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Vector Fields and Chern Numbers

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Introduction

The purpose of this note is to expose an elementary Kozsul complex argument which underlies the work of Bott [3, 4] and Baum-Bott [5]. Their results relate global Chern invariants of X to local residue calculations. The treatment of their results given by the present approach is valid for both the analytic and algebraic categories, and we feel that it clarifies the role of the Grothendieck Residue symbol in the formulae. A more important consequence of the present approach is that for Kähler manifolds having a vector field with non-trivial zeroes one can prove that the Chern *classes* (and not merely the Chern numbers) of X may be computed on the zero set. Furthermore, we study systematically the notion of bundles equivariant with respect to a $\mathscr{W}$-valued vector field V, i.e. a section of $\Theta\otimes\mathscr{W}$, where Θ is the holomorphic tangent bundle and $\mathscr{W}$, a vector bundle. One proves easily that the Chern numbers of equivariant bundles are determined on the zeroes of V (as are the Chern classes for X Kähler, $\mathscr{W}$ trivial, and zero $(V)\neq\phi$). The notion of equivariance is extremely fruitful since it vastly increases the applicability of the theorem. All line bundles are equivariant if X is Kähler, $\mathscr{W}$ is trivial and zero (V) is nonempty. Further if $\mathscr{W}$ is taken to be a sufficiently ample line bundle then any given bundle $\mathscr{E}$ will be equivariant for all $\mathscr{W}$-valued vector fields, which will exist, moreover, in abundance (see § 1).

A section V of $\Theta\otimes\mathscr{W}=(\Omega^1)^*\otimes\mathscr{W}$ is viewed as defining a map $i(V)$: $\Omega^1\to\mathscr{W}$ from the holomorphic forms to $\mathscr{W}$. We obtain therefore a derivation $V:\mathscr{O}_X\to\mathscr{W}$ by $V(f)=i(V)(df)$. A bundle $\mathscr{E}$ is called V-equivariant if one may lift V to $\tilde{V}:\mathscr{E}\to\mathscr{W}\otimes\mathscr{E}$ such that:

$$\tilde{V}(fs)=V(f)\otimes s+f\,\tilde{V}(s)$$

for f a function, and s a section of $\mathscr{E}$. If one restricts to Z, the locus of points at which V vanishes, then the formula above shows that $\tilde{V}$ defines a *linear* map

$$\tilde{V}_Z\in H^0(Z,\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W}).$$

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If one gives a degree d polynomial map p: $\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W}\to G$, then one obtains $p(\tilde{V}_Z)\in H^0(Z,\mathscr{G})$. Our results relate this local class to global invariants obtained by applying p to the Atiyah class $c(\mathscr{E})\in H^1(X,\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\Omega^1)$, which yields an element $p(c(\mathscr{E}))\in H^d(X,\Omega^d\otimes\mathscr{W}^{-d}\otimes\mathscr{G})$.

Main Theorem. *Let X be a complex manifold, $\mathscr{W}$ a line bundle and V a $\mathscr{W}$-valued holomorphic vector field with isolated zeroes Z. Given a V-equivariant bundle $\mathscr{E}$ and a polynomial mapping of degree d, p:* $\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W}\to\mathscr{G}$ *for some bundle $\mathscr{G}$, then*

a) *If $d=n=\dim X$ there exists a natural homomorphism*

$$e: H^0(Z,\mathscr{G})\to H^n(X,\Omega^n\otimes\mathscr{W}^{-n}\otimes\mathscr{G})$$

such that

$$e(p(\tilde{V}_Z))=p(c(\mathscr{E}))\,.$$

b) *If $d=n$, X is compact and $\mathscr{G}=\mathscr{W}^n$ so that $p(c(\mathscr{E}))\in H^n(X,\Omega^n)$ can be integrated to give a Chern number of $\mathscr{E}$, then one has*

$$\textstyle\int_X p(c(\mathscr{E}))=(2\pi i)^n\,\mathrm{Res}(p(\tilde{V}_Z))$$

where Res, *the Grothendieck Residue, is the canonical map of $H^0(Z,W^n)=\mathrm{Ext}^n(\mathcal{O}_Z,\Omega^n)$ to the scalars.*

c) *If $\mathscr{W}=\mathscr{G}=\mathcal{O}_X$ and $d<n$, then* $\mathrm{Res}(p(\tilde{V}_Z))=0$.

d) *If X is compact Kähler, $Z\neq\emptyset$, and $\mathscr{G}=\mathscr{W}=\mathcal{O}_X$ is trivial, then there exists a filtration F_i on $H^0(Z,\mathcal{O}_Z)$, with $F_i\supseteqq F_{i+1}$ and $F_i\cdot F_j\subseteqq F_{i+j}$, and isomorphisms*

$$e_d: F_{-d}/F_{-d+1}\to H^d(X,\Omega^d)$$

such that

$$e_d(p(\tilde{V}_Z))=p(c(\mathscr{E}))\,.$$

In particular for σ_d: $\mathrm{Hom}(E,E)\to\mathcal{O}_X$ *the d* th *elementary symmetric function one obtains that $\sigma_d(\tilde{V}_Z)\in H^0(Z,\mathcal{O}_Z)$ computes the d* th *Chern class of E.*

In Section 4 we indicate how one may explicitly calculate the cohomology rings of Grassmanians by using the equivariance of the canonical quotient and subbundle with respect to flows on the Grassmanian.

