beta,\gamma_\beta\rangle}D_X(\eta|K(u)\,k\,y)\cdot\sum_{P^*\in\mathcal{P}(A^*)}\frac{\langle\lambda^*,Y_{j_\beta(P^*)}-H_{P^*}(\eta)\rangle^{p-1}}{\prod_{\gamma\in\Phi_{P^*}}\langle\lambda^*,\gamma\rangle}.$$

In view of Lemma 1.3, this expression is just the product of

$$\frac{\langle\gamma_\beta,H_\beta(u)\rangle}{\langle\gamma_\beta,\gamma_\beta\rangle^{\frac{1}{2}}}$$

with

$$D_X\,v^*(K(u)\,k\,y,\mathcal{Y}^*)=D_X\,v^*(y,\mathcal{Y}^*).$$

It follows that the limit as z approaches 0 of the sum of (6.2) and (6.3) exists and is independent of λ^*.

Let us denote the terms (6.2) and (6.3) by $\psi^-(z, k y)$ and $\psi^+(z, K(u) k y)$ respectively. Each of these functions is meromorphic in the first variable and smooth in the second. For any z

$$\int_{\Gamma} |e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1} u^{-1} \cdot \exp H \cdot u k) \psi^-(z, k y) \, du \, dk \, dy$$

equals

$$e^{-t} \int_{\Gamma} \int_{K_\beta} \int_{U_\beta} f(y^{-1} \zeta k^{-1} \exp H u k y) \psi^-(z, k y) \, du \, dk \, dy.$$

Since Γ and K_β are compact, the constant term of the Laurent expansion about $z=0$ of this function extends to a smooth function of $H \in \bar{\Omega}$. Next, examine the contribution the term (6.3) makes to (6.1). This is

$$\begin{aligned} &\int_{\Gamma} |e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1} u^{-1} \exp H u k) \psi^+(z, K(u) k y) \, du \, dk \, dy \\ &\quad = \int_{\Gamma} |e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1}(uK(u)^{-1})^{-1} \exp H \; (uK(u)^{-1}) k) \psi^+(z, k y) \, du \, dk \, dy. \end{aligned} \tag{6.6}$$

Now

$$u K(u)^{-1} = \exp H_\beta(u) \cdot N(u).$$

As may be seen by direct calculation on $SL(2, \mathbb{R})$, the map

$$u \to N(u), \qquad u \in U_\beta,$$

is a diffeomorphism from U_β to N_β which preserves the Haar measures. It follows that (6.6) equals

$$e^t \int_{\Gamma} \int_{K_\beta} \int_{N_\beta} f(y^{-1} \zeta k^{-1} \exp H n k y) \psi^+(z, k y) \, dn \, dk \, dy.$$

Once again, the constant term of the Laurent expansion about $z=0$ of this function extends to a smooth function of $H \in \bar{\Omega}$.

We have so far shown that $S_f^\beta(\zeta, H : \mathscr{Y} : X)$ is the integral over $y \in \Gamma$ of the product of

$$\tilde{\Delta}_0(\zeta, H) D_X v^*(y, \mathscr{Y}^*)$$

with

$$|e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi_y(k^{-1} u^{-1} \cdot \exp H \cdot u k) \left(\tau(H) - \frac{\langle \gamma_\beta, H_\beta(u) \rangle}{\langle \gamma_\beta, \gamma_\beta \rangle^{\frac{1}{2}}} \right) du \, dk.$$

This last expression is just

$$n_\beta(A) S_{\varphi_y}(H).$$

Define

$$F_{\varphi_y}^*(H^*) = (e^{\frac{1}{2}(A\beta)(H^*)} - e^{-\frac{1}{2}(A\beta)(H^*)}) \int_{G_\beta} \varphi_y(x^{-1} \exp H^* x) \, dx,$$

for $H^* \in \mathfrak{t}_{\text{reg}}^*(\zeta)$.

Lemma 6.2. *For each $u \in S(\mathfrak{t}_{\mathbb{C}})$ the limits $S_{\varphi_y}(H_0;\partial(u))^{\pm}$ and*

$$\lim_{\theta \to \pm 0} F^*_{\varphi_y}(H_0+\theta(X'_\beta - Y'_\beta);\partial(\Lambda u))$$

all exist uniformly for $H_0 \in \Omega_0$ and $y \in \Gamma$. Moreover the two sided limit

$$\lim_{\theta \to 0} F^*_{\varphi_y}(H_0+\theta(X'_\beta - Y'_\beta);\quad \partial(\Lambda s_\beta u)-\partial(\Lambda u))$$

exists and equals

$$S_{\varphi_y}(H_0;\partial(u))^+ - S_{\varphi_y}(H_0;\partial(u))^-.$$

Let us assume the proof of this lemma for the moment while we complete the proof of the theorem. Define

$$\bar{\Omega}^* = \{H^* = H_0+\theta(X'_\beta - Y'_\beta): H_0 \in \Omega_0, \qquad \theta \in (-\varepsilon,\varepsilon)\},$$

and let Ω^* be the complement of Ω_0 in $\bar{\Omega}^*$. Suppose $\tilde{\Delta}^*(\zeta, H^*)$ is our usual function associated with $\mathfrak{t}^*$. The function

$$\tilde{\Delta}^*_0(\zeta, H^*) = \tilde{\Delta}^*(\zeta, H)\,(e^{\frac{1}{2}(\Lambda\beta)(H^*)} - e^{-\frac{1}{2}(\Lambda\beta)(H^*)})^{-1}, \qquad H^* \in \Omega^*,$$

extends smoothly to $\bar{\Omega}^*$. In order to reduce the proof of the theorem to Lemma 6.2, we must compare the differential operators

$$\partial(u)\circ \tilde{\Delta}_0(\zeta, H), \qquad H \in \bar{\Omega}, \tag{6.7}$$

and

$$\partial(\Lambda s_\beta u - \Lambda u)\circ \tilde{\Delta}^*_0(\zeta, H^*), \qquad H^* \in \bar{\Omega}^*. \tag{6.8}$$

By Leibnitz' rule there are elements

$$\{u_i, u^i : 1 \leqq i \leqq n\}$$

in $S(\mathfrak{t}_{\mathbb{C}})$ such that

$$\partial(u)\circ \tilde{\Delta}_0(\zeta, H) = \sum_i \tilde{\Delta}_0(\zeta, H;\partial(u_i))\,\partial(u^i),$$

$$\partial(\Lambda u)\circ \tilde{\Delta}^*_0(\zeta, H^*) = \sum_i \tilde{\Delta}^*_0(\zeta, H^*;\partial(\Lambda u_i))\,\partial(\Lambda u^i),$$

and

$$\partial(\Lambda s_\beta u)\circ \tilde{\Delta}^*_0(\zeta, H^*) = \sum_i \tilde{\Delta}^*_0(\zeta, H^*;\partial(\Lambda s_\beta u_i))\,\partial(\Lambda s_\beta u^i).$$

Let R^β_+ be the set of roots of $(\mathfrak{g},\mathfrak{t})$ which do not equal β or $-\beta$ and which do not vanish on $\mathfrak{t}_{\mathbb{R}}$. Then for $H \in \bar{\Omega}$, $\tilde{\Delta}_0(\zeta, H)$ equals

$$\Delta_I(H) \prod_{\gamma \in R^\beta_+} |1-\xi_\gamma(\zeta)^{-1}\, e^{-\gamma(H)}|^{\frac{1}{2}}.$$

Let γ be a real root in R^β_+ which is positive on $\bar{\Omega}$. Then $\xi_\gamma(\zeta) = \xi_{-\gamma}(\zeta) = \pm 1$. Therefore the contribution of γ and $-\gamma$ to the above product is

$$e^{\frac{1}{2}\gamma(H)} - \xi_\gamma(\zeta)^{-1}\, e^{-\frac{1}{2}\gamma(H)}.$$

Suppose that γ is a complex root in R_+^β. Then the root

$$\bar{\gamma} = -\theta\gamma$$

is different from γ, and

$$\xi_{\bar{\gamma}}(\zeta) = \xi_\gamma(\zeta)^{-1} = \overline{\xi_\gamma(\zeta)}.$$

The contribution of γ, $-\gamma$, $\bar{\gamma}$ and $-\bar{\gamma}$ to the above product is

$$(\xi_\gamma(\zeta)\, e^{\frac{1}{2}\gamma(H)} - e^{-\frac{1}{2}\gamma(H)})\,(\xi_{\bar{\gamma}}(\zeta)\, e^{\frac{1}{2}\bar{\gamma}(H)} - e^{-\frac{1}{2}\bar{\gamma}(H)}).$$

It follows that $\tilde{\Delta}_0(\zeta, H)$ is a polynomial in the variables

$$x_\gamma = e^{\frac{1}{2}\gamma(H)},$$

indexed by the roots of $(\mathfrak{g}, \mathfrak{t})$. We claim that $\tilde{\Delta}_0^*(\zeta, H^*)$ is the same polynomial, but with each x_γ replaced by

$$y_\gamma = e^{\frac{1}{2}(\Lambda\gamma)(H^*)}.$$

For any γ we have

$$\xi_\gamma(\zeta) = \zeta_{\Lambda\gamma}(\zeta).$$

If $\Lambda\gamma$ is an imaginary root of $(\mathfrak{g}, \mathfrak{t}^*)$, $\xi_{\Lambda\gamma}(\zeta) = 1$. These two facts follow from the definition of $Z(\mathfrak{t}^*)$, and they suffice to establish our claim.

