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A Measure Space Without the Strong Lifting Property

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If (X, Σ, λ) is a measure space, we consider the algebra M^∞ of bounded, measurable functions on X. A map $l: M^\infty \to M^\infty$ is called a *lifting*, if it is linear, multiplicative, satisfies $l(1)=1$, any two functions which agree λ-a.e. have the same image and $l(f)$ equals f λ-a.e. If X is a compact topological space, λ a Radon measure, whose support is the whole space, Σ the σ-algebra of λ-measurable sets, the lifting l is called *strong*, if $l(f)=f$ holds for all continuous functions f (cp. [5, p. 105]). The problem of the existence of a strong lifting has been studied by several authors (see [1, 2, 4–7]). In [5] the existence was proved for compact, second countable spaces, in [1] a slightly weaker existence theorem was given for compact spaces with an open basis of cardinality not greater than $\aleph_1$. Under assumption of the continuum hypothesis, the existence theorem for such spaces follows also from [7].

We give an example of a space without the strong lifting property and furthermore this space has an open basis of cardinality $\aleph_2$.

Construction of the Space

Let S be a compact, zero dimensional topological space, which is metrizable and not discrete. Let ν be a probability measure on S whose support is the whole space and which is not concentrated on the isolated points of S (one can take e.g. $S=\{0,1\}^{\mathbb{N}}$ and ν the product of the measures which assign equal weight to 0 and 1.

It is well known (see e.g. [8], 19.9.4) that for any such measure there exists a nowhere dense subset of positive measure. Choose a fixed closed, nowhere dense subset M_1 of S with $\nu(M_1)>0$ and a clopen subset M_2 of S such that $M_2 \neq \emptyset$ and $M_3 = S \backslash M_2 \neq \emptyset$. We consider the family $\mathscr{M}$ of all sets in the algebra generated by M_1, M_2, M_3 which have positive measure and are either clopen or nowhere dense. Let J be an index set, whose cardinality is at least $\aleph_2$. Put $T=S^J$ and write μ for the product of the measures ν.

For $j \in J$ we write S_j for the j-th factor in the product S^J. If $J_1 \subseteq J$ we will often not distinguish between a subset A of S^{J_1} and the subset $A \times S^{J \backslash J_1}$ which depends only on the coordinates from J_1, e.g. if $j_1, \ldots, j_k \in J$ and $A_{j_i} \subseteq S_{j_i}$ then $A_{j_1} \times \ldots \times A_{j_k}$

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stands for $A_{j_1}\ldots A_{j_k}\times S^{J\setminus\{j_1,\ldots,j_k\}}$. (The subscripts will always refer to the coordinate to which the set corresponds.) Put

$$I=\{A_{j_1}\times\ldots\times A_{j_k}: j_1,\ldots,j_k\in J, A_{j_i}\in\mathscr{M},\ \text{at least one } A_{j_i} \text{ is nowhere dense}\}\ . \tag{1}$$

Finally define $X=T\times T^I$.

T^I will be identified with $S^{I\times J}$. We will use the same convention as above for sets which depend only on a subset of I resp. $I\times J$. In the representation $A\times B_{C_1}\times\ldots\times B_{C_n}$ $(C_i\in I)$ the term A corresponds to the first factor T in the definition of X.

X is compact and zero dimensional. If A and B_{C_k} are clopen subsets of T (resp. T_{C_k}) we define a measure λ on X by

$$\lambda(A\times B_{C_1}\times\ldots\times B_{C_n}) = \sum_{\Gamma\subset\mathbb{N}_n}\mu\Big(A\cap\bigcap_{k\in\mathbb{N}_n\setminus\Gamma}(C_k\cap B_{C_k})\cap\bigcap_{l\in\Gamma}\mathscr{C}C_l\Big)\cdot\prod_{l\in\Gamma}\mu(B_{C_l}) \tag{2}$$

($\mathbb{N}_n=\{1,\ldots,n\}$, $\mathscr{C}C_k$ stands for the complement of C_k in T. A more detailed description of this measure will be given in Lemma 1. In Lemma 3 it will be shown that $\operatorname{supp}\lambda=X$).

Theorem. *The space (X,λ) admits no strong lifting.*

Remark. The construction of X and λ is based on the following observation: if A is a nowhere dense subset of T with $\mu(A)>0$ put $T_1=T\times T$, $\mu_1=c_{T\setminus A\times T}\cdot\mu\otimes\mu+c_{A\times T}\cdot\mu_d$, where $c_{A\times T}$ stands for the characteristic function of $A\times T$ and μ_d denotes the diagonal measure on $T\times T$: $\int f(x,y)d\mu_d(x,y)=\int f(x,x)d\mu(x)$. Then $\operatorname{supp}\mu_1=T_1$ (since A is nowhere dense) and the projection of μ_1 onto the first coordinate is μ. If l is a strong lifting of (T,μ), then the function $l_1(A\times T)=l(A)\times T$ cannot be extended to a strong lifting of T_1: if U is clopen in T, $U\subsetneqq T\setminus A$, one has $\mu_1(A\times U)=0$ and it would follow that $\emptyset=l_1(A\times U)=l(A)\times U$. This means that in order to get a strong lifting on T_1, the lifting of sets of the form $A\times T$ must depend also on the second coordinate. If one iterates this construction and considers $T\times T^I$ instead of $T\times T$ [Definition (2) is the generalization of the above definition of μ_1], one gets an uncountable number of conditions for the lifting of $A\times T^I$ (see Lemma 6 for the explicit fashion). On the other hand, it turns out that the measurable sets in X depend in a certain sense only on countably many coordinates (see Lemma 5) and this brings the result that no measurable solution to these conditions exists, if the index set I is properly chosen.

