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# Modular Forms of Half-Integral Weight on $\Gamma_0(4)$

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## Introduction

For each positive squarefree integer, Shimura [11] and Niwa [9] constructed a lifting of cusp forms of weight $k+1/2$ for $\Gamma_0(4N)$ with character $\chi$ to cusp forms of weight $2k$ for $\Gamma_0(2N)$ with character $\chi^2$; here $k$ denotes an integer $\geqq 3$. Now one can look for a subspace which under the above or similar liftings corresponds to the space of cusp forms of weight $2k$ for $\Gamma_0(N)$. The present paper investigates this problem in the simplest case, where $N=1$ and $\chi$ is trivial. Probably our results can be generalized to arbitrary level $N$. However, we have not checked this as yet.

## Notation

If $z\in\mathbb{C}^*$ and $x\in\mathbb{C}$, we put $z^x=e^{x\log z}$, where $\log z=\log|z|+i\arg z$ and the argument is determined by $-\pi<\arg z\leqq\pi$. The letter $\mathfrak{H}$ stands for the upper half-plane $\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$. For $z\in\mathfrak{H}$ we set $q=e^{2\pi iz}$.

The symbol $\left(\frac{c}{d}\right)$ defined for $c,d\in\mathbb{Z}$, $d\neq 0$, is used as in [11].

If $K$ is a quadratic number field, we denote by $\mathcal{O}_K$ its ring of integers and by $\delta_K$ its different. We write $v'$ for the conjugate of $v$ in $K$. If $v$ is totally positive, we write $v\gg 0$.

Throughout the paper we assume that $k$ is an integer. We write $M_k(1)$ and $S_k(1)$ for the space of modular forms and cusp forms of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$, respectively. The space of modular forms (cusp forms) of weight $k+1/2$ for $\Gamma_0(4)$ is denoted by $M_{k+1/2}(4)$ $(S_{k+1/2}(4))$.

## 1. Statement of Results

We have subdivided our results into two propositions and three theorems.

Define $M^+_{k+1/2}(4)$ as the subspace of $M_{k+1/2}(4)$ consisting of modular forms whose $n$-th Fourier coefficients vanish whenever $(-1)^k n\equiv 2,3 \pmod 4$, and put $S^+_{k+1/2}(4)=M^+_{k+1/2}(4)\cap S_{k+1/2}(4)$.

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We let $\theta=\sum_{n\in\mathbb{Z}} q^{n^2}$ be the standard theta function, which is in $M_{1/2}^+(4)$. Furthermore, if $k\geqq 2$, $H_{k+1/2}$ denotes the uniquely determined linear combination of the Eisenstein series of weight $k+1/2$ on $\Gamma_0(4)$, which is contained in $M_{k+1/2}^+(4)$ and equals 1 at infinity. This series was introduced and studied by Cohen [3].

**Proposition 1.** *If k is even the spaces* $M_k(1)\oplus M_{k-2}(1)$ *and* $M_{k+1/2}^+(4)$ *are isomorphic under the map* $(g(z),h(z))\mapsto g(4z)\theta(z)+h(4z)H_{5/2}(z)$. *If k is odd the spaces* $M_{k-3}(1)\oplus M_{k-5}(1)$ *and* $M_{k+1/2}^+(4)$ *are isomorphic under the map* $(g(z),h(z))\mapsto g(4z)H_{7/2}(z)+h(4z)H_{11/2}(z)$. *One has* $\dim M_{k+1/2}^+(4)=\dim M_{2k}(1)$ *and* $\dim S_{k+1/2}^+(4)=\dim S_{2k}(1)$. *For* $k\geqq 2$ *we have* $M_{k+1/2}^+(4)=\mathbb{C}H_{k+1/2}\oplus S_{k+1/2}^+(4)$.

Now define operators $U_4$ and $W_4$ acting on $M_{k+1/2}(4)$ by

$$(f|U_4)(z)=\frac{1}{4}\sum_{v \bmod 4} f\left(\frac{z+v}{4}\right),$$

$$(f|W_4)(z)=(-2iz)^{-k-1/2}f\left(-\frac{1}{4z}\right).$$

These operators leave $S_{k+1/2}(4)$ stable. Niwa [10] proved that $U_4W_4$ is hermitian on the Hilbert space $S_{k+1/2}(4)$ (with respect to the Petersson scalar product), and that it satisfies the equation $(U_4W_4-\alpha_1)(U_4W_4-\alpha_2)=0$ with $\alpha_1=\left(\frac{2}{2k+1}\right)2^k$ and $\alpha_2=-\frac{1}{2}\alpha_1$. Thus we have an orthogonal decomposition

$$S_{k+1/2}(4)=\bigoplus_{v=1,2} S_{k+1/2}^{(v)}(4),$$

where $S_{k+1/2}^{(v)}(4)$ is the eigenspace for the eigenvalue $\alpha_v$.

**Proposition 2.** *One has* $S_{k+1/2}^+(4)=S_{k+1/2}^{(1)}(4)$.

Let $p$ be a prime. If $f=\sum_{n\geqq 0} a(n)q^n$ is an element of $M_{k+1/2}^+(4)$, define

$$f|T_{k+1/2}^+(p^2)=\sum_{\substack{n\geqq 0\\(-1)^k n\equiv 0,1 \,(\mathrm{mod}\, 4)}}\left(a(p^2n)+\left(\frac{(-1)^k n}{p}\right)p^{k-1}a(n)+p^{2k-1}a\left(\frac{n}{p^2}\right)\right)q^n$$

[for a number-theoretic function $a(n)$ we put $a(x)=0$ if $x\notin\mathbb{N}\cup\{0\}$]. Note that, if $p$ is odd, $T_{k+1/2}^+(p^2)$ is the restriction of Shimura's Hecke operator of degree $p^2$ to $M_{k+1/2}^+(4)$ (cf. [11]), and that the definition of $T_{k+1/2}^+(4)$ is already implicitly contained in Shintani's paper [12].

We denote by $T_{2k}(p)$ the Hecke operator of degree $p$ acting on $M_{2k}(1)$ by

$$\sum_{n\geqq 0} c(n)q^n|T_{2k}(p)=\sum_{n\geqq 0}\left(c(pn)+p^{2k-1}c\left(\frac{n}{p}\right)\right)q^n.$$

Our first main result says that $S_{k+1/2}^+(4)$ and $S_{2k}(1)$ are isomorphic as modules over the Hecke algebra.

**Theorem 1.** i) *The operators* $T_{k+1/2}^+(p^2)$ *preserve* $M_{k+1/2}^+(4)$ *and* $S_{k+1/2}^+(4)$. *On* $S_{k+1/2}^+(4)$ *they are hermitian.*

ii) *The space* $S^+_{k+1/2}(4)$ *has an orthogonal basis of common eigenfunctions for all* $T^+_{k+1/2}(p^2)$, *unique up to multiplication with non-zero complex numbers. If* $f$ *is such an eigenform, and* $f|T^+_{k+1/2}(p^2)=\lambda_p f$, *then there is an eigenform* $F\in S_{2k}(1)$, *uniquely determined up to multiplication with a non-zero complex number, which satisfies* $F|T_{2k}(p)=\lambda_p F$ *for all primes* $p$. *The Fourier expansions of* $f$ *and* $F$ *are related as follows: if* $f=\sum_{n\geq 1} a(n)q^n$ *and* $F=\sum_{n\geq 1} A(n)q^n$, *and if* $D$ *is a fundamental discriminant (i.e.* $D$ *equals* 1 *or is the discriminant of a quadratic field) such that* $(-1)^k D>0$, *then*

$$L\left(s-k+1,\left(\frac{D}{\cdot}\right)\right)\sum_{n\geq 1} a(|D|n^2)n^{-s}=a(|D|)\sum_{n\geq 1} A(n)n^{-s}.$$

iii) *If* $D$ *is as in* ii) *and* $(D,k)\neq(1,0)$, *the map* $\mathcal{S}^+_{D,k}$ *defined by*

$$\sum_{n\geq 0} b(n)q^n\mapsto\frac{b(0)}{2}\mathrm{L}\left(1-k,\left(\frac{D}{\cdot}\right)\right)+\sum_{n\geq 1}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}b\left(\frac{n^2}{d^2}|D|\right)\right)q^n$$

*maps* $M^+_{k+1/2}(4)$ *to* $M_{2k}(1)$ *and* $S^+_{k+1/2}(4)$ *to* $S_{2k}(1)$ *and commutes with the action of Hecke operators. There exists a linear combination in the* $\mathcal{S}^+_{D,k}$ *which is an isomorphism.*

The proofs of Propositions 1 and 2 and of Theorem 1 are based on a result of Niwa's [10], who using a trace formula of Shimura's, showed that $S_{k+1/2}(4)$ and the space of cusp forms of weight $2k$ on $\Gamma_0(2)$ are isomorphic as modules over the Hecke algebra.