If $E \in \mathfrak{t}_{\mathbb{C}}$,

$$\partial(E)\, x_\gamma = \tfrac{1}{2}\gamma(E)\, x_\gamma,$$

while

$$\partial(\Lambda E)\, y_\gamma = \tfrac{1}{2}\Lambda\gamma(\Lambda E)\, y_\gamma = \tfrac{1}{2}\gamma(E)\, y_\gamma.$$

It follows inductively that for any $v \in S(\mathfrak{t}_{\mathbb{C}})$, $\tilde{\Delta}_0^*(\zeta, H^*; \partial(\Lambda v))$ may be obtained from $\tilde{\Delta}_0(\zeta, H; \partial(v))$ by replacing each variable x_γ by y_γ. Now set both H and H^* equal to H_0, a fixed point in Ω_0. Then each $x_\gamma = y_\gamma$. Therefore

$$\tilde{\Delta}_0^*(\zeta, H_0; \partial(\Lambda v)) = \tilde{\Delta}_0(\zeta, H_0; \partial(v)).$$

On the other hand,

$$\tilde{\Delta}_0(\zeta, s_\beta H) = \tilde{\Delta}_0(\zeta, H), \qquad H \in \bar{\Omega}.$$

Consequently, for any $v \in S(\mathfrak{t}_{\mathbb{C}})$,

$$\tilde{\Delta}_0^*(\zeta, H_0; \partial(\Lambda s_\beta v)) = \tilde{\Delta}_0(\zeta, H_0; \partial(s_\beta v)) = \tilde{\Delta}_0(\zeta, H_0; \partial(v)).$$

Applying these remarks to the elements $\{u_i : 1 \leq i \leq n\}$, we see that the local expressions at $H^* = H = H_0$ of the differential operators (6.7) and (6.8) are

$$\sum_i \tilde{\Delta}_0(\zeta, H_0; \partial(u_i))\, \partial(u^i)$$

and

$$\sum_i \tilde{\Delta}_0(\zeta, H_0; \partial(u_i))\, \partial(\Lambda s_\beta u^i - \Lambda u^i)$$

respectively.

It follows that all the assertions of Lemma 6.2 remain valid when the differential operators $\partial(u)$ and $\partial(\Lambda s_\beta u - \Lambda u)$ are replaced by (6.7) and (6.8) respectively. It is in this form that the lemma leads to the proof of Theorem 6.1. To transform the data given by the lemma to data required by the theorem, we merely multiply by

$$n_\beta(A)\, D_X\, v^*(y, \mathscr{Y}^*)$$

and integrate over $y \in \Gamma$. With the observation that

$$\int_\Gamma \tilde{\Delta}_0^*(\zeta, H^*)\, F_{\varphi_y}^*(H^*)\, D_X\, v^*(y, \mathscr{Y}^*)\, dy$$

equals $R_f^{T^*, A^*}(\zeta, H^* : \mathscr{Y}^* : X)$, the proof of the theorem is complete. □

We still have to prove Lemma 6.2. The existance of the limits involving $F_{\varphi_y}^*$ follows from a general result of Harish-Chandra [2(i), Theorem 9.1], so we can concentrate on those assertions that concern S_{φ_y}. Any $u \in S(\mathfrak{t}_\mathbb{C})$ is a sum of elements of the form

$$u_1 = \gamma_\beta(z)\, J,$$

where z is in $\mathscr{Z}_\beta$ and J equals either 1 or H'_β. From Corollary 5.3 we know that, modulo a smooth function of $y \in \Gamma$ and $H \in \bar{\Omega}$, $S_{\varphi_y}(H; \partial(u_1))$ equals $S_{z\varphi_y}(H; \partial(J))$. Therefore the first statement of the lemma would be proved if we could establish the uniform existance of the limits $S_\varphi(H_0; \partial(J))^\pm$ for any function $\varphi \in C_c^\infty(G_\beta)$ which varies continuously with a parameter $y \in \Gamma$. Note that

$$\Lambda(s_\beta u_1 - u_1) = \gamma_\beta^*(z)\, \Lambda(s_\beta J - J),$$

where γ_β^* is the isomorphism from $\mathscr{Z}_\beta$ to the invariants of $S(\mathfrak{t}_\mathbb{C}^*)$. It follows from the differential equations satisfied by $F_{\varphi_y}^*$ that

$$F_{z\varphi_y}^*(H^*; \partial(\Lambda s_\beta J - \Lambda J)) = F_{\varphi_y}^*(H^*; \partial(\Lambda s_\beta u_1 - \Lambda u_1)),$$

for any $H^* \in \Omega^*$. It is therefore enough to prove the lemma for $u = J$.

It would be possible to extract what remains to be proved of Lemma 6.2 from the results for groups of real rank 1 in [1(b)]. However, it is perhaps safer to proceed directly. To simplify matters, we replace φ_y by an arbitrary $\varphi \in C_c^\infty(G_\beta)$ which we assume varies continuously with a parameter $y \in \Gamma$. It will be clear that the limits we establish will be uniform in y, so we will not allude to this point again.

Let dk be the Haar measure on K_β for which the volume of K_β is 1. Let du and da be the Haar measures on U_β and $T_\mathbb{R}$ respectively, obtained via the exponential map from the Euclidean measures on the corresponding Lie algebras (with respect to the norm $\| \ \|$). The Haar measure on G_β is of course defined by the product measure $da\, du\, dk$. If $H = H_0 + t H'_\beta$ belongs to Ω,

$$S_\varphi(H) = |e^t - e^{-t}| \int_{K_\beta} \int_{U_\beta} \varphi(k^{-1} u^{-1} \cdot \exp H \cdot u k) \cdot \{\tfrac{1}{2} \|H'_\beta\| (\log |e^t - e^{-t}|^2 - \beta(H_\beta(u)))\}\, du\, dk.$$

For $x \in \mathbb{R}$ define

$$u(x) = \exp(x Y'_\beta).$$

The restriction of the form B to $\mathfrak{g}_\beta$ is a multiple of the Killing form B_β on $\mathfrak{g}_\beta$. Since

$$B_\beta(X'_\beta, \theta X'_\beta) = B_\beta(Y'_\beta, \theta Y'_\beta) = \tfrac{1}{2} B_\beta(H'_\beta, H'_\beta),$$

we have

$$\|H'_\beta\|^2 = 2\|X'_\beta\|^2 = 2\|Y'_\beta\|^2.$$

In particular, for any function $\rho \in C_c^\infty(U_\beta)$,

$$\int_{U_\beta} \rho(u)\, du = \frac{\|H'_\beta\|}{\sqrt{2}} \int_{-\infty}^{\infty} \rho(u(x))\, dx.$$

We have

$$\begin{aligned} e^{-\beta(H_\beta(u(x)))} &= \|X'_\beta\|^{-1} \|\mathrm{Ad}(u(x)^{-1}) X'_\beta\| \\ &= \|X'_\beta\|^{-1} \|\exp(-x \,\mathrm{ad}\, Y'_\beta) X'_\beta\| \\ &= \|X'_\beta\|^{-1} \|X'_\beta + x H'_\beta + x^2 Y'_\beta\| \\ &= 1 + x^2. \end{aligned}$$

Therefore,

$$\log|e^t - e^{-t}|^2 - \beta(H_\beta(u(x))) = \log\{1 + x^2)(1 - e^{-2t})^2\} + 2t.$$

From these facts it follows easily that $S_\varphi(H)$ equals

$$\int_{-\infty}^{\infty} \varphi(H:x)(\log\{(1 - e^{-2t})^2 + x^2\} + 2t)\, dx, \tag{6.9}$$

where

$$\varphi(H:x) = \frac{e^t}{2\sqrt{2}} \|H'_\beta\|^2 \int_{K_\beta} \varphi(k^{-1} u(x) \exp H\, k)\, dk.$$

$\varphi(H:x)$ is smooth in H, and as a function of x, is smooth and compactly supported. In particular, (6.9) is continuous for H in $\bar{\Omega}$. This proves the lemma for $u = J = 1$.

The only other case left to prove is for $u = J = H'_\beta$. The operator $\partial(H'_\beta)$ is just differentiation with respect to t. The function

$$2t \int_{-\infty}^{\infty} \varphi(H:x)\, dx$$

is smooth for $H \in \bar{\Omega}$. Therefore we can omit the factor $2t$ from (6.9). Moreover, the function

$$\int_{-\infty}^{\infty} \varphi(H:x) \log(x^2)\, dx$$

is also smooth in H. We therefore have only to consider the limit as t approaches $+0$ of the difference of

$$4 \int_{-\infty}^{\infty} \varphi(H_0 + tH'_\beta : x) \frac{e^{-2t}(1-e^{-2t})}{(1-e^{-2t})^2 + x^2} dx \tag{6.10}$$

and

$$4 \int_{-\infty}^{\infty} \varphi(H_0 - tH'_\beta : x) \frac{e^{2t}(1-e^{2t})}{(1-e^{2t})^2 + x^2} dx. \tag{6.11}$$

The term (6.10) equals

$$4e^{-2t} \int_{-\infty}^{\infty} \varphi(H_0 + tH'_\beta : (1-e^{-2t})\, x) \frac{1}{1+x^2} dx.$$

The limit of this expression exists uniformly in H_0. By the dominated convergence theorem the limit is

$$4\pi \varphi(H_0 : 0).$$

By a similar argument the limit of (6.11) exists uniformly in H_0, and equals

$$-4\pi \varphi(H_0 : 0).$$

We have shown that

$$S_\varphi(H_0; \partial(H'_\beta))^+ - S_\varphi(H_0; \partial(H'_\beta))^- \tag{6.12}$$

equals

$$2\sqrt{2}\, \pi \, \|H'_\beta\|^2 \, \varphi(\exp H_0).$$

To this last expression we apply a general limit formula of Harish-Chandra. In fact, according to [2(i), Lemma 17.5 and Theorem 37.1]

$$\lim_{\theta \to 0} F^*_\varphi(H_0 + \theta(X'_\beta - Y'_\beta);\ \partial(\Lambda\beta)) = -2\sqrt{2}\, \pi\, \varphi(\exp H_0).$$

Here of course β is to be regarded as a vector in $\mathfrak{t}$. But

$$\beta = 2\, \|H'_\beta\|^{-2}\, H'_\beta,$$

so that

$$\Lambda(s_\beta H'_\beta) - \Lambda(H'_\beta) = \|H'_\beta\|^2 (\Lambda\beta).$$

It follows that (6.12) equals

$$\lim_{\theta \to 0} F^*_\varphi(H_0 + \theta(X'_\beta - Y'_\beta); \partial(\Lambda(s_\beta H'_\beta - H'_\beta))).$$

This completes the proof of Lemma 6.2. □

The distributions $R(\zeta, H : \mathscr{Y} : X)$ also satisfy boundary conditions at any singular imaginary root. Here the situation is the same as for the invariant integrals.