In order to carry out this program, one needs some more information on the measure space (X,λ), in particular on the structure of the sets of measure zero. This will be done in the following lemmas.

Let Σ_ν be the measure algebra for (S,ν), that is, the set of equivalence classes of measurable subsets of S modulo sets of ν-measure zero. Let $S^\wedge$ be the Stone representation space of the Boolean algebra Σ_ν. We identify $S^\wedge$ with the set of all ultrafilters in Σ_ν, any such ultrafilter corresponds to an ultrafilter in the algebra of measurable subsets of S, which contains only sets of positive measure.

Put $T_0=S^{\frown J}$ and $X_0=T_0\times T^I$. We consider the subalgebra Σ_μ^0 of Σ_μ which is generated by all sets of the form $A_{j_1}\times\ldots\times A_{j_k}$, such that A_{j_i} is a measurable subset of S_{j_i}; $j_1,\ldots,j_k\in J$. To each element $(\mathscr{F}_j)_{j\in J}\in T_0$ we associate the filter $\mathscr{F}$ in Σ_μ^0 which is generated by the sets $A_{j_1}\times\ldots\times A_{j_k}$, where $A_{j_i}\in\mathscr{F}_{j_i}$. It is easily seen that any element $B\in\Sigma_\mu^0$ can be represented as a disjoint union of sets of the form $B_{j_1}\times\ldots\times B_{j_k}$. It follows that $\mathscr{F}$ is an ultrafilter in Σ_μ^0 and conversely every ultrafilter $\mathscr{F}$ on Σ_μ^0 determines a unique element $(\mathscr{F}_j)_{j\in J}\in T_0$. This establishes a bijective correspondence between T_0 and the representation space of the algebra Σ_μ^0 (see [9, p. 37]). In the following we will identify the elements of T with filters on Σ_μ^0.

Since S is compact and $\operatorname{supp}\nu=S$, each $\mathscr{F}\in S^\frown$ has a unique limit point $p_S(\mathscr{F})\in S$. This defines a continuous, surjective map $p_S:S^\frown\to S$. In a canonical fashion, p_S induces continuous, surjective maps $p_T:T_0=S^{\frown J}\to T=S^J$ and $p:X_0=T_0\times T^I\to X=T\times T^I$.

Each clopen subset $A^\frown$ of T_0 corresponds to some measurable subset A of T and as in (2) we can define a measure λ_0 on X_0 by

$$\begin{aligned}&\lambda_0(A^\frown\times\mathrm{B}_{C_1}\times\ldots\times\mathrm{B}_{C_n})\\&=\mu\Big(A\cap\bigcap_{k\in\mathbb{N}_n\setminus\Gamma}(C_k\cap B_{C_k})\cap\bigcap_{l\in\Gamma}\mathscr{C}C_l\Big)\cdot\prod_{l\in\Gamma}\mu(B_{C_l})\,.\end{aligned}\tag{2'}$$

It will turn out that the measure space (X_0,λ_0) can be handled easier than the original space (X,λ).

For $t\in T_0$ put

$$\begin{aligned}X_t&=\{(p_T(t),t_C)_{C\in I}:C\in t\text{ implies }t_C=p_T(t)\}\,,\\X_{0t}&=\{(t,t_C)_{C\in I}:C\in t\text{ implies }t_C=p_T(t)\}\,,\\B_t&=B\cap X_t\text{ for }B\subseteqq X\,,\quad B_t=B\cap X_{0t}\text{ for }B\subseteqq X_0\,.\end{aligned}\tag{3}$$

(We use here the identification of elements $t\in T_0$ with filters on Σ_μ^0). X_t (resp. X_{0t}) is a closed subset of X (resp. X_0) and both are homeomorphic to $T^{I\setminus t}$. μ_t shall be the measure on X (resp. X_0) which corresponds to the product measure of μ on $T^{I\setminus t}$.

Since each clopen subset of $S^\frown$ corresponds to a measurable subset of S, the measure ν on S induces a measure $\nu^\frown$ on $S^\frown$. Let μ_0 be the product measure of $\nu^\frown$ on $T_0=S^{\frown J}$.

Lemma 1. *Consider the space of all Radon measures on X (resp. X_0) with the weak topology with respect to continuous functions on X (X_0). Then the map $t\mapsto\mu_t$ is continuous on T_0. We have*

$$\lambda(B)=\int_{T_0}\mu_t(B_t)\,d\mu_0(t)$$

for each λ-measurable subset B of X.

$$\lambda_0(B)=\int_{T_0}\mu_t(B_t)\,d\mu_0(t)$$

for each λ_0-measurable subset B of X_0.

If B is clopen in $X(X_0)$, then $\mu_t(B_t)$ takes only finitely many values.

Proof. We will give the proof for λ, the argument for λ_0 is similar.

