Inventiones math. 16, 237–248 (1972)

Satake Compactification and Extension of Holomorphic Mappings

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1. Introduction

Let Y be a complex space and M a complex subspace of Y such that

(1) M is hyperbolic, i.e., the intrinsic pseudo-distance d_M is a distance (see [3]);

(2) the closure $\bar{M}$ of M in Y is compact;

(3) given a point p of the boundary $\partial M = \bar{M} - M$ and a neighborhood U of p in Y, there exists a neighborhood V of p in Y such that $\bar{V} \subset U$ and the distance between $M \cap (Y - U)$ and $M \cap V$ with respect to d_M is positive[1].

As in [2] we say that M is *hyperbolically imbedded* if these three conditions are satisfied.

Let X be a complex space and A a closed complex subspace of X. We consider the problem of extending a holomorphic mapping $X - A \to M$ to a holomorphic mapping $X \to Y$. Kwack [5] has shown that this is possible if M is compact (so that conditions (2) and (3) are vacuous) and X is non-singular. In [3] the problem was affirmatively solved when both X and A are non-singular (whether M is compact or not). This result was further extended in [2] to the case where X is non-singular and the singularities of A are normal crossing. On the other hand, there are simple examples [3; p. 100] which answer the problem negatively in general when X is singular.

The purpose of the present paper is to solve the problem affirmatively when $X - A$ is the quotient of a symmetric bounded domain $\mathscr{D}$ by an arithmetic discrete group Γ and X is its compactification in the sense of Satake, Baily and Borel, and Pyatetzki-Shapiro.

On the other hand, it was shown in [4] that if M is also the quotient of a symmetric bounded domain $\mathscr{D}'$ by an arithmetic discrete group Γ' and Y is its compactification, then M is hyperbolically imbedded in Y.

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* Partially supported by NSF Grant GP 16651.

1 Condition (3) is equivalent to the following: (3'). If p and q are boundary points of M and if $\{p_n\}$ and $\{q_n\}$ are sequences in M such that $p_n \to p$, and $q_n \to q$, and $d_M(p_n, q_n) \to 0$, then $p = q$.

(Technically speaking, when the action of Γ' on $\mathcal{D}'$ is not free, the distance d_M in conditions (1) and (3) has to be replaced by the distance d'_M coming from the intrinsic distance $d_{\mathcal{D}'}$ of the domain $\mathcal{D}'$ as explained in [4].) As an immediate consequence we obtain the following result. Every holomorphic mapping of $\mathcal{D}/\Gamma$ into $\mathcal{D}'/\Gamma'$ which is (locally) liftable to a mapping from $\mathcal{D}$ into $\mathcal{D}'$ can be extended to a holomorphic mapping of the compactification $\mathcal{D}^*/\Gamma$ into the compactification $\mathcal{D}'^*/\Gamma'$. Moreover, the extended mapping sends each boundary component of $\mathcal{D}^*/\Gamma$ into a boundary component of $\mathcal{D}'^*/\Gamma'$. This generalizes the result of Satake in [8] where the given mapping $\mathcal{D} \to \mathcal{D}'$ is assumed to come from a homomorphism between the automorphism groups of the domains $\mathcal{D}$ and $\mathcal{D}'$.

2. Siegel Domains [6]

To fix our notation we review quickly Pyatetzki-Shapiro's theory of Siegel domains. Let V be an n-dimensional real vector space and Ω a convex cone in V, i.e., an open non-empty convex subset such that i) $t\,y \in \Omega$ whenever $y \in \Omega$ and $t > 0$, and ii) it contains no straight lines.

The open subset T_Ω of $V_{\mathbb{C}} = V + iV$ defined by

$$T_\Omega = \{x + i\,y \in V_{\mathbb{C}};\ y \in \Omega\}$$

is called the *tube domain* or the *Siegel domain of the first kind* associated to Ω.

An *Ω-hermitian form* on an m-dimensional complex vector space W is a mapping H: $W \times W \to V_{\mathbb{C}}$ such that

i) $H(\alpha u + \beta v, w) = \alpha(H u, w) + \beta H(v, w)$ for $u, v, w \in W$ and $\alpha, \beta \in \mathbb{C}$;

ii) $H(u, v) = \overline{H(v, u)}$ for $u, v \in W$;

iii) $H(u, u) \in \bar{\Omega}$ for $u \in W$,

where $\bar{\Omega}$ denotes the topological closure of Ω;

iv) $H(u, u) = 0$ only if $u = 0$.

The open subset $\mathcal{D}(H, \Omega)$ of $V_{\mathbb{C}} \times W$ defined by

$$\mathcal{D}(H, \Omega) = \{(x + i\,y, w) \in V_{\mathbb{C}} \times W;\ y - H(w, w) \in \Omega\}$$

is called the *Siegel domain of the second kind* associated to H and Ω.

In order to define the Siegel domain of the third kind, let $\mathcal{K}$ be the set of all complex antilinear mappings p: $W \to W$ such that

i) $H(p u, v) = H(p v, u)$ for $u, v \in W$;

ii) $H(u, u) - H(p u, p u) \in \bar{\Omega}$ for $u \in W$;

iii) $H(u, u) \neq H(p u, p u)$ if $u \neq 0$.

