Inventiones math. 5, 85—105 (1968)

Weyl's Character Formula for Algebraic Groups

T. A. SPRINGER (Utrecht)

Introduction

H. WEYL's formula for the character of irreducible representations of compact semisimple Lie groups, proved by him by transcendental methods ([13], p. 358) is actually a statement of a purely algebraic nature. Moreover there is an obvious variant which makes sense for the rational representations of connected semisimple algebraic groups over an algebraically closed field of characteristic 0, which one derives easily from WEYL's original formula, for instance by invoking the "Lefschetz principle". Of course a proof of an algebraic statement along such lines is not too satisfactory.

The algebraic proof of WEYL's formula given by FREUDENTHAL ([5], see also [8], Ch. VIII, §3) has the disadvantage that it operates with the Lie algebra, so that there still is a transition to be made from Lie algebra to group, which one would rather avoid in dealing with algebraic groups.

In the present note a "global" proof of WEYL's character formula will be given, which operates with the group itself. The ideas used in this proof are quite familiar, in one form or another. The main tool is an identity (Proposition 2.9) involving the Casimir operator which is the algebraic analog of one due to HARISH-CHANDRA (and which is also implicit in [5]).

An advantage of our method is, that it does not completely break down in characteristic $p>0$, so that one can extract information about certain irreducible representations in characteristic $p>0$ (viz. those for which p is "large with respect to the highest weight", in a sense made more precise by 4.3). The results are, however, far from conclusive and it does not seem likely that the methods of this note are sufficient to obtain a complete solution of the problem of finding a general character formula in characteristic p.

The author is indebted to J. TITS for various useful remarks.

1. Preliminaries

1.1. Let k be an algebraically closed field of characteristic p. Let G be a connected linear algebraic group, defined over k. We may and shall identify it with its group of k-rational points. We refer to [4] for the relevant facts about algebraic groups.

The ring of regular functions on G is denoted by $k[G]$. The group G acts on $k[G]$ by left (right) translations $\gamma(g)$ (resp. $\delta(g)$), which are defined by

$$\gamma(g)f(x)=f(g^{-1}x), \quad \delta(g)f(x)=f(xg) \quad (g,x\in G,\ f\in k[G]).$$

We have $\gamma(g)\circ\delta(h)=\delta(h)\circ\gamma(g)$ for $g,h\in G$, moreover γ and δ are representations of G in $k[G]$.

A k-derivation X of $k[G]$ which commutes with all $\gamma(g)$ is called *left invariant*. The left invariant k-derivations form the Lie algebra $\mathfrak{g}$ of G. Using the fact that for $p>0$ the p^{th} power of a left invariant derivation is again one, one gets for $p>0$ a structure of restricted Lie algebra (or p-algebra) on $\mathfrak{g}$. We shall identify $\mathfrak{g}$ with the tangent space to G at the neutral element e. This tangent space being the space of k-linear functions φ on $k[G]$ satisfying $\varphi(fg)=f(e)\varphi(g)+\varphi(f)g(e)$, the element $X\in\mathfrak{g}$ is identified with the element $f\mapsto Xf(e)$ of the tangent space. We refer to [1] (§1) for a more complete discussion of these matters. For $g\in G$, $X\in\mathfrak{g}$ we put $\mathrm{Ad}(g)X=\delta(g)\circ X\circ\delta(g)^{-1}$. It is readily checked that $\mathrm{Ad}(g)X\in\mathfrak{g}$. We get in this manner a rational representation Ad of G in $\mathfrak{g}$, the *adjoint* representation. Its differential is the adjoint representation ad of $\mathfrak{g}$, defined by $\mathrm{ad}\,X(Y)=[X,Y]$ (see [1] for details).

1.2. *Invariant Differential Operators.* In the situation of 1.1 let $\mathscr{U}$ denote the universal enveloping algebra of $\mathfrak{g}$ for $p=0$ and the restricted universal enveloping algebra of $\mathfrak{g}$ for $p>0$ (called u-algebra in [8], p. 192). From the definition of $\mathfrak{g}$ as a Lie algebra of derivations it follows that there is a homomorphism of k-algebras $h: \mathscr{U}\to\mathrm{Hom}_k(k[G], k[G])$. Any element of $h(\mathscr{U})$ is called a left invariant differential operator in $k[G]$. An *invariant differential operator* on $k(G)$ is an element of $h(\mathscr{U})$ which commutes with all $\delta(g)$ for $g\in G$. The invariant differential operators form a k-algebra. If G is semisimple of rank l and if $p=0$, results of Harish-Chandra and Chevalley (see e.g. [7]) imply that the algebra of invariant differential operators is a commutative polynomial algebra in l generators. For $p>0$ no such results seem to be known. We will use in this note for G semisimple only one particular invariant differential operator, viz. the Casimir operator.

1.3. *Example.* Let T be an algebraic torus over k, let $X(T)$ denote the group of rational characters of T (written additively). $k[T]$ is then isomorphic to the group ring of $X(T)$ over k. In order to avoid a confusion between additive and multiplicative notations, we denote for $a\in X(T)$ by $e(a)$ the element of $k[T]$ defined by a, so that we have for $t\in T$

$$e(a)t=t^a \quad \text{(value of } a \text{ in } t).$$

Then $e(a+b)=e(a)e(b)$.

It is easily checked that there is a bijection α of $\mathrm{Hom}_{\boldsymbol{Z}}(X, k)$ onto the Lie algebra $\mathfrak{t}$ of T, α being given by $(\alpha\lambda)e(a)=\lambda(a)e(a)$. As an example of an invariant differential operator on $k[T]$ we mention the following one. Let F be a $\boldsymbol{Z}$-valued polynomial function on $X(T)$, define D by $De(a)=F(a)e(a)$. Then D is an invariant differential operator.

1.4. Now let G be connected semisimple of rank l. Fix an l-dimensional maximal torus T of G and let Σ be its root system. This is a finite set of vectors in $V=X(T)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ with familiar properties. We fix an order in Σ and denote by $\Delta=\{r_1, \dots, r_l\}$ the corresponding set of simple roots. Let $N(T)$ be the normalizer of T. Then the Weyl group $W=N(T)/T$ acts in Σ and in $k[T]$. Denote by $\langle\ ,\ \rangle$ a positive definite scalar product on V which is W-invariant, so that $c_{rs}=2\langle s, s\rangle^{-1}\langle r, s\rangle$ is an integer for $r, s\in\Sigma$. We put $c_{ij}=c_{r_i, r_j}$ $(1\leqq i, j\leqq l)$.

The scalar product is normalized as follows. First let Σ be a simple root system. Then $\langle\ ,\ \rangle$ is unique up to a scalar factor, which we normalize by requiring that the minimum value of $\langle r, r\rangle$ for $r\in\Sigma$ equals 1. In that case $\langle r, r\rangle$ is the length of r (as in [3], p. 17) and is 1, 2 or 3. If Σ is not simple, the normalization is carried out in each of the simple components of Σ.

For $r\in\Sigma$ there exists an isomorphism x_r of the additive group $\boldsymbol{G}_a$ onto a closed subgroup G_r of G such that $tx_r(\lambda)t^{-1}=x_r(t^r\lambda)$ for $\lambda\in k$. G_r and G_{-r} generate a subgroup P_r of G. If G is simply connected, then P_r is isomorphic to $\boldsymbol{SL}_2$ ([4], exp. 23, prop. 2).