The underlying idea of our proof is that the Atiyah class $c(\mathscr{E})$ can be thought of as coming from a class $\tilde{c}(\mathscr{E})$ in the hypercohomology of an $i(V)$-Kozsul complex. This hypercohomology lives on Z, in particular, for isolated Z, $\tilde{c}(\mathscr{E})$ will be identifiable with $\tilde{V}_Z\in H^0(Z,\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W})$. Now, by construction $c(\mathscr{E})$ arises from $\tilde{c}(\mathscr{E})$ via a spectral sequence edge morphism e, and this morphism commutes with the evaluation of p in view of the multiplicative character of the Kozsul complex. The lack of multiplicative structure for the generalized Kozsul complexes is a first obstacle to the simple treatment of higher dimensional foliations, cf. [6], by an analogous argument. We remark in §3 that statement b) of our theorem follows easily from a standard commutative diagram in residue theory. The analogous result for the higher dimensional foliations needs to be uncovered—it is clearly implicit in the fibered residue calculations of [6].

Finally, we remark that for higher dimensional foliations the hypothesis of *integrability* of Baum and Bott is precisely the hypothesis that the quotient sheaf

$Q = \Theta/\mathscr{W}$ is *equivariant*, and "explains" the calculation of Q-Chern numbers rather than Θ-Chern numbers in their work. Since Q is in general only *coherent* rather than locally free it is necessary to extend our spectral sequence argument to this broader context to cover the Baum-Bott results—this is achieved in §6.

§1. V-Equivariant Sheaves

Throughout our discussion X will denote a (not necessarily compact) complex manifold of dimension n, $\mathscr{W}$ will denote a locally free, rank one sheaf of $\mathscr{O}_X$ modules and $V \in H^0(X, \Theta \otimes \mathscr{W})$ a "$\mathscr{W}$-valued" holomorphic vector field on X. (Note that since $\Theta \otimes \mathscr{W} = \mathrm{Hom}(\mathscr{W}^*, \Theta)$, one may equivalently view V as a map $\mathscr{W}^{a*} \to \Theta$. We indicate by "$V$", or by "$V\otimes$", any map $\mathscr{W}^* \otimes \mathscr{F} \to \Theta \otimes \mathscr{F}$ produced from V by linearity. Dually we denote by $i(V)$ maps produced from $i(V)$: $\Omega^1 \to \mathscr{W}$ by tensorisation.)

A sheaf of $\mathscr{O}_X$ modules $\mathscr{E}$ is called *V-equivariant* if there exists a C-linear map, $\tilde{V}: \mathscr{E} \to \mathscr{W} \otimes \mathscr{E}$, lifting the derivation $V: \mathscr{O}_X \to \mathscr{W}$, that is

$$\tilde{V}(f \cdot s) = V(f) \otimes s + f \cdot \tilde{V}(s) \tag{1.1}$$

where f (resp. s) is a local section of $\mathscr{O}_X$ (resp. $\mathscr{E}$). For vector fields (i.e. when W is trivial) the sheaves Θ and Ω^p are equivariant via Lie bracket and Lie derivative. In general Θ will not be equivariant for $\mathscr{W}$-valued fields. However, if one defines Q by the sequence

$$0 \to \mathscr{W}^{a*} \xrightarrow{V} \Theta \xrightarrow{\pi} Q \to 0 \tag{1.2}$$

then one obtains a natural

$$\tilde{V}: Q \to \mathscr{W} \otimes Q = \mathrm{Hom}(W^*, Q)$$

by taking $\tilde{V}(q)$ to be the map: $s \to \pi([V(s), \tilde{q}])$ where s is a section of $\mathscr{W}^{a*}$ and $\tilde{q}$ is any π-lift of q. (Independence of the choice of $\tilde{q}$ is due to the *integrability* of rank 1 subbundles of Θ. For higher rank $\mathscr{W}$, one may assume integrability to guarantee equivariance of Q, as in [6].)

We denote by Z the subvariety of X defined by the vanishing of V, i.e. by the sheaf of ideals I_Z which is the image of $i(V): \Omega^1 \otimes \mathscr{W}^{a*} \to \mathscr{O}_X$. Note that by definition $V(f) \in I_Z$ for all f. If $\mathscr{E}$ is any equivariant bundle we see therefore that $\tilde{V}: \mathscr{E} \to \mathscr{E} \otimes \mathscr{W}$ maps $I_Z \cdot \mathscr{E}$ to $I_Z \cdot (\mathscr{E} \otimes \mathscr{W})$ and modulo I_Z induces an $\mathscr{O}_Z$ linear map $\tilde{V}_Z$, important in the sequel.

For general $\mathscr{E}$ one may measure the obstruction to constructing a $\tilde{V}$ by a class $\delta(\mathscr{E}) \in H^1(X, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{W})$. Namely, locally one may construct $\tilde{V}_\alpha$ satisfying (1.1) and $\tilde{V}_\alpha$ is determined by to a linear map in $\mathrm{Hom}(\mathscr{E}, \mathscr{W} \otimes \mathscr{E})$, whence the patching obstruction for global $\tilde{V}$.

Proposition (1.1). *Given a locally free sheaf $\mathscr{E}$ of $\mathscr{O}_X$-modules, the element $\delta(V)$ obstructing the V-equivariance of $\mathscr{E}$ is $i(V)$ $c(\mathscr{E}) \in H^1(X, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{W})$ where $c(\mathscr{E}) \in H^1(X, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \Omega^1)$ is the Atiyah Chern class of $\mathscr{E}$.*

Proof. The result follows from noting that $c(\mathscr{E})$ obstructs the existence of a connexion $\nabla: \mathscr{E} \to \Omega^1 \otimes \mathscr{E}$ and that given ∇ one may define $\tilde{V}$ as $i(V) \cdot \nabla$.