*Remark 1.* Shimura's main theorem in [11] in case of level 4 and trivial character is very similar to our theorem. Examples, however, show that a "multiplicity 1 theorem" does not hold for $S_{k+1/2}(4)$.

*Remark 2.* By using Theorem 1 and the methods of [5] one can prove that if $g=\sum_{n\geq 0} c(n)q^n\in M_k(1)$ and $K$ is a real quadratic field of discriminant $D$, then

$$g|\iota_k=\frac{c(0)}{2}\mathrm{L}\left(1-k,\left(\frac{D}{\cdot}\right)\right)+\left(\sum_{\substack{\nu\in\delta_K^{-1}\\ \nu\gg 0}}\sum_{\substack{d\in\mathbb{N}\\ d|(\nu)\delta_K}}\left(\frac{D}{d}\right)d^{k-1}c\left(\frac{\nu\nu'}{d^2}D\right)\right)e^{2\pi i(\nu z+\nu' z')}$$

$(z,z'\in\mathfrak{H})$

is a Hilbert modular form of weight $k$ for $\mathrm{SL}_2(\mathcal{O}_K)$. Note that $g|\iota_k$ is the Doi-Naganuma lifting of $g$, cf. [6, 14] and [16, Sect. 6].

The next theorem gives a relationship between the map $\mathcal{S}^+_{1,k}$ and the non-vanishing of the Dirichlet series attached to a Hecke eigenform of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$ at the real point of the critical line. The existence of such a relationship is indicated by Shintani's paper [12].

Let $g=\sum_{n\geq 1} c(n)q^n$ be a normalized Hecke eigenform of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$ and denote by $\mathrm{L}_g(s)=(2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} c(n)n^{-s}$ $(\mathrm{Re}\, s\gg 0)$ the associated Dirichlet series completed with the gamma factor. Recall that $L_g(s)$ has a holomorphic continuation to the entire complex plane and satisfies the functional equation $\mathrm{L}_g(2k-s)=(-1)^k \mathrm{L}_g(s)$.

**Theorem 2.** *Let $k$ be even. The image of the restriction of $\mathscr{S}_{1,k}^{+}$ to $S_{k+1/2}^{+}(4)$ is generated by those normalized Hecke eigenforms $g$ in $S_{2k}(1)$ which satisfy $\mathrm{L}_g(k) \neq 0$. The map $\mathscr{S}_{1,k}^{+}$ is an isomorphism if and only if $\mathrm{L}_g(k) \neq 0$ for all normalized Hecke eigenforms $g$ in $S_{2k}(1)$.*

**Corollary.** *Let $k$ be even, and let $r$ be the dimension of $S_{2k}(1)$. Then of the $r$ normalized Hecke eigenforms $g \in S_{2k}(1)$, at least $[\sqrt{2r}]$ satisfy $L_g(k) \neq 0$.*

The heart of the proof of Theorem 2 consists of an application of a result of Zagier's [16], which is based on Rankin's convolution idea. It is not difficult to see that $\dim S_{k+1/2}^{+}(4) | \mathscr{S}_{1,k}^{+} \geqq [\sqrt{2r}]$, hence the corollary.

*Remark.* As was communicated to me by Zagier, Buhler verified by a numerical computation that for $k \leqq 200$ the Hecke algebra on $S_{2k}(1)$ is irreducible over $\mathbb{Q}$. From this it follows easily that if $k \leqq 200$, $\mathrm{L}_g(k) \neq 0$ for all normalized Hecke eigenforms $g$ in $S_{2k}(1)$.

The last theorem of this paper gives congruences for the Hecke-Eisenstein series associated to real quadratic fields and is a consequence of the correspondence, first discovered by Cohen [4, 5], between liftings of modular forms of half-integral weight to modular forms of integral weight in one variable and liftings of the latter to Hilbert modular forms in two variables.

Fix a fundamental discriminant $D > 1$ and put $K = \mathbb{Q}(\sqrt{D})$. Suppose that $k$ is even and $\geqq 2$. The Hecke-Eisenstein series $g_k^K(z,z')\,(z,z' \in \mathfrak{H})$ of weight $k$ for $K$ is defined by

$$g_k^K(z,z') = \tfrac{1}{4}\zeta_K(1-k) + \sum_{\substack{\nu \in \delta_K^{-1} \\ \nu \gg 0}} \Bigg( \sum_{\mathfrak{A} | (\nu)\delta_K} \mathcal{N}(\mathfrak{A})^{k-1} \Bigg) e^{2\pi i(\nu z + \nu' z')},$$

where $\zeta_K$ is the Dedekind zeta function of $K$, and where the inner sum runs over all integral ideals $\mathfrak{A}$ in $\mathcal{O}_K$ that divide the integral ideal $(\nu)\delta_K$. The series $g_k^K$ is a Hilbert modular form of weight $k$ for $\mathrm{SL}_2(\mathcal{O}_K)$ (cf. e.g. [7], Kap. 20); its Fourier coefficients, except for the constant term, are by definition rational integers. Thus the restriction to the diagonal $G_{2k}^K(z) = g_k^K(z,z)$ is contained in the $\mathbb{Z}$-module $M_{2k}^{\mathbb{Z}}$ consisting of modular forms of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z})$ whose $q$-coefficients, apart from the constant term, are all integral. The module $M_{2k}$ is free of rank $\dim M_{2k}(1)$, and $M_{2k}(1) = M_{2k}^{\mathbb{Z}} \bigotimes_{\mathbb{Z}} \mathbb{C}$.

**Theorem 3.** *For $2 \leqq k \leqq 10$ and $K$ not belonging to the finite set*

$$(*) \quad \{\mathbb{Q}(\sqrt{2})\} \cup \{\mathbb{Q}(\sqrt{p}) \,|\, p \text{ prime}, (p-1) \nmid k, (p-1) | 2k\}$$

*the function $G_{2k}^K$ is contained in the lattice $M_{2k}^{HE} \subset M_{2k}^{\mathbb{Z}}$ given by the following table (in which $Q$ and $R$ denote the normalized Eisenstein series in $M_4(1)$ and $M_6(1)$, respectively):*

| $k$ | Basis for $M_{2k}^{HE}$ | $[M_{2k}^{\mathbb{Z}}:M_{2k}^{HE}]$ |
|---|---|---|
| 2 | $\frac{1}{24}Q$ | $2.5 \quad = \quad 10$ |
| 4 | $\frac{1}{240}Q^2$ | $2$ |
| 6 | $\frac{1}{24}Q^3$, $\frac{5}{504}R^2$ | $2^4 \cdot 3^2 \cdot 5^2 \cdot 13 \quad = \quad 46{,}800$ |
| 8 | $\frac{7}{480}Q^4$, $\frac{5}{12}QR^2$ | $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \quad = \quad 171{,}360$ |
| 10 | $\frac{147}{8}Q^5$, $\frac{5}{264}Q^2R^2$ | $2^2 \cdot 3^4 \cdot 5^3 \cdot 7^2 \quad = 7{,}938{,}000$ |

This result was conjectured on the basis of extensive numerical evidence by Zagier [15]. He also conjectured that there are similar results for higher weights and that the lattice $M_{2k}^{HE}$ cannot be made smaller by enlarging the finite set (*), but up to now we do not see a way to attack these problems. Note that (as described in detail in [15]) the statement for $k=2$ and $k=4$ and part of the statement for larger $k$ follow from the results of Fresnel, Serre, and Deligne-Ribet on the denominator of $\zeta_K(1-k)$ (these imply that $G_{2k}^K$ lies in a certain sublattice of $M_{2k}^{\mathbb{Z}}$, denoted $M_{2k}^{Se}$ in [15], which for $k=6, 8$, and 10 has index 130, 34, and 50, respectively). The point of the above theorem is that the restrictions of the Hecke-Eisenstein series to the diagonal satisfy many congruences above and beyond those needed to give the right denominator for the constant term. It is also of interest that the set (*) of exceptional fields for the congruence $G_{2k}^K \subset M_{2k}^{HE}$ is the same (i.e. no larger) than the set of fields which must be expected to get the best bound on $\zeta_K(1-k)$.