1.5. In the situation of 1.4. let G_0 be the Chevalley-Demazure scheme corresponding to G. This is an affine group scheme, which is of finite type and smooth over $\boldsymbol{Z}$. So $G_0=\mathrm{Spec}(A_0)$ and $k[G]=A_0\otimes_{\boldsymbol{Z}}k$. Moreover denoting by $\mathfrak{g}_0$ the Lie algebra of G_0 we have $\mathfrak{g}=\mathfrak{g}_0\otimes_{\boldsymbol{Z}}k$. Then $\mathfrak{g}_0$ is a direct sum

$$\mathfrak{g}_0=\Gamma+\sum_{r\in\Sigma}\boldsymbol{Z}E_r,$$

where Γ is a lattice in $\mathfrak{g}_0$ of rank l, such that $\Gamma\otimes k$ is the Lie algebra $\mathfrak{t}$ of T. We have CHEVALLEY's rules

$$[E_r E_s]=N_{rs}E_{r+s} \qquad (r, s\in\Sigma,\ r+s\neq 0),$$

where the integers N_{rs} are as in [3] (Th. 1, p. 24). Moreover

$$[E_r E_{-r}]=F_r\in\Gamma, \quad [FE_r]=\mu_r(F)E_r(F\in\Gamma),$$

where μ_r is a $\boldsymbol{Z}$-valued linear function on Γ such that $\mu_r(F_s)=c_{rs}$ $(=2\langle s, s\rangle^{-1}\langle r, s\rangle)$. Let Δ be the sublattice of Γ spanned by all F_r. Put $f=|\det(c_{ij})|$. Then the index $(\Gamma:\Delta)$ is $\leqq f$ and we have $(\Gamma:\Delta)=1$ (resp. f) if G_0 is of adjoint type (resp. of simply connected type).

If Σ is simple and not of type A_l, then $1 \leqq f \leqq 4$ (see [3], p. 63–64 for the precise values of f).

Returning to $\mathfrak{g}$, we put $X_r = E_r \otimes 1$, $H_r = F_r \otimes 1$. Denote by λ_r the k-linear function on $\mathfrak{t} = \Gamma \otimes k$ determined by μ_r. Then λ_r is the differential of the morphism $t \mapsto t^r$ of T into the multiplicative group G_m (the tangent space to G_m in 1 being identified with k). We need the following simple result, which is contained in [3] (p. 22, line 17 from below).

1.6. Lemma. *If $r, s, r+s \in \Sigma$ then we have*

$$\langle s, s \rangle N_{rs} + \langle r+s, r+s \rangle N_{r,-r-s} = 0.$$

1.7. Let R be the lattice in $V = X(T) \otimes \boldsymbol{R}$ generated by the elements of Σ. Any set of simple roots $\{r_1, \ldots, r_l\}$ is a basis of R. Let P be the lattice in V generated by the weights, i.e. the elements $a \in V$ such that $2\langle r, r \rangle^{-1} \langle a, r \rangle$ is an integer for all $r \in \Sigma$. We have $R \subset P$ and $(P:R) = f$ $(= |\det(c_{ij})|)$.

The weight d is *dominant* with respect to a given order, if $\langle d, r \rangle \geqq 0$ for all $r > 0$. A dominant weight is an integral linear combination

$$d = \sum_{i=1}^{l} n_i d_i$$

with $n_i \in \boldsymbol{Z}$, $n_i \geqq 0$ of the fundamental weights d_i, defined by

$$2\langle r_j, r_j \rangle^{-1} \langle d_i, r_j \rangle = \delta_{ij}.$$

Put $\rho = \sum_{i=1}^{l} d_i$. It is well-known that $2\rho = \sum_{r>0} r$.

Finally, let us recall that we have $R \subset X(T) \subset P$ and that $X(T) = R$ (resp. P) if G is adjoint (resp. simply connected).

2. The Casimir Operator

We first prove some lemmas.

2.1. Lemma. *Let G be a linear algebraic group over k, let H be an algebraic subgroup of G. Denote by φ: $k[G] \to k[H]$ the canonical projection and by i: $\mathfrak{h} \to \mathfrak{g}$ the canonical injection of Lie algebras. We then have for $X \in \mathfrak{h}$, $h \in H$, $f \in k[G]$*

$$(iX) f(h) = X \varphi(f)(h).$$

Proof. By left invariance it suffices to establish this for $h = e$. Then the assertion is a direct consequence of the identification of $\mathfrak{g}$ ($\mathfrak{h}$) with the tangent space to G (H) at e (recalled in 1.1).

The point of this lemma is that the left hand side of the equality can be computed in $k[H]$.

2.2. Lemma. *Let G be a linear algebraic group over k. Then for $X \in \mathfrak{g}$, $g \in G$, $f \in k[G]$ we have that $X \mapsto (Xf)(g)$ is the differential in e of the morphism of algebraic varieties $\psi: x \mapsto f(gx)$ of G into the additive group G_a, the tangent space to G_a in 0 being identified with k.*

Proof. There exists a finite family $\{f_i, g_i\}$ of elements of $k[G]$ such that

$$f(gx) = \sum_i f_i(g)\, g_i(x).$$

By left invariance of X we have

$$(Xf)(gx) = \sum_i f_i(g)\, X\, g_i(x), \quad \text{so} \quad (Xf)(g) = \sum_i f_i(g)\, X\, g_i(e).$$

The right hand side also equals the differential of ψ in e.

In the next lemmas G is a connected semisimple group. We use the notations of §1. We denote by α the canonical homomorphism $k[G] \to k[T]$. Moreover $C[G]$ will denote the subalgebra of $k[G]$ consisting of the *class-functions*, i.e. the $f \in k[G]$ such that $f(gxg^{-1}) = f(x)$ $(g, x \in G)$.

2.3. Lemma. *Let G be a connected semisimple algebraic group over k. Let $f \in C[G]$, put*

$$\alpha f = \sum_{a \in X(T)} m(a)\, e(a) \qquad \big(m(a) \in k\big).$$

Then we have for $r \in \Sigma$

$$(1) \qquad \alpha(X_r X_{-r} f) = \sum_{i \geqq 0,\, a \in X(T)} 2\langle r, r\rangle^{-1} \langle a + ir, r\rangle\, m(a + ir)\, e(a).$$

Notice that $2\langle r, r\rangle^{-1}\langle a + ir, r\rangle$ is an integer!

Proof. We reduce to the case $G = \boldsymbol{SL}_2$.

(a) Let G' be the universal covering group of G. Then $k[G]$ can be identified with a subalgebra of $k[G']$ and $C[G]$ with a subalgebra of $C[G']$. One checks that the validity of (1) in G' implies the validity in G. Hence we may suppose that G is *simply connected.*

(b) With the notations of 1.4, let Q_r denote the subgroup of G generated by G_r, G_{-r} and T. Let U denote the identity component of the kernel of r, then we have $Q_r = P_r \cdot U$ where (as in 1.4) P_r is the subgroup generated by G_r and G_{-r}, which is isomorphic to $\boldsymbol{SL}_2$ (G being simply connected). From 2.1 we conclude that it suffices to prove the formula corresponding to (1), where G is replaced by Q_r and α by the canonical homomorphism $k[Q_r] \to k[T]$. Now there exists a central isogeny $P_r \times U \to P_r \cdot U = Q_r$. As in (a) it follows that we may replace Q_r by $P_r \times U$. Since $k[P_r \times U] = k[P_r] \otimes k[U]$ and since X_r, X_{-r} are tangent to P_r, they act trivially on $1 \otimes k[U]$, as derivations of $k[P_r \otimes U]$. So it suffices to consider the action on elements of $k[P_r] \otimes 1$. This means that we may replace G by P_r, i.e. we may assume $G = \boldsymbol{SL}_2$.