An important application of this is

Proposition (1.2). *Given X a projective manifold and $\mathscr{E}$ locally free, there exists a line bundle $\mathscr{L}$ on X such that $\mathscr{E}$ is equivariant with respect to any section of $H^0(X, \Theta\otimes\mathscr{L})$ and such that $\Theta\otimes\mathscr{L}$ has sections with isolated zeros.*

Proof. When $\mathscr{L}$ is sufficiently positive, $H^1(X, \mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{L})=0$ and $\Theta\otimes\mathscr{L}$ will be very ample, hence will have sections in general position.

As another example, we have

Proposition (1.3). *If X is a compact Kähler manifold and $V\in H^0(X,\Theta)$ has zeros, then every invertible sheaf $\mathscr{L}$ on X is equivariant.*

Proof. For then, $i(V)$: $H^1(X,\Omega^1)\to H^1(X,\mathcal{O}_X)$ is zero. This is essentially Lichnerowicz's Lemma [7].

§ 2. The Fundamental Kozsul Complex

Given $V\in H^0(X,\Theta\otimes\mathscr{W})$ we may extend the canonical map $i(V)$: $\Omega^1\otimes\mathscr{W}^*\to\mathcal{O}_X$ to define a Kozsul complex. Namely, we define: $K^{-p}=\Lambda^p(\Omega^1\otimes\mathscr{W}^*)=\Omega^p\otimes\mathscr{W}^{-p}$, where $\mathscr{W}^{-p}=(\mathscr{W}^*)^{\otimes p}$. We have the fundamental Kozsul complex of sheaves:

$$0\to K^{-n}\to\ldots\to K^{-1}\to K^0\to 0 \tag{2.1}$$

in which the differential is contraction with V, $i(V)$. Given any locally free $\mathscr{F}$ we shall denote by $K(\mathscr{F})$ the complex obtained by tensoring (2.1) with $\mathscr{F}$, over $\mathcal{O}_X$.

We will be analysing the two spectral sequences of hypercohomology for these complexes

$$E_2^{p,q}=H^p(X,\mathscr{H}^q(F))\Rightarrow \boldsymbol{H}^{p+q}(X,K(\mathscr{F})) \tag{2.2}$$

$$E_1^{p,q}=H^q(X,\Omega^{-p}\otimes\mathscr{F})\Rightarrow \boldsymbol{H}^{p+q}(X,K(\mathscr{F})) \tag{2.3}$$

where $\mathscr{H}(\mathscr{F})$ denotes the cohomology sheaves of the complex $K(\mathscr{F})$.

The hypercohomology $\boldsymbol{H}(X,K(\mathscr{F}))$ may be computed as total cohomology of a double complex—for example fixing a Leray covering $\mathscr{U}$ for X, one may form the double Čech complex $\check{C}^p(\mathscr{U},\Omega^q\otimes F)$ with differentials $i(V)$ and δ, (the Cech coboundary) and with total differential $\delta+(-1)^p i(V)$. Alternatively one can use the double complex $A^{p,q}(F)$ of F valued C^∞ forms of type p, q with differentials $\bar\partial$ and $i(V)$. In fact this double complex is implicit in the projector calculations of Bott. The remarks we make below concerning hypercohomology may be easily followed in the explicit representation given by either of these double complexes. The spectral sequences above are the two spectral sequences of the double complex.

We recall that (2.1) is exact off the zeroes Z of V, and the cohomology sheaves $\mathscr{H}(\mathscr{F})$ are in fact coherent sheaves of $\mathcal{O}_Z$ modules in particular are supported on Z. When $\dim Z=0$ then $\mathscr{H}^q=0$ for $q>0$ and $\mathscr{H}^0(\mathscr{F})=\mathscr{F}_Z$ by the well known theory of the Kozsul complex.

Remark 2.4. The sequence (2.2) shows that $\boldsymbol{H}(K(\mathscr{F}))$ is determined by contributions on Z. One sees immediately,

a) If $Z=\phi$ then $\boldsymbol{H}^r(X,K(\mathscr{F}))=0$ for all r.

b) If $\dim Z=0$ then $\boldsymbol{H}^r(X, K(\mathscr{F}))=0$ for $r\neq 0$ and $\boldsymbol{H}^0(X, K(\mathscr{F}))=H^0(Z, \mathscr{F}_Z)$. where $\mathscr{F}_Z=\mathscr{F}\otimes_{\mathcal{O}_X}\mathcal{O}_Z$.

Moreover we remark on the multiplicative nature of these spectral sequences. Namely, given a bilinear map $A\times B\to C$ one extends it to a natural bilinear map

$$K^p(A)\times K^q(B)\to K^{p+q}(C) \tag{2.5}$$

by using the natural exterior product in $K^{\cdot}=\Lambda^{\cdot}K^1$. This is a pairing of complexes in view of the contraction identity

$$i(V)(k_1\wedge k_2)=(i(V)\,k_1)\wedge k_2+(-1)^p\,k_1\wedge i(V)\,k_2\,.$$

Consequently one obtains a bilinear pairing

$$\boldsymbol{H}^p(K(A))\times \boldsymbol{H}^q(K(B))\to \boldsymbol{H}^{p+q}(K(C))$$

of hypercohomology. Further, if $F.\,\boldsymbol{H}$ denotes the filtration on $\boldsymbol{H}$ defined by the spectral sequence (2.3) then one has $F_r\boldsymbol{H}\times F_s\boldsymbol{H}\to F_{r+s}\boldsymbol{H}$.