## 2. Proofs

### *2.1. Preliminaries*

For details on modular forms of half-integral weight the reader is referred to [11] and [3].

We introduce the group $\mathfrak{G}$ consisting of all pairs $(A, \phi(z))$, where $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(z)$ is a complex valued function holomorphic on $\mathfrak{H}$ satisfying $|\phi(z)|=(\det A)^{-1/4}|cz+d|^{1/2}$, with group law defined by $(A, \phi(z))(B, \psi(z)) =(AB,\ \phi(Bz)\,\psi(z))$. If $f:\mathfrak{H}\to\mathbb{C}$ and $\xi=(A, \phi(z))\in\mathfrak{G}$, we put $f|_{k+1/2}\xi=f|\xi =\phi(z)^{-2k-1}f(Az)$. Then $f|\xi_1|\xi_2=f|\xi_1\xi_2$. We have a monomorphism $\Gamma_0(4)\to\mathfrak{G}$ given by $A\mapsto A^*:=(A, \mathrm{j}(A,z))$, where $\mathrm{j}(A,z)=\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz+d)^{1/2}$ if $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Recall that $M_{k+1/2}(4)$ consists of all complex valued functions $f$ holomorphic on $\mathfrak{H}$ which satisfy $f|A^*=f$ for every $A\in\Gamma_0(4)$, and which are holomorphic at the cusps, while $S_{k+1/2}(4)$ is the subspace of $M_{k+1/2}(4)$ consisting of those $f$ which vanish at the cusps.

The Riemann-Roch theorem gives

$$\dim M_{k+1/2}(4) = \sup\left\{0, 1+\left[\frac{k}{2}\right]\right\},$$

$$\dim S_{k+1/2}(4) = \sup\left\{0, -1+\left[\frac{k}{2}\right]\right\}.$$

Put $F_2 = \sum_{\substack{n\geqq 1\\ n\,\text{odd}}} \sigma_1(n)q^n$. Then $F_2$ is a modular form of weight 2 for $\Gamma_0(4)$, and $\left\{\theta^a F_2^b \,\middle|\, a, b\in\mathbb{N}, \frac{a}{2}+2b=k+1/2\right\}$ is a basis of $M_{k+1/2}(4)$.

For $k\geqq 2$ we have $M_{k+1/2}(4)=\mathbb{C}E^{i\infty}_{k+1/2}\oplus\mathbb{C}E^{0}_{k+1/2}\oplus S_{k+1/2}(4)$, where $E^{i\infty}_{k+1/2}=\sum_A \mathrm{j}(A,z)^{-2k-1}$ is an Eisenstein series for the cusp $i\infty$ $\left[\text{summation over a system of representatives for the action of } \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\,\middle|\, n\in\mathbb{Z}\right\} \text{ on } \Gamma_0(4)\right]$, and $E^0_{k+1/2} =(-1)^k iz^{-k-1/2}E^{i\infty}_{k+1/2}\left(-\frac{1}{4z}\right)$ is an Eisenstein series for the cusp 0. Define

$$H_{k+1/2}=E^{i\infty}_{k+1/2}+2^{-2k-1}(1-(-1)^k i)E^0_{k+1/2}.$$

One has (cf. [3])

$$H_{k+1/2}=1+\sum_{\substack{n\geqq 1\\ (-1)^k n\equiv 0,1 (\mathrm{mod}\, 4)}} h_{k+1/2}(n)q^n,$$

where

$$h_{k+1/2}(n)=\left(\mathrm{L}\left(1-k,\left(\frac{D}{\cdot}\right)\right)\Big/\zeta(1-2k)\right)\sum_{d\mid f}\mu(d)\left(\frac{D}{d}\right)d^{k-1}\sigma_{2k-1}\left(\frac{f}{d}\right),$$

if $(-1)^k n\equiv Df^2$ and $D$ is the discriminant of $\mathbb{Q}(\sqrt{(-1)^k n})/\mathbb{Q}$.

Let $f=\sum_{n\geqq 0} a(n)q^n$ be an element of $M_{k+1/2}(4)$. If $p$ is an odd prime, define

$$f\,|\,T_{k+1/2}(p^2)=\sum_{n\geqq 0}\left(a(p^2n)+\left(\frac{(-1)^k n}{p}\right)p^{k-1}a(n)+p^{2k-1}a\left(\frac{n}{p^2}\right)\right)q^n.$$

The Hecke operators $T_{k+1/2}(p^2)$ map cusp forms to cusp forms and are hermitian on $S_{k+1/2}(4)$.

### 2.2. *Proof of Propositions 1 and 2*

We shall prove Propositions 1 and 2 in four steps.

i) *The maps defined in Proposition 1 are injective. One has* $\dim M_{2k}(1)\leqq\dim M^+_{k+1/2}(4)$.

*Proof.* If $g(4z)\theta(z)+h(4z)H_{5/2}(z)$ or $g(4z)H_{7/2}(z)+h(4z)H_{11/2}(z)$ is identically zero and $h\neq 0$, the function $H_{5/2}(z)/\theta(z)=1-12q+\dots$ or $H_{11/2}(z)/H_{7/2}(z)$

$=1-144q^2+\dots$ would be invariant under $z\mapsto z+\frac{1}{4}$, a contradiction. Hence our maps are injective. In particular we conclude

$$\dim M^+_{k+1/2}(4) \geqq \begin{cases} \dim M_k(1)\oplus M_{k-2}(1) & \text{if } k \text{ is even} \\ \dim M_{k-3}(1)\oplus M_{k-5}(1) & \text{if } k \text{ is odd}. \end{cases}$$

Note that the number on the right-hand side is precisely $\dim M_{2k}(1)$, as follows from the well-known formula

$$\dim M_k(1) = \begin{cases} \sup\left\{0, \left[\frac{k}{12}\right]\right\} & \text{if } k \text{ is even, } k\equiv 2(\text{mod}\,12) \\ \sup\left\{0, 1+\left[\frac{k}{12}\right]\right\} & \text{if } k \text{ is even, } k\not\equiv 2(\text{mod}\,12). \end{cases}$$

ii) One has $M^+_{k+1/2}(4)\subset M^{(1)}_{k+1/2}(4) := \{f\in M_{k+1/2}(4) \mid f|U_4W_4=\alpha_1 f\}$.

*Proof*. Let $f$ be an element of $M^+_{k+1/2}(4)$. By definition

$$f|U_4W_4=f_1+f_2,$$

where

$$f_1=2^{k-2+1/2}\left(f\left|\left(\begin{pmatrix}1 & 1\\ 0 & 4\end{pmatrix}, 2^{1/2}\right)+f\right|\left(\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}, 2^{1/2}\right)\right)\Bigg| W_4$$

and

$$f_2=\frac{1}{4}\left(f\left(\frac{z}{4}\right)+f\left(\frac{z+2}{4}\right)\right)\Bigg| W_4.$$

We have

$$\begin{aligned} 2^{-k+2-1/2}f_1 &= f\left|\left(\begin{pmatrix}4 & -1\\ 16 & 0\end{pmatrix}, 2(-iz)^{1/2}\right)+f\right|\left(\begin{pmatrix}12 & -1\\ 16 & 0\end{pmatrix}, 2(-iz)^{1/2}\right) \\ &= f\left|\begin{pmatrix}1 & 0\\ -4 & 1\end{pmatrix}^*\right|\left(\begin{pmatrix}4 & -1\\ 16 & 0\end{pmatrix}, 2(-iz)^{1/2}\right) \\ &\quad + f\left|\begin{pmatrix}-1 & 1\\ -4 & 3\end{pmatrix}^*\right|\left(\begin{pmatrix}12 & -1\\ 16 & 0\end{pmatrix}, 2(-iz)^{1/2}\right) \\ &= f\left|\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, e^{-\pi i/4}\right)+f\right|\left(\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix}, e^{\pi i/4}\right) \\ &= 2^{-1/2}(i^k(1+i)f(z-\tfrac{1}{4})+i^{-k}(1-i)f(z+\tfrac{1}{4})). \end{aligned}$$

Since the $n$-th $q$-coefficients of $f$ vanish for $(-1)^k n\equiv 2,3(\text{mod}\,4)$, we have

$$i^k(1+i)f\left(z-\frac{1}{4}\right)+i^{-k}(1-i)f\left(z+\frac{1}{4}\right)=2\left(\frac{2}{2k+1}\right)f(z)$$

$\left[\text{note that } \left(\frac{2}{2k+1}\right)=i^{k^2+k}\right]$ and

$$f\left(\frac{z}{4}\right)+f\left(\frac{z+2}{4}\right)=2f|U_4.$$

Hence

$$f_1=\left(\frac{2}{2k+1}\right)2^{k-1}f,$$

$$f_2=\frac{1}{2}f|U_4W_4.$$

From this we get $f|U_4W_4=\alpha_1 f$.

iii) *One has* $M^{(1)}_{k+1/2}(4)=\mathbb{C}H_{k+1/2}\oplus S^{(1)}_{k+1/2}(4)$.