The set of all complex antilinear mappings p satisfying only (i) forms a complex vector space in which $\mathcal{K}$ is a bounded domain. If $p \in \mathcal{K}$ and I denotes the identity transformation of W, then $I+p$ is a real linear automorphism of W and we can define a mapping L_p: $W \times W \to V_{\mathbb{C}}$ by setting

$$L_p(u, v) = H\big(u, (I+p)^{-1} v\big) \quad \text{for } u, v \in W.$$

Let $\mathcal{F}$ be a bounded domain in a complex vector space U and φ a holomorphic mapping from $\mathcal{F}$ into $\mathcal{K}$. The open subset $\mathcal{D}(H, \Omega, \mathcal{F}, \varphi)$ of $U \times V_{\mathbb{C}} \times W$ defined by

$$\mathcal{D}(H, \Omega, \mathcal{F}, \varphi) = \big\{(t, z, w) \in U \times V_{\mathbb{C}} \times W;\ t \in \mathcal{F}, \operatorname{Im}(z) - \operatorname{Re}\big(L_{\varphi(t)}(w, w)\big) \in \Omega\big\}$$

is called the Siegel domain of the third kind associated to $H, \Omega, \mathcal{F}$, and φ.

Let $\mathcal{O}$ be an open set in $\mathcal{F}$. For each fixed element r of Ω, a *cylindrical set* with base $\mathcal{O}$ is defined by

$$\mathcal{D}_r(\mathcal{O}) = \big\{(t, z, w) \in \mathcal{D};\ t \in \mathcal{O}, \operatorname{Im}(z) - \operatorname{Re}\big(L_{\varphi(t)}(w, w)\big) - r \in \Omega\big\}.$$

The natural projection $U \times V_{\mathbb{C}} \times W \to U$ induces a fibering of $\mathcal{D} = \mathcal{D}(H, \Omega, \mathcal{F}, \varphi)$ over $\mathcal{F}$. Let G be the group of holomorphic automorphisms of $\mathcal{D}$ that preserve this fibering. Let G' be the group of holomorphic transformations of the base $\mathcal{F}$. We denote the natural homomorphism from G to G' by h. Let Z be the subgroup of G consisting of *parallel translations*, i.e., automorphisms of the following type:

$$\begin{aligned} &t \to t \\ &z \to z + a + 2iH(w, b) + iH\big((I + \varphi(t))\, b, b\big) \\ &w \to w + b + \varphi(t)\, b, \end{aligned}$$

where $a \in V$, $b \in W$.

Let Γ be a discrete subgroup of the largest connected group of holomorphic transformations of $\mathcal{D}$. The base domain $\mathcal{F}$ is said to be *Γ-rational* if $Z/(Z \cap \Gamma)$ is compact and $h(G \cap \Gamma)$ is a discrete subgroup of G'.

3. Compactification [1, 6, 7, 9]

Let $\mathcal{D}$ be a symmetric bounded domain in $\mathbb{C}^N$ in the so-called Harish-Chandra realization. Let $\bar{\mathcal{D}}$ be the topological closure of $\mathcal{D}$ and put $\partial\mathcal{D} = \bar{\mathcal{D}} - \mathcal{D}$. The topological boundary $\partial\mathcal{D}$ is a disjoint union of the so-called boundary components. Each boundary component $\mathcal{F}$ is also a symmetric bounded domain. If $\mathcal{F}'$ is another boundary component of $\mathcal{D}$ and if $\mathcal{F}' \subset \partial\mathcal{F}$, then $\mathcal{F}'$ is a boundary component of $\mathcal{F}$ also. With respect to each fixed boundary component $\mathcal{F}$ of $\mathcal{D}$, the domain $\mathcal{D}$ can be biholomorphically identified with a Siegel domain of the third kind $\mathcal{D}(H, \Omega, \mathcal{F}, \varphi)$.

Let Γ be an arithmetically defined discrete subgroup of the largest connected group of holomorphic automorphisms of $\mathscr{D}$. A boundary component $\mathscr{F}$ of $\mathscr{D}$ is said to be *Γ-rational* if it is Γ-rational as a base domain of the fibering $\mathscr{D} \to \mathscr{F}$ (see § 2). Let $\mathscr{B}$ denote the union of all Γ-rational boundary components of $\mathscr{D}$ and set

$$\mathscr{D}^* = \mathscr{D} \cup \mathscr{B}.$$

The action of Γ on $\mathscr{D}$ extends to $\mathscr{D}^*$ in a natural manner and

$$\mathscr{D}^*/\Gamma = (\mathscr{D}/\Gamma) \cup (\mathscr{B}/\Gamma).$$

Let $\eta: \mathscr{D}^* \to \mathscr{D}^*/\Gamma$ denote the natural projection. We shall now introduce a topology in $\mathscr{D}^*/\Gamma$. For each point of $\mathscr{D}/\Gamma$, a basis of its neighborhood system is given by its neighborhood system in $\mathscr{D}/\Gamma$ with the usual quotient topology. For a point p in $\mathscr{B}/\Gamma$, we construct a basis of its neighborhood system as follows. Assume $p \in \eta(\mathscr{F})$ and let $\tilde{p} \in \mathscr{F}$ be a point such that $\eta(\tilde{p}) = p$. Consider the family of all Γ-rational boundary components $\mathscr{E}$ of $\mathscr{D}$ such that $\mathscr{F} \subset \partial\mathscr{E}$. It is known that there are only a finite number of Γ-equivalence classes in this family. Let $\mathscr{F}_1, \ldots, \mathscr{F}_m$ be a system of representatives for these Γ-equivalence classes. Thus the family