(c) We make now an explicit computation, which will be briefly indicated. Identify T with the group of diagonal matrices in G. We may take X_r and X_{-r} to be the tangent vectors defined by the homomorphisms $G_a \to G$ which send $\lambda \in k$ into

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \left(\text{resp. } \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \right).$$

Moreover $k[G] = k[X, Y, Z, U]/(XU - YZ - 1)$. Denoting by D_X etc. partial derivation in the polynomial ring $k[X, Y, Z, U]$, it follows from 2.2 that X_r is the derivation of $k[G]$ induced by $XD_Y + ZD_U$ and X_{-r} the one induced by $YD_X + UD_Z$. Define $a \in X(T)$ by

$$a\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right) = \lambda.$$

Then $e(a)$, $e(-a)$ generate $k[T]$ and it suffices to prove (1) for f such that $\alpha f = e(a)^n + e(-a)^n$ $(n \geqq 0)$. Let F_n be the polynomial with integral coefficients in one indeterminate such that $F_n(X + X^{-1}) = X^n + X^{-n}$. The above f is then defined by

$$f\left(\begin{pmatrix} x & y \\ z & u \end{pmatrix} \right) = F_n(x+u).$$

Using the explicit expressions for X_r, X_{-r} one sees that

$$\alpha(X_r X_{-r} f) = e(a) F_n'(e(a) + e(-a)).$$

The right hand side of (1) is now

$$n \sum_{\substack{-n < k \leqq n \\ n-k \text{ even}}} e(a)^k.$$

(1) then follows from the identity

$$X F_n'(X + X^{-1}) = n \sum_{\substack{-n < k \leqq n \\ n-k \text{ even}}} X^k.$$

2.4. Lemma. *Let G be a connected semisimple group over k. Let $f \in k[G]$, $a \in X(T)$, $\alpha f = e(a)$. For $r \in \Sigma$, let $H_r = [X_r X_{-r}]$ be as in 1.5. Then*

(2) $$\alpha(H_r f) = 2\langle r, r\rangle^{-1} \langle a, r\rangle e(a).$$

The proof is similar to that of 2.3. One reduces this to the case $G = \boldsymbol{SL}_2$, in the same manner as before. Using the notations of part (c) of the proof of 2.3 we have $H_r = XD_X - YD_Y + ZD_Z - UD_U$ and it suffices to check (2) in $\boldsymbol{SL}_2$ for f defined by

$$f\left(\begin{pmatrix} x & y \\ z & u \end{pmatrix} \right) = x^m \quad \text{or} \quad u^n \qquad (m, n \geqq 0),$$

which is easy.

With the notations of 1.4 and 1.5, let u be the smallest positive integer such that $u\langle a, a\rangle \in \boldsymbol{Z}$ for all $a \in X(T)$. It is easily seen that u divides $2f$; $u=1$ if G is of adjoint type.

We define invariant differential operators L and Δ on $k[T]$ by

$$Le(a)=2u\langle\rho, a\rangle e(a), \qquad \Delta e(a)=u\langle a, a\rangle e(a),$$

where ρ is defined as in 1.4.

They extend canonically to left invariant differential operators on $k[G]$, denoted by the same symbols. With these notations we have

2.5. Lemma.

$$[L, X_r]=\langle 2u\rho, r\rangle X_r,\ [\Delta, X_r]=-u\langle r, r\rangle X_r+u\langle r, r\rangle H_r X_r.$$

Proof. Put $H_j=H_{r_j}$. It follows from 2.1 and 2.4 that we have

$$L=\sum_j \langle 2u\rho, d_j\rangle H_j,$$

where d_j is as in 1.7. The first relation is then a consequence of

$$[H_j, X_r]=2\langle r_j, r_j\rangle^{-1}\langle r, r_j\rangle X_r$$

(see 1.5).

To prove the second relation, it is convenient first to assume that $p=0$. Define $\Delta'=u^{-1}\Delta$. Then

$$\Delta'=\sum_{1\leq i,j\leq l}\langle d_i, d_j\rangle H_i H_j.$$

Using again the previous formula we find $[\Delta', X_r]=-\langle r, r\rangle X_r+\langle r, r\rangle H_r X_r$.

Multiplying this by u, we get a relation with integral coefficients, which is true in any characteristic.

2.6. Lemma.

$$[X_r, \sum_{s>0}\langle s, s\rangle X_s X_{-s}]=\langle r, r\rangle H_r X_r-\langle 2\rho, r\rangle X_r-\langle r, r\rangle X_r.$$

Proof. We have

$$\begin{aligned}
&[X_r, \sum_{s\in\Sigma}\langle s, s\rangle X_s X_{-s}]\\
&=\sum_{s\neq -r}\langle s,s\rangle N_{rs}X_{r+s}X_{-s}+\sum_{s\neq r}\langle s,s\rangle N_{r,-s}X_s X_{r-s}+\langle r,r\rangle(H_r X_r+X_r H_r)\\
&=\sum_{s\neq -r}(\langle s,s\rangle N_{rs}+\langle r+s, r+s\rangle N_{r,-r-s})X_{r+s}X_{-s}\\
&\quad+2\langle r,r\rangle H_r X_r-2\langle r,r\rangle X_r.
\end{aligned}$$

The sum equals 0 by 1.6. Moreover

$$\sum_{s\in\Sigma}\langle s,s\rangle X_s X_{-s}=2\sum_{s>0}\langle s,s\rangle X_s X_{-s}-\sum_{s>0}\langle s,s\rangle H_s.$$

This gives

$$2\Big[X_r,\sum_{s>0}\langle s,s\rangle X_s X_{-s}\Big]=2\langle r,r\rangle H_r X_r-2\langle r,r\rangle X_r+\sum_{s>0}\langle s,s\rangle [X_r H_s].$$

The last sum equals $-\langle 4\rho, r\rangle X_r$. This establishes the asserted formula for $p\neq 2$. Since it is an identity with integral coefficients, it is then also true in characteristic 2.

With the previous notations, we put

$$C=\sum_{s>0}u\langle s,s\rangle X_s X_{-s}-L+\Delta.$$

This is a left invariant differential operator on $k[G]$. We call C the *Casimir operator*.

2.7. Proposition. *C is an invariant differential operator.*

Proof. It follows from 2.5 and 2.6 that $[X_r, C]=0$. This implies that if $p=0$, C commutes with all $\delta(x_r(\lambda))$ for $r\in\Sigma$. Since the groups G_r generate G, it follows that C is invariant in characteristic 0. Invariance of C being expressible in terms of polynomial identities with integral coefficients, the assertion is true for arbitrary p once it has been proved in characteristic 0.

2.8. We will now establish an important formula for the Casimir operator. The corresponding analytic result is due to HARISH-CHANDRA ([6], Th. 2, p. 125). First some preparations. Let ρ be as in 1.7 and suppose that $\rho\in X(T)$ (which is the case, for example, if G is simply connected). Define $h\in k[T]$ by

$$h=\sum_{w\in W}\varepsilon(w)\,e(w\rho),$$

where $\varepsilon(w)$ is the sign of $w\in W$. A well-known result of WEYL ([13], p. 355) asserts that

$$h=e(\rho)\prod_{r>0}(1-e(r)^{-1}).$$

The formula mentioned before is contained in the following proposition.