*Proof.* By ii), $H_{k+1/2}$ is in $M^{(1)}_{k+1/2}(4)$. If $f\in M^{(1)}_{k+1/2}(4)$, there exists $\lambda\in\mathbb{C}$ such that $g:=f-\lambda H_{k+1/2}$ vanishes at infinity. Since $g|W_4=\frac{1}{\alpha_1}g|U_4$, we conclude that $g$ vanishes at the cusp 0, too, hence is a cusp form. This proves iii).

iv) *We have* $\dim M^{(1)}_{k+1/2}(4)\leqq\dim M_{2k}(1)$ and $\dim S^{(1)}_{k+1/2}(4)\leqq\dim S_{2k}(1)$.

*Proof.* Using basis elements the first formula is easy to check for $k<2$. For $k\geqq 2$ one has $\dim M^{(1)}_{k+1/2}(4)=1+\dim S^{(1)}_{k+1/2}(4)$ and $\dim M_{2k}(1)=1+\dim S_{2k}(1)$, hence it suffices to prove that $\dim S^{(1)}_{k+1/2}(4)\leqq\dim S_{2k}(1)$.

Note that the Hecke operators $T_{k+1/2}(p^2)$ commute with $U_4$ and $W_4$, hence preserve the space $S^{(1)}_{k+1/2}(4)$. Since they generate a commutative $\mathbb{C}$-algebra of hermitian operators, $S^{(1)}_{k+1/2}(4)$ has an orthogonal basis $\{f_i\}$ of common eigenfunctions for all $T_{k+1/2}(p^2)$.

Write $S_{2k}(2)$ for the space of cusp forms of weight $2k$ for $\Gamma_0(2)$. On $S_{2k}(2)$ we have Hecke operators $T_{2k}(p)$ ($p$ an odd prime) and $U_2$ defined by

$$\sum_{n\geqq 1}c(n)q^n|T_{2k}(p)=\sum_{n\geqq 1}(c(pn)+p^{2k-1}c(n/p))q^n,$$

$$\sum_{n\geqq 1}c(n)q^n|U_2\quad=\sum_{n\geqq 1}c(2n)q^n.$$

According to Niwa (theorem in [10], Sect. 1) there exists an isomorphism $\psi:S_{k+1/2}(4)\to S_{2k}(2)$ satisfying $U_4\psi=\psi U_2$ and $T_{k+1/2}(p^2)\psi=\psi T_{2k}(p)$ for all odd primes $p$ (in [10] this is proved for $k\geqq 2$; note that for $k<2$ we have $S^{(1)}_{k+1/2}(4)=\{0\}=S_{2k}(2)$).

We now apply $\psi$ to the basis $\{f_i\}$ of $S^{(1)}_{k+1/2}(4)$. We claim that $f_i|\psi$ cannot be a new form (for the theory of new forms cf. [1], Sect. 4, in particular Theorem 5, and [8], Chap. VIII). Indeed, if it were, $f_i|\psi$ would be an eigenfunction of $U_2$ for the eigenvalue $\pm 2^{k-1}$, hence $\pm 2^{k-1}f_i=f_i|U_4$, which implies

$$\alpha_1 f_i=f_i|U_4W_4=\pm 2^{k-1}f_i|W_4,$$

a contradiction since $W_4^2=1$.

So $f_i|\psi$ is old, and we have $f_i|\psi\in\mathbb{C}F_i(z)\oplus\mathbb{C}F_i(2z)$, where $F_i\in S_{2k}(1)$ is a (uniquely determined) normalized eigenform of $T_{2k}(p)$ for all primes $p$. To complete the proof we shall show that the association $f_i\mapsto F_i$ extends to an injective linear map $\psi^+:S^{(1)}_{k+1/2}(4)\to S_{2k}(1)$. This follows from the following

**Lemma.** *Suppose* $f$ *and* $f'$ *are two non-zero elements of* $S^{(1)}_{k+1/2}(4)$ *which are eigenfunctions of* $T_{k+1/2}(p^2)$ *for all odd primes* $p$ *with the same eigenvalues. Then* $\mathbb{C}f=\mathbb{C}f'$.

*Proof.* Put $h=f|\psi$, $h'=f'|\psi$. Assume $\mathbb{C}h \neq \mathbb{C}h'$. Then we may suppose without loss of generality that $h(z)=F(z)$ and $h'(z)=F(2z)$, where $F \in S_{2k}(1)$ is a Hecke eigenform. Thus $h=h'|U_2$, which implies $f=f'|U_4$. Hence

$$\alpha_1 f' = f'|U_4W_4 = f|W_4 = \frac{1}{\alpha_1} f|U_4 = \frac{1}{\alpha_1} f'|U_4^2,$$

from which it follows that

$$2^{2k}h' = h'|U_2^2,$$

i.e.

$$2^{2k}F(2z) = (F|U_2)(z).$$

Let $F|T_{2k}(2)=\lambda F$. We obtain

$$\lambda F(z) = (F|U_2)(z) + 2^{2k-1}F(2z) = (2^{2k}+2^{2k-1})F(2z),$$

which clearly implies $F=0$, a contradiction. Therefore we must have $\mathbb{C}h=\mathbb{C}h'$, hence $\mathbb{C}f=\mathbb{C}f'$.

Propositions 1 and 2 obviously follow from i)–iv).

### 2.3. *Proof of Theorem 1*

We shall first prove that $S_{k+1/2}^+(4)$ has an orthogonal basis of common eigenfunctions of the operators $T_{k+1/2}^+(p^2)$.

If $p$ is an odd prime, $T_{k+1/2}^+(p^2)$ is the restriction of $T_{k+1/2}(p^2)$ to $M_{k+1/2}^+(4)$.

Assume $f=\sum_{n\geqq 0} a(n)q^n$ is in $M_{k+1/2}^+(4)$. We wish to prove that

$$f|T_{k+1/2}^+(4) = (f|W_4)|(U_4W_4-\alpha_2)$$

[note that $U_4W_4-\alpha_2$ is up to a constant factor the orthogonal projection of $S_{k+1/2}^{(1)}(4)$ to $S_{k+1/2}^+(4)=S_{k+1/2}^{(1)}(4)$]; this implies that $T_{k+1/2}^+(4)$ maps $M_{k+1/2}^+(4)$ and $S_{k+1/2}^+(4)$ to themselves; furthermore it follows that $T_{k+1/2}^+(4)$ is hermitian on $S_{k+1/2}^+(4)$, since $U_4W_4$ is hermitian and $W_4$ is an unitary involution.