$$\{\gamma(\mathscr{F}_i);\ \gamma \in \Gamma \text{ and } i = 1, \ldots, m\}$$

exhausts the rational boundary components $\mathscr{E}$ of $\mathscr{D}$ such that $\mathscr{F} \subset \partial\mathscr{E}$. Let $\mathscr{O}$ be an open neighborhood of $\tilde{p}$ in $\mathscr{F}$. Considering $\mathscr{D}$ as a Siegel domain $\mathscr{D}(H, \Omega, \mathscr{F}, \varphi)$ of the third kind, we consider a cylindrical set $\mathscr{D}_r(\mathscr{O})$ in $\mathscr{D}$, where $r \in \Omega$. Since each $\mathscr{F}_i$ is also a Siegel domain $\mathscr{F}_i = \mathscr{D}(H_i, \Omega_i, \mathscr{F}, \varphi_i)$ of the third kind, we can speak of the cylindrical set $\mathscr{F}_{i, r_i}(\mathscr{O})$ in $\mathscr{F}_i$, where $r_i \in \Omega_i$. Put

$$\tilde{\mathscr{U}} = \mathscr{O} \cup \mathscr{D}_r(\mathscr{O}) \cup \mathscr{F}_{1, r_1}(\mathscr{O}) \cup \cdots \cup \mathscr{F}_{m, r_m}(\mathscr{O})$$

and

$$\mathscr{U} = \eta(\tilde{\mathscr{U}}).$$

We take the family of $\mathscr{U}$ with varying $\mathscr{O}, r, r_1, \ldots, r_m$ as a basis for the open neighborhood system for p.

The topology thus introduced in $\mathscr{D}^*/\Gamma$ by Pyatetzki-Shapiro is easily seen to be at least as coarse as the one defined by Baily and Borel. It has been recently established by Borel that the two topologies coincide, i.e., the topology of Pyatetzki-Shapiro is also Hausdorff. This fact is essential for Theorem 2 since our proof is based on Pyatetzki-Shapiro's topology. (For Theorem 1, it suffices to know that Pyatetzki-Shapiro's topology is at least as coarse as the one defined by Baily and Borel.)

4. Lemmas on Siegel Domains

Lemma 4.1. *Let Ω be a convex cone in an n-dimensional real vector space V and T_Ω the tube domain associated to Ω. Let $x_1, y_1, y \in V$ such that $y \in \Omega$ and $y_1 \in \bar{\Omega}$. Define two curves σ_1 and σ_2 in T_Ω by*

$$\sigma_1(s) = i\,y_1 + i\,s\,y, \qquad \sigma_2(s) = x_1 + i\,y_1 + i\,s\,y \qquad \text{for } s > 0.$$

Then

$$\lim_{s\to\infty} d_{T_\Omega}(\sigma_1(s), \sigma_2(s)) = 0,$$

where d_{T_Ω} denotes the intrinsic distance of the domain T_Ω.

Proof. We shall first prove the lemma in the special case where $y_1 = 0$. For s large, define a mapping h_s from the open unit disk $\Delta = \{\zeta \in \mathbb{C};\ |\zeta| < 1\}$ into $V_{\mathbb{C}}$ by

$$h_s(\zeta) = i\,s\,y + i\,c\,s\,\zeta\,x_1, \qquad \zeta \in \Delta,$$

where c is a positive constant such that $y \pm c\,x_1 \in \Omega$. Since

$$\operatorname{Im}(h_s(\zeta)) = s(y + c(\operatorname{Re}\zeta)\,x_1)$$

is in Ω, $h_s(\zeta)$ is in T_Ω for all $\zeta \in \Delta$. Since h_s is distance-decreasing, we have

$$d_{T_\Omega}(\sigma_1(s), \sigma_2(s)) = d_{T_\Omega}(h_s(0), h_s(-i/c\,s)) \leqq d_\Delta(0, -i/c\,s) = \log\frac{c\,s+1}{c\,s-1}.$$

The general case will be reduced to the special case $y_1 = 0$. Define a mapping h from T_Ω into itself by

$$h(z) = z + i\,y_1.$$

Since h is distance-decreasing, we have

$$\begin{aligned} d_{T_\Omega}(\sigma_1(s), \sigma_2(s)) &= d_{T_\Omega}(h(i\,s\,y), h(x_1 + i\,s\,y)) \leqq d_{T_\Omega}(i\,s\,y, x_1 + i\,s\,y) \\ &\leqq d_\Delta(0, -i/c\,s) = \log\frac{c\,s+1}{c\,s-1}. \quad \text{QED.} \end{aligned}$$

Lemma 4.2. *With the same notations as in Lemma 4.1, let $z_1 \in T_\Omega$ and $y \in \Omega$. Define two curves σ_1 and σ_2 in T_Ω by*

$$\sigma_1(s) = i\,s\,y, \qquad \sigma_2(s) = z_1 + i\,s\,y \qquad \text{for } s > 0.$$

Then

$$\lim_{s\to\infty} d_{T_\Omega}(\sigma_1(s), \sigma_2(s)) = 0.$$

Proof. Put $z_1 = x_1 + i\,y_1$ with $x_1 \in V$ and $y_1 \in \Omega$. Consider the curve σ_2' defined by

$$\sigma_2'(s) = i\,y_1 + i\,s\,y.$$

Since $d_{T_\Omega}(\sigma_2'(s), \sigma_2(s)) \to 0$ as $s \to \infty$ by Lemma 4.1, it suffices to prove that $d_{T_\Omega}(\sigma_1(s), \sigma_2'(s)) \to 0$ as $s \to \infty$. In other words, we can assume that $x_1 = 0$.