Lemma 6.3. *Suppose that* $\zeta \in Z(\mathfrak{t}^*)$, $u \in S(\mathfrak{t}_{\mathbb{C}})$ *and* $f \in C_c^\infty(G)$. *Then the limits*

$$\lim_{\theta \to +0} R_f^{T^*, A^*}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta);\, \partial(\Lambda u) : \mathscr{Y}^* : X)$$

and

$$\lim_{\theta \to -0} R_f^{T^*, A^*}(\zeta, H_0 + \theta(X'_\beta - Y'_\beta);\, \partial(\Lambda u) : \mathscr{Y}^* : X)$$

both exist, uniformly for $H_0 \in \Omega_0$. *The first limit minus the second one equals*

$$-\pi\sqrt{-1}\,\|H'_\beta\| \lim_{t \to 0} R_f^{T, A^*}(\zeta, H_0 + tH'_\beta;\, \partial(u) : \mathscr{Y}^* : X).$$

This lemma is proved by an argument similar to but easier than that used above. Proceeding as in the proof of Theorem 6.1 one shows that it is enough to treat the case that $G = G_\beta$. But in this case the lemma is well known. See [2(i), Theorem 9.1]. □

§ 7. A Growth Condition

In this section we estimate the growth of our distributions as the $\mathfrak{t}_{\mathbb{R}}$ component of H gets large. At the same time we shall show that the distributions are tempered. As usual, this leads to a study of inequalities.

In order to estimate the functions $v(x, \mathscr{Y})$ we will first verify a couple of easy and more or less standard facts. To state them it is useful to fix a *maximal* special subgroup ${}^{(0)}A$ of G, and a minimal parabolic subgroup

$${}^{(0)}P = {}^{(0)}N\, {}^{(0)}A\, {}^{(0)}M$$

in $\mathscr{P}({}^{(0)}A)$. We assume that $T_{\mathbb{R}}$ is contained in ${}^{(0)}A$. Recall that the Schwartz space, $\mathscr{C}(G)$, is the space of all f in $C^\infty(G)$ such that

$$\sup_{x \in G} |f(g_1; x; g_2)|\, \Xi(x)^{-1}\, (1 + \sigma(x))^r < \infty$$

for any g_1 and g_2 in $\mathscr{G}$ and r in $\mathbb{R}$. Here Ξ is defined as in [2(g), § 7] and σ is defined by

$$\sigma(k_1 \cdot \exp H \cdot k_2) = \|H\|, \qquad H \in {}^{(0)}\mathfrak{a},\ k_1, k_2 \in K.$$

Suppose that π is an irreducible finite dimensional representation of G, which by convention here we always take to act on the right. It is possible to fix an inner product on the space on which π acts so that different root spaces are orthogonal and so that for any $x \in G$,

$$\pi(x)^* = \pi(\theta x^{-1}).$$

If $x \in G$, we write $\|\pi(x)\|_2$ for the Hilbert-Schmidt norm of $\pi(x)$. We can always write such an x as

$$k_1 \exp H\, k_2, \qquad k_1, k_2 \in K,\ H \in c_{{}^{(0)}P}({}^{(0)}\mathfrak{a}).$$

Then

$$\|\pi(x)\|_2 = \|\pi(\exp H)\|_2.$$

If $\Phi(\pi)$ is the set of weights of π with respect to $(\mathfrak{g}, {}^{(0)}\mathfrak{a})$ and μ^+ is the highest weight,

$$\|\pi(x)\|_2^2 = \sum_{\mu \in \Phi(\pi)} e^{2\langle \mu, H\rangle} \leqq \dim \pi \cdot e^{2\langle \mu^+, H\rangle}.$$

In particular $\|\pi(x)\|_2$ is no less than 1. Therefore we can find a constant c_π such that

$$\log(1+\|\pi(x)\|_2) \leqq c_\pi(1+\sigma(x)), \qquad x \in G. \tag{7.1}$$

The inequality remains valid if we remove the hypothesis of irreducibility of π, since

$$\log(1+t_1+t_2) \leqq \log(1+t_1) + \log(1+t_2)$$

for two positive numbers t_1 and t_2.

Next we will check that there is a finite dimensional representation π and a constant c such that

$$\|{}^{(0)}H(x)\| \leqq c \log(1+\|\pi(x)\|_2), \qquad x \in G, \tag{7.2}$$

where we have put

$${}^{(0)}H(x) = H_{(0)_P}(x).$$

It is clearly enough to prove the formula obtained by replacing the left side of (7.2) by

$$|\langle \mu, {}^{(0)}H(x)\rangle|,$$

where μ is any arbitrary element in $\hat{\Phi}_{(0)_P}$. Let π be an irreducible representation whose highest weight on ${}^{(0)}\mathfrak{a}$ equals $k\mu$, for k a positive real number. Let φ be a highest weight vector of unit length. We have

$$e^{k\langle \mu, {}^{(0)}H(x)\rangle} = \|\varphi \pi(x)\| = (\varphi \pi(x) \pi(x)^*, \varphi)^{\frac{1}{2}}.$$

The right hand side of this formula is clearly bounded by $\|\pi(x)\|_2$. If $\langle \mu, {}^{(0)}H(x)\rangle$ is positive we are done. Suppose that $\langle \mu, {}^{(0)}H(x)\rangle$ is negative. Let π' be the irreducible representation of G whose lowest weight is $-k\mu$, and let φ' be a lowest weight vector of unit length. Note that

$$\varphi' \pi'(x) - e^{-k\langle \mu, {}^{(0)}H(x)\rangle} \varphi'$$

is orthogonal to φ'. It follows that

$$e^{-k\langle \mu, {}^{(0)}H(x)\rangle} \leqq \|\varphi' \pi'(x)\| \leqq \|\pi'(x)\|_2,$$

which leads to the inequality

$$|\langle \mu, {}^{(0)}H(x)\rangle| \leqq \frac{1}{k} \log(1+\|\pi'(x)\|).$$

This establishes (7.2).

Lemma 7.1. *There is a constant c such that for every* $X \in {}^{(0)}\mathfrak{n}$,

$$\log(1+\|X\|) \leqq c(1+\sigma(\exp X)).$$

Proof. Embed ${}^{(0)}\mathfrak{a}$ in a θ-stable Cartan subalgebra, ${}^{(0)}\mathfrak{t}$, of $\mathfrak{g}$. Let R denote the set of roots α of $(\mathfrak{g}, {}^{(0)}\mathfrak{t})$ such that the root space $\mathfrak{g}_\alpha$ of α lies in ${}^{(0)}\mathfrak{n}$. For $X \in {}^{(0)}\mathfrak{n}$ and $\gamma \in R$, let X_γ be the component of X in $\mathfrak{g}_\gamma$ relative to the decomposition

$${}^{(0)}\mathfrak{n} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha .$$

It would be enough to prove that there is a c such that for any $X \in {}^{(0)}\mathfrak{n}$,

$$\log(1+\|X_\gamma\|) \leqq c(1+\sigma(\exp X)). \tag{7.3}$$

Let r be the height of γ relative to some order for which every root in R is positive. We shall verify the above assertion by induction on r, starting at $r=0$ where of course there is nothing to prove.

Let π be any finite dimensional representation of G for which there are weights on ${}^{(0)}\mathfrak{t}$ of the form μ and $\mu+\gamma$. Let φ and φ_1 be corresponding unit weight vectors. Then

$$\xi(Y) = (\varphi\pi(Y), \varphi_1), \qquad Y \in \mathfrak{g}_\gamma,$$

is a nonzero linear functional on the one dimensional space $\mathfrak{g}_\gamma$. Define

$$P_{r-1}(X) = \sum_\alpha X_\alpha, \qquad X \in {}^{(0)}\mathfrak{n},$$

where the sum is over all roots α of R of height less than r. There is a polynomial q on $P_{r-1}({}^{(0)}\mathfrak{n})$ such that

$$(\varphi\pi(\exp X), \varphi_1) = \xi(X_\gamma) + q(P_{r-1}(X)), \qquad X \in {}^{(0)}\mathfrak{n}.$$

Therefore there are constants c_1 and c_2 such that

$$1+\|X_\gamma\| \leqq 1 + c_1|q(P_{r-1}(X))| + c_2|(\varphi\pi(\exp X), \varphi_1)|,$$

for each $X \in {}^{(0)}\mathfrak{n}$. Now

$$|(\varphi\pi(\exp X), \varphi_1)| \leqq \|\pi(\exp X)\|_2 .$$

In addition, we can find constants c_3 and d such that

$$1 + c_1|q(P_{r-1}(X))| \leqq c_3(1+\|P_{r-1}(X)\|)^d.$$

Formula (7.3) now follows from (7.1) and our induction hypothesis. □

Consider the set of numbers

$$\{|1-\xi_\alpha(\zeta \exp H)^{-1}|\}, \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta),$$

indexed by the roots α of $(\mathfrak{g}, \mathfrak{t})$ which do not vanish on $\mathfrak{a}$. Our distributions are not defined when any of these numbers equals 0. This shows up in the growth conditions. Define $L(\zeta \exp H)$ to be the absolute value of the logarithm of the smallest of these numbers.

For each N, define

$$I_N(\zeta, H) = |\tilde{\Delta}(\zeta, H)| \int_{T_{\mathbb{R}} \backslash G} \Xi(x^{-1} \zeta \exp H x)(1+\sigma(x^{-1} \zeta \exp H x))^{-N}.$$

$$|D_X v(x, \mathcal{Y})| \, dx, \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta).$$

If $H \in \mathfrak{t}$, we put

$$H = H_I + H_{\mathbb{R}}, \qquad H_I \in \mathfrak{t}_{\mathbb{R}}, \quad H_{\mathbb{R}} \in \mathfrak{t}_{\mathbb{R}}.$$

Lemma 7.2. *For each integer n we can find constants c and N such that*

$$I_N(\zeta, H) \leqq c(1 + L(\zeta \exp H))^p (1 + \|H_R\|)^{-n}, \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta).$$

Proof. Fix a parabolic subgroup $P = NAM$ in $\mathscr{P}(A)$. For $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$, $I_N(\zeta, H)$ equals

$$|\tilde{\Delta}(\zeta, H)| \int_M \int_{\mathfrak{n}} U_N((\exp R)^{-1} \mu \, (\exp R)) \, |D_X v(\exp R, \mathcal{Y})| \, dR \, dm,$$

where we write

$$\mu = m^{-1} \zeta \exp H \, m$$

and

$$U_N(x) = \Xi(x)(1+\sigma(x))^{-N}, \qquad x \in G.$$

Consider the polynomial map

$$\varphi_\mu : R \to \log(\exp(-\mathrm{Ad}(\mu^{-1}) R) \exp R), \qquad R \in \mathfrak{n},$$

of $\mathfrak{n}$ to itself. Since $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$,

$$A_\mu : R \to -\mathrm{Ad}(\mu^{-1}) R + R, \qquad R \in \mathfrak{n},$$

is a linear isomorphism of $\mathfrak{n}$. It follows from [2(f), Lemma 10] that the inverse of φ_μ exists and is also a polynomial function from $\mathfrak{n}$ to itself.