2.9. Proposition. *Suppose that $\rho\in X(T)$. Then if $f\in C[G]$ we have*

$$h\cdot\alpha(Cf)=\Delta(h\cdot\alpha f)-u\langle\rho,\rho\rangle h\cdot\alpha f.$$

Proof. Put

$$\alpha f=\sum_{a\in X(T)}m(a)\,e(a).$$

Using the definitions of C, Δ, L and 2.3 we obtain

$$(3)\qquad \alpha(Cf)=\sum_{\substack{i>0,\,r>0\\ a\in X(T)}} 2u\langle a,r\rangle\, m(a)\, e(a-ir)+L(\alpha f)+\Delta(\alpha f).$$

First let $p=0$. Extend $k[T]$ to a k-algebra A containing all formal power series in the $e(-r_j)$ $(1\leqq j\leqq l)$ and such that A is integral over the subalgebra generated by such power series and by the $e(r_j)$. The k-derivations of $k[T]$ extend canonically to k-derivations of A. The algebra A contains for $r>0$ the element

$$\log(1-e(-r))=-\sum_{i=1}^{\infty} i^{-1}e(-ir).$$

Put

$$v=\prod_{r>0}(1-e(-r)),\qquad \log v=\sum_{r>0}\log(1-e(-r)).$$

Then $h=e(\rho)v$. For $f, g\in A$, put $D(f,g)=\Delta(fg)-f\cdot\Delta g-\Delta f\cdot g$.

For fixed g, $f\mapsto D(f,g)$ is a derivation of A. From the definition of Δ it follows that the sum in (3) equals $D(\log v, \alpha f)$, which by the remark just made is $v^{-1}D(v,\alpha f)$. Similarly $uL(\alpha f)=e(\rho)^{-1}D(e(\rho),\alpha f)$. Thus we get from (3)

$$\begin{aligned}\alpha(Cf)&=v^{-1}D(v,\alpha f)+e(\rho)^{-1}D(e(\rho),\alpha f)+\Delta(\alpha f)\\ &=h^{-1}D(h,\alpha f)+\Delta(\alpha f)=h^{-1}\Delta(h\cdot\alpha f)-u\langle\rho,\rho\rangle\alpha f.\end{aligned}$$

This is the asserted formula.

If $p>0$ we use that $C[G]$ is generated, as a vector space over k, by the $f\in C[G]$ such that αf equals a sum $\sum e(a)$, a running over the distinct conjugates of an element of $X(T)$ under W (this is implicit in [12], proof of 6.3, p. 62). For this f the identity to be proved is again a universal one with integral coefficients, so it holds in arbitrary characteristic if it is true in characteristic 0 (notice that by 2.3 and 2.4 $\alpha(Cf)$ depends only on $\alpha(f)$).

3. The Character Formula (Characteristic 0)

3.1. Let G be a connected semisimple group over the algebraically closed field k. Until further notice, the characteristic p of k is arbitrary. We keep the previous notations.

The elements of the character group $X(T)$ are ordered as follows: $a>b$ if $a-b$ is a positive linear combination of positive roots. Let $\pi\colon G\to \boldsymbol{GL}(V)$ be an irreducible rational representation of G. The weights of π with respect to T are the elements of $X(T)$ which occur in the restriction of π to T. There is a unique maximal dominant weight g (for the order on $X(T)$), which we also call the *highest weight of* π (in another terminology, what is called here the highest weight of π is called *the* dominant weight of π; in the present note the latter notion is always used in the sense of 1.7). Its multiplicity is 1. We have $d<g$ for any other dominant weight of π. An arbitrary weight of π is of the form wd, where d is dominant and $w\in W$ (see [4], vol. 2 for these results on representations). The following lemma is known (see [5], p. 373 or [8], Lemma 3, p. 248), we include a proof for completeness.

3.2. Lemma. *Let g be the highest weight of the irreducible representation π of G. Let ρ be as in 1.7.*

(a) *For any weight $a\neq g$ of π we have $\langle a+\rho, a+\rho\rangle < \langle g+\rho, g+\rho\rangle$;*

(b) *If $g\neq 0$ then $\langle \rho, \rho\rangle < \langle g+\rho, g+\rho\rangle$.*

Proof. If $w\in W$ then $2w\rho$ is the sum of all positive roots in Σ, for some order. It follows that

$$2w\rho = \sum_{r>0} \varepsilon(r)\, r,$$

where $\varepsilon(r)=\pm 1$. Hence $w\rho \leqq \rho$ for all $w\in W$, equality holding only if $w=1$. Let a be a weight of π, let $a=w^{-1}d$ ($w\in W$, d dominant). Then

$$\langle a+\rho, a+\rho\rangle = \langle d+w\rho, d+w\rho\rangle = \langle d+\rho, d+\rho\rangle - 2\langle \rho - w\rho, d\rangle.$$

The weight d being dominant, we see that $\langle \rho - w\rho, d\rangle \geqq 0$, whence $\langle a+\rho, a+\rho\rangle \leqq \langle d+\rho, d+\rho\rangle$, equality holding only if a is dominant. If a is dominant, then

$$\langle a+\rho, a+\rho\rangle = \langle g+\rho, g+\rho\rangle - 2\langle g-a, \rho\rangle - \langle g-a, g-a\rangle.$$

Since $g-a\geqq 0$ and since ρ is a dominant weight $\left(\text{for } \rho=\sum_{i=1}^{l} d_i \text{ with the notations of 1.7}\right)$, we have $\langle g-a, \rho\rangle \geqq 0$. This implies (a).

As to (b), we have $\langle g+\rho, g+\rho\rangle - \langle \rho, \rho\rangle = 2\langle g, \rho\rangle + \langle g, g\rangle$ and $\langle g, \rho\rangle \geqq 0$ since g is dominant and since 2ρ is a sum of positive roots.

Let π be a (non necessarily irreducible) rational representation $G\to \boldsymbol{GL}(V)$. In this situation we will say that V is a G-space. Let V' be the dual space of V, let (,) denote the canonical pairing of V and V'. For $x\in G$, $v\in V$, $v'\in V'$ the matrix element $x\mapsto (\pi(x)v, v')$ is in $k[G]$. Let D be a left invariant differential operator on $k[G]$. Define a linear transformation $D\pi(x)$ of V by

$$\big(D\pi(x)\, v, v'\big) = D\big(\pi(x)\, v, v'\big) \qquad (x\in G,\ v\in V,\ v'\in V').$$

π being as above, we call *irreducible constituents* of π the irreducible representations of G in the composition factors of a composition series of the G-space V. With these notations we have:

3.3. Proposition. (a) *If D is an invariant differential operator, then there exists a linear transformation $A(\pi, D)$ of V such that $D\pi(x) = A(\pi, D)\pi(x) = \pi(x)A(\pi, D)$ $(x \in G)$.*

(b) *If π is irreducible, there exists $\lambda(\pi, D) \in k$ such that $A(\pi, D) = \lambda(\pi, D) \cdot id$. $\lambda(\pi, D)$ depends only on the equivalence class of π.*

(c) *The eigenvalues of $A(\pi, D)$ are the $\lambda(\vartheta, D)$, ϑ running through the irreducible constituents of π.*

Proof. (a) By the invariance of D we have

$$D\pi(xy) = \pi(x) \cdot D\pi(y) = D\pi(x) \cdot \pi(y) \qquad (x, y \in G).$$

This implies (a), with $A(\pi, D) = D\pi(e)$. The first assertion in (b) is then a consequence of SCHUR's lemma and the second one is obvious.