If y_1 and y are linearly independent, let P be the real 2-dimensional plane in V spanned by y_1 and y. If y_1 and y are linearly dependent, let P be any plane in V containing y_1 and y. Then $P \cap \Omega$ is a convex cone in the 2-dimensional vector space P. If we take two independent vectors p_1 and p_2 on the boundary of the cone $P \cap \Omega$, then

$$P \cap \Omega = \{\alpha p_1 + \beta p_2;\ \alpha > 0, \beta > 0\}.$$

Let $\mathscr{H}$ be the upper-half plane in $\mathbb{C}$. Define a mapping $g\colon \mathscr{H} \times \mathscr{H} \to T_\Omega$ by

$$g(\zeta_1, \zeta_2) = \zeta_1 p_1 + \zeta_2 p_2 \qquad \text{for } (\zeta_1, \zeta_2) \in \mathscr{H} \times \mathscr{H}.$$

Then g is an injective holomorphic mapping which sends the cone

$$\{(\zeta_1, \zeta_2) \in \mathscr{H} \times \mathscr{H};\ \operatorname{Re}(\zeta_1) = \operatorname{Re}(\zeta_2) = 0\}$$

onto the cone $i(P \cap \Omega)$. Let

$$i\,\tilde{y}_1 = g^{-1}(i\,y_1), \qquad i\,\tilde{y} = g^{-1}(i\,y).$$

Then

$$i\,\tilde{y}_1 = (i\,\alpha, i\,\beta), \qquad i\,\tilde{y} = (i\,\gamma, i\,\delta),$$

where α, β, γ, and δ are positive numbers determined by

$$y_1 = \alpha p_1 + \beta p_2, \qquad y = \gamma p_1 + \delta p_2.$$

Let

$$\tilde{\sigma}_1(s) = g^{-1}(\sigma_1(s)) = i\,s\,\tilde{y} = (i\,s\,\gamma, i\,s\,\delta),$$
$$\tilde{\sigma}_2(s) = g^{-1}(\sigma_2(s)) = i\,\tilde{y}_1 + i\,s\,\tilde{y} = \big(i(\alpha + s\,\gamma), i(\beta + s\,\delta)\big).$$

Since g is distance-decreasing, it suffices to show that

$$\lim_{s \to \infty} d_{\mathscr{H} \times \mathscr{H}}(\tilde{\sigma}_1(s), \tilde{\sigma}_2(s)) = 0.$$

But

$$\begin{aligned} d_{\mathscr{H} \times \mathscr{H}}(\tilde{\sigma}_1(s), \tilde{\sigma}_2(s)) &= d_{\mathscr{H} \times \mathscr{H}}\{(i\,s\,\gamma, i\,s\,\delta), \big(i(\alpha + s\,\gamma), i(\beta + s\,\delta)\big)\} \\ &= \operatorname{Max}\{d_{\mathscr{H}}(i\,s\,\gamma, i(\alpha + s\,\gamma)), d_{\mathscr{H}}(i\,s\,\delta, i(\beta + s\,\delta))\} \\ &= \operatorname{Max}\left\{\log \frac{\alpha + s\,\gamma}{s\,\gamma}, \log \frac{\beta + s\,\delta}{s\,\delta}\right\}. \end{aligned}$$

Therefore, $d_{\mathscr{H} \times \mathscr{H}}(\tilde{\sigma}_1(s), \tilde{\sigma}_2(s)) \to 0$ as $s \to \infty$. QED.

Lemma 4.3. *Let $\mathscr{D} = \mathscr{D}(H, \Omega, \mathscr{F}, \varphi)$ be a Siegel domain of the third kind. Given $y \in \Omega$, $t' \in \mathscr{F}$, and $(t'', z, w) \in \mathscr{D}$, define curves:*

$$\sigma_1(s) = (t', i\,s\,y, 0), \qquad \sigma_2(s) = (t', z + i\,s\,y, 0)$$
$$\sigma_3(s) = (t'', z + i\,s\,y, 0), \quad \sigma_4(s) = (t'', z + i\,s\,y, w) \quad \textit{for } s > 0.$$

Then

(1) $\lim_{s \to \infty} d_{\mathscr{D}}(\sigma_j(s), \sigma_{j+1}(s)) = 0$ *for $j = 1$ and $j = 3$;*

(2) $d_{\mathscr{D}}(\sigma_2(s), \sigma_3(s)) = d_{\mathscr{F}}(t', t'')$ *for all $s > 0$.*

Proof. (1) To prove this for $j=1$, define a mapping $h\colon T_\Omega \to \mathscr{D}$ by

$$h(z') = (t', z', 0) \quad \text{for } z' \in T_\Omega.$$

Since h is distance-decreasing, the result follows immediately from Lemma 4.2.