By definition

$$(f|W_4)|(U_4W_4-\alpha_2) = \sum_{0\leqq \nu \leqq 4} s_\nu(f),$$

where

$$s_\nu(f) = 2^{k-2+1/2}(f|W_4)\left|\left(\begin{pmatrix}1 & \nu\\ 0 & 4\end{pmatrix}, 2^{1/2}\right)\right| W_4 \quad \text{for} \quad 0\leqq \nu \leqq 3$$

and

$$s_4(f) = \left(\frac{2}{2k+1}\right) 2^{k-1} f|W_4.$$

We have [compare with Sect. 2.2ii)]

$$(s_1+s_3+s_4)(f)=2^{k-2+1/2}\sum_{v=1,3}\left(\frac{1}{\alpha_1}f|U_4\right)\left|\left(\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix},2^{1/2}\right)\right|W_4+s_4(f)$$

$$=\left(\frac{2}{2k+1}\right)\frac{1}{4}\left(i^k(1+i)(f|U_4)\left(z-\frac{1}{4}\right)+i^{-k}(1-i)(f|U_4)\left(z+\frac{1}{4}\right)\right.$$

$$\left.+2\left(\frac{2}{2k+1}\right)(f|U_4)(z)\right)$$

$$=\sum_{\substack{n\geqq 0\\ (-1)^k n\equiv 0,1(\mathrm{mod}\,4)}} a(4n)q^n. \tag{1}$$

Obviously

$$s_0(f)=2^{2k-1}\sum_{n\geqq 0}a\left(\frac{n}{4}\right)q^n. \tag{2}$$

Finally let us compute $s_2(f)$. Suppose that $k$ is even. Assume $f(z)=g(4z)\theta(z)$ with $g\in M_k(1)$. Then

$$s_2(f)=2^{2k}s_2(\theta)\,g\left(\frac{16z}{-8z+1}\right)(1-8z)^{-k}.$$

The reader will easily verify that $\theta|W_4|(U_4W_4-\alpha_2)=\theta|T^+_{1/2}(4)$. Thus applying (1) and (2) to $\theta$, we see that

$$s_2(\theta)=\frac{1}{2}\sum_{n\geqq 1}\left(\frac{n^2}{2}\right)2q^{n^2}$$

$$=2^{-2-1/2}\sum_{v\,\mathrm{mod}\,8}\left(\frac{v}{2}\right)\theta\left(z+\frac{v}{8}\right)$$

$\left[\text{recall that } \sum_{v\,\mathrm{mod}\,8}\left(\frac{v}{2}\right)e^{2\pi i v(n/8)}=2^{-1-1/2}\left(\frac{n}{2}\right)\right].$

On the other hand

$$g\left(\frac{16z}{-8z+1}\right)=g\left(\frac{-1}{\frac{1}{2}-\frac{1}{16z}}\right)=g\left(\frac{1}{2}-\frac{1}{16z}\right)\left(\frac{1}{2}-\frac{1}{16z}\right)^k$$

$$=2^{-k}g\left(\frac{4z+\frac{1}{2}-1}{2(4z+\frac{1}{2})-1}\right)\left(\frac{8z-1}{8z}\right)^k$$

$$=2^{-k}g(4z+\tfrac{1}{2})(8z-1)^k.$$

Thus we obtain

$$s_2(f)=2^{k-2-1/2}\sum_{v\,\mathrm{mod}\,8}\left(\frac{v}{2}\right)g\left(4\left(z+\frac{v}{8}\right)\right)\theta\left(z+\frac{v}{8}\right)$$

$$=2^{k-2-1/2}\sum_{v\,\mathrm{mod}\,8}\left(\frac{v}{2}\right)f\left(z+\frac{v}{8}\right)$$

$$=2^{k-1}\sum_{n\geqq 0}\left(\frac{n}{2}\right)a(n)q^n.$$

If $f(z)=h(4z)H_{5/2}(z)$ with $h\in M_{k-2}(4)$, then using $H_{5/2}|W_4|(U_4W_4-\alpha_2)=H_{5/2}|T^+_{5/2}(4)$, a similar argument gives again $s_2(f)=\sum_{n\geq 0}\left(\frac{n}{2}\right)a(n)q^n$. Since according to Proposition 1 any $f\in M^+_{k+1/2}(4)$ can be written as $f(z)=g(4z)\theta(z)+h(4z)H_{5/2}(z)$ with $g\in M_k(1)$ and $h\in M_{k-2}(1)$, we are through.

If $k$ is odd an analogous argument gives $s_2(f)=\sum_{n\geq 0}\left(\frac{(-1)^k n}{2}\right)a(n)q^n$.

The $T^+_{k+1/2}(p^2)$ generate a commutative algebra of hermitian operators on the complex Hilbert space $S^+_{k+1/2}(4)$; hence $S^+_{k+1/2}(4)$ has an orthogonal basis of common eigenfunctions for all $T^+_{k+1/2}(p^2)$. We have already proved [cf. lemma in Sect. 2.2iv) and Proposition 2] that such an eigenfunction $f$ is uniquely determined by its eigenvalues up to multiplication with a non-zero complex number.

Assume $f=\sum_{n\geq 1}a(n)q^n$ and $f|T^+_{k+1/2}(p^2)=\lambda_p f$, and let $D$ be a fundamental discriminant such that $(-1)^kD>0$. A formal calculation as in [11], p. 452 shows that

$$\sum_{n\geq 1}a(|D|n^2)n^{-s}=\left(1-\left(\frac{D}{p}\right)p^{k-1-s}\right)(1-\lambda_p p^{-s}+p^{2k-1-2s})^{-1}\cdot\sum_{\substack{n\geq 1\\(n,p)=1}}a(|D|n^2)n^{-s}$$

for every prime $p$ [if $p=2$ we have to use the fact that, by definition, $a(n)=0$ for $(-1)^kn\equiv 2,3\,(\mathrm{mod}\,4)$]. From this it follows that

$$\sum_{n\geq 1}a(|D|n^2)n^{-s}=a(|D|)\prod_p\left(1-\left(\frac{D}{p}\right)p^{k-1-s}\right)(1-\lambda_p p^{-s}+p^{2k-1-2s})^{-1},$$

i.e.

$$\mathrm{L}\left(s-k+1,\left(\frac{D}{\cdot}\right)\right)\sum_{n\geq 1}a(|D|n^2)n^{-s}=a(|D|)\prod_p(1-\lambda_p p^{-s}+p^{2k-1-2s})^{-1}.$$

We will prove now the statements about the maps $\mathscr{S}^+_{D,k}$. If $f$ is an element of $M^+_{k+1/2}(4)$, a formal calculation shows that $f|\mathscr{S}^+_{D,k}T_{2k}(p)=f|T^+_{k+1/2}(p^2)\mathscr{S}^+_{D,k}$ for all primes $p$; we leave the details to the reader [if $p=2$, we again have to use the fact that the $n$-th Fourier coefficients of $f$ are zero whenever $(-1)^kn\equiv 2,3\,(\mathrm{mod}\,4)$].

We shall next show that for $(D,k)\neq(1,0)$, $\mathscr{S}^+_{D,k}$ maps $M^+_{k+1/2}(4)$ to $M_{2k}(1)$. If $k<0$ or $k=1$ we have $M^+_{k+1/2}(4)=\{0\}$, and nothing is to prove. Recall that $M^+_{1/2}(4)=\mathbb{C}\theta$ and note that $\theta|\mathscr{S}^+_{D,0}=\frac{1}{2}\mathrm{L}\left(1,\left(\frac{D}{\cdot}\right)\right)\in M_0(1)$ for $D\neq 1$.

Now suppose $k\geq 2$. Then we have the decomposition $M^+_{k+1/2}(4)=\mathbb{C}H_{k+1/2}\oplus S^+_{k+1/2}(4)$. Using the $q$-expansion of $H_{k+1/2}$, it is a simple exercise to verify that $H_{k+1/2}|\mathscr{S}^+_{D,k}=\frac{1}{2}L\left(1-k,\left(\frac{D}{\cdot}\right)\right)E_{2k}$, where $E_{2k}=1+\frac{1}{\zeta(1-2k)}\sum_{n\geq 1}\sigma_{2k-1}(n)q^n$ is the normalized Eisenstein series in $M_{2k}(1)$. Thus it remains to show that $\mathscr{S}^+_{D,k}$ maps $S^+_{k+1/2}(4)$ to $S_{2k}(1)$. We may suppose $k\geq 3$.

First assume that $D\equiv 0\,(\mathrm{mod}\,4)$. Write $D=4d$ with $d$ square-free and $d\equiv 2,3\,(\mathrm{mod}\,4)$. Let $f=\sum_{n\geq 1}a(n)q^n\in S^+_{k+1/2}(4)$. According to [11] and [9] the function $f|\mathscr{S}_{d,k}:=\sum_{n\geq 1}\left(\sum_{j|n}\left(\frac{4d}{j}\right)j^{k-1}a\left(\frac{n^2}{j^2}|d|\right)\right)q^n$ is in $S_{2k}(2)$; if $n$ is odd, its $n$-th $q$-coefficient is zero. Hence $f|\mathscr{S}^+_{D,k}=(f|\mathscr{S}_{d,k})|U_2$ is in $S_{2k}(1)$ (cf. e.g. [8], Chap. VIII, Sect. 4, Lemma 7).