Let $B = A \times B_{C_1} \times \ldots \times B_{C_n}$ be a clopen subset of X. First we will show that $t \mapsto \mu_t(B_t)$ is continuous and that the equation stated above holds for B. This will be done by evaluating $\mu_t(B_t)$ according to (3).

Fix some element $t \in T_0$. Since A is clopen, $p_T(t) \in A$ holds iff $A \in t$. Put $\Gamma = \{k : C_k \notin t\}$.

If $(p_T(t), t_C) \in B_t$ and $k \notin \Gamma$, then $t_{C_k} = p_T(t)$, consequently $p_T(t) \in B_{C_k}$ and $B_{C_k} \in t$. It follows that $A \cap \bigcap_{k \in \mathbb{N}_n \setminus \Gamma} (C_k \cap B_{C_k}) \cap \bigcap_{l \in \Gamma} \mathscr{C} C_l \in t$. Conversely, if this set belongs to t, then $p_T(t) \in B_{C_k}$ for $k \notin \Gamma$, and for $l \in \Gamma$ $t_{C_l} \in B_{C_l}$ can be arbitrarily chosen. It follows that in this case B_t corresponds to the set $\prod_{l \in \Gamma} B_{C_l}$ and $\mu_t(B_t) = \prod_{l \in \Gamma} \mu(B_{C_l})$. This means that to each choice of $\Gamma \subsetneqq \mathbb{N}_n$ corresponds a possible value of $\mu_t(B_t)$ which is taken on a clopen subset of T_0. The μ_0-measure of this subset is $\mu\left(A \cap \bigcap_{k \notin \Gamma} (C_k \cap B_{C_k}) \cap \bigcap_{l \in \Gamma} \mathscr{C} C_l\right)$ and in this way one gets formula (2).

The continuity of $t \mapsto \int f d\mu_t$ for continuous f follows now easily from an approximation argument. The formula $\lambda(B) = \int_{T_0} \mu_t(B_t) d\mu_0(t)$ follows e.g. from [3], § 3.4, Theorem 1, p. 21. The equation shall also indicate that B_t is μ_t-measurable μ_0-a.e.

Lemma 2. $\lambda = p(\lambda_0)$.

Proof. λ and λ_0 are regular measures on X resp. X_0. By (3) $p(X_{0t}) = X_t$ consequently $p^{-1}(X_t) \supseteq X_{0t}$. If $B \subseteq X$, then $p^{-1}(B_t) \cap X_{0t} = p^{-1}(B) \cap X_{0t} = p^{-1}(B)_t$. Assume that B is Borel measurable. Since p is continuous $p^{-1}(B)$ is also Borel measurable and since μ_t is concentrated on X_{0t} one gets $\mu_t(p^{-1}(B)_t) = \mu_t(B_t)$ for all $t \in T_0$. By Lemma 1 $\lambda(B) = \lambda_0(p^{-1}(B))$ and by extension this holds for arbitrary λ-measurable sets B.

Lemma 3. $\operatorname{supp} \lambda = X$,

$$\operatorname{supp} \lambda_0 = \{(t, t_C)_{C \in I} : C \in t \textit{ implies } t_C = p_T(t)\} = \bigcup_{t \in T_0} X_{0t} .$$

Proof. To prove the first formula consider a clopen subset $A \times B_{C_1} \times \ldots \times B_{C_n}$ of X. Take $\Gamma = \emptyset$ in (2) to get $\lambda(A \times B_{C_1} \times \ldots \times B_{C_n}) \geqq \mu\left(A \cap \bigcap_{l=1}^{n} \mathscr{C} C_l\right) \cdot \prod_{l=1}^{n} \mu(B_{C_l})$. Since $\operatorname{supp} \mu = T$ and each C_l is nowhere dense in T [see (1)], this is positive if A and B_{C_l} are nonempty.

In order to determine the support of λ_0 we consider a clopen subset B of X. The function $t \mapsto \mu_t(B_t)$ is continuous by Lemma 1. If $\lambda_0(B) = 0$, then $\mu_t(B_t) = 0$ for all t, since $\operatorname{supp} \mu_0 = T$ (this holds since $\operatorname{supp} \nu\hat{} = S\hat{}$). Since $\operatorname{supp} \mu_t = X_{0t}$, $B_t = \emptyset$ for all t. It follows that $B \cap X_{0t} = \emptyset$ for all $t \in T_0$. On the other hand we have clearly $B_t \neq \emptyset$ for some t if $\lambda_0(B) > 0$.

This shows that $\bigcup X_{0t}$ is dense in the support of λ_0. If $x = (t, t_C) \in X_0$ is the limit of $x^i = (t^i, t^i_C) \in \bigcup X_{0t}$, then t is the limit of t^i and $C \in t$ implies $C \in t^i$ for $i \geqq i_0$ ($\{t : C \in t\}$

is clopen in T_0). Consequently $t^i_C = p_T(t^i)$ for $i \geqq i_0$ and $p_T(t^i)$ converges to $p_T(t)$ since p_T is continuous. This gives $t_C = p_T(t)$ and so $\bigcup X_{0t}$ is closed in X_0.

