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Stable Vector Bundles of Rank 2 on $\mathbb{P}^3$

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Contents

§0. Introduction

In my survey article [17, §6] I described the general problem of the existence of indecomposable vector bundles on $\mathbb{P}^n$, with emphasis on the theme that bundles of small rank on $\mathbb{P}^n$ for n large seem to be very rare. Since then, a number of other results have been obtained supporting this point of view [12], [25], [39], [40], [41], [44], [45].

Working in another direction, several authors [43], [10], [28] have introduced a notion of *stability* for vector bundles on projective varieties, generalizing the concept of stability suggested by Mumford and used by Narasimhan and Seshadri [35] and others in their study of vector bundles on curves. For stable vector bundles they have proved the existence of a variety of moduli. Thus one can ask for the structure of the variety of moduli of stable bundles on $\mathbb{P}^n$ whenever they exist. For rank 2 bundles, this work was begun by Barth [6], [7].

My own philosophy for the last couple of years has been to concentrate on the study of rank 2 vector bundles on $\mathbb{P}^3$, with the hope that a good understanding of

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them will help in tackling the problem on $\mathbb{P}^n$ with $n>3$. In this case there is a close connection with the old problem of classifying space curves, which I reported on in my Kyoto lectures [18]. Also the recent work of Peskine and Szpiro [36] and others [37], [14] on liaison is very relevant here.

A striking recent development was the discovery [2] that the theory of stable rank 2 bundles on $\mathbb{P}^3_{\mathbb{C}}$ is intimately related to the solution of the Yang-Mills equation which comes up in modern elementary particle physics. I have discussed this relationship and the resulting problem in *real* algebraic geometry in my paper [20], and will say no more about it here except to point out that it provides further motivation, from a completely unexpected direction, for the study of stable vector bundles on $\mathbb{P}^3$.

This paper, then, is devoted to the study of stable rank 2 vector bundles on $\mathbb{P}^3$ over an arbitrary algebraically closed ground field k. There are several methods in current use to study such vector bundles. One is the technique, so successfully used by Barth and Van de Ven [4], [5], [6], [7], of restricting the bundle to a line and studying the decomposition of the restricted bundle as the line moves. Another method uses the notion of a *monad*, first introduced by Horrocks [22], and more recently refined by Horrocks and Barth, and used by Atiyah et al. [3] in their study of the bundles coming from instantons.

In this paper however, we will stick to the older method of associating to a rank 2 vector bundle on $\mathbb{P}^3$ a curve in $\mathbb{P}^3$, and relating properties of the bundle to properties of the curve. If s is a sufficiently general global section of a rank 2 bundle $\mathscr{E}$ on $\mathbb{P}^3$, then the zero set Y of s is a curve. Serre, when studying projective modules over polynomial rings [42], was the first to notice that the bundle $\mathscr{E}$ could be recovered from Y. This fact was used implicitly by Horrocks [23], and appears to have been rediscovered independently by Barth and Van de Ven [4] and by Grauert and Mülich [11].

In order to apply this method to an arbitrary vector bundle $\mathscr{E}$, one must first twist $\mathscr{E}$ by a sufficiently high multiple $\mathcal{O}(t)$ of the hyperplane bundle, so that the twisted bundle $\mathscr{E}(t)$ will have global sections. Then one can in principle classify the bundles by classifying the possible curves associated to a section of $\mathscr{E}(t)$. An important problem is to find a good bound for the least integer t (as a function of the Chern classes c_1 and c_2 of $\mathscr{E}$) so that $H^0(\mathscr{E}(t))\neq 0$. In case $c_1=0$, we conjecture that $t>\sqrt{3c_2+1}-2$ will do. However, we are only able to prove a somewhat weaker result (8.2) with $t\sim(\frac{3}{2}c_2^2)^{1/3}$. This is one of the main results of this paper. The idea of the proof is to restrict $\mathscr{E}$ to a suitable plane $H\subseteq\mathbb{P}^3$, and associate $\mathscr{E}|_H$ with a finite set of points Z in $\mathbb{P}^2$. Then we obtain estimates on $\dim H^1(\mathscr{I}_Z(l))$ for all $l\in\mathbb{Z}$ in §5. In §7 these estimates are used to get estimates for $\dim H^1(\mathscr{E}|_H(l))$. These estimates, together with the Riemann-Roch theorem for $\mathscr{E}$ on $\mathbb{P}^3$ and some exact sequences, give the result. As a byproduct of this method, in §6 we get a new proof of a theorem of Harris giving an upper bound on the genus of a curve of degree d in $\mathbb{P}^3$ not contained in a surface of given degree. (In fact, the methods of §5 first arose in connection with Harris's theorem, and were only later applied to vector bundles.)

In §1–4 we set up basic techniques and establish some preliminary results. The connection between vector bundles and curves is explained in §1. In §2 we relate the numerical invariants c_1, c_2 of $\mathscr{E}$, and the α-invariant of Atiyah and Rees [1], here defined over fields of arbitrary characteristic, to invariants of the curve Y. Using a theorem of Ferrand about bundles associated to multiplicity 2 structures on curves, we establish the existence, for each c_1, c_2, α such that c_1c_2 is even, of families of arbitrarily large dimension of nonisomorphic bundles with given invariants. In §§3, 4 we give basic properties of stable bundles and some results about the variety of moduli. Here we also give many examples of stable bundles. For $c_1=0$, the Riemann-Roch theorem shows that the variety of moduli will have dimension $8c_2-3$ if H^2 of the sheaf of endomorphisms of the bundles is 0. However, we give an example (4.3.6) of a family of stable bundles of dimension $>8c_2-3$, showing that the H^2 does not always vanish.

In §9, we give a complete analysis of stable bundles with $c_1=0$ and $c_2=2$. In this case we can describe the structure of these bundles quite explicitly. We find that the variety of moduli is irreducible and nonsingular of dimension 13, and we get an explicit description of the divisor of jumping lines. In particular, we show that such a bundle is uniquely determined by its divisor of jumping lines (except in characteristic 3).

I would like to thank M. Atiyah, J. Harris, M. Maruyama, D. Mumford, and A. Ogus for many stimulating conversations during the preparation of this work. In particular, (3.3) is due to Maruyama; (5.2) and (5.3) were communicated to me by Mumford; and (6.1) is due to Harris. I would also like to thank W. Barth for generously sharing his published and unpublished ideas about vector bundles. They were very valuable to me. Finally, I would like to thank Pete Wever who over a period of two years worked through many of these ideas with me. In particular, (1.3), (3.0.2), (8.4.1), and the calculations in (4.3.1) and (4.3.3) are due to him.

§ 1. The Correspondence Between Vector Bundles and Curves

In this section we discuss our main tool for studying vector bundles, which is the correspondence between a vector bundle of rank 2 on $\mathbb{P}^3$ and a curve in $\mathbb{P}^3$, obtained by taking the zeros of a global section of the bundle. This correspondence allows us to reduce many questions about vector bundles to questions about curves; it also provides a method for constructing families of vector bundles.

We work over a fixed algebraically closed field k. A *vector bundle* on a scheme X of finite type over k will mean a locally free coherent sheaf on X. If $\mathscr{E}$ is a vector bundle on X, and if $s\in H^0(X,\mathscr{E})$ is a global section of $\mathscr{E}$, then s determines a map $\mathcal{O}_X \xrightarrow{s} \mathscr{E}$. Taking duals (i.e., applying the functor $\mathscr{H}om(\cdot,\mathcal{O}_X)$), we get a map $\mathscr{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_X$, whose image will be a sheaf of ideals $\mathscr{I}$ in $\mathcal{O}_X$. The corresponding closed subscheme Y of X is called the *scheme of zeros* of s, and is denoted by $(s)_0$.

Remark 1.0.1. Now let $\mathscr{E}$ be a vector bundle of rank 2 on $P=\mathbb{P}^n_k$, $n\geqq 2$, let $s\in H^0(P,\mathscr{E})$ be a nonzero global section, and let Y be its scheme of zeros. It may happen that Y is empty. In that case the map $\mathscr{E}^\vee\to\mathcal{O}_P$ is surjective, so $\mathscr{E}$ is an extension of line bundles, which implies that $\mathscr{E}$ is actually a direct sum of two line bundles, since on $\mathbb{P}^n$ for $n\geqq 2$ there are no nontrivial extensions of line bundles. It may also happen that Y has a component D of codimension 1. In that case D is a divisor, and the section s lies in the subspace $H^0(P, \mathscr{E}(-D))$ of $H^0(P, \mathscr{E})$. Then considering s as a section of $\mathscr{E}(-D)$, its scheme of zeros will have codimension $\geqq 2$. Therefore we will usually exclude these two cases by assuming that Y is nonempty and has codimension $\geqq 2$. Finally, note that since $\mathscr{E}$ is locally free of rank 2, the ideal sheaf $\mathscr{I}_Y$ is locally generated by two elements. This implies that Y has codimension exactly 2, and that it is a locally complete intersection subscheme of $\mathbb{P}^n$. In case $n=3$, Y is a *curve*, by which we mean a 1-dimensional closed subscheme of $\mathbb{P}^3$. It may be reducible, disconnected, and may have nilpotent elements.

Thus given a vector bundle $\mathscr{E}$ of rank 2 on $P=\mathbb{P}^3$, and given a section $s \in H^0(P, \mathscr{E})$, whose scheme of zeros has codimension 2, we obtain a curve $Y=(s)_0$. In this case we say the bundle $\mathscr{E}$ *corresponds* to the curve Y. Our first main result is to characterize the curves Y which occur in this way, and to show how to recover the bundle $\mathscr{E}$ from the curve.

For any curve $Y \subseteqq P$, let $\omega_Y = \mathscr{E}xt^2_P(\mathscr{O}_Y, \omega_P)$ denote its *dualizing sheaf* [AG, III, 7.5].[1]

Theorem 1.1. *Let $P=\mathbb{P}^3_k$. A curve Y in P occurs as the scheme of zeros of a section of a vector bundle $\mathscr{E}$ of rank 2 on P if and only if Y is a local complete intersection and ω_Y is isomorphic to the restriction to Y of some invertible sheaf on P. More precisely, for any fixed invertible sheaf $\mathscr{L}$ on P, there is a bijective correspondence between* (i) *and* (ii):

(i) *the set of triples $\langle \mathscr{E}, s, \varphi \rangle$ modulo the equivalence relation $\sim$, where $\mathscr{E}$ is a vector bundle of rank 2 on P; $s \in H^0(P, \mathscr{E})$ is a global section whose scheme of zeros Y has codimension 2; $\varphi: \wedge^2 \mathscr{E} \xrightarrow{\sim} \mathscr{L}$ is an isomorphism; and $\langle \mathscr{E}, s, \varphi \rangle \sim \langle \mathscr{E}', s', \varphi' \rangle$ if there is an isomorphism $\psi: \mathscr{E} \xrightarrow{\sim} \mathscr{E}'$ and an element $\lambda \in k$, $\lambda \neq 0$, such that $s' = \lambda \psi(s)$ and $\varphi' = \lambda^2 \varphi \circ (\wedge^2 \psi)^{-1}$.*

(ii) *the set of pairs $\langle Y, \xi \rangle$, where Y is a locally complete intersection curve in P, and $\xi: \mathscr{L} \otimes \omega_P \otimes \mathscr{O}_Y \to \omega_Y$ is an isomorphism.*