According to [2(f), Lemma 11], the diffeomorphism φ_μ transforms the Haar measure on $\mathfrak{n}$ by the factor

$$\delta(\zeta \exp H) = |\det(1 - \mathrm{Ad}(\zeta \exp H)^{-1})_{\mathfrak{n}}|.$$

This is just

$$\prod_\alpha |1 - \xi_\alpha(\zeta \exp H)^{-1}| = e^{-\rho(H)} \prod_\alpha |e^{\frac{\alpha(H)}{2}} - \xi_\alpha(\zeta) e^{\frac{-\alpha(H)}{2}}|,$$

where the product is taken over all roots α of $(\mathfrak{g}, \mathfrak{t})$ whose corresponding root spaces lie in $\mathfrak{n}$, and

$$\rho(H) = \tfrac{1}{2} \operatorname{tr}(\operatorname{ad} H)_{\mathfrak{n}}.$$

It follows easily that $|\tilde{\Delta}(\zeta, H)|$ is the product of

$$\delta(\zeta \exp H)^{-1} e^{\rho(H)}$$

with

$$|\tilde{\Delta}_M(\zeta, H)|,$$

the function we have defined earlier, but associated with the group M. Therefore $I_N(\zeta, H)$ equals

$$|\tilde{\Delta}_M(\zeta, H)|\, e^{\rho(H)} \int_M \int_{\mathfrak{n}} U_N(\mu \exp R)\, |D_X v(\exp \varphi_\mu^{-1} R, \mathscr{Y})|\, dR\, dm.$$

A closer look at the proof of [2(f), Lemma 10] reveals that, relative to a suitable fixed basis of $\mathfrak{n}$, the coefficients of the polynomial function φ_μ^{-1} can be bounded by a polynomial in the Hilbert-Schmidt norms of A_μ and A_μ^{-1}. But the matrix coefficients of A_μ^{-1} are just polynomials in the matrix coefficients of A_μ, divided by the determinant of A_μ. The absolute value of this determinant is

$$\delta(\zeta \exp H).$$

For $R \in \mathfrak{n}$, define

$$R\, \pi_M(\mu) = \mathrm{Ad}\,(\mu^{-1})\, R.$$

It follows that there are constants c_1 and d_1 such that for all R and μ under consideration

$$\|\varphi_\mu^{-1} R\| \leqq c_1((1+\delta(\zeta \exp H)^{-1})(1+\|\pi_M(\mu)\|)(1+\|R\|))^{d_1}. \tag{7.4}$$

Referring to formula (4.1), we see that there is a constant c_2 such that for all μ and R, $|D_X v(\exp \varphi_\mu^{-1} R, \mathscr{Y})|$ is bounded by

$$c_2 \sum_{P \in \mathscr{P}(A)} (1+\|H_P(\exp \varphi_\mu^{-1} R)\|)^p.$$

By (7.2) there is a finite dimensional representation π of G and a constant c_3 such that this last expression bounded by

$$c_3(1+\log(1+\|\pi(\exp \varphi_\mu^{-1} R)\|))^p.$$

Now the Hilbert-Schmidt norm

$$\|\pi(\exp S)\|, \qquad S \in \mathfrak{n},$$

can certainly be bounded by a polynomial in $\|S\|$, since

$$S \to \pi(\exp S), \qquad S \in \mathfrak{n},$$

is a polynomial on $\mathfrak{n}$. Therefore, in view of (7.4), there is a constant c_4 such that $|D_X v(\exp \varphi_\mu^{-1} E, \mathscr{Y})|$ is no greater than

$$c_4(1+\log(1+\delta(\zeta \exp H)^{-1})+\log(1+\|\pi_M(\mu)\|)+\log(1+\|R\|))^p.$$

Finally, applying (7.1) to the representation π_M of M, and referring to Lemma 7.1, we find that there is a constant c_5 such that $|D_X v(\exp \varphi_\mu^{-1} R, \mathscr{Y})|$ is bounded by the product of

$$c_5(1+\log(1+\delta(\zeta \exp H)^{-1}))^p \tag{7.5}$$

and

$$(1+\sigma(\exp R))^p\, (1+\sigma(\mu))^p.$$

Now

$$\begin{aligned}1+\sigma(\exp R) &\leqq 1+\sigma(\mu \exp R)+\sigma(\mu^{-1})\\ &=1+\sigma(\mu \exp R)+\sigma(\mu)\\ &\leqq (1+\sigma(\mu \exp R))(1+\sigma(\mu)),\end{aligned}$$

by [2(g), Lemma 10]. Therefore the second factor in the above product is no greater than

$$(1+\sigma(\mu \exp R))^p (1+\sigma(\mu))^{2p}.$$

We have shown so far that for $N \in \mathbb{R}$, and $H \in \mathfrak{t}_{\text{reg}}(\zeta)$, $I_N(\zeta, H)$ is bounded by the product of (7.5) with the integral over $m \in M$ of the product of

$$|\tilde{\Delta}_M(\zeta, H)| (1+\sigma(m^{-1} \zeta \exp H m))^{2p} \tag{7.6}$$

and

$$e^{\rho(H)} \int_{\mathfrak{n}} \Xi(m^{-1} \zeta \exp H \cdot m \exp R)(1+\sigma(m^{-1} \zeta \exp H \cdot m \exp R))^{-N+p} \, dR. \tag{7.7}$$

If N_1 is any real number, we can, by [2(g), Lemma, 21], choose N and c_6 such that for all m and H, (7.7) is no greater than

$$c_6 \, \Xi_M(m^{-1} \zeta \exp H \cdot m)(1+\sigma(m^{-1} \zeta \exp H \cdot m))^{-N_1}.$$

For suitable N_1, the product of this expression with (7.6) is integrable over M. In fact, by the results of [2(g)], we can, given n, choose N_1 and c_7 so that the integral over M of this product is bounded by

$$c_7(1+\|H_{\mathbb{R}}\|)^{-n}.$$

We still have the term (7.5). However, with a little manipulation of the formula for $\delta(\zeta \exp H)$ given above, one sees that for some constant c_8,

$$1+\log(1+\delta(\exp H)^{-1}) \leqq c_8(1+L(\zeta \exp H)).$$

This last inequality completes the proof of our lemma. □

Recall that a distribution is said to be *tempered* if it extends to a continuous linear functional on $\mathscr{C}(G)$. From the lemma we obtain immediately

Corollary 7.3. *For each $H \in \mathfrak{t}_{\text{reg}}(\zeta)$ the distribution $R(\zeta, H : \mathscr{Y} : X)$ is tempered. If $f \in \mathscr{C}(G)$ the integral*

$$\tilde{\Delta}(\zeta, H) \int_{T_{\mathbb{R}} \setminus G} f(x^{-1} \cdot \delta \exp H \cdot x) \, D_X v(x, \mathscr{Y}) \, dx$$

is absolutely convergent, and equals

$$\langle R(\zeta, H : \mathscr{Y} : X), f \rangle. \qquad \square$$

Corollary 7.4. *For every integer n there is a continuous semi-norm v on $\mathscr{C}(G)$ such that for all $f \in \mathscr{C}(G)$ and $H \in \mathfrak{t}_{\text{reg}}(\zeta)$,*

$$R_f(\zeta, H : \mathscr{Y} : X)| \leqq v(f)(1+L(\zeta \exp H))^p (1+\|H_{\mathbb{R}}\|)^{-n}. \qquad \square$$

§ 8. The Mapping $f \to S_f^\beta$

Our final task is to extend our earlier results to the Schwartz space. In particular, we must show that for $f \in \mathscr{C}(G)$ the functions $R_f(\zeta, H:\mathscr{Y}:X)$ and $S_f^\beta(\zeta, H:\mathscr{Y}:X)$ are smooth in $H \in \mathfrak{t}_{\mathrm{reg}}(\zeta)$. We must then prove that the earlier differential equations and boundary conditions apply to this more general setting. We neglected to do this in [1(b)]. However, it is not a serious problem in the case of real rank 1, essentially because one has Helgason's explicit formula for

$$v(n, \mathscr{Y}), \qquad n \in N, \quad P = NAM \in \mathscr{P}(A).$$

In the higher rank case there is no such explicit formula. The matter is further complicated by the fact that the group M is not compact. It would therefore seem unfeasible to prove what is needed by the method of [2(g), Lemma 22]. Instead we will use the technique by which Harish-Chandra proved [2(e), Theorem 3].