Proposition. *Let Y be an arbitrary compact topological space with a Radon measure λ. If $N \subseteq Y$ satisfies $\lambda(N) = 0$, then there exists a Baire set N_0 and closed Baire sets K_n $(n = 1, 2, \ldots)$ such that $\lambda(N_0) = 0$ and $N \subseteq N_0 \cup \bigcup_{n=1}^{\infty} \mathscr{n}(K_n)$ where*

$$\mathscr{n}(K_n) = K_n \setminus \operatorname{supp}(\lambda | K_n) .$$

If Y is zero dimensional, then $\mathscr{n}(K_n) = K_n \cap \bigcup \{F : F \text{ clopen in } Y, \lambda(F \cap K_n) = 0\}$.

Proof. By the regularity of λ there exist open sets U_n $(n = 1, 2, \ldots)$ such that $U_n \supseteqq U_{n+1} \supseteqq N$ and $\lim \lambda(U_n) = 0$. For each U_n there exists an open Baire set V_n such that $V_n \subseteqq U_n$ and $\lambda(U_n) = \lambda(V_n)$. Put $N_0 = \bigcap_{n=1}^{\infty} V_n$, $K_n = \mathscr{C} V_n$.

Then $N \subseteqq N_0 \cap \bigcup_{n=1}^{\infty} U_n \cap \mathscr{C} V_n$, $\lambda(N_0) = 0$ and $U_n \cap \mathscr{C} V_n \subseteqq \mathscr{n}(K_n)$ since U_n is open.

In Lemma 5 we will apply this result to the measure space (X_0, λ_0) to describe the sets of λ_0-measure zero.

If A is a subset of $X_0 = S^{\sim J} \times S^{I \times J}$ and I_1 and J_1 are subsets of I and J respectively, we will say that A depends only on I_1 and J_1, if it is the preimage of a set in $S^{\sim J_1} \times S^{I_1 \times J_1}$ via the canonical projection. (Note that this refers to both appearances of J in the decomposition of X_0.) Since by (1) any element A of I is a subset of S^J it makes also sense to speak of elements $A \in I$ which depend only on J_1.

Lemma 4. *Let J_1 be a subset of J, $I_1 \subseteqq I$ such that each $A \in I_1$ depends only on J_1. Assume that B is a subset of X which depends only on I_1 and J_1. If $t \in T_0 = S^{\sim J}$ we write $t = (t^1, t^2)$ with $t^1 \in S^{\sim J_1}$, $t^2 \in S^{\sim J \setminus J_1}$. Then the following holds: If the first coordinates of t and t' agree, then $\mu_t(B_t) = \mu_{t'}(B_{t'})$.*

Proof. Put $t = (t^1, t^2)$, $t' = (t^1, t^{2'})$, where $t^1 \in S^{\sim J_1}$, $t^2, t^{2'} \in S^{\sim J \setminus J_1}$. t and t' define ultrafilters on Σ^0_μ. It follows from the definition of these ultrafilters that a set $A \subseteqq S^J$ which depends only on $J_1 \subseteqq J$ belongs to t iff it belongs to t'. Consequently $I_1 \cap t = I_1 \cap t'$. By (3) $B_t \subseteqq X_t$ is homeomorphic to a subset of $T^{I \setminus t} = S^{(I \setminus t) \times J}$, but since B depends only on I_1 and J_1 it suffices to look at the coordinates from $(I_1 \setminus t) \times J_1 = (I_1 \setminus t^1) \times J_1$. Since $t' = (t^1, t^{2'})$, $B_{t'}$ depends on the same set of coordinates.

Since μ_t and $\mu_{t'}$ are both product measures, it suffices to show that the projections of B_t and $B_{t'}$ into $S^{(I_1 \setminus t^1) \times J_1}$ agree. Let $(s_i)_{i \in (I_1 \setminus t^1) \times J_1}$ be given and assume that there exists an element $(t, t_C)_{C \in I} \in B_t$ whose restriction to $(I_1 \setminus t^1) \times J_1$ coincides with (s_i). For $C \in t'$ put $t'_C = p_T(t')$, if $C \notin t'$ put $t'_C = t_C$. Then t_C and t'_C coincide in all coordinates from J_1. It follows that $(t, t'_C) \in B$. Since t and t' have the same image in $S^{\sim J_1}$ we conclude that $(t', t'_C) \in B$. By the definition of t'_C we have $(t', t'_C) \in B_{t'}$ [see (3)] and the restriction of this element to $(I_1 \setminus t^1) \times J_1$ is just (s_i) (since $I_1 \cap t = I_1 \cap t'$).

Lemma 5. *If N is a subset of X with $\lambda_0(N) = 0$, then there exist countable subsets $I_1 \subseteqq I$, $J_1 \subseteqq J$ and a subset N_1 of X_0 which depends only on I_1 and J_1 such that $\lambda_0(N_1) = 0$ and $N \cap \operatorname{supp} \lambda_0 \subseteqq N_1$.*

Proof. Consider the sets N_0, K_n $(n=1,2,\dots)$ constructed in the proposition. Since they are Baire sets, there exist countable subsets $I_1 \subseteq I$, $J_1 \subseteq J$ such that each of the sets N, K_n depends only on I_1 and J_1. We may also assume that each $A \in I_1$ depends only on J_1 [since each singular $A \in I_1$ depends only on a finite subset of J, see (1)].