Proof.[2] The first statement follows from the bijection between (i) and (ii), so it is sufficient to prove that. Given $\langle \mathscr{E}, s, \varphi \rangle$, we take Y to be the scheme of zeros of s as above. Since Y has codimension 2, locally the two generators of $\mathscr{I}_Y$ form a regular sequence in $\mathscr{O}_P$, so the local Koszul complexes glue together (see [AG, III, 7.11] and its proof) to give a resolution of $\mathscr{I}_Y$:

$$0 \to \wedge^2(\mathscr{E}^\vee) \to \mathscr{E}^\vee \xrightarrow{s} \mathscr{I}_Y \to 0. \tag{1}$$

Now φ gives an isomorphism of $\wedge^2(\mathscr{E}^\vee)$ with $\mathscr{L}^\vee$, so we obtain an exact sequence

$$0 \to \mathscr{L}^\vee \to \mathscr{E}^\vee \to \mathscr{I}_Y \to 0. \tag{2}$$

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1 [AG] refers to item 19 of the references

2 This theorem is due to Serre [42] in the affine case and Horrocks [23] implicitly in the projective case. All the ideas of the proof are present in Serre's paper. Other independent proofs have been given by Barth and Van de Ven [4], and Grauert and Mülich [11]. See also Ferrand [9] and Hartshorne [17, 6.1]

This extension determines an element $\xi \in \operatorname{Ext}^1_P(\mathscr{I}_Y, \mathscr{L}^\vee)$. Using the exact sequence of Ext applied to the short exact sequence of sheaves $0 \to \mathscr{I}_Y \to \mathcal{O}_P \to \mathcal{O}_Y \to 0$, and the fact that $H^1(P, \mathscr{L}^\vee) = H^2(P, \mathscr{L}^\vee) = 0$, we see that

$$\operatorname{Ext}^1_P(\mathscr{I}_Y, \mathscr{L}^\vee) \cong \operatorname{Ext}^2_P(\mathcal{O}_Y, \mathscr{L}^\vee).$$

Now, as in the proof of [AG, III, 7.4], or using the spectral sequence of local and global Ext, we see that

$$\operatorname{Ext}^2_P(\mathcal{O}_Y, \mathscr{L}^\vee) \cong H^0(P, \mathscr{E}xt^2_P(\mathcal{O}_Y, \mathscr{L}^\vee)).$$

Using the definition of ω_Y, this can be expressed as $H^0(Y, \omega_Y \otimes \omega_P^\vee \otimes \mathscr{L}^\vee)$ $= \operatorname{Hom}(\omega_P \otimes \mathscr{L} \otimes \mathcal{O}_Y, \omega_Y)$. Thus, finally, the element ξ above can be interpreted as giving a morphism

$$\xi : \omega_P \otimes \mathscr{L} \otimes \mathcal{O}_Y \to \omega_Y . \tag{3}$$

Next, we observe that this construction makes sense also if we restrict the sequence (1) to any open affine subset of P. If U is an open affine subset on which $\mathscr{E}$ is free, then $Y \cap U$ is a complete intersection, and (1) is an actual Koszul complex. In this case one sees easily that ξ is a generator of the corresponding Ext module [42, Prop. 1, p. 2–08], which implies that the morphism (3) is an isomorphism.

Note also that replacing s by λs gives a resolution (1′) isomorphic to (1) if we map $\mathscr{E}^\vee \to \mathscr{E}^\vee$ by λ and $\wedge^2(\mathscr{E}^\vee) \to \wedge^2(\mathscr{E}^\vee)$ by λ^2. So replacing φ by $\lambda^2 \varphi$ gives an extension (2′) equivalent to the extension (2), which therefore produces the same ξ. So we have constructed a map of sets (i)→(ii).

For the reverse direction, suppose given Y and ξ. Then via the identifications above, $\xi \in \operatorname{Ext}^1_P(\mathscr{I}_Y, \mathscr{L}^\vee)$, so it determines an extension

$$0 \to \mathscr{L}^\vee \to \mathscr{F} \to \mathscr{I}_Y \to 0 , \tag{4}$$

where $\mathscr{F}$ is a coherent sheaf. Now the fact that ξ is an isomorphism implies that locally, ξ is a generator of the corresponding Ext module, and this in turn implies that $\mathscr{F}$ is locally free of rank 2 [42, *loc. cit.*]. So we define $\mathscr{E} = \mathscr{F}^\vee$ and take s to be the section obtained by dualizing the map $\mathscr{F} \to \mathscr{I}_Y \subseteq \mathcal{O}_P$ and taking the image of $1 \in H^0(P, \mathcal{O}_P)$. Comparing (1) and (4) gives an isomorphism of $\wedge^2(\mathscr{E}^\vee)$ with $\mathscr{L}^\vee$ and hence an isomorphism $\varphi : \wedge^2 \mathscr{E} \xrightarrow{\sim} \mathscr{L}$. Note that ξ determines the extension (4) only up to equivalence of extensions. Thus $\mathscr{E}$ is determined up to isomorphism, but s is determined only up to a scalar λ, and φ up to the square of that λ.

Since we now have maps both ways between the sets (i) and (ii), and they are clearly inverse to each other, the theorem is proved.

Remark 1.1.1. The same proof applies to rank 2 vector bundles on $\mathbb{P}^n$ for any $n \geqq 3$, or more generally on any nonsingular projective variety X with respect to an invertible sheaf $\mathscr{L}$ for which $H^1(X, \mathscr{L}^\vee) = H^2(X, \mathscr{L}^\vee) = 0$. In that case Y is a codimension 2 locally complete intersection closed subscheme. In particular, it applies to Grassmann varieties. Note however that it does not apply to $\mathbb{P}^2$ without modification.

Corollary 1.2. *If a bundle $\mathscr{E}$ corresponds to a curve Y, then Y is a complete intersection if and only if $\mathscr{E}$ is a direct sum of line bundles.*

Proof. Indeed, if $\mathscr{E} \cong \mathscr{L}_1 \oplus \mathscr{L}_2$, then a section s is given by sections $s_i \in H^0(P, \mathscr{L}_i)$, $i=1, 2$, and Y is the intersection of the divisors D_1, D_2 corresponding to s_1 and s_2. Conversely, if Y is a complete intersection of two divisors D_1 and D_2, then one possibility for $\mathscr{E}$ is the direct sum of the invertible sheaves $\mathscr{L}(D_1) \oplus \mathscr{L}(D_2)$. On the other hand, for a complete intersection Y we have $H^0(\mathcal{O}_Y)=k$ [AG, III, Ex. 5.5], so ξ is unique up to a scalar λ, and hence $\mathscr{E}$ is uniquely determined.

As a first application, we give a criterion for distinct sections of a bundle to have distinct schemes of zeros.

Proposition 1.3 (Wever). *Let $\mathscr{E}$ be a rank 2 bundle on $P=\mathbb{P}^3$, and assume that for every nonzero $s \in H^0(P, \mathscr{E})$, the scheme of zeros $(s)_0$ has codimension 2. (This will be the case, for example, if $H^0(P, \mathscr{E}(-1))=0$.) Then two nonzero sections $s, s' \in H^0(P, \mathscr{E})$ have the same scheme of zeros if and only if $s'=\lambda s$ for some $\lambda \in k^*$.*

Proof. Clearly s and λs have the same scheme of zeros. Therefore the map $s \mapsto (s)_0$ defines a morphism of the projective space $(H^0(P, \mathscr{E})-\{0\})/k^*$ to the Hilbert scheme of closed subschemes of P. Now any morphism of a projective space either has finite fibres or has image a single point [AG, II, Ex. 7.3]. In the former case, since each fibre is a linear space, it must be a single point, so we are done. In the latter case, let Y be the common scheme of zeros of all $s \in H^0(P, \mathscr{E})$. Then by (1.1) we obtain a morphism of the same projective space $(H^0(P, \mathscr{E})-\{0\})/k^*$ to the space of isomorphisms of $\mathscr{L} \otimes \omega_P \otimes \mathcal{O}_Y$ with ω_Y, modulo k^*. If ξ is one such isomorphism, then any other is of the form $\xi'=a\xi$ for some $a \in H^0(Y, \mathcal{O}_Y^*)$. Thus this latter space is (noncanonically) isomorphic to $H^0(Y, \mathcal{O}_Y^*)/k^*$. If Y has connected components $Y_1, \ldots, Y_r$, let $\mathcal{N}_i$ be the sheaf of nilpotent elements on Y_i, and let

$$N_i=\ker(H^0(\mathcal{O}_{Y_i}^*) \to H^0((\mathcal{O}_{Y_i}/\mathcal{N}_i)^*)).$$

Since the global sections of a connected reduced scheme in P are just k, the sequence splits, and we see that $H^0(\mathcal{O}_{Y_i}^*) \cong k^* \times N_i$. Now in characteristic 0, the exponential map shows that $N_i \cong H^0(Y_i, \mathcal{N}_i)$; in characteristic p, N_i is a successive extension of the vector spaces $H^0(Y_i, \mathcal{N}_i^r/\mathcal{N}_i^{r+1})$. In any case,

$$H^0(Y, \mathcal{O}_Y^*)/k^* = \prod_{i=1}^{r} (k^* \times N_i)/k^*,$$

which is a product of affine varieties. The only way a projective space can be isomorphic to a product of affine varieties is if each is reduced to a point. We conclude that $\dim H^0(P, \mathscr{E})=1$, in which case the result is trivial.

Next we give a criterion, analogous to Bertini's theorem for divisors, for the scheme of zeros of a section to be nonsingular.

Proposition 1.4. *Let $\mathscr{E}$ be a rank 2 vector bundle on $P=\mathbb{P}^n$,*

(a) *If $\mathscr{E}(-1)$ is generated by global sections, then for all sufficiently general $s \in H^0(P, \mathscr{E})$, the scheme of zeros $(s)_0$ will be nonsingular (but not necessarily connected).*

(b) *If* char. $k=0$, *the same is true under the weaker hypothesis that $\mathscr{E}$ is generated by global sections.*

(c) *If $H^1(P, \mathscr{E}^\check{})=0$, and $n \geqq 3$, then $(s)_0$ is connected for any s.*

Proof. (a) This is a result of Kleiman [26, 3.6], since if $\mathscr{E}(-1)$ is generated by global sections, then $\mathscr{E}$ gives a "twisted embedding" of P into the appropriate Grassmann variety.

(b) Horrocks and Mumford [24, proof of 5.1] obtain this result over $\mathbb{C}$ as a consequence of Sard's theorem. The same proof works over any field of characteristic 0 using the theorem of generic smoothness [AG, III, 10.7]. Let Q be the affine space $H^0(P,\mathscr{E})$, and let $Z \subseteqq Q \times P$ be the set of pairs $\langle s,x\rangle$ such that $x \in (s)_0$. The scheme structure on Z is obtained by considering it as the scheme of zeros of the "diagonal" section of $Q \times \mathscr{E}$ on $Q \times P$. Now since $\mathscr{E}$ is generated by global sections, the projection $Z \to P$ is a fibre bundle, hence Z is nonsingular. Then by generic smoothness [*loc. cit.*], there is a nonempty open subset $U \subseteqq Q$ such that for every $s \in U$, the fibre Z_s, which is just the scheme of zeros $(s)_0$, is nonsingular.