Suppose that β is a fixed real root of $(\mathfrak{g}, \mathfrak{t})$. Adopt the notation § 4 and § 5. Let ζ be in $Z(\mathfrak{t}^*)$. It follows from Corollary 7.3 that the distribution

$$S^\beta(\zeta, H:\mathscr{Y}:X), \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta),$$

are tempered. Let $\mathfrak{t}_{\mathrm{reg}}(\zeta, \beta)$ be the union of $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ and $\mathfrak{t}_{0,\mathrm{reg}}(\zeta)$. It is an open subset of $\mathfrak{t}$. Suppose that Ω is an open subset of $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ which is relatively compact in $\mathfrak{t}_{\mathrm{reg}}(\zeta, \beta)$. We can certainly choose a c such that

$$(1 + L(\zeta \exp H))^p \leqq c\,|\beta(H)|^{-1}, \qquad H \in \Omega.$$

It follows that there is a continuous semi-norm ν on $\mathscr{C}(G)$ such that for all $f \in \mathscr{C}(G)$,

$$\sup_{H \in \Omega} |S_f^\beta(\zeta, H:\mathscr{Y}:X)\,\beta(H)| \leqq \nu(f). \tag{8.1}$$

Suppose that Ξ is an open set in a finite dimensional Euclidean space V. For any u in the symmetric algebra of V and any integer n, put

$$\|\varphi\|_{u,n} = \sup_{H \in \Xi} \{(1 + \|H\|)^n\,|\varphi(H; \partial(u))|\},$$

for $\varphi \in C^\infty(\Xi)$. Define $\mathscr{C}(\Xi)$ to be the set of all φ in $C^\infty(\Xi)$ such that for each u and n,

$$\|\varphi\|_{u,n} < \infty.$$

Let $\mathscr{S}(G)$ be the set of continuous semi-norms on $\mathscr{C}(G)$.

Lemma 8.1. *Suppose that $\zeta \in Z(\mathfrak{t}^*)$ and that Ω is an open subset of $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ which is relatively compact in $\mathfrak{t}_{\mathrm{reg}}(\zeta, \beta)$. Then the map*

$$f \to S_f^\beta(\zeta, H:\mathscr{Y}:X), \qquad H \in \Omega, \quad f \in C_c^\infty(G),$$

extends to a continuous linear map from $\mathscr{C}(G)$ to $\mathscr{C}(\Omega)$.

Proof. Suppose that $X \in \mathscr{G}_A(r)$. We shall prove the lemma by decreasing induction on r. If $r > p$ we have

$$D_X\, v(x, \mathscr{Y}) = 0,$$

in view of formula (4.1). Therefore fix r, $0 \leqq r \leqq p$, and assume that the lemma is valid for all Ω if X is replaced by any $X_1 \in \mathcal{G}_A(r_1)$, $r_1 > r$.

The first stage of the proof is to show that for any fixed $z \in \mathcal{Z}$, the map which sends $f \in C_c^\infty(G)$ to

$$d_f(H) = S_f^\beta(\zeta, H; \partial(\gamma(z)) : \mathcal{Y} : X) - S_{zf}^\beta(\zeta, H : \mathcal{Y} : X), \qquad H \in \Omega,$$

extends to a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{C}(\Omega)$. According to (5.1),

$$d_f(H) = -\sum_i S_f^\beta(\zeta, H; \tilde{\partial}^i(z) : \mathcal{Y} : X\,\tilde{X}_i).$$

We can choose a positive integer n so that for each i the differential operator

$$\beta(H)^n \, \tilde{\partial}^i(z)$$

has analytic coefficients on $\mathfrak{t}_{\mathrm{reg}}(\zeta, \beta)$. Each $\tilde{X}_i$ belongs to $\mathcal{G}_A(1)$ so we may apply the induction hypothesis. It follows that the map which sends $f \in C_c^\infty(G)$ to

$$e_f(H) = \beta(H)^n \, d_f(H), \qquad H \in \Omega,$$

extends to a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{C}(\Omega)$.

If the closure of Ω does not meet $\mathfrak{t}_0$, the first stage of our proof is established. Therefore, to complete this first stage, we may assume that

$$\Omega = \{H_0 + t H'_\beta : H_0 \in \Omega_0, \ -\varepsilon < t < \varepsilon, \ t \neq 0\}, \tag{8.2}$$

where Ω_0 is an open subset of $\mathfrak{t}_0$ and $\varepsilon > 0$.

For $\varphi \in \mathcal{C}(\Omega)$ and $H_0 \in \Omega_0$, define

$$\tilde{\varphi}(H_0 + t H'_\beta) = t^{-n} \left(\varphi(H_0 + t H'_\beta) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \lim_{s \to +0} \frac{d^k}{ds^k} \varphi(H_0 + s H'_\beta) \right),$$

if t belongs to $(0, \varepsilon)$. If t belongs to $(-\varepsilon, 0)$, define $\tilde{\varphi}(H_0 + t H'_\beta)$ in a similar way. It follows from the integral form of the remainder term in the Taylor expansion of a function of one variable that

$$\varphi \to \tilde{\varphi}, \qquad \varphi \in \mathcal{C}(\Omega),$$

is a continuous linear map of $\mathcal{C}(\Omega)$ to itself. Now it is a consequence of Theorem 6.1 that for any $f \in C_c^\infty(G)$, the function

$$d_f(H), \qquad H \in \Omega,$$

belongs to $\mathcal{C}(\Omega)$. This means that

$$\tilde{e}_f(H) = \beta(H)^{-n} \, e_f(H) = d_f(H), \qquad H \in \Omega.$$

We have shown, for Ω as in (8.2) and hence for all required Ω, that the map

$$f \to d_f(H), \qquad f \in C_c^\infty(H), \quad H \in \Omega,$$

extends to a continuous linear map from $\mathcal{C}(G)$ to $\mathcal{C}(\Omega)$. Combining this fact with (8.1), we find there are constants c and N such that

$$\sup_{H \in \Omega} |S_f^\beta(\zeta, H; \partial(\gamma(z)) : \mathcal{Y} : X)\, \beta(H)| \leqq c\, \nu_{z,N}(f), \qquad f \in C_c^\infty(G), \tag{8.3}$$

where

$$\nu_{z,N}(f)=\sup_{x\in G}|(zf)(x)\cdot\Xi(x)^{-1}\cdot(1+\sigma(x))^{-N}|$$

is a continuous semi-norm on $\mathscr{C}(G)$.

The second stage of our proof is to combine what we have shown so far with the proof given by Harish-Chandra for [2(e), Lemma 48]. In order to convince the reader that it actually applies to the situation at hand, we repeat the relevant portion of Harish-Chandra's argument. Suppose then that u is an arbitrary element of $S(\mathfrak{t}_{\mathbb{C}})$ of degree d. Let ω be the element in $S(\mathfrak{t}_{\mathbb{C}})$ such that $\partial(\omega)$ is the Laplacian on $\mathfrak{t}$, with respect to our Euclidean norm. For any integer $m\geqq 1$ we can choose an integer r and elements

$$\{z_j : 1\leqq j\leqq r\}$$

in $\mathscr{Z}$ such that

$$\omega^{mr}+\sum_{j=1}^{r}\gamma(z_j)\,\omega^{m(r-j)}=0.$$

If m is sufficiently large there is a function E_0 in $C^{2m(r-1)+d}(\mathfrak{t})$ which is of class C^∞ away from 0, such that $\partial(\omega)^{mr}E_0$ equals δ, the Dirac delta distribution at the origin [2(e), Lemma 57].

Define $E=\partial(u)^* E_0$, where the star denotes adjoint, and

$$E_j=-\partial(\omega)^{m(r-j)}E,\qquad 1\leqq j\leqq r.$$

Then E_j is of class $C^{2m(j-1)}$, and is of class C^∞ away from 0. From the relation

$$\partial(\omega)^{mr}+\sum_{j=1}^{r}\partial(\gamma(z_j))\,\partial(\omega)^{m(r-j)}=0$$

we obtain the formula

$$\partial(u)^*\,\delta=\sum_{j=1}^{r}\partial(\gamma(z_j))^*\,E_j.$$

Let ψ be a function in $C^\infty(\mathbb{R})$ which equals 0 on $(-\infty,0]$, 1 on $[1,\infty)$ and such that

$$0\leqq\psi(x)\leqq 1,\qquad x\in\mathbb{R}.$$

Given ε, $0<\varepsilon\leqq\frac{1}{3}$, define

$$\Psi_\varepsilon(H)=\psi(\varepsilon^{-1}\|H\|-2),\qquad H\in\mathfrak{t},$$

$$E_{j,\varepsilon}(H)=\Psi_\varepsilon(H)\,E_j(H),\qquad H\in\mathfrak{t},\ 1\leqq j\leqq r,$$

and

$$B_\varepsilon(H)=\sum_{j=1}^{r}\partial(\gamma(z_j))^*\,(E_{j,\varepsilon}(H)-E_j(H)),\qquad H\in\mathfrak{t}.$$

$\Psi_\varepsilon(H)$ equals 1 if $\|H\|\leqq 2\varepsilon$ and equals 0 if $\|H\|\geqq 3\varepsilon\cdot B_\varepsilon(H)$ vanishes unless

$$2\varepsilon\leqq\|H\|\leqq 3\varepsilon.$$

We can find positive integers b and n (which depend on u but not on ε) such that

$$\sup_{H \in \mathfrak{t}} |B_\varepsilon(H)| \leqq b\, e^{-n},$$

(see [2(e), p. 498]).