We have $N \subseteq N_0 \cup \bigcup_{n=1}^{\infty} \mathscr{n}(K_n)$. By definition $\mathscr{n}(K_n) = K_n \cap \bigcup \{F : F \text{ clopen in } X_0, \lambda_0(K_n \cap F) = 0\}$. Let $\mathscr{n}'(K_n)$ be the set which one gets, if the union is restricted to those clopen subsets that depend only on I_1 and J_1. We will show that $\mathscr{n}(K_n) \cap \operatorname{supp} \lambda_0 = \mathscr{n}'(K_n) \cap \operatorname{supp} \lambda_0$ and then $N_1 = N_0 \cup \bigcup_{n=1}^{\infty} \mathscr{n}'(K_n)$ satisfies our demands.

Recall that $X_0 = S^{\hat{}J} \times S^{I \times J}$. Assume that F is clopen in X_0, and that F is the product of subsets of $S^{\hat{}}$ and S. Then we may write $F = F^1 \times F^2 \times F^3$, where $F^1 \subseteq S^{\hat{}J_1} \times S^{I_1 \times J_1}$, $F^2 \subseteq S^{\hat{}J \setminus J_1}$, $F^3 \subseteq S^{(I \times J) \setminus (I_1 \times J_1)}$. We identify the sets F^1, F^2, F^3 with their preimages in X_0. Now assume that $\lambda_0(K_n \cap F) = 0$, for some fixed $n \in \mathbb{N}$. We want to replace F by some clopen set P which depends only on I_1 and J_1 and satisfies $\lambda_0(K_n \cap P) = 0$ and $F \cap \operatorname{supp} \lambda_0 \subseteq P$.

Put $P^1 = \{t \in T_0 : \mu_t((K_n \cap F^1)_t) > 0\}$, $P^3 = \{t \in T_0 : \mu_t(F_t^3) > 0\}$. For $t \in T_0$, $(K_n \cap F^1)_t$ depends on $(I_1 \setminus t) \times J_1$, if it is considered as a subset of $S^{(I \setminus t) \times J}$ [see (3)]. Similarly F_t^3 depends on $(I \times J) \setminus (I_1 \times J_1 \cup t \times J)$. It follows that these two sets are independent in $S^{(I \setminus t) \times J}$ and consequently $\mu_t((K_n \cap F)_t) = \mu_t((K_n \cap F^1)_t) \mu_t(F_t^3)$ for $t \in F^2$ (μ_t is the product measure on $S^{(I \setminus t) \times J}$). Since $\lambda_0(K_n \cap F) = 0$ it follows from Lemma 1 that $\mu_0(\{t \in T_0 : \mu_t((K_n \cap F)_t) > 0\}) = 0$. This means that

$$\mu_0(P^1 \cap F^2 \cap P^3) = 0 .$$

By Lemma 4 P^1 depends only on J_1 as a subset of $T_0 = S^{\hat{}J}$ By Lemma 1 P^3 is clopen in $S^{\hat{}J}$. Let P^4 be the projection of $F^2 \cap P^3$ onto $S^{\hat{}J_1}$. Since $F^2 \cap P^3$ is clopen, the same holds for P^4. Since μ_0 is the product measure with components $\nu^{\hat{}}$, $\operatorname{supp} \nu^{\hat{}} = S^{\hat{}}$ and $\mu_0(P^1 \cap F^2 \cap P^3) = 0$, it follows from Fubini's theorem that $\mu_0(P^1 \cap P^4) = 0$ (we identify P^4 with its preimage in $S^{\hat{}J}$). By construction we have $F^2 \cap P^3 \subseteq P^4$. Now put $P = F^1 \cap P^4$. P is a clopen set and depends only on I_1 and J_1.

We want to show that P has the required properties. Since F^3 is clopen, it follows from the definition of P^3 that $F^3 \cap \operatorname{supp} \lambda_0 \subseteq P^3 \times T^I$ (use Lemma 3). This gives combined with the inclusion relation for P^4:

$$F^2 \times F^3 \cap \operatorname{supp} \lambda_0 \subseteq F^2 \cap P^3 \times T^I \subseteq P^4 \times S^{\hat{}J \setminus J_1} \times T^I$$

and finally: $F \cap \operatorname{supp} \lambda_0 = F^1 \times F^2 \times F^3 \cap \operatorname{supp} \lambda_0 \subseteq F^1 \cap P^4 = P$.

P^4 depends only on the first factor T_0 of X_0. By Lemma 1, the definition of P^1 and the fact that $\mu_0(P^1 \cap P^4) = 0$, it follows that $\lambda_0(F^1 \cap P^4 \cap K_n) = 0$.

Since $P^4 \subseteq S^{\hat{}J_1}$, $F^1 \cap P^4$ depends only on I_1 and J_1. This gives the following result:

$$\begin{aligned}\mathscr{n}(K_n) \cap \operatorname{supp} \lambda_0 &\subseteq K_n \cap \bigcup \{P \subseteq S^{\hat{}J_1} \times S^{I_1 \times J_1} \text{ clopen}, \lambda_0(P \cap K_n) = 0\} \\ &= \mathscr{n}'(K_n) .\end{aligned}$$

We have obviously $\mathscr{n}'(K_n) \subseteq \mathscr{n}(K_n)$, and this finishes the proof of Lemma 5.