(c) From the exact sequence $0 \to \mathscr{L}^\vee \to \mathscr{E}^\vee \to \mathscr{I}_Y \to 0$ of (1.1) we have $H^1(P,\mathscr{E}^\vee) \cong H^1(P,\mathscr{I}_Y)$, provided $P = \mathbb{P}^n$ with $n \geqq 3$. If this is zero, then the natural map $H^0(P,\mathcal{O}_P) \to H^0(Y,\mathcal{O}_Y)$ is surjective. Therefore $H^0(Y,\mathcal{O}_Y) = k$, which implies that Y is connected.

Remark 1.4.1. If $\mathscr{E}$ is any vector bundle on P, then by Serre's theorems, for $m \gg 0$, $\mathscr{E}(m)$ will be generated by global sections and $H^1((\mathscr{E}(m))^\vee) = 0$. So after a suitable twist, any bundle of rank 2 corresponds to an irreducible nonsingular curve in P. On the other hand, it is sometimes preferable to consider the *least* integer m for which $H^0(\mathscr{E}(m)) \neq 0$, and study curves corresponding to sections of that twist. This is a more canonical procedure, but the curves we get may have more complicated scheme structure.

To conclude this section, we state a remarkable theorem of Ferrand, which implies in particular that *every* nonsingular curve in $P = \mathbb{P}^3$ admits a multiplicity 2 scheme structure which corresponds to a rank 2 vector bundle. This theorem will be used in the next section to construct large families of vector bundles.

Theorem 1.5 (Ferrand). *Let X be a locally complete intersection curve in $P = \mathbb{P}^3$ with $H^0(X,\mathcal{O}_X) = k$. Let $\mathscr{L}$ be an invertible sheaf on X, let $u: \mathscr{I}_X \to \mathscr{L} \to 0$ be a surjective map, and let Y be the scheme defined by $\mathscr{I}_Y = \ker u$. Let m be an integer. Then the following conditions are equivalent:*

(i) $\omega_Y \cong \mathcal{O}_Y(-m)$

(ii) $\mathscr{L} \cong \omega_X(m)$ *and the map*

$$\bar{u}: H^1(P,\mathscr{I}_X(-m)) \to H^1(X,\omega_X)$$

induced by u is the zero map.

Furthermore, if $m \geqq 0$ and $\omega_X \otimes (\mathscr{I}_X/\mathscr{I}_X^2)^\vee(m)$ is generated by global sections, then there is a map $u: \mathscr{I}_X \to \omega_X(m) \to 0$ satisfying the condition of (ii). *In particular, the corresponding Y satisfies* (i).

Proof. See Ferrand [8]. We put the additional hypothesis $H^0(X,\mathcal{O}_X) = k$ to make (i) and (ii) equivalent – Ferrand proves only the nontrivial direction (ii)⇒(i). Also note that if $m \geqq 0$, then $H^1(P,\mathscr{I}_X(-m)) = 0$ because of the hypothesis $H^0(X,\mathcal{O}_X) = k$. Therefore $\bar{u} = 0$ automatically.

Corollary 1.6. *For any locally complete intersection curve X in* $\mathbb{P}^3$ *with* $H^0(X, \mathcal{O}_X)=k$, *there is a scheme Y with support equal to X, which corresponds to a vector bundle of rank 2 on* $\mathbb{P}^3$.

Proof. Combine with (1.1).

Remark 1.6.1. Unlike the other results of this section, there is definitely *not* an analogue of this result on $\mathbb{P}^n$ for $n \geqq 4$. Indeed, we will see later that the bundles constructed by this method for $m \gg 0$ are always unstable, whereas a result of Grauert and Schneider [12] states that on $\mathbb{P}^n_{\mathbb{C}}$, for $n \geqq 4$, any indecomposable rank 2 bundle must be stable. (Grauert has informed me that the proof in [12] is incomplete, but that he is preparing a new proof, valid for $n \geqq 5$.)

§ 2. Numerical Invariants

Throughout this section $\mathscr{E}$ will denote a rank 2 vector bundle on $P=\mathbb{P}^3_k$. We will discuss the Chern classes c_1 and c_2 of $\mathscr{E}$, and the mod 2 invariant α of Atiyah and Rees. If $\mathscr{E}$ corresponds to a curve $Y \subseteq P$, we show how to compute c_1, c_2, and α in terms of Y. We show that $c_1 c_2 \equiv 0 \pmod 2$, and that conversely, for any given values of c_1, c_2, α satisfying $c_1 c_2 \equiv 0 \pmod 2$, there are families of arbitrarily large dimensions of nonisomorphic bundles with the given invariants.

For any $\mathscr{E}$ we have Chern classes c_1 and c_2 in the Chow ring of P. But since that Chow ring is isomorphic to $\mathbb{Z}[h]/h^4$, we will consider c_1 and c_2 as integers. From the general theory of Chern classes (see [AG, App. A] it follows that $\wedge^2 \mathscr{E} \cong \mathcal{O}(c_1)$. Also, for any integer m,

$$c_1(\mathscr{E}(m)) = c_1(\mathscr{E}) + 2m$$

$$c_2(\mathscr{E}(m)) = c_2(\mathscr{E}) + m c_1(\mathscr{E}) + m^2.$$

Also note that since $\mathscr{E}$ has rank 2, the natural map $\mathscr{E} \otimes \mathscr{E} \to \wedge^2 \mathscr{E}$ is a perfect pairing, whence $\mathscr{E}^{\vee} \cong \mathscr{E}(-c_1)$.

Proposition 2.1. *Let* $\mathscr{E}$ *correspond to a curve Y, and let Y have degree d and arithmetic genus* p_a. *Then* $d=c_2$ *and* $2p_a - 2 = c_2(c_1 - 4)$.

Proof. The fact that $d=c_2$ is a general property of Chern classes: the scheme of zeros of a section of a bundle represents the highest Chern class [AG, App. A, § 3, C6]. For the second statement we use the isomorphism $\mathscr{L} \otimes \omega_P \otimes \mathcal{O}_Y \cong \omega_Y$. Since $\mathscr{L} \cong \wedge^2 \mathscr{E} \cong \mathcal{O}_P(c_1)$ and $\omega_P \cong \mathcal{O}_P(-4)$, we have $\omega_Y \cong \mathcal{O}_Y(c_1 - 4)$. If Y is an irreducible nonsingular curve of genus g, then ω_Y is the canonical sheaf, which has degree $2g-2$. Therefore $2g-2 = d(c_1-4)$ and $g=p_a$ so we have our result.

To obtain the same result in the general case, we use the Hilbert polynomial of Y [AG, III, Ex 5.2]. For any integer m, it expresses the Euler characteristic of the sheaf $\mathcal{O}_Y(m)$ in terms of d and p_a:

$$\dim H^0(\mathcal{O}_Y(m)) - \dim H^1(\mathcal{O}_Y(m)) = md + 1 - p_a.$$

Applying this once with $m=0$ and using the duality $H^1(\mathcal{O}_Y) \perp H^0(\omega_Y)$ [AG, III, 7.7], and again with $m = c_1 - 4$ and using the duality $H^1(\omega_Y) \perp H^0(\mathcal{O}_Y)$, and adding the two, we find that $d(c_1 - 4) = 2p_a - 2$ as required.

Corollary 2.2. *If $\mathscr{E}$ has Chern classes c_1 and c_2, then $c_1c_2 \equiv 0 \pmod 2$.*

Proof. For suitable m, the twisted bundle $\mathscr{E}(m)$ will have global sections, so will correspond to a curve Y. Then from the equation $2p_a - 2 = c_2(c_1 - 4)$ of (2.1) it is clear that $c_1c_2 \equiv 0 \pmod 2$ for the Chern classes of $\mathscr{E}(m)$. But a glance at the formulas for $c_i(\mathscr{E}(m))$ in terms of $c_i(\mathscr{E})$ shows the same is true for $\mathscr{E}$.

Remark 2.2.1. Over $\mathbb{C}$, this result is known from homotopy theory for any continuous $\mathbb{C}^2$-bundle over $\mathbb{P}^3_{\mathbb{C}}$. For algebraic bundles it was proved by Schwarzenberger [21, p. 166] as a consequence of the Riemann-Roch theorem for $\mathscr{E}$ on P. Indeed, if c_1 is odd, we may assume $c_1 = -1$. In this case the Riemann-Roch theorem (8.1) says that $\chi(\mathscr{E}) = 1 - \frac{3}{2}c_2$. Since the Euler characteristic χ is an integer, c_2 must be even. In contrast to that, our proof of (2.2) is essentially the Riemann-Roch theorem on Y.

Next we come to the α-invariant of Atiyah and Rees [1]. They show that for any c_1 odd and c_2 even, there is a unique continuous $\mathbb{C}^2$-bundle on $\mathbb{P}^3_{\mathbb{C}}$ with Chern classes c_1, c_2. However, for c_1 even and any c_2, they show that there are exactly two continuous $\mathbb{C}^2$-bundles on $\mathbb{P}^3_{\mathbb{C}}$, distinguished by a certain mod 2 homotopy invariant. If $\mathscr{E}$ is an algebraic rank 2 bundle on $\mathbb{P}^3_{\mathbb{C}}$ with c_1 even, they show that α can be computed as the Euler semi-characteristic

$$\alpha = h^0(P, \mathscr{E}(-\tfrac{1}{2}c_1 - 2)) + h^1(P, \mathscr{E}(-\tfrac{1}{2}c_1 - 2)) \quad \pmod 2,$$

where $h^i = \dim H^i$. Then they show that for each choice of c_1 even, c_2, and $\alpha \in \mathbb{Z}/2\mathbb{Z}$, there is an algebraic rank 2 bundle on $\mathbb{P}^3_{\mathbb{C}}$ with the given invariant.

Over an arbitrary (algebraically closed) ground field k, we take this latter expression as the definition of α.

Definition. Let $\mathscr{E}$ be a rank 2 vector bundle on $P = \mathbb{P}^3_k$, with even first Chern class c_1. Let $\alpha \in \mathbb{Z}/2\mathbb{Z}$ be defined by

$$\alpha = h^0(P, \mathscr{E}(-\tfrac{1}{2}c_1 - 2)) + h^1(P, \mathscr{E}(-\tfrac{1}{2}c_1 - 2)) \quad \pmod 2.$$

Note from this definition that $\alpha(\mathscr{E}(m)) = \alpha(\mathscr{E})$ for any m.