The formula

$$\sum_{j=1}^{r} \partial(\gamma(z_j))^* E_{j,\varepsilon} = \partial(u)^* \delta + B_\varepsilon$$

follows from the definitions. Suppose H is a point in Ω. Choose $\varepsilon = \varepsilon_H$, with $0 < \varepsilon \leqq \frac{1}{3}$, such that the distance from H to any point in the complement of $\mathfrak{t}_{\mathrm{reg}}(\zeta)$ in $\mathfrak{t}$ is greater than 4ε. Then Ψ_ε is supported on $\mathfrak{t}_{\mathrm{reg}}(\zeta)$. It follows from integration by parts that for any $\varphi \in C^\infty(\mathfrak{t}_{\mathrm{reg}}(\zeta))$, $\varphi(H; \partial(u))$ equals

$$\sum_{j=1}^{r} \int_{\mathfrak{t}} \varphi(\tilde{H}; \partial(\gamma(z_j)))\, E_{j,\varepsilon}(H - \tilde{H})\, d\tilde{H} - \int_{\mathfrak{t}} \varphi(\tilde{H})\, B_\varepsilon(\tilde{H} - H)\, d\tilde{H}.$$

For each j,

$$|E_{j,\varepsilon}(H - \tilde{H})| \leqq |E_j(H - \tilde{H})|.$$

Using the above estimate for B_ε and (8.3), we can obtain a ν in $\mathscr{S}(G)$ such that for every $f \in C_c^\infty(G)$,

$$\sup_{H \in \Omega} |S_f^\beta(\zeta, H; \partial(u) : \mathscr{Y} : X)\, \beta(H)^n| \leqq \nu(f). \tag{8.4}$$

In the case that the closure of Ω does not meet $\mathfrak{t}_0$ this last inequality suffices to prove the lemma. Therefore, we may assume that Ω is of the form (8.2). The third and final stage of our proof is based on the method of [2(e), Lemma 49]. Choose elements

$$u_1 = 1, u_2, \ldots, u_r$$

in $S(\mathfrak{t}_{\mathbb{C}})$ such that

$$S(\mathfrak{t}_{\mathbb{C}}) = \sum_{i=1}^{r} u_i\, I(\mathfrak{t}_{\mathbb{C}}).$$

For each i there are elements

$$z_{ij}, \quad 1 \leqq j \leqq r,$$

in $\mathscr{Z}$ such that

$$H'_\beta\, u_i = \sum_{j=1}^{r} \gamma(z_{ij})\, u_j.$$

For $H \in \Omega$, put

$$\psi_{f,i}(H) = S_f^\beta(\zeta, H; \partial(u_i) : \mathscr{Y} : X), \quad f \in C_c^\infty(G),\ 1 \leqq i \leqq r.$$

From the first stage of our proof we know that there is a v_1 in $\mathscr{S}(G)$ such that for $f \in C_c^\infty(G)$, $H \in \Omega$ and $1 \leqq i \leqq r$,

$$|\psi_{f,i}(H; \partial(H'_\beta)) - \sum_{j=1}^{r} \psi_{z_{ij}f,j}(H)| \leqq v_1(f). \tag{8.5}$$

If $0 < t < \varepsilon$ and $H_0 \in \Omega_0$ write

$$\psi_{f,i}(H_0 + t H'_\beta) = \psi_{f,i}(H_0 + \varepsilon H'_\beta) - \int_t^\varepsilon \psi_{f,i}(H_0 + s H'_\beta; \partial(H'_\beta))\, ds.$$

It follows that there is a v_2 in $\mathscr{S}(G)$, which depends on ε but not on H_0 or t, such that

$$|\psi_{f,i}(H_0 + t H'_\beta)| \leqq v_2(f) + \int_t^\varepsilon \sum_j |\psi_{z_{ij}f,j}(H_0 + s H'_\beta)|\, ds. \tag{8.6}$$

It follows from (8.4) that there is a $v \in \mathscr{S}(G)$ and a positive integer N such that for all $g \in C_c^\infty(G)$, $H_0 \in \Omega_0$ and $s \in (0, \varepsilon)$,

$$|\psi_{g,j}(H_0 + s H'_\beta)| \leqq s^{-N} v(g), \qquad 1 \leqq j \leqq r. \tag{8.7}$$

Therefore $|\psi_{f,i}(H_0 + t H'_\beta)|$ is bounded by

$$v_2(f) + \int_t^\varepsilon (s)^{-N}\, ds \cdot \sum_j v(z_{ij}f).$$

But the semi-norm

$$f \to \sum_j v(z_{ij}f), \qquad f \in C_c^\infty(G),$$

is in $\mathscr{S}(G)$. It follows that in (8.7) we can replace s^{-N} by $s^{-(N-1)}$ if $N \geqq 2$, and by $\log(\varepsilon/s)$ if $N = 1$. Of course the new inequality would hold for some different $v \in \mathscr{S}(G)$. By induction there is a $v \in \mathscr{S}(G)$ such that for all g, H_0 and s,

$$|\psi_{g,j}(H_0 + s H'_\beta)| \leqq \log(\varepsilon/s)\, v(g), \qquad 1 \leqq j \leqq r.$$

Once again, we apply this inequality to the integrand in (8.6). We conclude that there is a $v_\varepsilon \in \mathscr{S}(G)$ such that for any $H_0 \in \Omega_0$, $t \in (0, \varepsilon)$, $f \in C_c^\infty(G)$ and $1 \leqq i \leqq r$,

$$|\psi_{f,i}(H_0 + t H'_\beta)| \leqq v_3(f). \tag{8.8}$$

By a similar argument we may assume that this inequality holds also for $t \in (-\varepsilon, 0)$.

Let u be an arbitrary element in $S(\mathfrak{t}_{\mathbb{C}})$. Then

$$u = \sum_i \gamma(z_i)\, u_i,$$

for elements $z_i \in \mathscr{Z}$. For $H \in \Omega$ and $f \in C_c^\infty(G)$, $|S_f^\beta(\zeta, H; \partial(u) : \mathscr{Y} : X)|$ is bounded by the sum of

$$\sum_i |\partial(u_i)(S_f^\beta(\zeta, H; \partial(\gamma(z_i)) : \mathscr{Y} : X) - S_{z_i f}^\beta(\zeta, H : \mathscr{Y} : X))|$$

and

$$\sum_i |S_{z_i f}^\beta(\zeta, H; \partial(u_i) : \mathscr{Y} : X)|.$$

We know from the first stage of our proof that there is a $\nu \in \mathscr{S}(G)$, independent of H, such that the first expression is bounded by $\nu(f)$. That the same is true for the second expression follows from (8.8). Our proof of the lemma is complete. □

Corollary 8.2. *Suppose that $\zeta \in Z(\mathfrak{t}^*)$ and that $f \in \mathscr{C}(G)$. Then the function*

$$S_f^\beta(\zeta, H:\mathscr{Y}:X), \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta),$$

is infinitely differentiable. Moreover the differential equations given in formula (5.1) *remain valid.*

Proof. It is enough to prove the corollary on any open set Ω which is relatively compact in $\mathfrak{t}_{\mathrm{reg}}(\zeta)$. Fix $f \in \mathscr{C}(G)$, and let f_n be a sequence of functions in $C_c^\infty(G)$ which converge to f in the topology of $\mathscr{C}(G)$. By the lemma the sequence

$$S_{f_n}^\beta(\zeta, H:\mathscr{Y}:X), \qquad H \in \Omega, \tag{8.9}$$

is Cauchy in $\mathscr{C}(\Omega)$, so it must have a limit in $\mathscr{C}(\Omega)$. The limit function must equal the pointwise limit of (8.9), which by definition is $S_f^\beta(\zeta, H:\mathscr{Y}:X)$. In particular, this latter function is smooth. Suppose that $z \in \mathscr{Z}$. It follows from the lemma and what we have just proved that the map which sends $f \in \mathscr{C}(G)$ to

$$S_f^\beta(\zeta, H; \partial(\gamma(z)):\mathscr{Y}:X) - S_{zf}^\beta(\zeta, H:\mathscr{Y}:X) - \sum_j S_f^\beta(\zeta, H; \tilde{\partial}^i(z):\mathscr{Y}:X\tilde{X}_j), \qquad H \in \Omega,$$

is a continuous linear map from $\mathscr{C}(G)$ to $\mathscr{C}(\Omega)$. Since it is zero on the dense subspace $C_c^\infty(G)$, it must be identically zero. □

Lemma 8.3. *Suppose that $\zeta \in Z(\mathfrak{t}^*)$ and that Ω^* is an open subset of $\mathfrak{t}_{\mathrm{reg}}^*(\zeta)$ which is relatively compact in the union of $\mathfrak{t}_{\mathrm{reg}}^*(\zeta)$ and $\mathfrak{t}_{0,\mathrm{reg}}(\zeta)$. Then the map*

$$f \to R_f^{T^*,A^*}(\zeta, H^*:\mathscr{Y}^*:X), \qquad H^* \in \Omega^*,\ f \in C_c^\infty(G),$$

extends to a continuous linear map from $\mathscr{C}(G)$ to $\mathscr{C}(\Omega^*)$.

This lemma is proved exactly the same way as Lemma 8.1, except that the role of Theorem 6.1 is played by Lemma 6.3. □

Corollary 8.4. *The statement of Theorem* 6.1 *remains true for $f \in \mathscr{C}(G)$.*

Proof. This corollary follows directly from Lemmas 8.1 and 8.3. □

Lemma 8.5. *Suppose that $\zeta \in Z(\mathfrak{t})$ and that $f \in \mathscr{C}(G)$. Then the function*

$$R_f(\zeta, H:\mathscr{Y}:X), \qquad H \in \mathfrak{t}_{\mathrm{reg}}(\zeta),$$

is infinitely differentiable. Moreover, the differential equations given in Theorem 5.1 *remain valid.*

Proof. If Ω is an open relatively compact set in $\mathfrak{t}_{\mathrm{reg}}(\zeta)$, the map

$$g \to R_g(\zeta, H:\mathscr{Y}:X), \qquad H \in \Omega,\ g \in C_c^\infty(G),$$

extends to a continuous linear map from $\mathscr{C}(G)$ to $\mathscr{C}(\Omega)$.

This is verified by a repetition of a part of the proof of Lemma 8.1. One then argues as in Corollary 8.2 to prove the lemma. □

§ 9. The Main Theorem

For this section suppose that the rank of G equals the rank of K. In particular the split component of the center of G is trivial. Let $\mathcal{E}_2(G)$ denote the set of equivalence classes of irreducible unitary square integrable representations of G. For any $\omega \in \mathcal{E}_2(G)$ let $\mathscr{C}_\omega(G)$ be the closed subspace of $\mathscr{C}(G)$ spanned by the K-finite matrix coefficients of any representation in the class ω. For each such ω we let Θ_ω be the character of ω. It is a tempered distribution on G which coincides with a locally integrable function, also denoted by Θ_ω.

As always A is a special subgroup of G of dimension p, T is a θ-stable Cartan subgroup of G containing A, and $\mathscr{Y}$ is an A-orthogonal set. We are ready to state and prove our main theorem.