In particular when $\mathscr{F}=\mathcal{O}_X$, the hypercohomology of $K^{\cdot}=K(\mathcal{O}_X)$ has a natural ring structure, which is compatible via (2.3) with the wedge product pairings of the groups $E^{-p,q}=H^q(X, \Omega^p\otimes\mathscr{W}^{-p})$. In case $\mathscr{W}=\mathcal{O}_X$ and X is a Kähler manifold, the $H^q(X, \Omega^p)$ are the Hodge components of the cohomology ring of X. In this case we recall the main result of [7]:

Theorem 2.6. *If X is a Kähler manifold, V a holomorphic vector field with nonempty zero set, Z, then the spectral sequence* (2.3) *has all differentials vanishing, so that the cohomology ring of X is determined on Z as the associated graded ring for the filtered $F_p\boldsymbol{H}^{p+q}(X, K^{\cdot})$.*

Remark 2.7. In particular if $\dim Z=0$, the cohomology ring of X is the associated graded of a filtration

$$H^0(Z, \mathcal{O}_Z)=F_{-n}\supseteqq F_{-n+1}\supseteqq\cdots\supseteqq F_{-1}\supseteqq F_0=0$$

on the ring of global functions on Z, cf. 2.4 b) above. Note that the F_i are not in general ideals in this ring, but do satisfy $F_i\cdot F_j\subseteqq F_{i+j}$.

An example with $X=P^n$ is done explicitly in §4, below.

§ 3. Principal Results

We next turn to the definition of hyper-Chern classes for equivariant bundles. Given any bundle we note that the Atiyah Chern class $c(\mathscr{E})$ defines an element of $H^1(X, \mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\Omega^1)$, which group may be interpreted as the $E_1^{-1,1}$ term in the $\boldsymbol{H}(K(\mathscr{F}))$ spectral sequence (2.3), where $\mathscr{F}=\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W}$. We note that $d_1(c(\mathscr{E}))=0$ if and only if $\mathscr{E}$ is V-equivariant (Proposition 1.1). Thus since $E_1^{r-1,2-r}$ vanishes if $r>1$, we find that $\mathscr{E}$ is equivariant if and only if $c(\mathscr{E})$ defines a hypercohomology class $\tilde{c}(\mathscr{E})$ lying in $F_{-1}\boldsymbol{H}^0(X, K(\mathrm{Hom}(\mathscr{E},\mathscr{E})\otimes\mathscr{W})$. [This $\tilde{c}(\mathscr{E})$ is well defined only up to $F_0\boldsymbol{H}^0$, and modulo F_0, $\tilde{c}(\mathscr{E})$ is the class in $E_\infty^{-1,1}$ given by $c(\mathscr{E})$ modulo all spectral sequence boundaries.] We call $\tilde{c}(\mathscr{E})$ a hyper-Chern class of $\mathscr{E}$.

Given $\mathcal{O}_X$-modules A and B, a mapping $p: A\to B$ will be called a polynomial of degree d if it is obtained by composing the diagonal map $A\to\otimes^d A$, $(a\to a\otimes a\ldots\otimes a)$, with a linear map $\otimes^d A \to B$. Given such a polynomial one obtains canonically $p: \boldsymbol{H}(K(A)) \to \boldsymbol{H}(K(B))$ by composing the diagonal map $\boldsymbol{H}(K(A)) \to \otimes^d \boldsymbol{H}(K(A))$ with the map $\otimes^d \boldsymbol{H}(K(A)) \to \boldsymbol{H}(K(B))$ induced by the linear mapping assumed above. Note that p respects the filtration of the spectral sequence (2.3), i.e. $p(F_r \boldsymbol{H}^q(A)) \subseteq F_{dr} \boldsymbol{H}^{dq}(B)$, and one finds immediately (in view of the multiplicative character of our spectral sequences):

Proposition 3.1. *Given* p: $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \otimes \mathscr{W} \to \mathscr{G}$ *a degree d polynomial, the class* $p(c(\mathscr{E}))$ *in* $H^d(X, \Omega^d \otimes \mathscr{W}^{-d} \otimes \mathscr{G})$ *is associated to the class* $p(\tilde{c}(\mathscr{E}))$ *in* $\boldsymbol{H}^0(X, K(\mathscr{G}))$. *More precisely,* $p(\tilde{c}(\mathscr{E})) \in F_{-d} \boldsymbol{H}^0(X, K(G))$ *and modulo* F_{-d+1} *it defines in* $E_\infty^{-d,d}$ *the class which is the image of* $p(c(\mathscr{E})) \in E_1^{-d,d}(K(G))$ *modulo* $B^{-d,d}$ *(the set of all spectral sequence boundaries in* $E_1^{-d,d}$*).*

In general the vanishing of $p(\tilde{c}(\mathscr{E}))$ implies only that $p(c(\mathscr{E}))$ is a spectral sequence boundary, and will not imply the vanishing of $p(c(\mathscr{E}))$. In particular cases where there are *no boundaries* $p(c(\mathscr{E}))$ is completely determined by $p(\tilde{c}(\mathscr{E}))$. Since this latter class is in $\boldsymbol{H}(K(\mathscr{G}))$ it is determined on the zero set Z in view of (2.4). Thus, we have results of Bott type:

Theorem 3.2. *Given* $V \in H^0(X, \Theta \otimes \mathscr{W})$ *where* X *is a (not necessarily compact) complex manifold, and* $\mathscr{W}$ *is a complex line bundle, then for any V-equivariant bundle E, and any polynomial* p: $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \otimes \mathscr{W} \to \mathscr{G}$ *of degree* $n = \dim X$:

$$e(p(\tilde{c}(\mathscr{E}))) = p(c(\mathscr{E}))$$

where e: $\boldsymbol{H}^0(X, K(\mathscr{G})) \to H^n(X, \Omega^n \otimes W^{-n} \otimes G)$ *is the edge morphism of the spectral sequence* (2.3). *Thus* $p(c(\mathscr{E}))$ *is determined by contributions on* $Z = \mathit{zero}(V)$, *vanishing if* $Z = \phi$.