To prove this for $j=3$, define a mapping $h_s\colon \Delta \to \mathscr{D}$ by

$$h_s(\zeta) = (t'', z + i\, s\, y, c\sqrt{s}\,\zeta\, w) \quad \text{for } \zeta \in \Delta,$$

where c is some positive constant such that

$$y - c^2 \operatorname{Re}(L_{\varphi(t'')}(w, w)) \in \Omega.$$

(This condition insures that for large s, $h_s(\Delta)$ is in $\mathscr{D}$.) Then

$$d_{\mathscr{D}}(\sigma_3(s), \sigma_4(s)) = d_{\mathscr{D}}(h_s(0), h_s(1/c\sqrt{s})) \leqq d_\Delta(0, 1/c\sqrt{s}) = \log \frac{c\sqrt{s}+1}{c\sqrt{s}-1}.$$

(2) Define a mapping $h\colon \mathscr{F} \to \mathscr{D}$ by

$$h(t) = (t, z + i\, s\, y, 0) \quad \text{for } t \in \mathscr{F}.$$

Since h is distance-decreasing,

$$d_{\mathscr{D}}(\sigma_2(s), \sigma_3(s)) \leqq d_{\mathscr{F}}(t', t'').$$

Since the projection from $\mathscr{D}$ to $\mathscr{F}$ is distance-decreasing,

$$d_{\mathscr{D}}(\sigma_2(s), \sigma_3(s)) \geqq d_{\mathscr{F}}(t', t''). \quad \text{QED.}$$

Lemma 4.4. *Let $\mathscr{D} = \mathscr{D}(H, \Omega, \mathscr{F}, \varphi)$ be a Siegel domain of the third kind. Let Γ be a discrete subgroup of the largest connected group of holomorphic transformations of the domain $\mathscr{D}$. If $\mathscr{F}$ is Γ-rational, there exists a vector y_0 in Ω such that the translation T_{y_0} defined by*

$$T_{y_0}(t, z, w) = (t, z + y_0, w) \quad \textit{for } (t, z, w) \in \mathscr{D}$$

is an element of the group Γ.

Proof. The n-dimensional real vector space V may be considered as a closed subgroup of the group Z of parallel translations since each element $a \in V$ defines a parallel translation $(t, z, w) \mapsto (t, z + a, w)$. Since $Z/(Z \cap \Gamma)$ is compact, $V/(V \cap \Gamma)$ is also compact. Therefore $V/(V \cap \Gamma)$ is a real torus of dimension n and $V \cap \Gamma$ is a lattice in V. Clearly this lattice meets the cone Ω. (If $(V \cap \Gamma) \cap \Omega$ were empty, the natural projection $V \to V/(V \cap \Gamma)$ would map Ω injectively into $V/(V \cap \Gamma)$. But this is impossible since the volume of Ω is infinite.) QED.

Lemmas 4.3 and 4.4 will be used in the proof of Theorem 1 in §5. We shall now prove a lemma which will be used in the proof of Theorem 2. First, we quote the following result from [4; Proposition 2.5].

Lemma 4.5. *Let $\mathscr{D}=\mathscr{D}(H.\ \Omega, \mathscr{F}, \varphi)$ be a Siegel domain of the third kind. For $r\in\Omega$, denote by $\mathscr{D}_r$ the cylindrical set $\mathscr{D}_r(\mathscr{F})$ with base $\mathscr{F}$. Then*

$$d_{\mathscr{D}}(a, b)\geqq \log s \qquad \textit{for } a\in\mathscr{D}_{sr},\ b\in\mathscr{D}-\mathscr{D}_r,\ s>1,\ r\in\Omega.$$

We say that a sequence $\{a_m\}$ of points of $\mathscr{D}$ converges to a point t_1 of $\mathscr{F}$ if, for every open neighborhood $\mathscr{O}$ of t_1 in $\mathscr{F}$ and every cylindrical set $\mathscr{D}_r(\mathscr{O})$ with base $\mathscr{O}$, there exists an integer M such that $a_m\in\mathscr{D}_r(\mathscr{O})$ for $m>M$.

Lemma 4.6. *Let $\mathscr{D}=\mathscr{D}(H, \Omega, \mathscr{F}, \varphi)$ be a Siegel domain of the third kind.*

(1) *Given two points t_1 and t_2 of $\mathscr{F}$, there exist sequences $\{a_m\}$ and $\{b_m\}$ of points of $\mathscr{D}$ such that* $\lim a_m=t_1$, $\lim b_m=t_2$ and $d_{\mathscr{D}}(a_m, b_m)$ *is bounded (by a number independent of m).*

(2) *If $\{a_m\}$ and $\{b_m\}$ are sequences of points of $\mathscr{D}$ such that $d_{\mathscr{D}}(a_m, b_m)$ is bounded and if* $\lim a_m=t_1\in\mathscr{F}$, *then $\{b_m\}$ contains a subsequence which converges to a point of $\mathscr{F}$ provided F is $d_{\mathscr{F}}$-complete.*