Theorem 9.1. *Fix $\omega \in \mathcal{E}_2(G)$ and $f \in \mathscr{C}_\omega(G)$. Then if $\mathfrak{a} \neq \mathfrak{t}_{\mathbb{R}}$,*

$$r_f(h:\mathscr{Y}) = 0, \qquad h \in T_{\mathrm{reg}}.$$

If $\mathfrak{a} = \mathfrak{t}_{\mathbb{R}}$,

$$r_f(h:\mathscr{Y}) = (-1)^p\, \Theta_\omega(f)\, \Theta_\omega(h), \qquad h \in T_{\mathrm{reg}}.$$

Define $\varepsilon(T, A)$ to be 1 if $\mathfrak{a} = \mathfrak{t}_{\mathbb{R}}$ and to be 0 otherwise. We shall actually prove the following

Theorem 9.1*. *For any $X \in \mathscr{G}_A$ and $h \in T_{\mathrm{reg}}$,*

$$r_f(h:\mathscr{Y}:X) = \varepsilon(T, A)\, c_0(X)\, (-1)^p\, \Theta_\omega(f)\, \Theta_\omega(h).$$

Proof. We will prove Theorem 9.1* by induction on p. Suppose that $p = 0$. Then $A = \{1\}$, $\mathscr{G}_A = G$, and

$$v(x, \mathscr{Y}) = 1, \qquad x \in G.$$

It follows that

$$D_X\, v(x, \mathscr{Y}) = c_0(X).$$

Therefore $r_f(h:\mathscr{Y}:X)$ equals the product of $c_0(X)$ and

$$r_f(h) = \int_{T_{\mathbb{R}} \backslash G} f(x^{-1} h x)\, dx.$$

If T is not compact we know from [2(g), Theorem 11] that

$$r_f(h) = 0.$$

If T is compact, we appeal to [2(g), Theorem 14] to see that

$$r_f(h) = \Theta_\omega(f)\, \Theta_\omega(h).$$

(For these last two formulae see also [2(i), Lemma 8.2, and the corollary to Lemma 27.4].) The theorem is thus valid for $p = 0$.

Fix $p > 0$. Suppose the theorem is true for any T_1 and A_1 with $\dim A_1 < p$. Let X belong to $\mathscr{G}_A(r)$. To prove the theorem for T and A we shall use a second induction, this time a decreasing induction on r. If $r>p$, we have

$$D_X v(x,\mathscr{Y})=0,$$

from formula (4.1). Therefore fix r, $0\leqq r\leqq p$, and assume that the theorem is valid with X replaced by any $X_1\in\mathscr{G}_A(r_1)$, $r_1>r$.

Fix $\zeta\in Z(\mathfrak{t})$. For any $H\in\mathfrak{t}_{\mathrm{reg}}(\zeta)$ define

$$\tilde{\Phi}(\zeta,H)=\tilde{\Delta}(\zeta,H)\,\Theta_\omega(\zeta\exp H).$$

Let $R_{\mathbb{R}}(\zeta)$ be the set of real roots β of $(\mathfrak{g},\mathfrak{t})$ such that $\xi_\beta(\zeta)=1$. Let $\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta)$ be the set of points in $\mathfrak{t}_{\mathbb{R}}$ on which no root in $R_{\mathbb{R}}(\zeta)$ vanishes. Suppose that U is a connected open subset of $\mathfrak{t}_I$ such that for any non real root β of $(\mathfrak{g},\mathfrak{t})$

$$\xi_\beta(\zeta\exp H)\neq 1$$

for all $H\in U$. The set

$$\zeta\exp(U+\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta))$$

is contained in T_{reg}, and the union over all ζ and all such U of the corresponding sets is dense in T_{reg}. Therefore, to complete the proof of the theorem we must show that for all $H\in U+\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta)$,

$$\Psi(H)=R_f(\zeta,H:\mathscr{Y}:X)-\varepsilon(T,A)\,c_0(X)(-1)^p\,\Theta_\omega(f)\,\tilde{\Phi}(\zeta,H)$$

equals 0. We will prove this by combining the differential equations of § 5, the boundary conditions of § 6 and the growth condition of § 7.

From Harish-Chandra's characterization of the discrete series [2(g), Theorem 16] and [2(i), § 27] we know that there is a regular linear functional ν on $\mathfrak{t}_{\mathbb{C}}$ such that for every $q\in I(\mathfrak{t}_{\mathbb{C}})$,

$$\tilde{\Phi}(\zeta,H;\partial(q))=q(\nu)\,\tilde{\Phi}(\zeta,H).$$

If $z\in\mathscr{Z}$ and $\gamma(z)=q$ then ν also has the property that

$$zf=q(\nu)f,$$

since $f\in\mathscr{C}_\omega(G)$. Now look at the equations satisfied by $R(\zeta,H:\mathscr{Y}:X)$. If X_i belongs to $\mathscr{G}_A(1)$, XX_i belongs to $\mathscr{G}_A(r+1)$. By our induction hypothesis on r,

$$R_f(\zeta,H:\mathscr{Y}:XX_i)=0.$$

It follows from Lemma 8.5 that if z and q are as above,

$$R_f(\zeta,H;\partial(q):\mathscr{Y}:X)=R_{zf}(\zeta,H:\mathscr{Y}:X)=q(\nu)\,R_f(\zeta,H:\mathscr{Y}:X).$$

We have shown that

$$\Psi(H;\partial(q))=q(\nu)\,\Psi(H),\quad q\in I(\mathfrak{t}_{\mathbb{C}}),\quad H\in U+\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta). \tag{9.1}$$

Fix a real root $\beta\in R_{\mathbb{R}}(\zeta)$. We continue to use the notation related to β which we set up in § 4 and § 5. Then $\zeta\in Z(\mathfrak{t}^*)$. Fix a point H_0 in $U+\mathfrak{t}_{\mathbb{R}}$ such that for any root α of $(\mathfrak{g},\mathfrak{t})$,

$$\xi_\alpha(\zeta\exp H_0)$$

equals 1 if and only if α equals $\pm\beta$. Define

$$\Phi(\zeta, H)=\Delta(\zeta, H)\,\Theta_\omega(\zeta \exp H)=\varepsilon_{\mathbb{R}}(H)\,\tilde{\Phi}(\zeta, H), \qquad H\in \mathfrak{t}_{\mathrm{reg}}(\zeta),$$

and let

$$\Phi^*(\zeta, H^*)=\varepsilon_{\mathbb{R}}^*(H^*)\,\tilde{\Phi}^*(\zeta, H^*), \qquad H^*\in \mathfrak{t}^*_{\mathrm{reg}}(\zeta),$$

be the corresponding function associated with $\mathfrak{t}^*$. It is known [3, p. 283] that the limits

$$\lim_{t\to 0}\Phi(\zeta, H_0+t\,H'_\beta;\partial(H'_\beta))$$

and

$$\lim_{\theta\to 0}\Phi^*(\zeta, H_0+\theta(X'_\beta-Y'_\beta);\partial(\Lambda H'_\beta))$$

exist and are equal. Now for t and θ sufficiently close to 0,

$$\varepsilon_{\mathbb{R}}(H_0+t\,H'_\beta)=\operatorname{sign} t\cdot\varepsilon^*_{\mathbb{R}}(H_0+\theta(X'_\beta-Y'_\beta)).$$

It follows that

$$\tilde{\Phi}(\zeta, H_0;\partial(H'_\beta))^+-\tilde{\Phi}(\zeta, H_0;\partial(H'_\beta))^-=2\lim_{\theta\to 0}\tilde{\Phi}^*(\zeta, H_0+\theta(X'_\beta-Y'_\beta);\partial(\Lambda H'_\beta)). \tag{9.2}$$

We shall use this fact to show that $\Psi(H_0+tH'_\beta)$ is continuously differentiable at $t=0$. First of all, we should see that it is continuous. By our induction hypothesis on p

$$R_f(\zeta, H_0+tH'_\beta:\mathscr{Y}:X)=S_f^\beta(\zeta, H_0+t\,H'_\beta:\mathscr{Y}:X).$$

By Corollary 8.4, this function is continuous at $t=0$. It is known that the same is true for

$$\Phi(\zeta, H_0+t\,H'_\beta).$$

Therefore $\Psi(H_0+t\,H'_\beta)$ is continuous at $t=0$. We have only to check that the left and right derivatives of $\Psi(H_0+t\,H'_\beta)$ are equal at $t=0$. Since

$$\Lambda(s_\beta H'_\beta)-\Lambda(H'_\beta)=-2\Lambda(H'_\beta),$$

we obtain from Corollary 8.4 the equality of

$$R_f(\zeta, H_0;\partial(H'_\beta):\mathscr{Y}:X)^+-R_f(\zeta, H_0;\partial(H'_\beta):\mathscr{Y}:X)^- \tag{9.3}$$

and

$$-2n_\beta(A)\lim_{\theta\to 0}R_f^{T^*,A^*}(\zeta, H_0+\theta(X'_\beta-Y'_\beta);\partial(\Lambda H'_\beta):\mathscr{Y}^*:X).$$

It is clear that $\varepsilon(T,A)=\varepsilon(T^*,A^*)$ and that $p^*=p-1$. Therefore by our induction hypothesis on p, (9.3) equals

$$2n_\beta(A)\,\varepsilon(T,A)\,c_0(X)(-1)^p\,\Theta_\omega(f)\lim_{\theta\to 0}\tilde{\Phi}^*(\zeta, H_0+\theta(X'_\beta-Y'_\beta);\partial(\Lambda H'_\beta)).$$

Combining this with (9.2) we find that

$$\Psi(H_0;\partial(H'_\beta))^+-\Psi(H_0;\partial(H'_\beta))^-$$

equals

$$2(n_\beta(A)-1)\,\varepsilon(T,A)\,c_0(X)(-1)^p\,\Theta_\omega(f)\lim_{\theta\to 0}\tilde{\Phi}^*(\zeta,H_0+\theta(X'_\beta-Y'_\beta);\partial(AH'_\beta)).$$

If $\varepsilon(T,A)=0$ this expression is 0. If $\varepsilon(T,A)=1$ then $\mathfrak{a}=\mathfrak{t}_{\mathbb{R}}$, so that $n_\beta(A)=1$. Again the expression is 0. We have shown that $\Psi(H_0+tH'_\beta)$ extends to a continuously differentiable function at $t=0$.