The concluding remark of the theorem is a reference to the spectral sequence (2.2). The local contributions are made explicit in (3.4) below. The key point in (3.2) is the non-existence of boundaries in $E_1^{-n,n}$ since $K^r = 0$ if $r < -n$. We have similarly

Theorem 3.3. *Let* X *be compact Kähler, and* V *a vector field with nonempty zero set* Z, *and let* p:$\mathrm{Hom}(\mathscr{E},\mathscr{E}) \to \mathcal{O}_X$ *be any degree d polynomial, then the Chern class* $p(c(\mathscr{E})) \in H^d(X, \Omega^d)$ *is* $e(p(\tilde{c}(\mathscr{E})))$ *where*

$$e: F_{-d} \boldsymbol{H}^0(K) \to H^d(X, \Omega^d)$$

arises from the totally degenerate spectral sequence (2.3), (*cf.* 2.6, *and* [7].)

One cannot conclude the vanishing of Chern *classes* for equivariant bundles on Kähler manifolds when $Z = \emptyset$, since the spectral sequences will not degenerate when $Z = \emptyset$. As an example, let $X = P^1 \times T$ with T an elliptic curve. Take V to be translation on T and $\mathscr{E} = \pi_1^*(\mathcal{O}(1))$. One has $p(\tilde{c}(\mathscr{E})) \in \boldsymbol{H}(K)) = 0$, while clearly for $p = \mathrm{tr}$: $\mathrm{Hom}(\mathscr{E},\mathscr{E}) \to \mathcal{O}_X$ one has $0 \neq c_1(\mathscr{E}) \in H^1(X, \Omega^1)$.

We turn next to the proof of the main theorem. The hypothesis that Z have *isolated* zeroes yields an identification $\boldsymbol{H}^0(X, K(\mathrm{Hom}(\mathscr{E},\mathscr{E}) \otimes \mathscr{W})) \xrightarrow{\sim} H^0(Z, \mathrm{Hom}(\mathscr{E},\mathscr{E}) \otimes \mathscr{W})$ in view of the degeneracy of spectral sequence (2.2), cf. (2.4). The statements a), c), of our main theorem correspond to Theorems 3.2, and 3.3, once one verifies that $\tilde{c}(\mathscr{E})$ and $\tilde{V}_Z$ correspond under the above isomorphism. Fixing an acyclic covering $\mathscr{U}$ of X by polydiscs U_α we employ the Cech double complex $C^p(\mathscr{U}, K^q)$ to calculate $\boldsymbol{H}$. Note that if D_α: $\mathscr{E} \to \mathscr{E} \otimes \Omega^1$ is a connexion for $\mathscr{E}$ on U_α then $D_\alpha - D_\beta = \Theta_{\alpha\beta} \in C^1(\mathscr{U}, K^{-1})$ is the Cech representative for the Atiyah class. Consider $L_\alpha = \tilde{V} - i(V) \cdot D_\alpha$ which yields an element of $C^0(\mathscr{U}, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{W})$ such that $L_\beta - L_\alpha = i(V)(\Theta_{\alpha\beta})$. Thus we see that the cochain $\{\Theta_{\alpha\beta}\} \oplus \{L_\alpha\} \in \bigoplus_{p+q=0} C^p(\mathscr{U}, K^q(\mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{W}))$ is a total cocycle (i.e. is annihilated by $\delta + (-1)^p i(V)$, the total differential) and represents the class $\tilde{c}(\mathscr{E})$. The corresponding class in $H^0(Z, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{W})$ is $\{L_\alpha\}|_Z$. But $L_\alpha|_Z = \tilde{V}_Z$ since $i(V)$ vanishes on Z, and we obtain our assertion.

To obtain statement b) of the main theorem concerning the Grothendieck residue, note that since $\dim Z = 0$

$$K = 0 \to \Omega^n \otimes \mathscr{W}^{-n} \to \dots \to \Omega^1 \otimes \mathscr{W}^{-1} \to \mathscr{O}_X$$

is a resolution of $\mathscr{O}_Z$ and since one has a natural identification $K(\mathscr{W}^n) = \mathrm{Hom}(K, \Omega^n)$ via

$$K^{-p}(\mathscr{W}^n) = \Omega^p \otimes \mathscr{W}^{n-p} = \mathrm{Hom}(\Omega^{n-p} \otimes \mathscr{W}^{p-n}, \Omega^n)$$

so that one may identify $\boldsymbol{H}^0(K(\mathscr{W}^n))$ and $\mathrm{Ext}^n(\mathscr{O}_Z, \Omega^n)$. Our claim is:

$$\begin{array}{ccc} \mathrm{Ext}^n(\mathscr{O}_Z, \Omega^n) & \xrightarrow{e} & H^n(X, \Omega^n) \\ & {\scriptstyle \mathrm{Res}} \searrow \quad \swarrow & \\ & \boldsymbol{C} & \end{array} \tag{3.4}$$

is commutative, where Res is the Grothendieck Residue, e is the edge morphism in the spectral sequence, and $H^n(X, \Omega^n) \to \boldsymbol{C}$ is the *canonical* map (called Tr in [8], and $\int$ in [9]). [This latter notation is misleading, since in fact the map Tr is given by $1/(2\pi i)^n \int_X$ under the standard identification of $H^n(X, \Omega^n)$ and $H^{2n}(X, \boldsymbol{C})$.] The commutativity of the above diagram is essentially the definition of Res, see [9] or [8, Chapter III, § 1] for a proof of commutativity.

c) Follows immediately from the fact that $e(p\tilde{c}(\mathscr{E})) = 0$ if $\deg p < n$.