(c) Let $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ be a composition series of the G-space V. Take a basis $(e_1, \ldots, e_n)$ of V such that $(e_1, \ldots, e_{n_i})$ (where $n_i = \dim V_i$) is a basis of V_i $(1 \leqq i \leqq r)$. It is readily seen that with respect to this basis, $A(\pi, D)$ is represented by a triangular matrix, in the diagonal of which only the $\lambda(\vartheta, D)$ occur. This implies (c).

We come now to WEYL's character formula. Let π be an irreducible rational representation of G with highest weight g. The character f_π of π is a class function on G.

We assume, as in 2.8, that $\rho \in X(T)$. For $a \in X(T)$ we then define the element h_a of $k[T]$ by

$$h_a = \sum_{w \in W} \varepsilon(w)\, e(w(a+\rho)),$$

so that h_0 is the h of 2.8. We then have (α denoting again the canonical homomorphism $k[G] \to k[T]$)

3.4. Theorem (WEYL'S *character formula*). *If k has characteristic 0, then $\alpha f_\pi = h^{-1} \cdot h_g$.*

Proof. Let C be the Casimir operator. By 3.3(b) we have $C\pi(x) = \lambda(\pi, C)\pi(x)$, whence $Cf_\pi = \lambda(\pi, C) f_\pi$. Using 2.9 we see that

$$\Delta(h \cdot \alpha f_\pi) = \big(u\langle\rho, \rho\rangle + \lambda(\pi, C)\big)\, h \cdot \alpha f_\pi. \tag{4}$$

From now on the argument is well-known (see [5], p. 376).

Let

$$h \cdot \alpha f_\pi = \sum_{a \in X(T)} m(a)\, e(a).$$

Then $m(wa)=\varepsilon(w)m(a)$ $(w\in W)$ and $m(a)\neq 0$ only if $a=b+w\rho$, where b is a weight of π and $w\in W$. Moreover $m(g+\rho)=1$. This implies that $\lambda(\pi, C)=u(\langle g+\rho, g+\rho\rangle-\langle\rho,\rho\rangle)$. From (4) we now infer that $m(a)\neq 0$ only if $a=b+w\rho$ with $\langle b+w\rho, b+w\rho\rangle=\langle g+\rho, g+\rho\rangle$, which by 3.2(a) implies $b=wg$. 3.4 then readily follows.

3.5. Proposition. (a) *(p arbitrary). Let π be an irreducible representation of G with highest weight g. Then $\lambda(\pi, C)=u(\langle g+\rho, g+\rho\rangle-\langle\rho,\rho\rangle) \bmod p$.*

(b) ($p=0$). *Let π be an irreducible representation of G of degree >1. Then $C\pi=\lambda\pi$ with $\lambda\in k$, $\lambda\neq 0$.*

Proof. (a) The argument of the proof of 3.4 gives this result in any characteristic. (b) is then a consequence of 3.2(b). (N.B. We still suppose $\rho\in X(T)$.)

We next indicate how in characteristic 0 the preceding results may be used to obtain some well-known theorems (e.g. the theorem of complete reducibility). We keep the same notations.

3.6. Proposition (char $k=0$). *Let V be the subspace of $k[G]$ spanned by the matrix elements of irreducible rational representations of degree >1. Then $k[G]$ is the direct sum of k and V (as a vector space).*

Proof. Suppose that $1\in V$, so that

$$1=\sum_i \alpha_i f_i$$

where the f_i are matrix elements of irreducible representations of degree >1, with $\alpha_i\in k^*$. We may assume the f_i to be linearly independent. Applying the Casimir operator to both sides we get from 3.5(b) a relation

$$0=\sum_i \alpha_i\lambda_i f_i,$$

where $\lambda_i\in k^*$. This is a contradiction. Hence the sum $k+V$ is direct. $k+V$ is invariant under left and right translations in $k[G]$. Let W be a subspace of $k[G]$, containing $k+V$, such that $k+V$ has finite codimension in W and that G acts irreducibly in $W/k+V$ via right translations (the existence of W follows from [4], exp. 4, no 1). Suppose that the cosets f_i mod $k+V$ span $W/k+V$. Then

$$f_i(gx)=\sum_j u_{ij}(x)f_j(g),$$

where the u_{ij} are matrix elements of an irreducible rational representation of G. Taking $g=e$ we conclude that $f_i\in k+V$. Hence $W=k+V$, which implies that $k[G]=k+V$.

3.7. *Invariant Means.* Let now G be a linear algebraic group defined over the field k, which need not be semisimple. For the moment, we do not make any hypotheses about k. A k-linear function μ on $k[G]$ is called an *invariant mean* on $k[G]$ if $\mu(1)=1$ and if μ is invariant under left and right translations. This can be expressed as follows: let $\bar{k}$ be an algebraic closure of k, then the canonical extension $\bar{\mu}$ of μ to $\bar{k}[G]=k[G]\otimes\bar{k}$ satisfies $\bar{\mu}(\gamma(g)f)=\bar{\mu}(\delta(g)f)=\bar{\mu}(f)$ for $g\in G$, $f\in\bar{k}[G]$. From 3.6 one derives the existence of an invariant mean in the situation of that proposition. However we will sketch the proof of a somewhat more general result.

3.8. Proposition. *Let G be a linear algebraic group defined over the field k. There is an invariant mean on $k[G]$ in the following cases:*

(a) *G is a torus;*

(b) char $k=0$ *and G is reductive.*

Proof. (a) Suppose that G is a torus which splits over k. Let

$$f=\sum_{a\in X(T)} m(a)\,e(a)$$

be an element of $k[G]$. Then one checks easily that $\mu(f)=m(0)$ is an invariant mean. In general, G splits over a finite separable extension l of k, which we may assume to be normal (see e.g. [2], 1.5, p. 61). One checks that the invariant mean, defined above, on $l[G]=k[G]\otimes_k l$ is invariant under the Galois group $\mathrm{Gal}(l/k)$, from which one obtains an invariant mean on $k[G]$.

(b) We recall that G is called reductive if the radical of its identity component is a torus.

First let G be connected semisimple, with k algebraically closed (and of characteristic 0). Let $V\subset k[G]$ be as in 3.6. Then if we put for $f\in k[G]$, $f=\mu(f)+v$ with $\mu(f)\in k$, $v\in V$, it follows from 3.6 that μ is an invariant mean on $k[G]$. Next if G is connected semisimple and k arbitrary, then an argument as in (a) shows that one can descend the invariant mean on $\bar{k}[G]$ to one on $k[G]$.

If G is connected reductive, let T be the radical of G. The quotient G/T is then connected semisimple and $k[G/T]$ is the subalgebra of $k[G]$ consisting of all $f\in k[G]$ such that $f(gt)=f(g)$ $(t\in T, g\in G)$ (see [9], Th. 1, p. 218). Let $\mu_{G/T}$, μ_T denote invariant means on $k[G/T]$, $k[T]$, respectively. Let β denote the canonical homomorphism $k[G]\to k[T]$. Let $f\in k[G]$,

$$f(xy)=\sum_i f_i(x)\,g_i(y) \qquad (x,y\in G).$$

Then

$$f' = \sum_i f_i \mu_T(\beta g_i)$$

lies in $k[G/T]$. Putting $\mu(f) = \mu_{G/T}(f')$ we obtain an invariant mean on $k[G]$. Finally, if G is arbitrary, let G be its identity component. There is an invariant mean on $k[G_0]$. One then easily constructs one on $k[G]$ by averaging over the finite algebraic group G/G_0.