Corollary. *Let B be a λ_0-measurable subset of X_0, N a set of λ_0-measure zero. I_1, J_1 and N_1 are given as in Lemma 5 and assume that B depends only on I_1 and J_1. Put $B_1 = B \backslash N_1$. Then $B_1 \cap \operatorname{supp} \lambda_0 \subseteqq B \backslash N$, $\lambda_0(B \backslash B_1) = 0$ and B_1 depends only on I_1 and J_1.*

Now assume that l is a strong lifting of (X, λ). Put $l_1(A) = p^{-1}(l(A))$ $(p: X_0 \to X$ denotes the canonical map).

If B is a measurable subset of S we write $B^\wedge = \{s \in S^\wedge : B \in s\}$. If F is clopen in S, then $F^\wedge = p_S^{-1}(F)$, if K is closed $K^\wedge \subseteqq p_S^{-1}(K)$. It follows that $v^\wedge(p_S^{-1}(K)) \geqq v^\wedge(K^\wedge) = v(K)$ for K closed and by complementation: $v^\wedge(p_S^{-1}(U)) \leqq v^\wedge(u^\wedge) = v(U)$ for U open. Since v and $v^\wedge$ are regular, one gets $v^\wedge(K^\wedge) = v^\wedge(p_S^{-1}(K))$. It follows that $B^\wedge \sim p_S^{-1}(B)$ (i.e. they differ only by a set of $v^\wedge$-measure zero) for any measurable subset of S. The same argument holds for sets of the form $A_{j_1} \times \ldots \times A_{j_n} \subseteqq T$ for which we define $(A_{j_1} \times \ldots \times A_{j_n})^\wedge = A_{j_1}^\wedge \times \ldots \times A_{j_n}^\wedge$ [this agrees with the representation of T_0 as a space of filters described before (2′)].

Recall that $X = T \times T^I$. In the following, we will study the lifting of sets which depend only on the first factor T and have the form $A_{j_1} \times \ldots \times A_{j_n}$ [in particular sets from the index set I; see (1)]. If A is such a set, then $A \sim l(A)$ and consequently $l_1(A) \sim p^{-1}(A) \sim A^\wedge$. We apply Lemma 5 with $N = A^\wedge \backslash l_1(A)$ and get some countable subsets $I_1(A)$, $J_1(A)$. $A^\wedge$ depends only on finitely many coordinates, consequently we may assume that these coordinates belong to $J_1(A)$. Put $B = A^\wedge$, then the preceding corollary produces a set $B_1 = d(A)$ which depends only on $I_1(A)$ and $J_1(A)$ and satisfies $B_1 \cap \operatorname{supp} \lambda_0 \subseteqq B \backslash N$ and $B_1 \sim B$. It follows that $d(A) \cap \operatorname{supp} \lambda_0 \subseteqq A^\wedge \backslash N \subseteqq l_1(A)$. This gives the following conclusion:

> there exist countable subsets $I_1(A) \subseteqq I$, $J_1(A) \subseteqq J$ and a subset $d(A) \subseteqq X_0$, which depends only on $I_1(A)$ and $J_1(A)$ such that $d(A) \cap \operatorname{supp} \lambda_0 \subseteqq l_1(A)$ and $A^\wedge \sim d(A)$. (4)

In Lemma 6 we will derive a condition which has to be fulfilled by the lifting because of the special form of our measure. It is essentially the same argument which was sketched at the beginning.

Lemma 6. *Assume that A and B belong to I and that the sets $d(A)$ and $d(B)$ of* (4) *have been constructed. Assume that (t, t_C), $(t', t_C) \in X_0$ are given such that $(t, t_C) \in d(B) \cap \operatorname{supp} \lambda_0$, $\mathscr{C}A \cap B \in t$, $(t', t_C) \in \operatorname{supp} \lambda_0$, $p_T(t') = p_T(t)$ and $A \cap \mathscr{C}B \in t'$.*

Then $(t', t_C) \notin d(A)$.

Proof. We assume that $(t', t_C) \in d(A)$ and we will show that this leads to a contradiction.

Put $t_0 = p_T(t) = p_T(t')$. By (4) and the definition of l_1 we have $(t_0, t_C) \in l(A) \cap l(B)$. It follows that $l(A \cap B) \neq \emptyset$, in particular that $\mu(A \cap B) > 0$.

Condition (4) remains true if we enlarge the sets I_1 and J_1. So we may take $I_1 \supseteqq I_1(A) \cup I_1(B)$, $J_1 \supseteqq J_1(A) \cup J_1(B)$ and we may assume that $A, B \in I_1$ and that each element of I_1 depends only on J_1.

A and B are subsets of $T = S^J$ which depend only on J_1 and we may consider them as subsets of S^{J_1}. Since J is uncountable, there exists $j \in J \backslash J_1$. We form the sets

$C_1=(A\cap B)\times M_{2j}$, $C_2=(A\cap B)\times M_{3j}$ [M_2, M_3 are the clopen subsets of S used in (1) and M_{2j} means that the set stands in the j-th position].