Proposition 2.3. *Suppose that $\mathscr{E}$ has even c_1 and corresponds to a curve Y. Then*

$$\alpha(\mathscr{E}) \equiv \begin{cases} h^0(\mathscr{O}_Y(\tfrac{1}{2}c_1 - 2)) + 1 & \pmod 2 \quad \text{if } c_1 \equiv 4 \pmod 8 \\ h^0(\mathscr{O}_Y(\tfrac{1}{2}c_1 - 2)) & \pmod 2 \quad \text{otherwise.} \end{cases}$$

Proof. We use the exact sequence (1) of (1.1), which can be written

$$0 \to \mathscr{O}(-c_1) \to \mathscr{E}(-c_1) \to \mathscr{I}_Y \to 0.$$

Twisting by $\frac{1}{2}c_1 - 2$ and taking cohomology, we get

$$0 \to H^0(\mathscr{O}_P(-\tfrac{1}{2}c_1 - 2)) \to H^0(\mathscr{E}(-\tfrac{1}{2}c_1 - 2) \to H^0(\mathscr{I}_Y(\tfrac{1}{2}c_1 - 2)) \to 0$$

and

$$H^1(\mathscr{E}(-\tfrac{1}{2}c_1 - 2)) \stackrel{\sim}{\to} H^1(\mathscr{I}_Y(\tfrac{1}{2}c_1 - 2)).$$

Combining with the cohomology of the sequence

$$0 \to \mathscr{I}_Y \to \mathcal{O}_P \to \mathcal{O}_Y \to 0$$

twisted by $\frac{1}{2}c_1 - 2$, we find that

$$\alpha \equiv h^0(\mathcal{O}_Y(\tfrac{1}{2}c_1 - 2)) + h^0(\mathcal{O}_P(\tfrac{1}{2}c_1 - 2)) + h^0(\mathcal{O}_P(-\tfrac{1}{2}c_1 - 2)) \pmod 2.$$

If $c_1 \geqq 0$, then $h^0(\mathcal{O}_P(-\frac{1}{2}c_1 - 2)) = 0$ and

$$h^0(\mathcal{O}_P(\tfrac{1}{2}c_1 - 2)) = \tfrac{1}{6}(\tfrac{1}{2}c_1 + 1)\tfrac{1}{2}c_1(\tfrac{1}{2}c_1 - 1).$$

A congruence for this (mod 2) is given by a congruence for c_1 (mod 8). Substituting $c_1 = 0, 2, 4, 6$, we get 0, 0, 1, 4 respectively. This gives the statement of the proposition for $c_1 \geqq 0$. A similar calculation gives the result for $c_1 \leqq 0$.

Corollary 2.4. *Provided* char. $k \neq 2$, *$\alpha(\mathscr{E})$ is invariant under deformations of $\mathscr{E}$.*

Proof. Over $\mathbb{C}$, this is a consequence of the topological definition of α by Atiyah and Rees [1]. We give an independent proof, valid over any field k of characteristic $\neq 2$. If $\{\mathscr{E}_t\}$ is a flat family of bundles, by (1.4) we can twist by a suitable m so that the bundles $\mathscr{E}_t(m)$ correspond to a flat family of irreducible nonsingular curves Y_t. Then $\alpha(\mathscr{E}_t)$ is determined by $h^0(\mathscr{L}_t)$ (mod 2) where $\mathscr{L}_t = \mathcal{O}_{Y_t}(\frac{1}{2}c_1 - 2)$. But $\mathscr{L}_t$ is an invertible sheaf with the property that $\mathscr{L}_t^{\otimes 2} \cong \omega_{Y_t}$, and a theorem of Mumford [34], generalizing classical results of Riemann and Atiyah, asserts that for such sheaves, the quantity $h^0(Y_t, \mathscr{L}_t)$ (mod 2) is a deformation invariant.

Now we come to the main result of this section. Horrocks [23] first showed that for any integers c_1 and c_2 satisfying $c_1 c_2 \equiv 0 \pmod 2$, there is an algebraic rank 2 bundle $\mathscr{E}$ on $\mathbb{P}_k^3$ with Chern classes c_1 and c_2. Then Atiyah and Rees [1], using the bundles constructed by Horrocks, verified that for c_1 even, both values of α occur. We strengthen these results by showing that for each c_1, c_2, α, the family of rank 2 bundles $\mathscr{E}$ on $\mathbb{P}^3$ with the given invariants is unbounded. This complements a theorem of Maruyama [27] which shows the existence of large families of indecomposable bundles of any rank $r \geqq \dim X$ for any nonsingular projective variety X of dimension $\geqq 2$.

Theorem 2.5. *For each choice of integers c_1, c_2 satisfying $c_1 c_2 \equiv 0 \pmod 2$, and, if c_1 is even, for each choice of $\alpha \in \mathbb{Z}/2\mathbb{Z}$, and for each $N > 0$ (independent of c_1, c_2, α) there exists a family $\{\mathscr{E}_t\}_{t \in T}$ of mutually nonisomorphic rank 2 bundles $\mathscr{E}_t$ on $P = \mathbb{P}_k^3$, with the given invariants c_1, c_2, α, parametrized by a variety T of dimension $> N$.*

Proof. We use the theorem of Ferrand (1.5). Let X be an irreducible nonsingular curve of degree d and genus g in P. Since X is nonsingular, there is an exact sequence of locally free sheaves on X [AG, II, 8.17]

$$0 \to \mathscr{I}_X/\mathscr{I}_X^2 \to \Omega_P \otimes \mathcal{O}_X \to \Omega_X \to 0.$$

In particular, taking duals, there is a surjective map

$$\mathscr{T}_P \otimes \mathcal{O}_X \to (\mathscr{I}_X/\mathscr{I}_X^2)^\vee \to 0$$

where $\mathcal{T}_P$ is the tangent sheaf on P. Now $\mathcal{T}_P$, being a quotient of $\mathcal{O}_P(1)^4$, is generated by global sections. Therefore $(\mathcal{I}_X/\mathcal{I}_X^2)^\vee$ is generated by global sections on X.

On the other hand, if $g \geqq 1$, the sheaf of differentials ω_X is generated by global sections, and if $g=0$, it has degree -2 on X. Thus if $g \geqq 1$ and $m \geqq 0$, or if $g=0$, $d \geqq 2$, and $m \geqq 1$, the sheaf $\omega_X \otimes (\mathcal{I}_X/\mathcal{I}_X^2)^\vee(m)$ is generated by global sections. So for these values of d, g, m, we can apply Ferrand's theorem, and we obtain the existence of a curve Y with support equal to X, and $\omega_Y \cong \mathcal{O}_Y(-m)$. Its degree is $2d$. Furthermore, by construction, there is an exact sequence

$$0 \to \omega_X(m) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0. \tag{5}$$

From this we can compute the arithmetic genus p_a of Y. If $\chi(\mathcal{F})$ denotes the Euler characteristic $\sum(-1)^i \dim H^i(\mathcal{F})$ of a sheaf $\mathcal{F}$, then p_a is determined by $\chi(\mathcal{O}_Y) = 1-p_a$. But from the exact sequence we have $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)+\chi(\omega_X(m))$. Applying the Riemann-Roch theorem on X, this gives

$$\chi(\mathcal{O}_Y)=1-g+2g-2+md+1-g.$$

Therefore $p_a=1-md$. Surprisingly, it is independent of g.

Now by (1.1), Y corresponds to a vector bundle $\mathcal{F}$ with Chern classes $c_1(\mathcal{F}) = 4-m$ and $c_2(\mathcal{F})=2d$. For any integer k, let $\mathcal{E}=\mathcal{F}(k)$. Then

$$c_1(\mathcal{E})=4-m+2k$$

$$c_2(\mathcal{E})=2d+(4-m)k+k^2.$$

The next step is to show that we can obtain any given c_1, c_2 satisfying $c_1c_2 \equiv 0 \pmod 2$ as the Chern classes of $\mathcal{E}$, for suitable choice of d, m, and k. First pick $m \geqq 0$ such that $m \equiv c_1+2c_2 \pmod 4$. Then let

$$k=\tfrac{1}{2}(c_1+m-4)$$

and

$$d=\tfrac{1}{2}(c_2-c_1k+k^2).$$

These values of m, k, d give the required c_1 and c_2. Note that the congruences $c_1c_2 \equiv 0 \pmod 2$ and $m \equiv c_1+2c_2 \pmod 4$ guarantee that k and d are integers. Note also that $m \gg 0$ implies $k \gg 0$ and $d \gg 0$.

Now we construct the required families. Given $N>0$, and given c_1, c_2, pick d_0 sufficiently large so that for any $d \geqq d_0$ there are families $\{X_t\}$ and $\{X_t'\}$ of dimension $>N$ of curves in P of degree d and genus 0 and 1, respectively. To see that this is possible, let Z be a fixed abstract curve of genus 0 or 1, and let $P_0 \in Z$ be a fixed point. Then any sufficiently general 4-dimensional subspace V of $H^0(Z, \mathcal{L}(dP_0))$ gives an embedding of Z in $\mathbb{P}^3$, and different subspaces give different embeddings. Since $\dim H^0(Z, \mathcal{L}(dP_0))=d+1$ (respectively, d), the choice of V is a choice of a point in the Grassmann variety $G(4, d+1)$ (respectively, $G(4,d)$), which had dimension $4(d-3)$ (respectively, $4(d-4)$). In either case, this dimension grows with d, so we can get families of arbitrarily large dimension of these curves. To get the dimension of the family of image curves in $\mathbb{P}^3$ one should subtract 3 (respectively, 1) for the automorphisms of Z. Still, the dimension grows with d.

Then pick $m \geqq 0$ such that $m \equiv c_1 + 2c_2 \pmod 4$ and m is sufficiently large that the corresponding d is $\geqq d_0$. Use the families $\{X_t\}$ and $\{X'_t\}$ to construct families of curves $\{Y_t\}$ and $\{Y'_t\}$ as above, and hence families of bundles $\{\mathscr{E}_t\}$ and $\{\mathscr{E}'_t\}$. These families then have dimension $>N$, and have the given Chern classes c_1 and c_2.

We compute the α invariant in case c_1 is even using (2.3) applied to Y, since $\alpha(\mathscr{E}) = \alpha(\mathscr{F})$. Since Y corresponds to $\mathscr{F}$ and $c_1(\mathscr{F}) = 4 - m$, we must compute $h^0(\mathscr{O}_Y(-\frac{1}{2}m))$. From the exact sequence (5) we get

$$0 \to \omega_X(\tfrac{1}{2}m) \to \mathscr{O}_Y(-\tfrac{1}{2}m) \to \mathscr{O}_X(-\tfrac{1}{2}m) \to 0.$$

For $m > 0$ this gives

$$h^0(\mathscr{O}_Y(-\tfrac{1}{2}m)) = h^0(\omega_X(\tfrac{1}{2}m)).$$

Furthermore, for $m > 0$, $\omega_X(\frac{1}{2}m)$ is nonspecial on X, so by Riemann-Roch on X,

$$\begin{aligned} h^0(\omega_X(\tfrac{1}{2}m)) &= 2g - 2 + \tfrac{1}{2}md + 1 - g \\ &= g - 1 + \tfrac{1}{2}md. \end{aligned}$$

This shows that keeping m and d fixed, the two values $g = 0, 1$ give both values of α. So one of the families $\{\mathscr{E}_t\}$, $\{\mathscr{E}'_t\}$ will have $\alpha = 0$, the other $\alpha = 1$.

Finally, we need to show that the bundles $\mathscr{E}_t$, $\mathscr{E}'_t$ in the families we have constructed are all mutually nonisomorphic. Indeed, we claim that for $m \gg 0$, $\dim H^0(\mathscr{F}) = 1$. It follows then from (1.1) that Y is uniquely determined by $\mathscr{F}$. Thus nonisomorphic curves X_t give nonisomorphic bundles $\mathscr{E}_t$.

To prove the claim, we use the exact sequence of (1.1), twisted by $c_1(\mathscr{F}) = 4 - m$:

$$0 \to \mathscr{O}_P \to \mathscr{F} \to \mathscr{I}_Y(4-m) \to 0.$$

For $m \geqq 5$, $H^0(\mathscr{I}_Y(4-m)) \subseteqq H^0(\mathscr{O}_P(4-m)) = 0$, so $H^0(\mathscr{F}) \cong H^0(\mathscr{O}_P)$ which has dimension 1. q.e.d.