3.9. The existence of an invariant mean on $k[G]$ has several consequences. We state some of them, without going into the details (which are standard and may be left to the reader):

(a) The theorem of complete reducibility of the rational representations of G which are defined over k. One obtains thus, in particular from 3.8a "global" proof of this theorem for reductive groups in characteristic 0.

(b) The triviality of extensions $1 \to N \to E \to G \to 1$ of algebraic groups over k, in the case that N is connected unipotent. An argument of ROSENLICHT ([10], Th. 1, p. 99) gives a regular cross-section for the homomorphism $E \to G$, from which one concludes that the extension may be described by SCHREIER's method. One then uses a reduction to the case of an abelian N.

(c) Orthogonality relations for matrix elements of irreducible representations (along the lines of [14], p. 115).

3.10. *Multiplicities.* Let G be a connected semisimple group over the algebraically closed field k of characteristic 0. Let π be an irreducible representation of G with highest weight g (in the previous notations). We denote by $m(g, a)$ the multiplicity of the weight a in π (i.e. the multiplicity of the character a of T in the restriction of π to T). One then has the following formula, due to FREUDENTHAL ([5], formula 3.1, p. 372)

$$\begin{aligned}(\langle g+\rho, g+\rho\rangle - \langle a+\rho, a+\rho\rangle)\, m(g,a) \\ = 2 \sum_{i>0,\, r>0} \langle a+i r, r\rangle\, m(g, a+i r).\end{aligned} \tag{5}$$

It is a direct consequence of (3) of §2, using the value of $\lambda(\pi, C)$ found in the proof of 3.4

(5) can be used effectively to compute multiplicities. We will use (5) in §4 for the case of G_2.

4. The Character Formula (Characteristic $p > 0$)

4.1. In this § we assume that G is a connected semisimple group over the algebraically closed field k of characteristic $p > 0$. We use the notations of the preceding paragraphs.

Let π be an irreducible representation of G, with highest weight g, let f_π be again the character of π. The argument used to prove 3.4 may be used now to show that WEYL's formula for f_π still holds, provided that $p > u\langle g+\rho, g+\rho\rangle$. However this is a rather weak result: it gives only congruences mod p for the multiplicities. By a different argument one can establish a better result.

We use the fact, recalled in 1.5, that there is an affine group scheme G_0, of finite type and smooth over $\mathbf{Z}$, such that $G = G_0 \times_{\mathbf{Z}} k$. It is known that, if char $k=0$, any irreducible representation of G comes from a representation of G_0 over $\mathbf{Z}$. Now let π be as above, let π_0 be a representation of G_0 over $\mathbf{Z}$, corresponding to an irreducible representation in characteristic 0 with highest weight g.

Reduction of π_0 modulo p gives a representation ϑ of G (observe that $G = G_0 \times_{\mathbf{Z}} k$, so that reduction modulo p makes sense). For any weight a, let $m_p(g, a)$ be the multiplicity of a in π ($m_p(g, a)$ depends only on p and not on k). For a dominant weight d let $n_p(g, d)$ denote the number of times an irreducible representation π_d of G with highest weight d occurs as an irreducible constituent of ϑ (in other words: let $[\pi]$ denote the element of the Grothendieck group of rational representations of G defined by the representation π, then

$$[\vartheta] = \sum_{d \text{ dominant}} n_p(g, d)[\pi_d]).$$

Clearly $n_p(g, g) = 1$.

4.2. Theorem. *Suppose that $\rho \in X(T)$. If $n_p(g, d) > 0$ then*

$$u\langle g+\rho, g+\rho\rangle \equiv u\langle d+\rho, d+\rho\rangle \bmod p.$$

Proof. Let $\mathbf{Z}[G_0]$ denote the ring of G_0. The Casimir operator C of G is the extension of an invariant differential operator C_0 in $\mathbf{Z}[G_0]$. We have $C_0 \pi_0 = \lambda \pi_0$, with $\lambda = u(\langle g+\rho, g+\rho\rangle - \langle \rho, \rho\rangle)$, as follows by extending $\mathbf{Z}$ to an algebraically closed field of characteristic 0 and applying 3.5(a). Reduction mod p shows that $C\vartheta = \bar{\lambda}\vartheta$ where $\bar{\lambda}$ is λ mod p. The assertion now follows from 3.3(c) and 3.5(a).

Remark. The proof shows that if the condition $\rho \in X(T)$ is not fulfilled one still has a similar congruence, provided one replaces u by „the u of a covering of G for which the condition $\rho \in X(T)$ holds".

4.3. Corollary. *If $p > u(\langle g+\rho, g+\rho\rangle - \langle \rho, \rho\rangle)$ then ϑ is irreducible.*

Direct consequence of 4.2 and 3.2(a).

From 4.3 one concludes that under the assumption of 4.3, a formula analogous to WEYL's character formula holds for a suitable "formal" character.

4.4. *Examples.* (a) From the results given in [5] for the irreducible representations of E_8 in characteristic 0 (p. 491, table E, 3rd column), one obtains easily, using 4.2, that the 8 fundamental representations of E_8 (whose degrees are 248, 3,875, 30,380, 147,250, 2,450,240, 6,696,000, 146,325,270, 6,899,079,264, respectively) remain irreducible in characteristic $p>29$.

(b) Let k be a finite field of characteristic p, let G be a connected, semisimple, simply connected algebraic group defined over k. STEINBERG has shown how the irreducible p-modular representations of the group $G(k)$ of k-rational points of G can be obtained from the irreducible rational representations of G ([11], Theorems 7.4 and 9.3, p. 45, 49). 4.2 can then be used to determine the Brauer characters of certain modular representations. An example will be discussed in more detail in 4.9.

Next we establish a result about splitting of exact sequences of G-spaces if the characteristic p is positive. G is as before.

4.5. Proposition. *Let* $0\to V'\to V\to V''\to 0$ *be an exact sequence of G-spaces. Let* g' *(resp.* g''*) run through the highest weights of the irreducible constituents of the representations of G in* V' *(resp. in* V''*). Suppose that* $u\langle g'+\rho, g'+\rho\rangle \not\equiv u\langle g''+\rho, g''+\rho\rangle \bmod p$ *for all such pairs* $\{g', g''\}$*. Then the exact sequence splits.*

Proof. Let π' resp. π'' be the representations of G in V' resp. V''. Put $W=\mathrm{Hom}_k(V'', V')$. The group G acts on W on the left via π' and on the right via π''. A well-known argument shows that it suffices to prove the following: let f be a morphism (of algebraic varieties) of G into W such that

$$f(x\,y)=x\cdot f(y)+f(x)\cdot y \qquad (x, y\in G), \tag{6}$$

then there exists $w\in W$ with

$$f(x)=x\cdot w-w\cdot x.$$

Let C be the Casimir operator. With the notations of 3.3(a), put $A'=A(\pi', C)$, $A''=A(\pi'', C)$. From 3.3(a) it follows that $A'(x\cdot w)=x\cdot(A'w)$, $(w\cdot x)A''=(wA'')\cdot x$ ($x\in G$, endomorphisms of V' and V'' acting in W in the obvious manner). Also the assumption about weights, together with 3.3(c) and 3.5(a), shows that no eigenvalue of A' equals one of A''.