Now suppose $f$ is a non-zero Hecke eigenform. We claim that there exists a fundamental discriminant $D\equiv 0 \pmod 4$ with $(-1)^k D>0$ such that $a(|D|)\neq 0$. Suppose the contrary. Then $g:=f|U_4$ has the property that its $n$-th $q$-coefficients are zero for $n\equiv 2\pmod 4$, hence

$$(*)\quad g\left|\left(\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix},1\right)+g\right|\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix},1\right)=4^{-k/2-1/4}\cdot 2g|U_4\left|\left(\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},4^{-1/4}\right).\right.$$

The right hand side and both terms on the left hand side are in $S_{k+1/2}(16)$, the space of cusp forms of weight $k+1/2$ on $\Gamma_0(16)$. Let $\mathrm{Tr}: S_{k+1/2}(16)\to S_{k+1/2}(4)$ be the trace operator defined by $h|\mathrm{Tr}=\sum_j h|A_j^*$, where $\{A_j\}$ is a set of representatives for $\Gamma_0(16)\backslash\Gamma_0(4)$. Applying Tr on both sides of $(*)$ and noting $\left(\begin{pmatrix}4 & \pm 1\\ 0 & 4\end{pmatrix},1\right)\circ\mathrm{Tr}$ $=4^{-k/2+3/4}e^{\pm 2\pi i(2k+1)/8}U_4W_4$ and $\left(\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},4^{-1/4}\right)\circ\mathrm{Tr}=4^{-k/2+3/4}(U_4W_4)^2$ (cf. [10], p. 200f., proof of lemma) we obtain $\left(\frac{2}{2k+1}\right)2^k g|U_4W_4=g|(U_4W_4)^2$, hence because $U_4W_4$ is injective, $g|U_4W_4=\left(\frac{2}{2k+1}\right)2^k g$, i.e. $g$ is in $S^+_{k+1/2}(4)$. Since $U_4$ and $T_{k+1/2}(p^2)$ ($p$ odd) commute, $f$ and $g$ have the same eigenvalues for all $T^+_{k+1/2}(p^2)$ ($p$ odd), hence $f$ and $g$ differ only by a constant factor. Moreover, since $f|U_4=2^{-k}f|W_4$ and $W_4^2=1$ we conclude $g=\pm 2^{-k}f$. An easy computation [cf. Sect. 2.2iv)] then shows that $(f|\psi^+)|T_{2k}(2)=\pm(2^k+2^{k-1})(f|\psi^+)$, which according to Deligne's theorem, previously the Ramanujan conjecture, is impossible unless $f|\psi^+=0$, i.e. $f=0$, a contradiction.

Now let $f_1,\ldots,f_r$ be an orthogonal basis of common eigenfunctions of the $T^+_{k+1/2}(p^2)$, and write $f_j=\sum_{n\geq 1}a_j(n)q^n$. For every $f_j$ determine a fundamental discriminant $D_j\equiv 0\pmod 4$ such that $a_j((-1)^k D_j)\neq 0$. The complex polynomial

$$P(X_1,\ldots,X_r)=\prod_{1\leq j\leq r}(a_j(|D_1|)X_1+\ldots+a_j(|D_r|)X_r)$$

is non-zero, hence there exists $(c_1,\ldots,c_r)\in\mathbb{C}^r$ with $P(c_1,\ldots,c_r)\neq 0$. Define $\mathscr{S}^+_k=c_1\mathscr{S}^+_{D_1,k}+\ldots+c_r\mathscr{S}^+_{D_r,k}$. Then for every $j\in\{1,\ldots,r\}$, $f_j|\mathscr{S}^+_k$ is in $S_{2k}(1)$ and is a non-zero eigenform of all $T_{2k}(p)$. If $f_j|\mathscr{S}^+_k=f_l|\mathscr{S}^+_k$, then because $\mathscr{S}^+_k$ commutes with Hecke operators, $f_j$ and $f_l$ have the same eigenvalues for all $T^+_{k+1/2}(p^2)$, and hence $j=l$. From this we see that $\mathscr{S}^+_k$ is injective, hence bijective. It is clear that the $c_j$ can be determined such that $H_{k+1/2}|\mathscr{S}^+_k\neq 0$.

Now suppose $D\equiv 1\pmod 4$. We may assume $k\geq 6$ if $k$ is even and $k\geq 9$ if $k$ is odd. Let $g$ be a normalized eigenform in $S_{2k}(1)$ with $g|T_{2k}(p)=\omega_p g$ for all primes $p$. Then we have $g=\sum_{n\geq 1}\omega_n p^n$, where the $\omega_n$ are determined by $\sum_{n\geq 1}\omega_n n^{-s}$ $=\prod_p(1-\omega_p p^{-s}+p^{2k-1-2s})^{-1}$. Write $\phi^+$ for the inverse of $\mathscr{S}^+_k$ and put

$G=g|\phi^+\mathcal{S}^+_{D,k}$. The function $G$ is a power series in $q$ which converges on $\mathfrak{H}$ and satisfies $G|T_{2k}(p)=\omega_p G$ for all primes $p$. Hence it follows that the coefficient of $G$ at $q^n$ equals $c\omega_n$, where $c$ is the first $q$-coefficient of $g|\phi^+\mathcal{S}^+_{D,k}$, i.e. $(g|\phi^+)|\mathcal{S}^+_{D,k}=cg$. Since $\phi^+$ is bijective, we see that $\mathcal{S}^+_{D,k}$ maps $S^+_{k+1/2}(4)$ to $S_{2k}(1)$.

### 2.4. *Proof of Theorem 2 and Corollary*

If $f,f'\in M_\kappa(N)$ (where $N\in\{1,4\}$ and $\kappa\in\mathbb{Z}$ for $N=1$, $\kappa\in\frac{1}{2}+\mathbb{Z}$ for $N=4$), and at least one of them is a cusp form, their Petersson scalar product

$$\int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\,\overline{f'(z)}\,y^{\kappa-2}\frac{dx\,dy}{y^2}$$

will be denoted by $\langle f,f'\rangle$. We will suppose that $k\geqq 6$.

Let $S^0_{2k}(1)$ be the $\mathbb{C}$-linear space spanned by normalized eigenforms $g$ of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$ satisfying $\mathrm{L}_g(k)\neq 0$. We have to show that $S^0_{2k}(1)=S^+_{k+1/2}(4)|\mathcal{S}^+_{1,k}$. As already mentioned the key for the proof is a result of Zagier's ([16], Sect. 5, Proposition 5) based on Rankin's convolution idea, which we will state now only for the special case where we need it:

**Lemma.** *Let $N\in\{1,4\}$. Let $k_2\in\mathbb{Z}$. Let $k_1\in 2\mathbb{Z}$, if $N=1$, and $k_1\in\frac{1}{2}+2\mathbb{Z}$, if $N=4$, and suppose $k_2\geqq k_1+2>2$. Let $E_{k_2}(z)=\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}(cz+d)^{-k_2}$ be the normalized Eisenstein series of weight $k_2$ for $\Gamma_0(N)$ $\left[\text{summation over a system of representatives for the action of } \left\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}\middle| n\in\mathbb{Z}\right\} \text{ on } \Gamma_0(N)\right]$. Let $g=\sum_{n\geqq 0} b(n)q^n\in M_{k_1}(N)$ and $f=\sum_{n\geqq 1} a(n)q^n\in S_{k_1+k_2}(N)$. Then the Petersson product of $f$ and $gE_{k_2}$ is given by*

$$\langle f,gE_{k_2}\rangle=\frac{\Gamma(k_1+k_2-1)}{(4\pi)^{k_1+k_2-1}}\sum_{n\geqq 1}\frac{a(n)\overline{b(n)}}{n^{k_1+k_2-1}}.$$

We first show that $S^0_{2k}(1)$ is contained in $S^+_{k+1/2}(4)|S^+_{1,k}$. Let $G_k=\frac{1}{2}\zeta(1-k)+\sum_{n\geqq 1}\sigma_{k-1}(n)q^n$ be the Eisenstein series of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$ and let $g=\sum_{n\geqq 1} b(n)q^n$ be a normalized Hecke eigenfunction in $S_{2k}(1)$. Then using the identity

$$\sum_{n\geqq 1}\sigma_r(n)\,b(n)n^{-s}=\Bigl(\sum_{n\geqq 1} b(n)n^{-s}\Bigr)\Bigl(\sum_{n\geqq 1} n^r b(n)n^{-s}\Bigr)\zeta(2s-r-2k+1)^{-1}$$
$$(\mathrm{Re}\,s>r+k+\tfrac{1}{2})$$

we easily see from the above lemma that

$$\langle g,G_rG_{2k-r}\rangle=(-1)^{r/2}2^{-2k+1}\mathrm{L}_g(2k-1)\,\mathrm{L}_g(r) \tag{1}$$

for any even integer $r$ with $k+2\leqq r\leqq 2k-4$, a formula due to Rankin (cf. [16], p. 117 and p. 146).