Proof. (1) Fix an element y of Ω and define

$$a_m=(t_1, i\, m\, y, 0), \qquad b_m=(t_2, i\, m\, y, 0).$$

Applying the proof of (2) of Lemma 4.3, we obtain

$$d_{\mathscr{D}}(a_m, b_m)=d_{\mathscr{F}}(t_1, t_2).$$

(2) We denote by π the projection $\mathscr{D}\to\mathscr{F}$ induced by the natural projection $U\times V_{\mathbb{C}}\times W\to U$. Since π is distance-decreasing,

$$d_{\mathscr{F}}(\pi(a_m), \pi(b_m))\leqq d_{\mathscr{D}}(a_m, b_m).$$

Since $d_{\mathscr{D}}(a_m, b_m)$ is bounded and $\pi(a_m)$ converges to a point $t_1\in\mathscr{F}$, some subsequence of $\pi(b_m)$ converges to a point, say t_2, of $\mathscr{F}$. (We are assuming that $\mathscr{F}$ is complete with respect to the distance $d_{\mathscr{F}}$.) Taking a subsequence if necessary, we may assume that $\pi(b_m)$ converges to a point t_2 of $\mathscr{F}$. Assume that no subsequence of $\{b_m\}$ converges to a point of F. Then there exists an element $r\in\Omega$ such that the cylindrical set $\mathscr{D}_r=\mathscr{D}_r(\mathscr{F})$ contains none of the points of the sequence $\{b_m\}$. (Since $\pi(b_m)$ converges to $t_2\in\mathscr{F}$, it suffices to consider only the cylindrical sets with base $\mathscr{F}$.) Now take an arbitrarily large number s. Since $\{a_m\}$ converges to a point in $\mathscr{F}$, there exists an integer M such that $a_m\in\mathscr{D}_{sr}$ for $m>M$. By Lemma 4.5,

$$d_{\mathscr{D}}(a_m, b_m)\geqq \log s \qquad \text{for } m>M.$$

This contradicts the assumption that $d_{\mathscr{D}}(a_m, b_m)$ is bounded. QED.

5. Extension Theorems

We are now in a position to prove the main theorem.

Theorem 1. *Let $\mathscr{D}$ be a symmetric bounded domain and let Γ be an arithmetically defined discrete subgroup of the largest connected group of holomorphic transformations of $\mathscr{D}$. Let M be a complex space hyperbolically imbedded in a complex space Y. Then every holomorphic mapping $f\colon \mathscr{D}/\Gamma \to M$ extends to a holomorphic mapping $f\colon \mathscr{D}^*/\Gamma \to Y$, where $\mathscr{D}^*/\Gamma$ denotes the compactification of $\mathscr{D}/\Gamma$.*

Proof. Let $\mathscr{D}^*/\Gamma = (\mathscr{D}/\Gamma) \cup (\mathscr{B}/\Gamma)$ as in §3. Since $\mathscr{D}^*/\Gamma$ is a normal complex space, it suffices to show that f extends continuously. Let $p_0 \in \mathscr{B}/\Gamma$. Then $p_0 = \eta(t_0)$, where $\eta\colon \mathscr{D}^* \to \mathscr{D}^*/\Gamma$ is the projection and t_0 is a point of a Γ-rational boundary component $\mathscr{F}$. Identify $\mathscr{D}$ with a Siegel domain $\mathscr{D}(H, \Omega, \mathscr{F}, \varphi)$ of the third kind. By Lemma 4.4, there is a vector y_0 in the cone Ω such that the translation T_{y_0} defined by y_0 is in Γ. Let $\mathscr{H}$ denote the upper-half plane in $\mathbb{C}$ and define a mapping $\tilde{g}_0\colon \mathscr{H} \to \mathscr{D}$ by

$$\tilde{g}_0(\zeta) = (t_0, \zeta\, y_0, 0) \qquad \text{for } \zeta \in \mathscr{H}.$$

Let Δ^* be the punctured unit disk in $\mathbb{C}$, i.e., $\Delta^* = \{\zeta \in \mathbb{C};\ 0 < |\zeta| < 1\}$. Then $\mathscr{H}$ is a covering space of Δ^* with projection $\zeta \to e^{2\pi i \zeta}$ and with covering group $\mathbf{Z}$ acting on $\mathscr{H}$ by $(n, \zeta) \in \mathbf{Z} \times \mathscr{H} \to \zeta + n \in \mathscr{H}$. Since $\eta(\tilde{g}_0(\zeta + n)) = \eta(T_{n y_0}(\tilde{g}_0(\zeta))) = \eta(\tilde{g}_0(\zeta))$ for every integer n, $\tilde{g}_0$ induces a mapping

$$g_0\colon \Delta^* = \mathscr{H}/\mathbf{Z} \to \mathscr{D}/\Gamma.$$

Let $f_0 = f \circ g_0$. Then f_0 is a holomorphic mapping from the punctured disk Δ^* in M. By Theorem 3.6 in [3; p. 99], it extends to a holomorphic mapping f_0 from the disk Δ into Y. Put

$$q = f_0(0) \in Y.$$

Let $\{p_m\}$ be any sequence in $\mathscr{D}/\Gamma$ such that $p_m \to p_0$. We will be done if we can show that $f(p_m) \to q$.