Remark 2.5.1. One might think from this result that the problem of classification of rank 2 bundles on $\mathbb{P}^3$ was hopeless. But in fact, for most of the bundles we have constructed here, we have seen that $\dim H^0(\mathscr{F}) = 1$. Thus $\mathscr{F}$ uniquely determines a curve Y and an isomorphism $\omega_Y \cong \mathscr{O}_Y(c_1 - 4)$ up to a scalar. Therefore the space of all such bundles is fibred over a certain subset of the Hilbert scheme by the spaces $H^0(\mathscr{O}_Y^*)/k^*$. On the other hand, in the next section we will discuss stable and semistable bundles, for which there are good varieties of moduli, and we will see (3.4) that if $\mathscr{E}$ is a rank 2 bundle which is not semistable, and if m is the least integer for which $H^0(\mathscr{E}(m)) \neq 0$, then $\dim H^0(\mathscr{E}(m)) = 1$. So the classification problem for all rank 2 vector bundles on $\mathbb{P}^3$ reduces to the classification of stable and semistable bundles on the one hand, and the study of the Hilbert scheme and the spaces $H^0(\mathscr{O}_Y^*)/k^*$ on the other hand.

§ 3. Stable Bundles

In this section we give the definition and elementary properties and examples of stable rank 2 bundles on $\mathbb{P}^3$.

Definition. A vector bundle $\mathscr{E}$ of rank 2 on $\mathbb{P}^n$ is *stable* (respectively, *semistable*) if for every invertible subsheaf $\mathscr{L}$ of $\mathscr{E}$,

$$c_1(\mathscr{L})<\tfrac{1}{2}c_1(\mathscr{E})$$

(respectively, $\leqq$).

This definition is easily seen to be equivalent to the definition of Mumford and Takemoto [43] which requires that for every rank 1 torsion-free quotient sheaf $\mathscr{F}$ of $\mathscr{E}$, $c_1(\mathscr{F})>\frac{1}{2}c_1(\mathscr{E})$ (respectively, $\geqq$).

Remark 3.0.1. Let us make some elementary observations about this definition. First of all, a bundle $\mathscr{E}$ is stable if and only if $\mathscr{E}(m)$ is stable, for any m. Secondly, if $c_1(\mathscr{E})$ is odd, then equality cannot occur, so $\mathscr{E}$ is stable if and only if it is semistable.

Since twisting a rank 2 bundle by m changes its first Chern class by $2m$, we can twist any bundle so that its first Chern class becomes 0 or -1. In this case we will say that $\mathscr{E}$ is *normalized.* If $\mathscr{E}$ is normalized, then $\mathscr{E}$ is stable if and only if $H^0(\mathscr{E})=0$. Indeed, if $s\in H^0(\mathscr{E})$ is a nonzero section, it determines an injective map of $\mathscr{O}$ to $\mathscr{E}$, whence $\mathscr{E}$ is not stable; conversely, if $\mathscr{E}$ is not stable, it contains a subsheaf $\mathscr{O}(m)$ for some $m\geqq 0$, whence $H^0(\mathscr{E})\neq 0$. Similarly, in case $c_1=0$, $\mathscr{E}$ is semistable if and only if $H^0(\mathscr{E}(-1))=0$.

Finally, a bundle $\mathscr{E}$ on $\mathbb{P}^n$ is stable if and only if it is *simple*, i.e. the only homomorphisms of $\mathscr{E}$ to itself are scalar multiplications. For a proof of this, and more generalities about stable bundles, see Barth [6, § 3].

Remark 3.0.2. Recently Gieseker [10] and Maruyama [29], [30] have introduced a new definition of stability, using the Hilbert polynomial instead of the first Chern class in the inequality of the definition. While the old and new definitions are not equivalent in general, they are equivalent for rank 2 bundles on $\mathbb{P}^3$. The proof simply involves computing the Hilbert polynomials, and using the fact (8.4) that if $\mathscr{E}$ is stable (old definition) on $\mathbb{P}^3$, then $c_1^2-4c_2<0$. On the other hand, the old and new definitions of semistable are not equivalent on $\mathbb{P}^3$: the only (new) semistable bundle is $\mathscr{O}\oplus\mathscr{O}$ and its twists, while there are many (old) semistable bundles. In this paper we always use the old definition.

Next we give a criterion for a bundle to be stable, in terms of a curve associated to the bundle.

Proposition 3.1. *Let $\mathscr{E}$ be a rank 2 bundle on $\mathbb{P}^3$ corresponding to a curve Y in $\mathbb{P}^3$. Then $\mathscr{E}$ is stable (respectively, semistable) if and only if*

(1) $c_1(\mathscr{E})>0$ *(respectively, $c_1(\mathscr{E})\geqq 0$), and*

(2) *Y is not contained in any surface of degree $\leqq\frac{1}{2}c_1(\mathscr{E})$ (respectively $<\frac{1}{2}c_1(\mathscr{E})$).*

Proof. From (1.1) we have the exact sequence

$$0\to\mathscr{O}(-c_1)\to\mathscr{E}(-c_1)\to\mathscr{I}_Y\to 0.$$

For simplicity we suppose c_1 is even, the proof for c_1 odd being entirely analogous. In this case, the normalized bundle $\mathscr{E}'$ corresponding to $\mathscr{E}$ is $\mathscr{E}(-\frac{1}{2}c_1)$, so we have an exact sequence

$$0\to\mathscr{O}(-\tfrac{1}{2}c_1)\to\mathscr{E}'\to\mathscr{I}_Y(\tfrac{1}{2}c_1)\to 0.$$

Now $\mathscr{E}$ is stable if and only if $H^0(\mathscr{E}')=0$. This is equivalent to (1) $H^0(\mathcal{O}(-\frac{1}{2}c_1))=0$, which says that $c_1>0$, and (2) $H^0(\mathscr{I}_Y(\frac{1}{2}c_1))=0$, which says that Y is not contained in any surface of degree $\frac{1}{2}c_1$. Similarly, $\mathscr{E}$ is semistable if and only if $H^0(\mathscr{E}'(-1))=0$, and this is equivalent to saying $c_1\geqq 0$ and Y not contained in any surface of degree $\frac{1}{2}c_1-1$.

Example 3.1.1. Let Y be the disjoint union of r lines in $\mathbb{P}^3$. Then $\omega_Y\cong\mathcal{O}_Y(-2)$, so Y corresponds to a bundle $\mathscr{E}$ with $c_1=2$, $c_2=r$. For $r\geqq 2$, the curve Y is not contained in any plane, so the corresponding bundle $\mathscr{E}$ is stable. From (2.3) we see that the α-invariant of these bundles is 0. The corresponding normalized bundle $\mathscr{E}'$ has $c_1=0$, $c_2=r-1$, so this shows the existence of stable bundles with $c_1=0$, $\alpha=0$, and any $c_2>0$.

Example 3.1.2. Let Y be the disjoint union of r conics in $\mathbb{P}^3$. Then $\omega_Y\cong\mathcal{O}_Y(-1)$, so Y corresponds to a bundle $\mathscr{E}$ with $c_1=3$, $c_2=2r$. For $r\geqq 2$, the curve Y is not contained in any plane, so $\mathscr{E}$ is stable. The corresponding normalized bundle $\mathscr{E}$ has $c_1=-1$, $c_2=2r-2$, so this shows the existence of stable bundles with $c_1=-1$, and any even $c_2>0$.

Example 3.1.3. Let Y be a disjoint union of a nonsingular plane cubic curve and a nonsingular elliptic space curve of degree $r\geqq 4$. Then $\omega_Y\cong\mathcal{O}_Y$, so Y corresponds to a bundle $\mathscr{E}$ with $c_1=4$, $c_2=r+3$. For $r\geqq 4$, the second curve does not lie in a plane, so Y is not contained in any surface of degree 2, hence $\mathscr{E}$ is stable. From (2.3) we see that the α-invariant of $\mathscr{E}$ is 1. The corresponding normalized bundle $\mathscr{E}'$ has $c_1=0$, $c_2=r-1$. This shows the existence of stable bundles with $c_1=0$, $\alpha=1$, and any $c_2\geqq 3$.

Remark 3.1.4. We will see later (8.4) that the values of c_1, c_2, α in the above examples are the *only* possible values of c_1, c_2, α for normalized stable bundles on $\mathbb{P}^3$. The proof involves showing that $c_1^2-4c_2\leqq 0$ for any semistable bundle, which we do in this section, and then eliminating a few special cases, which we do later.

The remainder of this section is devoted to proving the inequality $c_1^2-4c_2\leqq 0$ for a semistable rank 2 bundle on $\mathbb{P}^3$. First we prove the analogous result on $\mathbb{P}^2$, using the Riemann-Roch theorem. Then we reduce the case of $\mathbb{P}^3$ to $\mathbb{P}^2$ using a result of Maruyama. Barth [6] has given a slightly different proof over $\mathbb{C}$, and also a proof of the stronger result that $c_1^2-4c_2<0$ for a stable bundle, which we prove later (8.4) by a different method.

Lemma 3.2 (Schwarzenberger). *If $\mathscr{E}$ is a stable (respectively, semistable) rank 2 bundle on $\mathbb{P}^2$, then $c_1^2-4c_2<0$ (respectively, $\leqq$). Furthermore, if $\mathscr{E}$ is stable, then $c_1^2-4c_2\neq -4$.*

Proof. The Riemann-Roch theorem for $\mathscr{E}$ on $\mathbb{P}^2$ says that

$$\chi(\mathscr{E})=\tfrac{1}{2}(c_1^2-2c_2+3c_1+4).$$

To discover this formula, it is not necessary to interpret the generalized Riemann-Roch theorem, computing Todd classes etc. It is enough to know that there exists some polynomial in $\mathbb{Q}[c_1,c_2]$ which gives $\chi(\mathscr{E})$ for any rank 2 bundle on $\mathbb{P}^2$, and then compute it in enough easy special cases. For example, let $\mathscr{E}=\mathcal{O}(r)\oplus\mathcal{O}(s)$, with $r,s\geqq 0$. Then $H^1(\mathscr{E})=H^2(\mathscr{E})=0$, so

$$\chi(\mathscr{E})=h^0(\mathscr{E})=\tfrac{1}{2}(r+2)(r+1)+\tfrac{1}{2}(s+2)(s+1).$$

Expanding, we get a symmetric function in r and s, which can be written uniquely in terms of $c_1=r+s$ and $c_2=rs$, which are the elementary symmetric functions in r and s. A short calculation gives the formula above.

Now let $\mathscr{E}$ be stable. Since $c_1^2-4c_2$ is invariant under twisting, we may assume that $\mathscr{E}$ is normalized. Then $H^0(\mathscr{E})=0$. By duality, $H^2(\mathscr{E})$ is dual to $H^0(\mathscr{E}(-c_1-3))$, and since $c_1=0$ or -1, this is also 0. So $\chi(\mathscr{E})=-h^1(\mathscr{E})\leqq 0$. In case $c_1=0$, substituting in the Riemann-Roch formula gives $c_2\geqq 2$. In case $c_1=-1$, we get $c_2\geqq 1$. In each case, $c_1^2-4c_2<0$ and $c_1^2-4c_2\neq -4$.

If $\mathscr{E}$ is semistable with $c_1=0$, a similar argument, using $\chi(\mathscr{E}(-1))$, shows that $c_2\geqq 0$. In this case $c_1^2-4c_2\leqq 0$.