We now show that (1) is also valid for $r=k$ (this is claimed in [16], p. 146). Indeed, the recurrence relation

$$\frac{(2k-6)(2k+1)}{12}\frac{1}{(2k-2)!}G_{2k}=\frac{1}{2!(2k-6)!}G_4G_{2k-4}+\frac{1}{4!(2k-8)!}G_6G_{2k-6}+\dots+\frac{1}{(2k-6)!2!}G_{2k-4}G_4,$$

valid for $k\geqq 4$, is well-known (cf. e.g. [13], p. 19). Using $\langle g, G_{2k}\rangle=0$ we obtain from (1)

$$\langle g, G_k^2\rangle=2^{-2k+1}\mathrm{L}_g(2k-1)\,2(k-2)!^2\left(-\frac{1}{2!(2k-6)!}\mathrm{L}_g(4)+\frac{1}{4!(2k-8)!}\mathrm{L}_g(6)\mp\dots+(-1)^{k/2-1}\frac{1}{(k-4)!\,k!}\mathrm{L}_g(k-2)\right).$$

That this is equal to $(-1)^{k/2}2^{-2k+1}\mathrm{L}_g(2k-1)\,\mathrm{L}_g(k)$ can be deduced from the "period relations" (cf. [8], Chap. V, Sect. 2, p. 73) by a simple computation. We omit the details.

Since $\mathrm{L}_g(2k-1)\neq 0$ and the Hecke algebra acts on $M_{2k}(1)$ with multiplicity 1, it follows that $G_k^2$ generates $\mathbb{C}G_{2k}\oplus S_{2k}^0(1)$ as a module over the Hecke algebra. But we have $G_k(4z)\,\theta(z)|\mathscr{S}_{1,k}^+=G_k^2(z)$ (this is easily checked; cf. the computations in [4], 2.4). Since $\mathscr{S}_{1,k}^+$ commutes with Hecke operators, it follows that $S_{k+1/2}^+(4)|\mathscr{S}_{1,k}^+$ contains $S_{2k}^0(1)$.

To prove the converse we shall again use the above lemma. Let $f=\sum_{n\geqq 1}a(n)q^n\in S_{k+1/2}^+(4)$ be a common eigenfunction of the $T_{k+1/2}^+(p^2)$, with $f|T_{k+1/2}^+(p^2)=\lambda_p f$. We want to compute $\langle f(z), G_k(4z)\,\theta(z)\rangle$. Let

$$G_k^{i\infty}(z)=-2^{-k}G_k(2z)+G_k(4z)$$

be an Eisenstein series of weight $k$ on $\Gamma_0(4)$ for the cusp $i\infty$. We have

$$\begin{aligned}G_k(2z)\,\theta(z)|(U_4W_4-\alpha_2)&=(G_k(z)|U_2)\,\theta(z)|W_4-\alpha_2G_k(2z)\,\theta(z)\\&=((1+2^{k-1})\,G_k(z)-2^{k-1}G_k(2z))\,\theta(z)|W_4-\alpha_2G_k(2z)\,\theta(z)\\&=(-1)^{k/2}2^k(1+2^{k-1})\,G_k(4z)\,\theta(z)\end{aligned}$$

and

$$G_k(4z)\,\theta(z)|(U_4W_4-\alpha_2)=(-1)^{k/2}\cdot 3\cdot 2^{k-1}G_k(4z)\,\theta(z).$$

Therefore

$$G_k^{i\infty}(z)\,\theta(z)|(U_4W_4-\alpha_2)=(-1)^{k/2}(2^k-1)\,G_k(4z)\,\theta(z),$$

and we conclude

$$\begin{aligned}(-1)^{k/2}(2^k-1)\langle f(z), G_k(4z)\,\theta(z)\rangle&=\langle f(z), G_k^{i\infty}(z)\,\theta(z)|(U_4W_4-\alpha_2)\rangle\\&=\langle f(z)|(U_4W_4-\alpha_2), G_k^{i\infty}(z)\,\theta(z)\rangle\\&=(-1)^{k/2}\cdot 3\cdot 2^{k-1}\langle f(z), G_k^{i\infty}(z)\,\theta(z)\rangle\end{aligned}$$

which by the above lemma is equal to

$$(-1)^{k/2}\cdot 3\cdot 2^{k-1}(1-2^{-k})\frac{1}{2}\zeta(1-k)\frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}}\sum_{n\geqq 1}\frac{2a(n^2)}{n^{2k-1}}.$$

In Sect. 2.3 we proved

$$\sum_{n\geqq 1} a(n^2)n^{-s} = a(1)\zeta(s-k+1)^{-1}\prod_p(1-\lambda_p p^{-s}+p^{2k-1-2s})^{-1}.$$

Therefore

$$\langle f(z), G_k(4z)\theta(z)\rangle = a(1)\frac{3}{2}\frac{\zeta(1-k)}{\zeta(k)}\frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}}\prod_p(1-\lambda_p p^{-2k+1}+p^{-2k+1})^{-1}$$

and hence one has $\langle f(z), G_k(4z)\theta(z)\rangle \neq 0$ if and only if the first $q$-coefficient of the Hecke eigenform $f$ does not vanish.

Now let $f_1, \ldots, f_r \in S^+_{k+1/2}(4)$ be an orthogonal basis of Hecke eigenforms. Write $f_\nu = \sum_{n\geqq 1} a_\nu(n)q^n$. Since the Hecke algebra acts on $M^+_{k+1/2}(4)$ with multiplicity 1, the function $G_k(4z)\theta(z)$ generates $\mathbb{C}H_{k+1/2} \oplus \left(\bigoplus_{a_\nu(1)\neq 0} \mathbb{C}f_\nu\right)$ as a module over the Hecke algebra. Furthermore, from the definition of $\mathscr{S}^+_{1,k}$ and the Euler product for $\sum_{n\geqq 1} a_\nu(n^2)n^{-s}$ we see that $f_\nu|\mathscr{S}^+_{1,k}=0$ if and only if $a_\nu(1)=0$. Therefore, from the equality $G_k(4z)\theta(z)|S^+_{1,k} = G_k^2(z)$ we conclude that $S^+_{k+1/2}(4)|\mathscr{S}^+_{1,k}$ is contained in $S^0_{2k}(1)$. Thus Theorem 2 is proved.

Let us now prove the corollary. Put $\psi_{k+1/2} = E_k(4z)\theta(z) - H_{k+1/2}(z)$. Then $\psi_{k+1/2} \in S^+_{k+1/2}(4)$. We claim that $\psi_{k+1/2}$ has order 1 at infinity. Indeed, this means that $2-\dfrac{\zeta(1-k)}{\zeta(1-2k)} \neq 0$ or equivalently $B_k \neq B_{2k}$, where $B_k$ is the $k$-th Bernoulli number, and it is well-known that $B_k = B_{2k}$ can only happen for $k=4$. From this and from the dimension formulae for $M^+_{k+1/2}(4)$ and $S^+_{k+1/2}(4)$ we see inductively that $M^+_{k+1/2}(4) = \mathbb{C}H_{k+1/2} \oplus \mathbb{C}\psi_{k+1/2} \oplus \Delta(4z)M^+_{k-12+1/2}(4)$, where $\Delta(z) = q\prod_{n\geqq 1}(1-q^n)^{24} \in S_{12}(1)$ is Ramanujan's function. In particular we conclude that $S^+_{k+1/2}(4)$ has a basis $h_1 = \sum_{n\geqq 1} b_1(n)q^n$, $h_4 = \sum_{n\geqq 1} b_4(n)q^n \ldots$ such that the matrix $b_i(j)\left[\text{where } i,j \equiv 0,1\ (\mathrm{mod}\,4) \text{ and } 1 \leqq i,j \leqq \begin{cases} 2r-1 & \text{if } r \text{ is odd} \\ 2r & \text{if } r \text{ is even}\end{cases}\right]$ is the unit matrix. The functions $h_1, h_4, \ldots, h_{\mu^2}$, where $\mu = [\sqrt{2r}]$, generate a subspace of dimension $\mu$, on which $\mathscr{S}^+_{1,k}$ is clearly injective. This proves the corollary.