To do this, it suffices to show that every sequence $\{p_m\}$ with $p_m \to p_0$ has a subsequence $\{p_{m_j}\}$ with $f(p_{m_j}) \to q$. Denote by $\mathscr{O}_m$ the open neighborhood of t_0 in $\mathscr{F}$ defined by

$$\mathscr{O}_m = \{t \in \mathscr{F};\ d_{\mathscr{F}}(t_0, t) < 1/2m\}.$$

Taking a subsequence of $\{p_m\}$ if necessary, we may assume that for each p_m, there exists a point $\tilde{p}_m = (t_m, z_m, w_m)$ in the cylindrical set $\mathscr{D}_{m y_0}(\mathscr{O}_m)$ such that $\eta(\tilde{p}_m) = p_m$. For each m, define a mapping $\tilde{g}_m\colon \mathscr{H} \to \mathscr{D}$ by

$$\tilde{g}_m(\zeta) = (t_m, z_m + \zeta\, y_0 - i\, m\, y_0, w_m).$$

As above, $\tilde{g}_m$ induces a mapping $g_m\colon \Delta^* = \mathscr{H}/Z \to \mathscr{D}/\Gamma$. Put

$$f_m = f \circ g_m.$$

We have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{H} & \xrightarrow{\tilde{g}_m} & \mathscr{D} & & \\ {\scriptstyle e^{2\pi i \cdot}}\downarrow & & \downarrow{\scriptstyle \eta} & & \\ \Delta^* & \xrightarrow{g_m} & \mathscr{D}/\Gamma & \xrightarrow{f} & M \subset Y. \end{array}$$

Let m be fixed. In Lemma 4.3, let

$$y = y_0, \quad t' = t_0 \quad \text{and} \quad (t'', z, w) = (t_m, z_m - i\, m\, y_0, w_m).$$

Choose $s_m > 0$ such that

$$d_{\mathscr{D}}(\sigma_1(s_m), \sigma_2(s_m)) < 1/4m, \qquad d_{\mathscr{D}}(\sigma_3(s_m), \sigma_4(s_m)) < 1/4m.$$

Then

$$\begin{aligned} &d_{\mathscr{D}}(\sigma_1(s_m), \sigma_4(s_m)) \\ &\quad \leqq d_{\mathscr{D}}(\sigma_1(s_m), \sigma_2(s_m)) + d_{\mathscr{D}}(\sigma_2(s_m), \sigma_3(s_m)) + d_{\mathscr{D}}(\sigma_3(s_m), \sigma_4(s_m)) \\ &\quad < \frac{1}{4m} + \frac{1}{2m} + \frac{1}{4m} = \frac{1}{m}. \end{aligned}$$

Put

$$\zeta_m = e^{-2\pi s_m} \in \Delta^*.$$

Since $\tilde{g}_0(i\, s_m) = (t_0, i\, s_m\, y_0, 0) = \sigma_1(s_m)$, we have

$$f_0(\zeta_m) = f \circ g_0(e^{-2\pi s_m}) = f \circ \eta \circ \tilde{g}_0(i\, s_m) = f \circ \eta \circ \sigma_1(s_m).$$

Similarly, since $\tilde{g}_m(i\, s_m) = (t_m, z_m + i\, s_m\, y_0 - i\, m\, y_0, w_m) = \sigma_4(s_m)$, we have

$$f_m(\zeta_m) = f \circ g_m(e^{-2\pi s_m}) = f \circ \eta \circ \tilde{g}_m(i\, s_m) = f \circ \eta \circ \sigma_4\,(s_m).$$

Therefore

$$\begin{aligned} d_M(f_0(\zeta_m), f_m(\zeta_m)) &= d_M(f \circ \eta \circ \sigma_1(s_m), f \circ \eta \circ \sigma_4(s_m)) \\ &\leqq d_{\mathscr{D}}(\sigma_1(s_m), \sigma_4(s_m)) < 1/m. \end{aligned}$$

Since $\lim_{m\to\infty} \zeta_m = 0 \in \Delta^*$, we have

$$\lim_{m\to\infty} f_0(\zeta_m) = f_0(0) = q.$$

Since $\lim_{m\to\infty} d_M(f_0(\zeta_m), f_m(\zeta_m)) = 0$ and M is hyperbolically imbedded in Y (see condition (3) in §1), we have

$$\lim_{m\to\infty} f_m(\zeta_m) = \lim_{m\to\infty} f_0(\zeta_m) = q.$$

Put

$$\zeta'_m = e^{-2\pi m} \in \Delta^*.$$

Since

$$\tilde{p}_m = (t_m, z_m, w_m) = \tilde{g}_m(i\, m),$$

we have

$$f(p_m)=f\circ\eta(\tilde{p}_m)=f\circ\eta\circ\tilde{g}_m(i\,m)=f\circ g_m(e^{-2\pi m})=f\circ g_m(\zeta'_m)=f_m(\zeta'_m).$$

To prove $\lim_{m\to\infty} f(p_m)=q$, it suffices therefore to show $\lim_{m\to\infty} f_m(\zeta'_m)=q$. Since $\lim_{m\to\infty} f_m(\zeta_m)=q$, this follows from the following

Lemma (see Theorem 1 in [2]). *Let M be hyperbolically imbedded in Y and let $f_m\colon \Delta^*\to M$ be a sequence of holomorphic mappings. Let ζ_m and ζ'_m be sequences in Δ^* converging to 0 and such that $\lim_{m\to\infty} f_m(\zeta_m)=q\in Y$. Then $\lim_{m\to\infty} f_m(\zeta'_m)=q$.*

This completes the proof of our main theorem.