Note the curious fact that $c_1=0$, $c_2=1$ is *not* possible for a stable bundle on $\mathbb{P}^2$, but it is possible on $\mathbb{P}^3$. Otherwise the possible values of c_1 and c_2 of stable bundles are the same on $\mathbb{P}^2$ and $\mathbb{P}^3$ provided $c_1c_2\equiv 0 \pmod 2$.

Theorem 3.3 (Maruyama)[3]. *If $\mathscr{E}$ is a semistable rank 2 bundle on $\mathbb{P}^3$, then for almost all planes $H\subsetneqq\mathbb{P}^3$, the restriction $\mathscr{E}|_H$ is semistable on H.*

Proof. We may assume that $\mathscr{E}$ is normalized. If $\mathscr{E}|_H$ is semistable for a single H, then from the criterion $H^0(\mathscr{E}(-1)|_H)=0$ and the semicontinuity of cohomology, we see that $\mathscr{E}|_H$ is semistable for almost all H. Thus if the theorem were false, $\mathscr{E}|_H$ would be not semistable for every plane $H\subsetneqq\mathbb{P}^3$. Before continuing, we need a lemma.

Lemma 3.4. *Let $\mathscr{E}$ be a normalized rank 2 bundle on $\mathbb{P}^n$, $n\geqq 1$, which is not semistable. Let $m<0$ be the least integer such that $H^0(\mathscr{E}(m))\neq 0$. Then $\dim H^0(\mathscr{E}(m))=1$. Furthermore, if $m'<0$ and $s\in H^0(\mathscr{E}(m'))$ is a nonzero section whose zero set has codimension 2, then $m'=m$.*

Proof. First note that if $s\in H^0(\mathscr{E}(m))$ is any nonzero section, then the zero set of s must have codimension 2, because of the minimality of m (1.0.1). Now let $m'<0$ and $s'\in H^0(\mathscr{E}(m'))$ be any nonzero section whose zero scheme Y has codimension 2. Then we have an exact sequence

$$0\to\mathcal{O}\to\mathscr{E}(m')\to\mathscr{I}_Y(c_1(\mathscr{E})+2m')\to 0.$$

Since $c_1(\mathscr{E})=0$ or -1, and $m'<0$, the sheaf on the right has no global sections. Therefore $H^0(\mathscr{E}(m'))\cong H^0(\mathcal{O})$, which has dimension 1.

Applying this result to the case $m'=m$ shows that $\dim H^0(\mathscr{E}(m))=1$. Applying it to any other m' shows that $m'=m$. For if $m'>m$, then the injective map

$$H^0(\mathscr{E}(m))\otimes H^0(\mathcal{O}(m'-m))\to H^0(\mathscr{E}(m'))$$

implies that $\dim H^0(\mathscr{E}(m'))>1$.

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3 This result and its proof were communicated to me by Maruyama. Barth [6] has proved the stronger result over $\mathbb{C}$ that if $\mathscr{E}$ is stable on $\mathbb{P}^3$, then $\mathscr{E}|_H$ is stable for almost all H, *unless* $\mathscr{E}$ is a twist of a bundle with $c_1=c_2=2$ corresponding to two lines (3.1.1). In this paper we use only the more elementary result of Maruyama, thus keeping our work independent of Barth's theorem

Proof of (3.3), continued. Let $\mathscr{E}$ be a normalized rank 2 bundle on $\mathbb{P}^3$, and assume that $\mathscr{E}|_H$ is not semistable for each plane $H \subseteq \mathbb{P}^3$. For each H, let $m(H)$ be the least integer m such that $H^0(\mathscr{E}(m)|_H) \neq 0$. Then because of the semicontinuity of cohomology, $m(H)$ is a lower semicontinuous function of H. Let $m_0 < 0$ be the maximum value of $m(H)$. Then $m(H) = m_0$ for almost all planes H.

Now take a plane H with $m(H) = m_0$. Let $s \in H^0(\mathscr{E}(m_0)|_H)$ be a nonzero section, and let $Y \subseteq H$ be its zero set, which is a finite set of points. Choose a line $L \subseteq H$ which does not meet Y. Then the section s induces a section $\bar{s} \in H^0(\mathscr{E}(m_0)|_L)$ whose zero set is empty (hence of codimension 2 in L). Therefore by the lemma, $m(L)$, the least integer m such that $H^0(\mathscr{E}(m)|_L) \neq 0$, is just m_0.

Next, consider any other plane H' containing L. By choice of m_0, we have $m(H') \leq m_0$. But since $L \subseteq H'$, we have also $m(L) \leq m(H')$. But $m(L) = m_0$, so $m(H') = m_0$ for all H' containing L.

The remainder of the proof is devoted to showing that $H^0(\mathscr{E}(m_0)) \neq 0$ on $\mathbb{P}^3$. The idea is to paste together the sections of $H^0(\mathscr{E}(m_0)|_{H'})$ as H' varies in the pencil of planes containing L. We carry this out formally by blowing up L in $\mathbb{P}^3$. This is similar to the "standard construction" of Barth [6, §4], but simpler in that we blow up a single line instead of using the entire incidence correspondence.

Let $\tilde{X}$ be $\mathbb{P}^3$ with the line L blown up, let $\pi: \tilde{X} \to \mathbb{P}^3$ be the projection, and let $p: \tilde{X} \to \mathbb{P}^1$ be the morphism given by the pencil of planes containing L.

$$\begin{array}{ccc} \pi^{-1}(L) \subseteq & \tilde{X} & \xrightarrow{p} \mathbb{P}^1 \\ \downarrow & \downarrow \pi & \\ L \quad \subseteq & \mathbb{P}^3 & \end{array}$$

For any $t \in \mathbb{P}^1$, let H_t be the corresponding plane in $\mathbb{P}^3$. Then H_t is the fibre of $\tilde{X}$ over t, and for any m,

$$H^0(p^{-1}(t), \pi^*\mathscr{E}(m)) = H^0(\mathscr{E}(m)|_{H_t}).$$

Taking $m = m_0$, all these cohomology groups have dimension 1, so by semicontinuity [AG, III, 12.9], $p_*\pi^*(\mathscr{E}(m_0))$ is locally free of rank 1 on $\mathbb{P}^1$.

Furthermore, for each t, the natural map

$$H^0(\mathscr{E}(m_0)|_{H_t}) \to H^0(\mathscr{E}(m_0)|_L)$$

is an isomorphism. This implies that the natural map

$$p_*\pi^*(\mathscr{E}(m_0)) \to p_*(\pi^*(\mathscr{E}(m_0))|_{\pi^{-1}(L)})$$

is also an isomorphism. But the latter is simply the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, because $H^0(\mathscr{E}(m_0)|_L)$ is independent of t. Thus we see that $p_*\pi^*(\mathscr{E}(m_0)) \cong \mathcal{O}_{\mathbb{P}^1}$.

In particular, this shows that $H^0(\pi^*(\mathscr{E}(m_0))) \neq 0$, and therefore also $H^0(\mathscr{E}(m_0)) \neq 0$ on $\mathbb{P}^3$. Thus $\mathscr{E}$ is not semistable.

Corollary 3.5. *If $\mathscr{E}$ is a semistable rank 2 bundle on $\mathbb{P}^3$, then $c_1^2 - 4c_2 \leq 0$.*

Proof. By (3.3), $\mathscr{E}|_H$ is semistable for some plane $H \subseteq \mathbb{P}^3$. Therefore $c_1^2 - 4c_2 \leq 0$ by (3.2).

§4. Variety of Moduli

In this section we discuss the existence of the variety of moduli of stable rank 2 vector bundles on $\mathbb{P}^3$. We show how to compute its dimension in certain cases. Also we will give some examples of families of stable bundles constructed from curves. The question whether these families represent *all* stable bundles with the given Chern classes will be deferred to Sections 8 and 9.

The general problem of moduli is this: having identified a certain class of objects, find a variety which parametrizes them in a suitable way. For stable torsion-free sheaves, Maruyama [30] has found a solution to this problem. Let X be a nonsingular projective variety over k, and let Σ be the set of isomorphism classes of stable torsion-free sheaves $\mathscr{E}$ of rank r, with a given Hilbert polynomial H. Then Σ has a *coarse moduli scheme* M, which is a separated scheme, locally of finite type over k. This means

(1) The closed points of M are in 1–1 correspondence with the elements of the set Σ;

(2) Whenever $\mathscr{F}$ is a flat family of sheaves $\mathscr{E}$ of Σ, parametrized by a scheme T (i.e., $\mathscr{F}$ is a coherent sheaf on $X \times T$, flat over T, whose fibres are in Σ), then there is a morphism $\varphi: T \to M$ such that for each closed point $t \in T$, $\varphi(t)$ is the point of M corresponding to the class of the sheaf $\mathscr{F}_t$ which is the fibre of $\mathscr{F}$ over t;

(3) The morphisms φ of (2) can be assigned functorially; and

(4) M is universal with properties (2) and (3).

An important question, left unanswered in general, is whether M is necessarily of finite type over k. This is equivalent to the question whether the family Σ is *bounded*. Maruyama has shown this is so if $\dim X \leqq 2$, and has announced that it is also so for the case of rank 2 on any X.

In our case, we take $X = \mathbb{P}^3$, and consider stable vector bundles of rank 2. The vector bundles form a subset of the set of all torsion-free sheaves, and our definition of stable agrees with his in this case (3.0.2). To specify the Hilbert polynomial of $\mathscr{E}$ is equivalent to giving its Chern classes. So we conclude from Maruyama's theorem that the set of stable rank 2 bundles $\mathscr{E}$ on $\mathbb{P}^3$ with given Chern classes c_1 and c_2 has a coarse moduli scheme $M(c_1, c_2)$ which is separated and locally of finite type over k. We will give an independent proof of boundedness in this case (8.3), so in fact M is of finite type over k.

The main goal of this paper is to describe the moduli schemes $M(c_1, c_2)$ explicitly.

We begin with the infinitesimal study of M. For any vector bundle $\mathscr{E}$, let $\mathscr{E}nd\,\mathscr{E}$ be the sheaf of *local endomorphisms* of $\mathscr{E}$, defined as $\mathscr{H}om(\mathscr{E}, \mathscr{E})$ or $\mathscr{E}^\vee \otimes \mathscr{E}$.

Proposition 4.1. *Let $\mathscr{E}$ be a stable bundle on a nonsingular projective variety X. Then $H^1(X, \mathscr{E}nd\,\mathscr{E})$ is naturally isomorphic to the Zariski tangent space of the moduli scheme M at the point corresponding to $\mathscr{E}$. If $H^2(X, \mathscr{E}nd\,\mathscr{E}) = 0$, then M is nonsingular at that point and its dimension is equal to* $\dim H^1(X, \mathscr{E}nd\,\mathscr{E})$.

Proof. This follows from Grothendieck's infinitesimal study of the scheme **Quot** [13] and the way the moduli scheme M is constructed [31, 6.7 and proof of 6.9].