Suppose that F is any connected component of $\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta)$. Then the restriction of Ψ to the open connected set $U+F$ is a smooth function which satisfies the equations (9.1). We repeat the argument used in [2(a), Theorem 3] to show that Ψ is actually analytic on this set. Let ω be the element in $S(\mathfrak{t}_{\mathbb{C}})$ such that $\partial(\omega)$ is the Laplacian on $\mathfrak{t}$ with respect to our Euclidean norm. We can find a positive integer n and elements

$$\{u_i: 0\leqq i\leqq n-1\}$$

in $I(\mathfrak{t}_{\mathbb{C}})$ such that

$$\omega^n+u_{n-1}\,\omega^{n-1}+\cdots+u_0=0.$$

It follows from (9.1) that

$$(\partial(\omega)^n+u_{n-1}(\nu)\cdot\partial(\omega)^{n-1}+\cdots+u_0(\nu))\,\Psi(H)=0,\qquad H\in U+F.$$

The restriction of Ψ to $U+F$ is a solution of a linear elliptic differential equation and is therefore analytic. This fact, combined with the differential equations (9.1), is exactly what is needed to apply another basic technique of Harish-Chandra. According to [2(b), p. 102] there are complex numbers

$$\{c_s: s\in W\},$$

indexed by the Weyl group of $(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$, such that for any $H\in U+F$,

$$\Psi(H)=\sum_{s\in W} c_s\, e^{s\nu(H)}.$$

Suppose that F' is another component of $\mathfrak{t}_{\mathbb{R},\mathrm{reg}}(\zeta)$ such that the chambers F and F' have in common a wall defined by a real root β in $R_{\mathbb{R}}(\zeta)$. We can assume that $\beta(F)$ is positive. If $V_{\mathbb{R}}$ is the interior of this common wall, then

$$V=U+V_{\mathbb{R}}$$

is an open subset of

$$\mathfrak{t}_0=\{H\in\mathfrak{t}:\beta(H)=0\}.$$

Any point H_0 in V satisfies the hypothesis we made above. Moreover if t is sufficiently close to 0, $\Psi(H_0+tH'_\beta)$ belongs to either F of F'. Suppose that W_0 is a set of representatives in W of the cosets $\{1,s_\beta\}\backslash W$. Let

$$\{c'_s: s\in W\}$$

be the set of constants associated to F'. Then for every point $H_0 \in V$ we can find a positive number $\varepsilon(H_0)$ such that

$$\Psi(H_0 + tH'_\beta) = \sum_{s \in W_0} (c_s\, e^{tsv(H'_\beta)} + c_{s_\beta s}\, e^{-tsv(H'_\beta)})\, e^{sv(H_0)},$$

if $0 < t < \varepsilon(H_0)$, and

$$\Psi(H_0 + tH'_\beta) = \sum_{s \in W_0} (c'_s\, e^{tsv(H'_\beta)} + c'_{s_\beta s}\, e^{-tsv(H'_\beta)})\, e^{sv(H_0)},$$

if $-\varepsilon(H_0) < t < 0$. It follows from the fact that $\Psi(H_0 + tH'_\beta)$ is continuously differentiable at $t = 0$ that

$$\sum_{s \in W_0} (c_s + c_{s_\beta s} - c'_s - c'_{s_\beta s})(s\, v(H'_\beta))\, e^{sv(H_0)} = 0$$

and

$$\sum_{s \in W_0} (c_s - c_{s_\beta s} - c'_s + c'_{s_\beta s})(s\, v(H'_\beta))\, e^{sv(H_0)} = 0$$

for all H_0 in V.

Suppose that $sv - s_1 v$ is orthogonal to $\mathfrak{t}_0$ for two elements s and s_1 in W_0. Then if $v_1 = s_1 v$, and $r = s s_1^{-1}$,

$$r v_1 = v_1 + x H'_\beta, \qquad x \in \mathbb{R}.$$

Since r is an orthogonal map, x must either equal 0 or $-2\dfrac{v_1(H'_\beta)}{\|H'_\beta\|^2}$. This second alternative is impossible because it would lead to the equation

$$s v = s_\beta s_1 v,$$

which, in view of the regularity of v, would mean that $s = s_\beta s_1$. Therefore s must equal s_1. From this it follows that the set of functions

$$H_0 \to e^{sv(H_0)}, \qquad H_0 \in V,$$

indexed by W_0, is linearly independent. On the other hand, the regularity of v implies that for all $s \in W_0$,

$$s v(H'_\beta) \neq 0.$$

We obtain, for each $s \in W_0$, the equations

$$c_s + c_{s_\beta s} - c'_s - c'_{s_\beta s} = 0$$

and

$$c_s - c_{s_\beta s} - c'_s + c'_{s_\beta s} = 0.$$

Thus for each $s \in W$, c_s equals c'_s. We have shown that the formula

$$\Psi(H) = \sum_{s \in W} c_s\, e^{sv(H)}$$

is valid for all H in the domain of Ψ.

Now we can apply Harish-Chandra's growth condition [2(i), Theorem 12.1] to the tempered, invariant, $\mathscr{Z}$-finite distribution Θ_ω. Combining this with the growth condition of Corollary 7.4, we obtain constants C and r such that

$$|\Psi(H)| \leqq C(1+L(\zeta \exp H))^p (1+\|H_{\mathbb{R}}\|)^r, \qquad H \in U + \mathfrak{t}_{\mathbb{R},\,\mathrm{reg}}(\zeta). \tag{9.4}$$

As in Corollary 7.4, we have put

$$H = H_I + H_{\mathbb{R}}, \qquad H_I \in \mathfrak{t}_I, \quad H_{\mathbb{R}} \in \mathfrak{t}_{\mathbb{R}}.$$

Let $\lambda_1, \ldots, \lambda_k$ be the distinct vectors obtained by projecting

$$\{s\nu : s \in W\}$$

onto $(\mathfrak{t}_{\mathbb{R}})_{\mathbb{C}}$. Each of these vectors is real. It is well known that, since G has a compact Cartan subgroup, there exists a real root of $(\mathfrak{g}, \mathfrak{t})$. Therefore by the regularity of ν,

$$\lambda_i \neq 0, \qquad 1 \leqq i \leqq k.$$

For $1 \leqq i \leqq k$, let W_i be the set of $s \in W$ such that the projection of $s\nu$ onto $(\mathfrak{t}_{\mathbb{R}})_{\mathbb{C}}$ is λ_i. If $s \in W_i$,

$$s\nu = \mu_s + \lambda_i,$$

where $\mu_s \in \sqrt{-1}\,\mathfrak{t}_I$. Then for $H \in U + \mathfrak{t}_{\mathbb{R},\,\mathrm{reg}}(\zeta)$,

$$\Psi(H) = \sum_{i=1}^{k} \Big(\sum_{s \in W_i} c_s\, e^{\mu_s(H_I)}\Big)\, e^{\lambda_i(H_{\mathbb{R}})}.$$

After a moment's reflection one realizes that the growth condition (9.4) will fail unless each coefficient function

$$\sum_{s \in W_i} c_s\, e^{\mu_s(H_I)}, \qquad H_I \in U, \tag{9.5}$$

vanishes. Distinct elements $s \in W_i$ give rise to distinct vectors μ_s. Therefore (9.5) is a linear combination of linearly independent functions. In other words,

$$c_s = 0, \qquad s \in W.$$

We have shown that

$$\Psi(H) = 0, \qquad H \in U + \mathfrak{t}_{\mathbb{R},\,\mathrm{reg}}(\zeta),$$

and thereby have completed the proof of Theorem 9.1*. □

Remark. Theorem 9.1 is at variance with the formula given in [1(b)]. The mistake there was the result of using two different Haar measures on the group

$$N_1 = \exp \mathbb{R}X',$$

(in the notation of [1(b)]. One measure, on page 579, was normalized by the restriction to $\mathfrak{n}_1$ of the Killing form of $\mathfrak{g}$, whereas the formula quoted on page 581, line 12, was based on the measure normalized by the Killing form of $\mathfrak{g}_1$. The measures differ by the factor $\frac{\sqrt{2r}}{4}$. It is this factor which should be removed from the formulae in Theorem 7.2 and Corollary 7.3.

References

1a. Arthur, J.: The Selberg trace formula for groups of F-rank one. Ann. of Math. **100**, 326–385 (1974)
1b. Arthur, J.: Some tempered distributions on semisimple groups of real rank one. Ann. of Math. **100**, 553–584 (1974)
2a. Harish-Chandra: The characters of semisimple Lie groups. Trans. Amer. Math. Soc. **83**, 98–163 (1956)
2b. Harish-Chandra: Differential operators on a semisimple Lie algebra. Amer. J. Math. **79**, 87–120 (1957)
2c. Harish-Chandra: Fourier transforms on a semisimple Lie algebra I. Amer. J. Math. **79**, 193–257 (1957)
2d. Harish-Chandra: A formula for semisimple Lie groups. Amer. J. Math. **79**, 733–760 (1957)
2e. Harish-Chandra: Invariant eigendistributions on a semisimple Lie group. Trans. Amer. Math. Soc. **119**, 457–508 (1965)
2f. Harish-Chandra: Two theorems on semisimple Lie groups. Ann. of Math. **83**, 74–128 (1966)
2g. Harish-Chandra: Discrete series for semisimple Lie groups II. Acta Math. **116**, 1–111 (1966)
2h. Harish-Chandra: On the theory of the Eisenstein integral. Lecture Notes in Math. **266**, 123–149. Berlin-Heidelberg-New York: Springer 1971
2i. Harish-Chandra: Harmonic analysis on real reductive groups I: The theory of the constant term. J. Funct. Analysis **19**, 104–204 (1975)
3. Hirai, T.: Explicit form of the characters of discrete series representations of semisimple Lie groups. Harmonic analysis on homogenous spaces. Proc. Sympos. Pure Math. (Amer. Math. Soc.), **26**, 281–288 (1972)
4a. Langlands, R. P.: On the functional equations satisfied by Eisenstein series. Mimeographed notes
4b. Langlands, R. P.: Eisenstein series. Algebraic groups and discontinuous subgroups. Proc. Sympos. Pure Math., (Amer. Math. Soc.) **9**, 225–252 (1966)
4c. Langlands, R. P.: Dimension of spaces of automorphic forms. Algebraic groups and discontinuous subgroups. Proc. Sympos. Pure Math. (Amer. Math. Soc.) **9**, 253–257 (1966)

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