§ 4. Examples

(I) The universal Chern classes

Let Grass (k, n) denote the set of k-planes in $\boldsymbol{C}^n$ viewed as the homogeneous space of left cosets G/H where $G = GL(n; \boldsymbol{C})$ and $H = GL(k, n-k; \boldsymbol{C})$ is the isotropy group of the k-plane $\boldsymbol{C}^k$ spanned by $e_1, \dots, e_k$ for the left action of G on Grass (k, n). For any $\tau \in G$, let $W(\tau)$ denote the first k columns of τ and for any sequence $I = \{1 \leqq i_1 < i_2 < \dots < i_k \leqq n\}$ denote by U_I the Zariski open H-invariant set in G defined by the requirement that $W_I(\tau)$, the submatrix of $W(\tau)$ formed by selecting rows I, be nonsingular. The universal subbundle $\mathscr{E}$ consisting of all pairs (S, x) where $S \in$ Grass (k, n) and $x \in S$ can be viewed as the homogeneous vector bundle $G \times \boldsymbol{C}^k/H$ where the right action of H is $(\tau, v)\,\phi = (\tau\phi, \phi^{-1} v)$. From this description it follows that $\mathscr{E}$ is trivial over $\tilde{U}_I = U_I/H$ and that the patching data on $\tilde{U}_I \cap \tilde{U}_J$ are $f_{IJ} = W_I W_J^{-J}$. An explicit representative of $c(\mathscr{E})$ in $C^1(\{U_I\}, \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \Omega^1_{\mathrm{Grass}})$ is $df_{IJ} f_{IJ}^{-1}$. First, computing this on G, gives

$$df_{IJ} f_{IJ}^{-1} = dW_I W_I^{-1} - f_{IJ} dW_J W_J^{-1} f_{IJ}^{-1}. \tag{4.1}$$

It is well known that a given $V \in H^0\ \Theta_{\mathrm{Grass}(k,n)}$ lifts to a linear vector field M on G; i.e. $M \in \mathrm{Hom}(C^n, C^n) \subset H^0(\Theta_G)$. Now (4.1) implies that the $\tilde{M}_I = i(M)\, dW_I W_I^{-1}$ provide a lift of M to a derivation on $\mathcal{O}_G^k$, consequently if the $\tilde{M}_I$ descend to $\tilde{V}_I$ on $\tilde{U}_I$ we will have established V-equivariance of $\mathscr{E}$. But

$$i(M) dW_I W_I^{-1} = (MW)_I W_I^{-1} = M_I W W_I^{-1},$$

hence M_I clearly does descend. We can thus represent $\tilde{c}(\mathscr{E})$ by

$$df_{IJ} f_{IJ}^{-1} - M_I W W_I^{-1}$$

and therefore the jth universal Chern class $c_j(\mathscr{E})$ is represented in $H^0(Z, \mathcal{O}_Z)$ by $(-1)^j \sigma_j(M_I W W_I^{-1})$.

To continue the example let $M = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in C$ are distinct complex numbers. Then the zeros of the induced vector field V on Grass (k, n) are all simple and these occur at the $\binom{n}{k}$ k-planes $\zeta_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$. Consequently $H^0(Z, \mathcal{O}_Z) = \oplus_I C$ is the ring of all $\binom{n}{k}$ square diagonal matrices and the computation of the contribution to $c_j(\mathscr{E})$ at the zero ζ_I is $(-1)^j \sigma_j(M W W_I^{-1}) = (-1)^j \sigma_j(\lambda_{i_1}, \ldots, \lambda_{i_k})$.

(II). *The Cohomology Ring of* P^{n-1}

The vector field

$$M = \begin{pmatrix} 0 & I_{n-1} \\ \vdots & \\ 0 & \ldots 0 \end{pmatrix}$$

on C^n vanishes along the line spanned by e_1 hence descends to $V \in H^0(\Theta_{P^{n-1}})$ with unique zero $\zeta = [1, 0, \ldots, 0]$. In inhomogeneous coordinates $w_1, \ldots, w_{n-1}$ about ζ

$$V = (w_2 - w_1^2)\, \partial/\partial w_1 + (w_3 - w_1 w_2)\, \partial/\partial w_2 + \ldots - w_1 w_{n-1}\, \partial/\partial w_{n-1}$$

It follows that $H^0(Z, \mathcal{O}_Z) = C[w_1]/(w_1^n)$. In the notation of the above example, $I = \{1\}$ and

$$W W_I^{-1} = \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}.$$

Therefore, $-M_I W W_I^{-1} = -w_1$ is the first Chern class of the universal subbundle $\mathcal{O}(-1)$ on P^{n-1}. Granting that $c_1(\mathcal{O}(-1))^{n-1} \neq 0$ it follows that the filtration degree of w^i is precisely $-i$, hence $H^{\cdot}(P^{n-1}, C)$ is the graded polynomial ring generated by the element $c_1(\mathcal{O}(-1))$ and truncated at degree n.