Apply C to both sides of (6). Using invariance one gets

$$C\,f(x\,y)=A'(x\cdot f(y))+C\,f(x)\cdot y=x\cdot C\,f(y)+(f(x)\cdot y)\,A'',$$

whence, putting $v=(Cf)(e)$,

$$A'f(x)-f(x)A''=x\cdot v-v\cdot x.$$

Now apply the following elementary lemma:

4.6. Lemma. *Let V', V'' be two finite dimensional vector spaces over the algebraically closed field k. Let A', A'' be linear transformations of V' and V'' such that no eigenvalue of A' equals one of A''. Then the linear transformation of* $\mathrm{Hom}_k(V'', V')$, *which maps T into $A'T-TA''$ is bijective.*

From this lemma one concludes that there exists $w\in W$ such that $v=A'w-wA''$ and that $f(x)=x\cdot w-w\cdot x$, finishing the proof of 4.5. As to the proof of the lemma, it suffices to prove injectivity. So let $T\in\mathrm{Hom}_k(V'', V')$, $A'T=TA''$. Let v be an eigenvector of A'' in V''. Then the assumption about A' and A'' implies that $Tv=0$. One then replaces V'' by V''/kv and uses induction on $\dim V''$.

4.7. Proposition. *Let π be a rational representation of G in V. Then the G-space V is a direct sum*

$$V=\sum_{i=1}^{n} V_i$$

of G-spaces V_i with the following properties: (a) *if g and g' are highest weights of two irreducible constituents of the restriction of π to V_i $(1\leqq i\leqq n)$ then $u\langle g+\rho, g+\rho\rangle\equiv u\langle g'+\rho, g'+\rho\rangle \bmod p$,* (b) *suppose $i\neq j$ and let g (resp. g') be a highest weight of an irreducible constituent of the restriction of π to V_i (resp. V_j). Then $u\langle g+\rho, g+\rho\rangle\not\equiv u\langle g'+\rho, g'+\rho\rangle \bmod p$.*

Proof. Induction on the degree of π. For an irreducible π there is nothing to prove. So suppose π reducible. Let V be the space of π. There is then an exact sequence $0\to V'\to V\to V''\to 0$ of G-spaces, where $V'\neq 0$, $V''\neq 0$, and where V'' is irreducible. Let g'' be the highest weight of the representation of G in V''.

By induction, there is a decomposition $V'=\Sigma V_i$, with the properties of the proposition. Let V_1' be the direct sum of those V_i, for which $u\langle g+\rho, g+\rho\rangle\not\equiv u\langle g''+\rho, g''+\rho\rangle \bmod p$ for all highest weights g of irreducible constituents of the restriction of π to V_i. We then have an exact sequence $0\to V_1'\to V\to V_1''\to 0$ of G-spaces, which splits by 4.5. This implies the assertion.

4.8. Corollary. *Let π be an indecomposable rational representation of G. Let g and g' be the highest weights of two irreducible constituents of π. Then $u\langle g+\rho, g+\rho\rangle\equiv u\langle g'+\rho, g'+\rho\rangle \bmod p$.*

4.9. *An Explicit Example.* To terminate, we give a detailed discussion of the irreducible representations of G_2 in characteristic 3. Much about this (in particular the degrees of the irreducible representations) is contained in [11]. We want to show what can be done in this particular case with the methods of the present note.

G_2 is simply connected, so we have $u=1$. The character group $X(T)$ is spanned by 3 elements x_0, x_1, x_2, whose sum is 0, such that $r_1=x_0$, $r_2=x_1-x_0$. Moreover we have $d_1=x_0+x_1=2r_1+r_2$, $d_2=x_1-x_2=3r_1+2r_2$. The roots are $\pm x_i$, x_i-x_j $(i\neq j)$. We have $\langle x_i, x_i\rangle=1$, $\langle x_i, x_j\rangle=-\frac{1}{2}$. The elements of W act on $X(T)$ as follows: $w\cdot x_i=ex_{\pi(i)}$, where π is a permutation of $\{0, 1, 2\}$ and $e=\pm 1$ (see [4], exp. 19, p. 11).

In order to find the characters of the irreducible representations of G_2 in characteristic 3, it suffices, by a theorem of STEINBERG ([11], Th. 6.1, p. 44) to determine those whose highest weights are $g=id_1+jd_2$ with $0\leqq i, j\leqq 2$.

We have first determined, in characteristic 0, the multiplicities $m(g, d)$ for such g and d dominant $\leqq g$. The method we follow is that of FREUDENTHAL ([5], II) based on formula (5) (of 3.10). The details of the calculation, which is not very laborious, are omitted. The results are given in Table 1. The rows and the first columns are labeled by the pairs ij, corresponding to the dominant weight id_1+jd_2. The value of $m(g, d)$ is in the intersection of row g and column d (zeros on the empty places). The last column contains the numbers $\langle g+\rho, g+\rho\rangle-\langle\rho, \rho\rangle$.

Table 1. *Multiplicities in characteristic 0*

	00	10	01	20	11	30	02	21	40	12	31	50	03	22	$\langle g+\rho, g+\rho\rangle$ $-\langle\rho, \rho\rangle$
00	1														0
10	1	1													6
01	2	1	1												12
20	3	2	1	1											14
11	4	4	2	2	1										21
30	5	4	3	2	1	1									24
02	5	3	3	2	1	1	1								30
21	9	8	6	5	3	2	1	1							32
40	8	7	5	5	3	2	1	1	1						36
12	10	10	7	7	5	3	2	2	1	1					42
31	16	14	12	10	7	6	4	3	2	1	1				45
50	12	11	9	8	6	5	3	3	2	1	1	1			50
03	9	7	7	5	4	4	3	2	1	1	1	0	1		54
22	21	19	16	15	11	9	7	6	4	3	2	1	1	1	56

Now let G be the algebraic group of type G_2 over an algebraically closed field k of characteristic 3. There is an inseparable isogeny $\sigma: G \to G$ of degree 3 such that $\sigma(T) = T$ and such that the induced homomorphism $\sigma^*: X(T) \to X(T)$ maps d_1 onto d_2 and d_2 onto $3d_1$ (see [4], exp. 21). Let π_{ij} be an irreducible representation of G of highest weight $id_1 + jd_2$, let e_{ij} be its degree. Clearly $\pi_{ij} \circ \sigma$ is then one with highest weight $3jd_1 + id_2$ and with the same degree. On the other hand, by STEINBERG's theorem, quoted above, we know that $\pi_{3j,i}$ is equivalent to the tensor product $s(\pi_{j0}) \otimes \pi_{0i}$, where s is the automorphism $\lambda \mapsto \lambda^3$ of k. It follows, in particular, that $e_{ij} = e_{j0} e_{0i}$, whence $e_{ij} = e_{i0} e_{j0}$. It follows also that e_{10} and e_{20} determine all degrees e_{ij} for $0 \leqq i, j \leqq 2$ (and even for arbitrary i and j, by STEINBERG's theorem). From 4.2 we conclude (using the last column of Table 1) that e_{20} is as in characteristic 0, and Table 1 then easily implies that $e_{20} = 27$. On the other hand it follows from Table 1 that e_{10} is either 6 or 7 (observe that d_1 has 6 conjugates under W). So the symmetric part of $\pi_{10} \otimes \pi_{10}$ has dimension 21 or 28. However this must contain the representation π_{20} of dimension 27 (by highest weights), whence $e_{10} = 7$ (as in characteristic 0).