Proposition 4.2. *Let $\mathscr{E}$ be a normalized stable rank 2 bundle on $\mathbb{P}^3$. Then*

$$h^1(\mathscr{E}nd\,\mathscr{E}) - h^2(\mathscr{E}nd\,\mathscr{E}) = \begin{cases} 8c_2 - 3 & \text{if } c_1 = 0 \\ 8c_2 - 5 & \text{if } c_1 = -1. \end{cases}$$

Proof. To compute these cohomology groups, we use the Riemann-Roch theorem for $\mathscr{E}nd\,\mathscr{E}$ on $\mathbb{P}^3$. As in the proof of (3.2), we can discover the Riemann-Roch formula by computing the same easy cases. Since the Chern classes of $\mathscr{E}nd\,\mathscr{E}$ are determined by the Chern classes c_1, c_2 of $\mathscr{E}$, the Riemann-Roch theorem implies that $\chi(\mathscr{E}nd\,\mathscr{E})$ is given by some polynomial in $\mathbb{Q}[c_1, c_2]$. So let $\mathscr{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then $\mathscr{E}nd\,\mathscr{E} = \mathcal{O}(a-b) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(b-a)$. Assuming $a > b$,

$$\chi(\mathscr{E}nd\,\mathscr{E}) = h^0(\mathcal{O}(a-b)) + 1 + 1 - h^3(\mathcal{O}(b-a)).$$

Using duality for h^3, and the known cohomology of $\mathbb{P}^3$,

$$\chi(\mathscr{E}nd\,\mathscr{E}) = \binom{a-b+3}{3} + 1 + 1 - \binom{a-b-1}{3}.$$

Now a short calculation gives

$$\chi(\mathscr{E}nd\,\mathscr{E}) = 2c_1^2 - 8c_2 + 4.$$

This is the Riemann-Roch formula.

To prove the proposition, note that since $\mathscr{E}$ is stable, it is simple (3.0.1), so $h^0(\mathscr{E}nd\,\mathscr{E}) = 1$. On the other hand, $H^3(\mathscr{E}nd\,\mathscr{E})$ is dual to $H^0((\mathscr{E}nd\,\mathscr{E})(-4))$, which must be 0 since $h^0(\mathscr{E}nd\,\mathscr{E}) = 1$. Thus only $h^1(\mathscr{E}nd\,\mathscr{E})$ and $h^2(\mathscr{E}nd\,\mathscr{E})$ are unknown. Substituting $c_1 = 0$ and $c_1 = -1$ gives the result.

Remark 4.2.1. We can expect that in good cases, $h^2(\mathscr{E}nd\,\mathscr{E})$ will be 0, in which case by (4.1) the moduli scheme will be nonsingular of dimension $8c_2 - 3$ (respectively, $8c_2 - 5$). However, we will show by example (4.3.6) that there may be components of bigger dimension.

Proposition 4.3. *Let $\mathscr{E}$ be a stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes c_1, c_2, corresponding to a curve Y. Assume*

(1) $H^1(\mathscr{I}_Y(-4)) = 0$,

(2) $H^1(\mathscr{I}_Y(c_1 - 4)) = 0$, *and*

(3) $H^1(\mathscr{N}_Y) = 0$, *where $\mathscr{N}_Y = (\mathscr{I}/\mathscr{I}^2)^\vee$ is the normal sheaf of Y.*

Then $H^2(\mathscr{E}nd\,\mathscr{E}) = 0$. Furthermore, if Y is nonsingular, then (1) *is automatically verified, and* (3) *can be replaced by*

(3′) $H^1(\mathcal{O}_Y(1)) = 0$.

Proof. Since $\mathscr{E}$ corresponds to Y we have the exact sequence

$$0 \to \mathcal{O}(-c_1) \to \mathscr{E}(-c_1) \to \mathscr{I}_Y \to 0.$$

Now $\mathscr{E}nd\,\mathscr{E} \cong \mathscr{E}^\vee \otimes \mathscr{E}$, and $\mathscr{E}^\vee \cong \mathscr{E}(-c_1)$, so tensoring with $\mathscr{E}$ gives an exact sequence

$$\ldots \to H^2(\mathscr{E}(-c_1)) \to H^2(\mathscr{E}nd\,\mathscr{E}) \to H^2(\mathscr{E} \otimes \mathscr{I}_Y) \to \ldots.$$

To show $H^2(\mathscr{E}nd\,\mathscr{E}) = 0$ it is sufficient to show that the two groups on either side are 0. Now $H^2(\mathscr{E}(-c_1))$ is dual to $H^1(\mathscr{E}(-4))$, which from the first sequence is isomorphic to $H^1(\mathscr{I}_Y(c_1-4))$. This is condition (2). On the other hand, from the sequence

$$0\to\mathscr{I}_Y\to\mathscr{O}\to\mathscr{O}_Y\to 0,$$

tensoring with $\mathscr{E}$, we get an exact sequence

$$\dots\to H^1(\mathscr{E}\otimes\mathscr{O}_Y)\to H^2(\mathscr{E}\otimes\mathscr{I}_Y)\to H^2(\mathscr{E})\to\dots.$$

Again, it is sufficient to show the two outside groups are 0. Now $\mathscr{E}\otimes\mathscr{O}_Y\cong\mathscr{N}_Y$, because tensoring the first sequence with $\mathscr{O}_Y$ gives an isomorphism

$$\mathscr{E}(-c_1)\otimes\mathscr{O}_Y\xrightarrow{\sim}\mathscr{I}_Y\otimes\mathscr{O}_Y=\mathscr{I}_Y/\mathscr{I}_Y^2.$$

So $H^1(\mathscr{E}\otimes\mathscr{O}_Y)=H^1(\mathscr{N}_Y)$, which is condition (3). Finally, $H^2(\mathscr{E})$ is dual to $H^1(\mathscr{E}(-c_1-4))$ which is isomorphic to $H^1(\mathscr{I}_Y(-4))$, which is condition (1).

Now if Y is nonsingular, then $H^0(\mathscr{O}_Y(-4))=0$, so (1) is automatic. On the other hand, $\mathscr{N}_Y$ is a quotient of the tangent bundle of $\mathbb{P}^3$ restricted to Y, which in turn is a quotient of $\mathscr{O}_Y(1)^4$. So it is sufficient to require that $H^1(\mathscr{O}_Y(1))=0$, which is condition (3′).

The rest of this section will be devoted to examples.

Example 4.3.1. Let Y be a union of r disjoint lines in $\mathbb{P}^3$, and let $\mathscr{E}$ be the corresponding bundle (3.1.1). Then $\mathscr{E}$ has Chern classes $c_1=2$ and $c_2=r$. The corresponding normalized bundle $\mathscr{E}'$ has $c_1=0$ and $c_2=r-1$. The conditions of (4.3) are immediately satisfied for Y, so the moduli space is nonsingular of dimension $8c_2(\mathscr{E}')-3=8r-11$ at the point corresponding to $\mathscr{E}$.

Now let us compute the actual dimension of the family of bundles $\mathscr{E}$ obtained by this construction. To give r lines in $\mathbb{P}^3$ requires $4r$ parameters, and to specify an isomorphism $\xi:\omega_Y\cong\mathscr{O}_Y(-2)$ requires r additional parameters, since $H^0(\mathscr{O}_Y^*)\cong(k^*)^r$. Therefore the set of pairs $\langle Y,\xi\rangle$ forms an irreducible family of dimension $5r$. By (1.1) this is the same as the family of triples $\langle\mathscr{E},s,\varphi\rangle$ modulo the equivalence relation $\sim$. For the purpose of counting dimensions, we can ignore φ, because it is determined in any case up to a scalar λ, and this ambiguity is eliminated by the equivalence relation $\sim$. We conclude that the family of isomorphism classes of vector bundles $\mathscr{E}$ obtained in this way is irreducible, and its dimension is equal to $5r$ minus the dimension of $H^0(\mathscr{E})$ for a general $\mathscr{E}$ in the family. This is because the section s can be any sufficiently general element of $H^0(\mathscr{E})$, and the difference in dimension of a surjective morphism of varieties is equal to the dimension of the general fibre.

To compute $H^0(\mathscr{E})$ we use the exact sequence

$$0\to\mathscr{O}\to\mathscr{E}\to\mathscr{I}_Y(2)\to 0,$$

which tells us that

$$h^0(\mathscr{E})=1+h^0(\mathscr{I}_Y(2)).$$

Now $H^0(\mathscr{I}_Y(2))$ is the space of quadratic polynomials corresponding to quadric surfaces containing Y. If $r=2$ its dimension is 4; if $r=3$ it is 1; if $r\geqq 4$ and the lines are in general position, it is 0. Thus $h^0(\mathscr{E})=5$, 2, 1 respectively, and the dimension of the family of bundles $\mathscr{E}$ is 5 if $r=2$; 13 if $r=3$; and $5r-1$ if $r\geqq 4$.

Expressing our results in terms of the normalized bundles $\mathscr{E}'$, the family of normalized bundles obtained from skew lines with $c_1=0$ and $c_2>0$ has dimension

$$\begin{array}{ll} 5 & \text{if } c_2=1 \\ 13 & \text{if } c_2=2 \\ 5c_2+4 & \text{if } c_2 \geqq 3. \end{array}$$

For $c_2=1$ and 2, the dimension of this family is equal to the dimension of the moduli space, which is $8c_2-3$. So in this case these families form an open subset of the moduli space. In fact, we will see later (8.4.1) and (9.6) that in these two cases, every stable bundle with these Chern classes is among the ones we have just constructed (except for the case $c_2=2$ over a field of characteristic 3). However, for $c_2 \geqq 3$, this family has dimension less than $8c_2-3$, so it does not include all stable bundles with the given Chern classes.

Example 4.3.2. Let Y be a union of r disjoint conics, $r \geqq 2$, and let $\mathscr{E}$ be the corresponding bundle (3.1.2). Then $\mathscr{E}$ has Chern classes $c_1=3$ and $c_2=2r$, and the corresponding normalized bundle $\mathscr{E}'$ has Chern classes $c_1=-1$ and $c_2=2r-2$. Again the conditions of (4.3) are immediately satisfied, so the moduli space is nonsingular of dimension $8c_2(\mathscr{E}')-5=16r-21$ at the corresponding point.

To compute the dimension of the family of bundles obtained, note that choosing r conics requires $8r$ parameters, and fixing the isomorphism ξ requires r more parameters. By an argument similar to the previous example, one can show that $h^0(\mathscr{E})=7$ if $r=2$; 2 if $r=3$, and 1 if $r \geqq 4$. Therefore the family of normalized bundles $\mathscr{E}'$ with $c_1=-1$ and $c_2>0$ obtained from conics has dimension

$$\begin{array}{ll} 11 & \text{if } c_2=2 \\ 25 & \text{if } c_2=4 \\ \frac{9}{2}c_2+8 & \text{if } c_2 \geqq 6. \end{array}$$

For $c_2=2$ this is the same as the dimension of the variety of moduli, so in this case we have an open subset of the moduli space. For $c_2 \geqq 4$ it is less, so it cannot be the whole moduli space.

Example 4.3.3. Let Y be a nonsingular elliptic curve of degree d in $\mathbb{P}^3$. Then $\omega_Y \cong \mathcal{O}_Y$, so Y corresponds to a bundle $\mathscr{E}$ with $c_1=4$ and $c_2=d$. $\mathscr{E}$ will be stable if Y is not contained in any quadric surface. For $d=3$, Y is a plane curve; for $d=4$ it is a complete intersection of two quadrics. But for $d \geqq 5$, Y cannot be contained in any quadric (it suffices to check the degree and genus of all curves on a quadric surface [AG, III, Ex. 5.6c; V, Ex. 2.9]). Therefore $\mathscr{E}$ is stable if $d \geqq 5$. By (2.3) the α-invariant of $\mathscr{E}$ is 0 since Y is irreducible so $h^0(\mathcal{O}_Y)=1$. The criterion of (4.3) is immediately satisfied, so the moduli variety at a corresponding point is nonsingular of dimension $8c_2-3$, where c_2 is the second Chern class of the normalized bundle $\mathscr{E}'$, which has $c_1=0$, $c_2=d-4$. So this gives another method, besides skew lines, of constructing stable bundles with $c_1=0$, $\alpha=0$, and any $c_2>0$.