Remark. The constructions employed in (I) are valid for any V-equivariant sheaf $\mathscr{E}$ on an arbitrary X. The appropriate role of G and M are respectively as the bundle $\mathscr{F}$ of frames of $\mathscr{E}$ and as equivariant lift of V to $\mathscr{F}$. The precise formulations are left to the reader.

§ 5. Higher Dimensional Foliations

Ideally, one would hope to treat the study of singular foliations $\mathscr{W}^* \to \Theta$ with $\operatorname{rank}(\mathscr{W}) > 1$ in a completely parallel manner, by employing generalized Kozsul complexes to simplify the argument of [6]. The simpler results of [4] are readily described in the present context:

Proposition 5.1. *Suppose* $i(V)$: $\Omega^1 \to \mathscr{W}$ *is a surjection and* $\mathscr{E}$ *is V-equivariant. Then for any* $\mathcal{O}_X$*-linear* p: $\operatorname{Hom}(\mathscr{E}, \mathscr{E})^{\otimes k} \to \mathcal{O}_X$, $p(c(\mathscr{E})) = 0$ *for* $\deg p = k > \operatorname{corank} \mathscr{W}$.

Proof. Following an idea of Sommese, consider the induced exact sequence of locally free sheaves

$$0 \to \operatorname{Hom}(E, E) \otimes \mathscr{F} \to \operatorname{Hom}(\mathscr{E}, \mathscr{E}) \otimes \Omega^1 \to \operatorname{Hom}(E, E) \otimes \mathscr{W} \to 0 .$$

Considering the H^1-level of the cohomology exact sequence one sees that V-equivariance implies $c(\mathscr{E}) \in H^1(X, \operatorname{Hom}(\mathscr{E}, \mathscr{E}) \otimes \mathscr{F})$. Hence $c(\mathscr{E})^{\otimes k} = 0$ if $k > \operatorname{rank} \mathscr{F}$.

Proposition 5.1, which is valid in all generality, explains in our context the foliation vanishing theorem of Bott [4] since, if $\mathscr{F}$ generates an ideal in $\Omega^{\cdot}$ closed under d, then $Q = \mathscr{F}^*$ is equivariant (see Section 1).

Proposition 5.2. *Suppose* $i(V)$: $\Omega^1 \to \mathscr{W}$ *is surjective and that* X *admits a line bundle* $\mathscr{L}$ *such that* $c_1(\mathscr{L})^k \neq 0$ *for some* $k > \operatorname{corank} \mathscr{W}$. *Then* $H^1(X, \mathscr{W}) \neq 0$.

Proof. If $H^1(X, \mathscr{W}) = 0$, then $\mathscr{L}$ is equivariant so Proposition 5.1 applies giving a contradiction.

§ 6. Coherent Sheaves, Equivariant Complexes

For general $\mathcal{O}_X$-modules $\mathscr{E}$, there are both local and global obstructions to equivariance. Denoting by $J^1(\mathscr{E})$ the $\mathscr{E}$-valued one jets, one has an exact sequence (see [1])

$$0 \to \Omega^1 \otimes \mathscr{E} \to J^1(\mathscr{E}) \to \mathscr{E} \to 0 . \tag{6.1}$$

The equivariance of $\mathscr{E}$ is equivalent to the existence of a linear map $J^1(\mathscr{E}) \to \mathscr{E}$ extending the map $i(V) \otimes 1$: $\Omega^1 \otimes \mathscr{E} \to \mathscr{E}$ and hence is obstructed by an element $\delta(V)$ of global $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$. The obstruction to splitting (6.1) is the generalized Atiyah class $c(\mathscr{E}) \in \operatorname{Ext}^1(\mathscr{E}, \Omega^1 \otimes \mathscr{E})$ and one obtains $i(V)\, c(\mathscr{E}) = \delta(V)$ under the natural map $i(V)$: $\operatorname{Ext}^1(\mathscr{E}, \Omega^1 \otimes \mathscr{E}) \to \operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$.

To be more explicit, assume

$$0 \to \mathscr{F}^{-n} \xrightarrow{\lambda} \mathscr{F}^{-n+1} \to \ldots \to \ldots \xrightarrow{\lambda} \mathscr{F}^0 \to \mathscr{E} \to 0 \tag{6.2}$$

is a global resolution of the coherent sheaf $\mathscr{E}$ by locally free $\mathscr{F}$. Then the Atiyah obstruction may be calculated in the following manner. Fix a Leray covering $\mathscr{U} = \{U_\alpha\}$ such that $\mathscr{F}^i$ is free on U_α. Fix local connexions d^i_α for $\mathscr{F}^i$ on U_α. Denote by $\operatorname{Hom}^{\cdot}(\mathscr{F}, \mathscr{F})$ the complex of sheaves with $\operatorname{Hom}^p(\mathscr{F}, \mathscr{F})$ the $\mathcal{O}_X$-linear maps