Theorem 2. *Let $\mathscr{D}$ (resp. $\mathscr{D}'$) be a symmetric bounded domain and Γ (resp. Γ') an arithmetically defined discrete subgroup of the largest connected group of holomorphic transformations of $\mathscr{D}$ (resp. $\mathscr{D}'$). Let $\mathscr{D}^*/\Gamma$ (resp. $\mathscr{D}'^*/\Gamma'$) be the compactification of $\mathscr{D}/\Gamma$ (resp. $\mathscr{D}'/\Gamma'$). Then every holomorphic mapping $f\colon \mathscr{D}/\Gamma\to\mathscr{D}'/\Gamma'$ that comes from a holomorphic mapping $\tilde{f}\colon \mathscr{D}\to\mathscr{D}'$ extends to a holomorphic mapping $f\colon \mathscr{D}^*/\Gamma\to\mathscr{D}'^*/\Gamma'$. Furthermore, the extended mapping sends each boundary component of $\mathscr{D}^*/\Gamma$ into a boundary component of $\mathscr{D}'^*/\Gamma'$.*

We observe that the condition that $f\colon \mathscr{D}/\Gamma\to\mathscr{D}'/\Gamma'$ be lifted to $\tilde{f}\colon \mathscr{D}\to\mathscr{D}'$ is satisfied if Γ' acts freely on $\mathscr{D}'$ since $\mathscr{D}$ is simply connected.

Proof. It was shown in [4] that $\mathscr{D}'/\Gamma'$ is hyperbolically imbedded in $\mathscr{D}'^*/\Gamma'$ provided that Γ' acts freely on $\mathscr{D}'$. In this case, the first statement follows immediately from Theorem 1. If Γ' is not acting freely on $\mathscr{D}'$, we replace the intrinsic pseudo-distance $d_{\mathscr{D}'/\Gamma'}$ by the distance $d'_{\mathscr{D}'/\Gamma'}$ induced from the intrinsic distance $d_{\mathscr{D}'}$ and modify conditions (3) in §1. Then as in [4], the first statement follows immediately from the proof of Theorem 1. It should perhaps be pointed out that $d_{\mathscr{D}'/\Gamma'}$ need not be a true distance if Γ' is not acting freely on $\mathscr{D}'$ and that Theorem 2 does not hold in general for a non-liftable mapping f.

To prove the last assertion, we fix a Γ-rational boundary component $\mathscr{F}$ of $\mathscr{D}$ and choose two arbitrary points t_1 and t_2 of $\mathscr{F}$. By (1) of Lemma 4.6, we can find sequences $\{a_m\}$ and $\{b_m\}$ of points of $\mathscr{D}$ such that $\lim a_m=t_1$, $\lim b_m=t_2$ and $d_{\mathscr{D}}(a_m, b_m)$ is bounded. (The convergence should be understood in terms of the neighborhood system defined by cylindrical sets; see the paragraph following Lemma 4.5.) Let $\eta\colon \mathscr{D}^*\to\mathscr{D}^*/\Gamma$ and $\eta'\colon \mathscr{D}'^*\to\mathscr{D}'^*/\Gamma'$ be the projections. Suppose $f\circ\eta(t_1)\in\eta'(\mathscr{F}')$, where $\mathscr{F}'$ is a Γ'-rational boundary component of $\mathscr{D}'$. We want to show that $f\circ\eta(t_2)\in\eta'(\mathscr{F}')$. Choose a point $t'_1\in\mathscr{F}'$ such that $\eta'(t'_1)=f\circ\eta(t_1)$. Since

$$\lim\eta'\circ\tilde{f}(a_m)=\lim f\circ\eta(a_m)=f\circ\eta(\lim a_m)=f\circ\eta(t_1)=\eta'(t'_1),$$

there exists a sequence γ'_m of elements of Γ' such that

$$\lim \gamma'_m \circ f(a_m) = t'_1 .$$

We put

$$a'_m = \gamma'_m \circ f(a_m), \qquad b'_m = \gamma'_m \circ f(b_m).$$

Then the sequence $\{a'_m\}$ converges to $t'_1 \in \mathscr{F}'$, and the distance $d_{\mathscr{D}'}(a'_m, b'_m)$ is bounded because

$$d_{\mathscr{D}'}(a'_m, b'_m) = d_{\mathscr{D}'}(\gamma'_m \circ f(a_m), \gamma'_m \circ f(b_m)) = d_{\mathscr{D}'}(f(a_m), f(b_m)) \leqq d_{\mathscr{D}}(a_m, b_m).$$

By (2) of Lemma 4.6, a suitable subsequence of $\{b'_m\}$ converges to a point, say t'_2, of $\mathscr{F}'$. On the other hand,

$$\begin{aligned} f \circ \eta(t_2) &= f \circ \eta(\lim b_m) = \lim f \circ \eta(b_m) = \lim \eta' \circ f(b_m) \\ &= \lim \eta' \circ \gamma'_m \circ f(b_m) = \lim \eta'(b'_m). \end{aligned}$$

Therefore,

$$f \circ \eta(t_2) = \eta'(t'_2) \in \eta'(\mathscr{F}'). \quad \text{QED.}$$

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(Received January 31, 1972)