One now concludes, by degrees, that π_{ij} is equivalent to $\pi_{i0} \otimes \pi_{0j}$. From this one derives easily a table of multiplicities in characteristic 3. Instead of this table, we give in Table 2 the (equivalent) table of "decomposition numbers" $n_3(g, d)$ (see 4.1).

Table 2. *Decomposition numbers*

	00	10	01	20	11	30	02	21	40	12	31	50	03	22
00	1													
10	0	1												
01	0	1	1											
20	0	0	0	1										
11	1	1	1	0	1									
30	0	1	2	0	1	1								
02	1	0	0	0	1	0	1							
21	0	0	0	0	0	0	0	1						
40	1	0	1	0	2	0	1	0	1					
12	0	0	2	0	1	1	1	0	0	1				
31	1	0	3	0	2	2	1	0	1	1	1			
50	0	0	0	0	0	0	0	1	0	0	0	1		
03	0	0	2	0	0	2	0	0	0	1	1	0	1	
22	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Let now G be the group of type G_2 defined over the field F_3 with 3 elements. Put $H = G(F_3)$, the finite group of rational points of G, with order $h = 3^6(3^2-1)(3^6-1) = 2^6 \cdot 3^6 \cdot 7 \cdot 13$. We will use the preceding results to determine the 3-modular characters (in BRAUER's sense) of H.

We first determine the 3-regular classes of H. According to [11] (§3) they come from the $t \in T$ such that $w \cdot t = t^3$ for some $w \in W$. We describe the elements of T by triples $(t^{x_0}, t^{x_1}, t^{x_2})$ in $(k^*)^3$, where the x_i are as in the beginning of this section. In this description, it is easy to work with the action of W on T. The result is that the 9 3-regular classes of H are described by the triples $(1, 1, 1)$, $(-1, -1, 1)$, $(\varepsilon_8^2, \varepsilon_8^2, \varepsilon_8^4)$, $(\varepsilon_8^2, \varepsilon_8^6, 1)$, $(\varepsilon_7, \varepsilon_7^2, \varepsilon_7^4)$, $(\varepsilon_8, \varepsilon_8^3, \varepsilon_8^4)$, $(\varepsilon_8, \varepsilon_8^5, \varepsilon_8^2)$, $(\varepsilon_{13}, \varepsilon_{13}^3, \varepsilon_{13}^9)$, $(\varepsilon_{13}^2, \varepsilon_{13}^6, \varepsilon_{13}^5)$, where ε_n denotes a primitive n^{th} root of unity in k^*. We label these classes by their orders as 1, 2, 4, 4′, 7, 8, 8′, 13, 13′ respectively. Denote by φ the isomorphism of the group of roots of unity of order $7 \cdot 8 \cdot 13$ in k^* into C^* defined by $\varphi(\varepsilon_n) = e^{2\pi i \cdot n^{-1}}$ for $n = 7, 8, 13$. The Brauer characters of H are computed using this homomorphism φ (and depend on it). Using the results of [11] (§7) one easily gets the modular character table from Tables 1 and 2. To each highest weight $id_1 + jd_2$ $(0 \leqq i, j \leqq 2)$ there corresponds such a character χ_{ij}. Moreover we have (as follows from the preceding remarks) $\chi_{11} = \chi_{10}\chi_{01}$, $\chi_{21} = \chi_{20}\chi_{01}$, $\chi_{12} = \chi_{10}\chi_{02}$, $\chi_{22} = \chi_{20}\chi_{02}$. So one only has to determine $\chi_{01}, \chi_{10}, \chi_{02}, \chi_{20}$. This is easy, using Tables 1 and 2. The result is given in Table 3.

Table 3. *Modular characters of* $G_2(F_3)$

Character	Class								
	1	2	4	4′	7	8	8′	13	13′
χ_{00}	1	1	1	1	1	1	1	1	1
χ_{10}	7	−1	−1	3	0	−1	1	$\frac{1}{2}(1+\sqrt{13})$	$\frac{1}{2}(1-\sqrt{13})$
χ_{01}	7	−1	3	−1	0	1	−1	$\frac{1}{2}(1-\sqrt{13})$	$\frac{1}{2}(1+\sqrt{13})$
χ_{11}	49	1	−3	−3	0	−1	−1	−3	−3
χ_{20}	27	3	−1	3	−1	1	−1	1	1
χ_{02}	27	3	3	−1	−1	−1	1	1	1
χ_{21}	189	−3	−3	−3	0	1	1	$\frac{1}{2}(1-\sqrt{13})$	$\frac{1}{2}(1+\sqrt{13})$
χ_{12}	189	−3	−3	−3	0	1	1	$\frac{1}{2}(1+\sqrt{13})$	$\frac{1}{2}(1-\sqrt{13})$
χ_{22}	729	9	−3	−3	1	−1	−1	1	1

Apart from sign, the values of χ_{02}, χ_{20} and χ_{22} are also contained in [11] (11.3). The modular characters depend on the choice of the isomorphism φ. If one makes a different choice of φ, then either the characters remain unchanged, or $\sqrt{13}$ gets replaced $-$by $\sqrt{13}$ throughout Table 3 ($\sqrt{13}$ occurs because of a Gaussian sum).

References

1. BOREL, A., and T. A. SPRINGER: Rationality properties of linear algebraic groups, II. (To appear.)
2. —, and J. TITS: Groupes réductifs. Publ. Math. I. H. E. S. No. 27, 55–151 (1965).

3. CHEVALLEY, C.: Sur certains groupes simples. Tôhoku Math. J. (2), **7**, 14—66. (1955).
4. — Séminaire sur la classification des groupes de Lie algébriques, 2 vol. Paris 1958.
5. FREUDENTHAL, H.: Zur Berechnung der Charaktere der halbeinfachen Lieschen Gruppen I. Proc. Kon. Ak. v. Wet. Amsterdam, Series A **57**, 369—376 (1954); II., ibid. 487—491.
6. HARISH-CHANDRA: The characters of semi-simple Lie groups. Trans. Amer. Math. Soc. **83**, 98—163 (1956).
7. — Differential operators on a semi-simple Lie algebra. Amer. J. of Math. **79**, 87—120 (1957).
8. JACOBSON, N.: Lie Algebras. Intersc. tracts in pure math., 10. New York: Interscience publ. 1962.
9. ROSENLICHT, M.: On quotient varieties and the affine embedding of certain homogeneous spaces. Trans. Amer. Math. Soc. **101**, 211—223 (1961).
10. — Questions of rationality for solvable algebraic groups over nonperfect fields. Annali di Mat. (IV), **61**, 97—120 (1963).
11. STEINBERG, R.: Representations of algebraic groups. Nagoya Math. J. **22**, 33—56 (1963).
12. — Regular elements of semisimple algebraic groups. Publ. Math. I. H. E. S. No. 25, 49—80 (1965).
13. WEYL, H.: Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen, in *Selecta*, p. 262—366. Basel-Stuttgart: Birkhäuser Verlag 1956.
14. — The classical groups. Princeton: Princeton University Press 1946.

T. A. SPRINGER
Mathematisch Instituut
der Rijksuniversiteit
Universiteitscentrum de Uithof
Utrecht, Niederlande

(Received October 20, 1967)