By the definition of $\mathcal{M}$ and I [see (1)] the intersection of two sets in I belongs again to I if it has positive measure. It follows from the preceding argument that $A\cap B\in I$ and [again by (1)] that $C_1, C_2\in I$. We have $C_1, C_2\notin I_1$ (since $j\notin J_1$). Our assumptions $\mathscr{C}A\cap B\in t$ and $A\cap\mathscr{C}B\in t'$ imply that $A\cap B\notin t, t'$, consequently $C_1, C_2\notin t, t'$.

Since C_1 and C_2 are nowhere dense subsets of T, there exists a nonempty clopen subset D of T such that $D\cap(C_1\cup C_2)=\emptyset$. We claim that the set $A\cap B\times D_{C_1}\times D_{C_2}$ has λ-measure zero.

By Lemma 2.1 and the argument given before (4)

$$\lambda(A\cap B\times D_{C_1}\times D_{C_2})=\lambda_0(A^\wedge\cap B^\wedge\times D_{C_1}\times D_{C_2})\,.$$

By (2′) the measure of the last set equals:

$$\begin{aligned}&\mu(A\cap B\cap C_1\cap D_{C_1}\cap C_2\cap D_{C_2})\\ &\quad+\mu(A\cap B\cap\mathscr{C}C_1\cap C_2\cap D_{C_2})\cdot\mu(D_{C_1})\\ &\quad+\mu(A\cap B\cap C_1\cap D_{C_1}\cap\mathscr{C}C_2)\cdot\mu(D_{C_2})\\ &\quad+\mu(A\cap B\cap\mathscr{C}C_1\cap\mathscr{C}C_2)\cdot\mu(D_{C_1})\cdot\mu(D_{C_2})\,.\end{aligned}$$

Since $M_3=\mathscr{C}M_2$, we have $\mathscr{C}C_1\cap\mathscr{C}C_2=\mathscr{C}(A\cap B)$ and $D\cap(C_1\cup C_2)=\emptyset$. So the four terms on the right are zero.

Since l is a strong lifting we get:

$$\emptyset=l(A\cap B\times D_{C_1}\times D_{C_2})=l(A)\cap l(B)\cap D_{C_1}\cap D_{C_2}$$

(on the right side D_{C_1} and D_{C_2} are regarded as subsets of X).

On the other hand we will now use the fact that $d(A)$ and $d(B)$ depend only on I_1 and J_1 to show that the set $l(A)\cap l(B)\cap D_{C_1}\cap D_{C_2}$ is nonempty, which brings the desired contradiction. Choose arbitrary elements t'_{C_1}, $t'_{C_2}\in D$ and put $t'_C=t_C$ for $C\neq C_1, C_2$, $C\in I$. Since $C_1, C_2\notin t, t'$, we have (t, t'_C), $(t', t'_C)\in\operatorname{supp}\lambda_0$ by Lemma 3. Since $C_1, C_2\notin I_1$ we have $(t, t'_C)\in d(A)$, $(t', t'_C)\in d(B)$ [by (4)]. But then it follows from (4) that $(t_0, t'_C)\in l(A)\cap l(B)\cap D_{C_1}\cap D_{C_2}$.

In the last step of the proof of the theorem we will now show that the property which was derived in Lemma 6 cannot be fulfilled if card $J\geqq\aleph_2$.

Proof of the Theorem

For each element $A\in I$ choose sets $I_1(A)$, $J_1(A)$, $d(A)$ according to (4). We may assume that $A\in I_1(A)$ and that each element of $I_1(A)$ depends only on $J_1(A)$.

Let J_0 be an arbitrary subset of J of cardinality $\aleph_1$. Consider the nowhere dense subset M_1 chosen at the beginning. For $j\in J$ M_{1j} shall denote the subset of S^J where M_1 stands in the j-th position.

Put $J_0^- = \bigcup_{j\in J_0} J_1(M_{1j})$. Since card $J_0^-=\aleph_1$ there exists an element $k\in J\backslash J_0^-$. Since $J_1(M_{1k})$ is countable, there exists $j\in J_0\backslash J_1(M_{1k})$.

Put $A=M_{1j}$, $B=M_{1k}$. We will show the existence of elements (t, t_C), $(t', t_C)\in X_0$ with the same properties as in Lemma 6 but with $(t', t_C)\in d(A)$.

By (4) $d(A)\sim A^{\hat{}}$ in X_0. Applying Lemma 1 gives

$$P_A=\{t\in T_0:\mu_t(d(A)_t)=1\}\sim A^{\hat{}} \text{ in } T_0 .$$

Similarly one has

$$P_B=\{t\in T_0:\mu_t(d(B)_t)=1\}\sim B^{\hat{}} \text{ in } T_0 .$$

Recall that $A^{\hat{}}$ equals the set of all elements $(s_i)_{i\in J}\in T_0$ such that $A\in s_j$ (s_j is considered as a filter of subsets of S_j). The measure μ_0 on $T_0=S^{\hat{}J}$ was defined as the product of the measures $\nu^{\hat{}}$. Since $A^{\hat{}}$ and $B^{\hat{}}$ are independent of each other, it follows that $A^{\hat{}}\cap P_A\cap B^{\hat{}}\cap P_B\sim A^{\hat{}}\cap B^{\hat{}}$ has positive measure, in particular it is nonempty. Fix an element $s=(s_i)_{i\in J}\in A^{\hat{}}\cap P_A\cap B^{\hat{}}\cap P_B$. It has the property that $A\in s_j$, $B\in s_k$ and $d(A)_s\cap d(B)_s\neq\emptyset$. Choose $(s,t'_C)_{C\in I}\in d(A)_s\cap d(B)_s$.