## 2.5. *Proof of Theorem 3*

The proof of Theorem 3 is rather technical and computational, and we will prove here only the case $k=6$. The cases $k=8$ and $k=10$ can be treated along the same lines.

Comparing $q$-coefficients we find

$$G_6(4z)\theta(z) = \tfrac{1}{691}(-65\zeta(-11)H_{13/2} + 3A_{13/2}), \tag{1}$$

where $A_{13/2} = (-\theta^5 + 2\theta F_2)\Delta_4 \in S_{13/2}(4)$ and $\Delta_4 = F_2(\theta^4 - 16F_2)$. Write $A_{13/2} = \sum_{n\geqq 1} a(n)q^n$. We have $A_{13/2}|\mathscr{S}^+_{D,6} \in S_{12}(1) = \mathbb{C}\Delta$, where $\Delta = \sum_{n\geqq 1}\tau(n)q^n$ is Ramanujan's function, hence because of $\tau(1)=1$, $A_{13/2}|\mathscr{S}^+_{D,k} = a(D)\Delta$. Furthermore $H_{13/2}|\mathscr{S}^+_{D,6} = \left(\mathrm{L}\left(-5, \left(\frac{D}{\cdot}\right)\right)\Big/\zeta(-11)\right)G_{12}$ (cf. proof of Theorem 2), and

$G_6(4z)\theta(z)|\mathscr{S}^+_{D,6}=G_{12}^{\mathbb{Q}(\sqrt{D})}$ (cf. [15], Sect. 3). Since $\Delta=(Q^3-R^2)/1728$ and $G_{12}=(441Q^3+250R^2)/65{,}520$, applying $\mathscr{S}^+_{D,6}$ to both sides of (1) we therefore see that

$$G_{12}^{\mathbb{Q}(\sqrt{D})}=\alpha_D \tfrac{1}{24}Q^3+\beta_D \tfrac{5}{504}R^2$$

with

$$\alpha_D=\frac{-2^2\cdot 3^2\cdot 7\mathrm{L}\left(-5,\left(\frac{D}{\cdot}\right)\right)+a(D)}{2^3\cdot 3\cdot 691},$$

$$\beta_D=\frac{-2^3\cdot 5^3\,\mathrm{L}\left(-5,\left(\frac{D}{\cdot}\right)\right)-7a(D)}{2^3\cdot 5\cdot 691},$$

and we have to show that $\alpha_D$, $\beta_D\in\mathbb{Z}$ for all positive fundamental discriminants $D\neq 1,5,8,13$. Thus, for the rest of the section, we will suppose $D\neq 1,5,8,13$.

That 691 does not divide the denominator of $\alpha_D$ and $\beta_D$ follows from the fact that $G_{12}^{\mathbb{Q}(\sqrt{D})}-\frac{1}{4}\zeta_{\mathbb{Q}(\sqrt{D})}(-5)$ has integral $q$-coefficients. Also, because of our assumption on $D$, $\mathrm{L}\left(-5,\left(\frac{D}{\cdot}\right)\right)$ is an even integer (cf. e.g. [15], Sect. 5). Thus we need only prove that $2^3\cdot 3\cdot 5\,|\,a(D)$.

Let us first prove that $8\,|\,a(D)$. Since $\theta^4=\left(1+2\sum_{n\geq 1} q^{n^2}\right)^4\equiv 1\ (\mathrm{mod}\,8)$ we have

$$A_{13/2}\equiv -\theta F_2+2\theta F_2^2\ (\mathrm{mod}\,8). \tag{2}$$

But

$$\theta F_2=-\frac{1}{2^6(1+i)}E^0_{5/2}, \tag{3}$$

and the $D$-th $q$-coefficient $e^0_{5/2}(D)$ of $E^0_{5/2}$ equals

$$e^0_{5/2}(D)=(1+i)2^3\frac{1-\left(\frac{D}{2}\right)2^{-2}}{1-2^{-4}}\zeta(-3)^{-1}\mathrm{L}\left(-1,\left(\frac{D}{\cdot}\right)\right)$$

(cf. [2]). Since $\zeta(-3)=\frac{1}{120}$, we have thus

$$e^0_{5/2}(D)=(1+i)2^8\left(4-\left(\frac{D}{2}\right)\right)\mathrm{L}\left(-1,\left(\frac{D}{\cdot}\right)\right),$$

and since $\mathrm{L}\left(-1,\left(\frac{D}{\cdot}\right)\right)$ is an even integer (cf. e.g. [15], Sect. 5), we see from (3) that the $D$-th $q$-coefficient of $\theta F_2$ is divisible by 8.

On the other hand we have

$$\theta^5F_2+\theta F_2^2=\frac{17}{1+i}2^{-12}E^0_{9/2},$$

and the $D$-th $q$-coefficient $e^0_{9/2}(D)$ of $E^0_{9/2}$ satisfies

$$\frac{17}{1+i}2^{-12}e^0_{9/2}(D)=17\cdot 2^{-12}\cdot 2^7(1-2^{-8})^{-1}\left(1-\left(\frac{D}{2}\right)2^{-4}\right)\zeta(-7)^{-1}\mathrm{L}\left(-3,\left(\frac{D}{\cdot}\right)\right)$$

(cf. [2]). Since $\zeta(-7)=\frac{1}{240}$, we obtain

$$\frac{17}{1+i}2^{-12}e_{9/2}^{0}(D)=2^{3}\left(2^{4}-\left(\frac{D}{2}\right)\right)\mathrm{L}\left(-3,\left(\frac{D}{\cdot}\right)\right),$$

and since $\mathrm{L}\left(-3,\left(\frac{D}{\cdot}\right)\right)$ is an (even) integer (cf. e.g. [15], Sect. 5), we conclude that the $D$-th $q$-coefficient of $\theta^5 F_2+\theta F_2^2$ is divisible by 8, too. But $\theta^5 F_2\equiv\theta F_2 \pmod 8$, and because the $D$-th $q$-coefficient of $\theta F_2$ is divisible by 8, we see that the same is true for the $D$-th $q$-coefficient of $\theta F_2^2$. Thus finally from (2) we get $8\,|\,a(D)$.

Now observe $\theta^4+F_2\equiv 1 \pmod 3$, hence $\Delta_4\equiv F_2(\theta^4-F_2)\equiv\theta^8-1 \pmod 3$, hence

$$\begin{aligned}A_{13/2}&=-\theta^5\Delta_4+2\theta F_2\Delta_4\equiv-\theta\Delta_4\equiv\theta-\theta^9\\&\equiv\left(1+2\sum_{n\geq 1}q^{n^2}\right)-\left(1+2\sum_{n\geq 1}q^{9n^2}\right)\pmod 3,\end{aligned}$$

which implies $3\,|\,a(D)$.

Furthermore, from (1) we get

$$a(D)\equiv 2\sum_r\sigma_5\left(\frac{D-r^2}{4}\right)\equiv 2\sum_r\sigma_1\left(\frac{D-r^2}{4}\right)\pmod 5.$$

But we have

$$\sum_r\sigma_1\left(\frac{D-r^2}{4}\right)=-5\mathrm{L}\left(-1,\left(\frac{D}{\cdot}\right)\right)$$

(cf. [4], Proposition 4.3.1). Therefore, because $\mathrm{L}\left(-1,\left(\frac{D}{\cdot}\right)\right)\in\mathbb{Z}$, we obtain $5\,|\,a(D)$.

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