$\mathscr{F}^i \to \mathscr{F}^{i+p}$ and with differential D: $\mathrm{Hom}^p \to \mathrm{Hom}^{p+1}$ defined by $D(\Phi) = \lambda \cdot \Phi + (-1)^p \Phi \cdot \lambda$ where λ denotes the differential in $\mathscr{F}^{\cdot}$. Note that

$$d^i_\alpha \cdot \lambda - \lambda \cdot d^{i-1}_\alpha \in C^0(\mathscr{U}, \mathrm{Hom}^1(\mathscr{F}, \mathscr{F}) \otimes \Omega^1)$$
$$d^i_\beta - d^i_\alpha \in C^1(\mathscr{U}, \mathrm{Hom}^0(\mathscr{F}, \mathscr{F}) \otimes \Omega^1) \tag{6.3}$$

define a 1-cocycle θ in the complex

$$C(\mathrm{Hom}(\mathscr{F}, \mathscr{F}) \otimes \Omega^1) = \bigoplus_{r+s=\cdot} C^r(\mathscr{U}, \mathrm{Hom}^s(\mathscr{F}, \mathscr{F}) \otimes \Omega^1) \tag{6.4}$$

in which the differential $C^r(\mathrm{Hom}^s) \to C^{r+1}(\mathrm{Hom}^s) \oplus C^r(\mathrm{Hom}^{s+1})$ is given by $\delta + (-1)^r D$. One checks easily that $\mathscr{E}$ has a connexion if and only if θ cobounds. In fact, the cohomology of the complex (6.4) is simply $\mathrm{Ext}^1(\mathscr{E}, \mathscr{E} \otimes \Omega^1)$ and θ represents the Atiyah class.

Given a vector field V, the obstruction to V-equivariance may be similarly calculated and is in fact the image of θ under

$$i(V)\colon C(\mathrm{Hom}(\mathscr{F}, \mathscr{F}) \otimes \Omega^1) \to C(\mathrm{Hom}(\mathscr{F}, \mathscr{F}))\,.$$

Assuming that $\mathscr{E}$ is V-equivariant, we obtain an element $L \in C^0(\mathrm{Hom}(\mathscr{F}, \mathscr{F}))$ whose total (δ, D) coboundary is $i(V)\theta$, and $\theta + L$ is therefore a total cocycle in the double complex

$$K^r(\mathrm{Hom}(\mathscr{F}, \mathscr{F})) = \bigoplus_{p-q=r} C^p(\mathrm{Hom}(\mathscr{F}, \mathscr{F}) \otimes \Omega^q)$$

in which the differential is $\delta \pm D$ in the p direction and $i(V)$ in the q direction. To define Chern numbers (or classes), one must give maps of *complexes* of sheaves $\otimes^p \mathrm{Hom}^{\cdot}(\mathscr{F}, \mathscr{F}) \to \mathscr{O}_X$ where $\otimes^p$ comes equipped with differential

$$D \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes D$$

and $\mathscr{O}_X$ is viewed as a complex concentrated in degree zero, i.e. maps are admissable if and only if they send boundaries in $\otimes^p \mathrm{Hom}^{\cdot}(\mathscr{F}, \mathscr{F})$ to zero. Note that when $p=1$, the boundaries in $\mathrm{Hom}^{\cdot}(\mathscr{F}, \mathscr{F})$ are the maps homotopic to zero hence admissable maps kill homotopies. As an example, the alternating sum of traces for $\Phi \in \mathrm{Hom}^0(\mathscr{F}, \mathscr{F})$ is admissable. One may calculate the Chern numbers of $\mathscr{E}$, i.e. the virtual Chern numbers of the complex $0 \to \mathscr{F}^{-n} \to \ldots \to \mathscr{F}^0 \to 0$ in the hypercohomology of the Koszul complex (2.1) associated to V by applying suitable admissable polynomials. Determination of these numbers on Z follows from our earlier remarks. This applies equally to the meromorphic case.

Note moreover that given any complex of locally free sheaves

$$\ldots \to \mathscr{L}^i \to \mathscr{L}^{i+1} \to \ldots$$

with $\mathscr{L}^i = 0$ for $|i|$ sufficiently great, one could define V-equivariance of $\mathscr{L}^{\cdot}$ by requiring that if θ is the cocycle defined as in (6.2) then $i(V)\theta$ cobounds and one may calculate Chern numbers of $\mathscr{L}^{\cdot}$ (or Chern classes if $Z \neq \emptyset$) by calculations on Z. N.B. The individual $\mathscr{L}^i$ need not be equivariant and their Chern numbers need not be calculable on Z in a natural way.

References

1. Atiyah, M. F.: Complex analytic connections in fibre bundles. Trans. Amer. Math. Soc. **85**, 181—207 (1957)
2. Atiyah, M. F., Singer, I.: The index of elliptic operators. III. Ann. Math. **87**, 546—604 (1968)
3. Bott, R.: Vector fields and characteristic numbers. Mich. Math. J. **14**, 231—244 (1967)
4. Bott, R.: On a topological obstruction to integrability. Proc. Sympos. Pure Math. **XVI**, AMS, 127—132 (1970)
5. Baum, P., Bott, R.: On the zeros of meromorphic vector fields. Essays on Topology and related topics (*Memoires dedies a Georges de Rham*), pp. 29—47. Berlin, Heidelberg, New York: Springer 1970
6. Baum, P., Bott, R.: Singularities of Holomorphic foliations. J. Diff. Geom. **7**, 279—342 (1972)
7. Carrell, J., Lieberman, D.: Holomorphic vector fields and Kaehler manifolds. Inventiones Math. **21**, 303—309
8. Hartshorne, R.: Residues and duality. Lecture notes in mathematics. No. 20. Berlin, Heidelberg, New York: Springer 1966
9. Verdier, J. L.: Base change for twisted inverse image of coherent sheaves. Algebraic geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), pp. 393—408. London: Oxford University Press 1969

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