Now we have the following situation: the set $d(A)$ depends only on $I_1(A)$ and $J_1(A)$. By our construction $k\notin J_1(A)$. If we replace s_k by an element t_k with $p_S(s_k)=p_S(t_k)$ and put $s_i=t_i$ for $i\neq k$, we get an element (t,t'_C) which belongs again to $d(A)$. In the same way we may replace s_j by an element t'_j with $p_S(s_j)=p_S(t'_j)$ and put $t'_i=s_i$ for $i\neq j$ to get an element of $d(B)$.

In order to fulfill the conditions which are needed in Lemma 6, two things have to be shown: we must find an element $t_k\in S_k$ with $\mathscr{C}B\in t_k$ (resp. $t'_j\in S_j$ with $\mathscr{C}A\in t'_j$) such that $p_S(t_k)=p_S(s_k)$ [resp. $p_S(t'_j)=p_S(s_j)$]. Secondly we have to show that it is possible to change the elements t'_C into a family (t_C) such that $(t,t_C)\in d(A)\cap\operatorname{supp}\lambda_0$ [resp. $(t',t_C)\in d(B)\cap\operatorname{supp}\lambda_0$].

Recall that $p_S(s_k)$ was defined as the limit of the filter s_k in S. The set of all complements of measurable, nowhere dense subsets of S forms a filter $\mathscr{F}_0$. If F is an arbitrary nonempty clopen subset of S, $M\in\mathscr{F}_0$, then $\nu(M\cap F)>0$ (since $\operatorname{supp}\nu=S$). It follows that for each point in S there exists an element of $S^{\hat{}}$ which is an extension of $\mathscr{F}_0$ and converges to that point. Let $t_k\in S^{\hat{}}$ be an extension of $\mathscr{F}_0$ such that $p_S(t_k)=p_S(s_k)$ and let $t'_j\in S^{\hat{}}$ be an extension of $\mathscr{F}_0$ such that $p_S(t'_j)=p_S(s_j)$. Then t_k and t'_j have the required properties since A and B *are nowhere dense.*

Now put $t_C=p_T(t)=p_T(t')$ for $C\in t\cup t'$ and $t_C=t'_C$ for $C\notin t\cup t'$. Then clearly (t,t_C), $(t',t_C)\in\operatorname{supp}\lambda_0$ by Lemma 3. We will show that $(t,t_C)\in d(A)$, a similar argument gives $(t',t_C)\in d(B)$.

We know that $(s,t'_C)\in d(A)_s\subseteqq\operatorname{supp}\lambda_0$ and it follows that $t'_C=p_T(s)=p_T(t)$ for all $C\in s$. By our construction $d(A)$ depends only on $I_1(A)$ in $T_0\times T^I$. Therefore it suffices to show that $I_1(A)\cap(t\cup t')\subseteqq s$. i.e. that we have changed only elements outside $I_1(A)$.

To show the inclusion $I_1(A)\cap t\subseteqq s$, recall that any element of $I_1(A)$ depends only on $J_1(A)$, i.e. $I_1(A)\subseteqq\mathscr{P}(S^{J_1(A)})$ [where $\mathscr{P}(\ )$ denotes the power set]. By construction, t and s differ only in the k-th coordinate, which does not belong to $J_1(A)$. Therefore $\mathscr{P}(S^{J_1(A)})\cap t\subseteqq s$ (recall that an element $A_{i_1}\times\ldots\times A_{i_n}$ belongs to t iff $A_{i_1}\in t_{i_1},\ldots,A_{i_n}\in t_{i_n}$).

For the proof of the inclusion $I_1(A)\cap t'\subseteqq s$, assume that $C\in I_1(A)\cap t'$ but $C\notin s$. By the same argument as above, the restriction of t' to $S^{J_1(A)\setminus\{j\}}$ equals that of s. Since $I_1(A)\subseteqq\mathscr{P}(S^{J_1(A)})$, it follows that $C=C^1\times C^2$, where $C^1\subseteqq S^{J_1(A)\setminus\{j\}}$, $C^2\subseteqq S_j$ and C^2 belongs to $\mathscr{M}$ [see (1)]. Now there are two possibilities for C^2. If C^2 is nowhere dense, then $\mathscr{C}C^2 \in \mathscr{F}_0 \subsetneqq t'_j$. This contradicts $C \in t'$. If C^2 is clopen, one has $p_S(t'_j) \in C^2$. Since $p_S(t'_j) = p_S(s_j)$, it follows that $C^2 \in s_j$. By our assumption $C \notin s$, i.e. we must have $C^1 \notin s$. But since the restrictions of s and t' to $S^{J_1(A)\setminus\{j\}}$ agree, it would follow that $C^1 \notin t'$, which contradicts $C \in t'$. This shows that $I_1(A) \cap t' \setminus s$ is empty and finishes the proof. of our theorem.

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