Now let us compute the dimension of the family of bundles obtained in this way. The choice of an abstract elliptic curve Y is 1 parameter (the j-invariant). The choice of an invertible sheaf $\mathscr{L}$ of degree d on Y is 1 parameter (a point in the Picard variety of Y, which is Y itself). The choice of a 4-dimensional linear subspace of $H^0(Y, \mathscr{L})$, which is d-dimensional, is $4(d-4)$ parameters. Then add automorphisms of $\mathbb{P}^3$ (15 parameters), subtract automorphisms of Y (1 parameter) and add the choice of ξ (1 parameter). This gives $4d+1$, from which we must subtract $h^0(\mathscr{E})$ to get the dimension of the family of bundles.

From the exact sequence

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_Y(4) \to 0$$

we see that $h^0(\mathscr{E}) = h^0(\mathscr{I}_Y(4)) + 1$. On the other hand, from the exact sequence

$$0 \to H^0(\mathscr{I}_Y(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_Y(4))$$

we see that $h^0(\mathscr{I}_Y(4)) \geqq h^0(\mathcal{O}_{\mathbb{P}^3}(4)) - h^0(\mathcal{O}_Y(4)) = 35 - 4d$.

Therefore $h^0(\mathscr{E}) \geqq \max(36-4d, 1)$. From this, we find that the dimension of the family of bundles constructed in this way is

$$\begin{array}{ll} \leqq 5 & \text{if } c_2 = 1 \\ \leqq 13 & \text{if } c_2 = 2 \\ \leqq 21 & \text{if } c_2 = 3 \\ \leqq 29 & \text{if } c_2 = 4 \\ \leqq 4c_2 + 16 & \text{if } c_2 \geqq 5. \end{array}$$

Since these numbers are equal to $8c_2 - 3$ for $c_2 = 1, 2, 3, 4$, it is reasonable to expect that most bundles with those Chern classes are obtained by this construction. But for $c_2 \geqq 5$ this number is less than $8c_2 - 5$, so we cannot get all bundles this way. Next, note that for $c_2 = 1, 2$ we get the same dimensions as in example (4.3.1), so we can expect that we get the same bundles this way: see (9.4.1). For $c_2 = 3, 4$ it appears that we get *more* bundles than in (4.3.1), but for $c_2 \gg 0$, the family has smaller dimension than in (4.3.1).

Remark and Conjecture 4.3.4. At the point in (4.3.3) where we derived the inequality $h^0(\mathscr{I}_Y(4)) \geqq \max(35-4d, 0)$, it seems reasonable to expect that for a *sufficiently general* elliptic curve Y of degree d in $\mathbb{P}^3$, we should have equality. For any $d \geqq 5$, any such Y is the projection of an elliptic curve Y' of degree d in $\mathbb{P}^{d-1}$, defined by the complete linear system on Y corresponding to $H^0(\mathcal{O}_Y(1))$. A theorem of Mumford [33] states that for any $d \geqq 2g+1$, any complete linear system of degree d on a curve of genus g gives a projectively normal embedding into projective space. Therefore Y' is projectively normal in $\mathbb{P}^{d-1}$, and so the map $H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(l)) \to H^0(\mathcal{O}_Y(l))$ is surjective for all l. So in our case, the equality $h^0(\mathscr{I}_Y(4)) = \max(35-4d, 0)$ would follow if we could prove the following *conjecture*: let $Y' \subsetneqq \mathbb{P}^n$ be a projectively normal curve, for some $n \geqq 3$. Then for any sufficiently general projection Y of Y' into $\mathbb{P}^3$, the natural maps $H^0(\mathcal{O}_{\mathbb{P}^3}(l)) \to H^0(\mathcal{O}_Y(l))$ should have *maximal rank*, for all l.

Example 4.3.5. Let Y be a projection into $\mathbb{P}^3$ of a canonical curve Y of genus g in $\mathbb{P}^{g-1}$. Then Y has degree $2g-2$, and $\omega_Y \cong \mathcal{O}_Y(1)$, so Y corresponds to a bundle $\mathscr{E}$ with $c_1 = 5$ and $c_2 = 2g-2$. The corresponding normalized bundle $\mathscr{E}'$ has $c_1 = -1$ and $c_2 = 2g-8$. The bundle $\mathscr{E}$ will be stable provided Y is not contained in any quadric surface. For $g = 3$, Y is a plane curve; for $g = 4$ it is a complete intersection of a quadric with a cubic surface; but for $g \geqq 5$ it is not contained in any quadric surface, so $\mathscr{E}$ is stable. The criterion of (4.3) fails, so we cannot tell if the variety of moduli is nonsingular.

To compute the dimension of the family of bundles obtained, we proceed as follows. The choice of an abstract curve of genus g is $3g-3$ parameters. The canonical embedding is then uniquely determined. The projection to $\mathbb{P}^3$ is given by a 4-dimensional subspace of $H^0(\omega_Y)$, which has dimension g, so this requires $4(g-4)$ parameters. Then we add the automorphisms of $\mathbb{P}^3$ (15 parameters), subtract the automorphisms of Y (0) and add the choice of ξ (1). This gives $7g-3$, from which we must subtract $h^0(\mathscr{E})$.

A calculation similar to that in (4.3.3) gives $h^0(\mathscr{E}) \geqq \max(66-9g, 1)$, from which we find that the dimension of the family of bundles obtained is

$$\begin{array}{ll} \leqq 11 & \text{if } c_2=2 \\ \leqq 27 & \text{if } c_2=4 \\ \leqq 43 & \text{if } c_2=6 \\ \leqq \frac{7}{2}c_2+24 & \text{if } c_2 \geqq 8. \end{array}$$

Furthermore, these inequalities could be replaced by equalities if we could prove the conjecture of (4.3.4), because by a theorem of Petri (see [38]), the canonical curve is projectively normal.

For $c_2=2,4,6$ these dimensions agree with the number $8c_2-5$ suggested by (4.2). So we might expect to get all bundles with $c_1=-1$ and $c_2=2,4,6$ by this construction, whereas for $c_2 \geqq 8$ it will not give all. Note also that for $c_2=2$ we get the same size family as in (4.3.2); for $c_2=4,6$, we get a larger family, but for $c_2 \gg 0$ we get a smaller family.

Example 4.3.6. Here we give an example of a family of stable bundles with $c_1=0$, $c_2=7$ and dimension 55, which is greater than the dimension $8c_2-3=53$ suggested by (4.2). If follows in this case that $h^2(\mathscr{E}nd\,\mathscr{E}) \neq 0$.

Let Z be the disjoint union of two curves Y_1, Y_2, each of which is a complete intersection $F_2 \cdot F_4$ of a quadric and a quartic surface. Then Z has degree 16 and $\omega_Z \cong \mathcal{O}_Z(2)$, so Z corresponds to a bundle $\mathscr{E}'$ with $c_1=6$ and $c_2=16$. The corresponding normalized bundle $\mathscr{E}'=\mathscr{E}(-3)$ has $c_1=0$ and $c_2=7$. $\mathscr{E}$ will be stable provided $H^0(\mathscr{E}(-3))=0$, which we will verify below.

Now we compute the dimension of the family. Let Y be a complete intersection of a quadric and a quartic surface. The quadric Q containing Y is uniquely determined, and its choice depends on 9 parameters. To determine Y, we must give a section of $H^0(\mathcal{O}_Q(4))$, up to scalar multiple. It is easy to compute $h^0(\mathcal{O}_Q(4))=25$, so Y depends on $9+25-1=33$ parameters. Therefore the choice of Z requires 66 parameters, and then ξ requires 2, since Z has two connected components. This makes 68, from which we must subtract $h^0(\mathscr{E})$. From the exact sequence

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_Z(6) \to 0$$

we have $h^0(\mathscr{E})=h^0(\mathscr{I}_Z(6))+1$, so we must compute $h^0(\mathscr{I}_Z(6))$. The same exact sequence shows that $h^0(\mathscr{E}(-3))=h^0(\mathscr{I}_Z(3))$, so to show $\mathscr{E}$ stable we must verify $h^0(\mathscr{I}_Z(3))=0$.

Let $I_1=(f_1, g_1)$ and $I_2=(f_2, g_2)$ be the homogeneous ideals of Y_1 and Y_2 in $S=k[x_0, x_1, x_2, x_3]$, where f_1, f_2 have degree 2 and g_1, g_2 have degree 4. Then the homogeneous ideal J of Z is $I_1 \cap I_2$. Since $Y_1 \cap Y_2 = \emptyset$, $I_1 + I_2$ is primary for the irrelevant ideal (x_0, x_1, x_2, x_3), from which it follows that f_1, g_1, f_2, g_2 form a *regular sequence* in S. From this it is easy to see that $J = (f_1 f_2, f_1 g_2, f_2 g_1, g_1 g_2)$ (see for example [16]).

Since $H^0(\mathscr{I}_Z(l))$ is simply the set of elements of degree l in J, we see immediately that $h^0(\mathscr{I}_Z(3)) = 0$, which proves that $\mathscr{E}$ is stable. In degree 6, J contains all expressions of the form $f_1 f_2 \cdot q + \alpha f_1 g_2 + \beta f_2 g_1$, where q is a quadratic form and α, β are scalars. Assuming for example that f_1, f_2, g_1, g_2 are irreducible and relatively prime, which is true in general, one sees easily that these expressions form a vector space of dimension $10+1+1=12$. Therefore $h^0(\mathscr{I}_Z(6)) = 12$ and $h^0(\mathscr{E}) = 13$. Thus the dimension of the family is $68 - 13 = 55$.

Note: I have learned from Barth that he has also constructed families of stable bundles with $c_1 = 0$ and dimension $> 8c_2 - 3$, for every odd value of $c_2 \geqq 5$, using an analogous method.

§ 5. Set of Points in $\mathbb{P}^2$

Our goal in the next few sections is to find a good bound for the least integer t such that $H^0(\mathscr{E}(t)) \neq 0$, where $\mathscr{E}$ is a stable bundle on $\mathbb{P}^3$ with given Chern classes. Our technique is to restrict $\mathscr{E}$ to a plane $\mathbb{P}^2$, and use various estimates of cohomology groups which we develop there.

In this section we will consider a finite set of points Z in $P = \mathbb{P}^2$. Given an integer l, we ask whether the points Z impose independent conditions on the curves of degree l passing through them. This means that

$$h^0(\mathscr{O}_P(l)) - h^0(\mathscr{I}_Z(l))$$

is equal to the number of points in the set Z. Equivalently, we ask whether the map

$$H^0(\mathscr{O}_P(l)) \to H^0(\mathscr{O}_Z(l))$$

is surjective. This in turn is equivalent to $H^1(\mathscr{I}_Z(l))$ being zero. So in this section we will be deriving estimates for $h^1(\mathscr{I}_Z(l))$.

We consider the following hypotheses on a set Z:

(∗) Z is a set of d distinct points in $\mathbb{P}^2$, contained in an irreducible curve C of degree k, but not contained in any curve of lower degree (irreducible or not).

Proposition 5.1. *Let $Z \subsetneqq \mathbb{P}^2$ satisfy (∗). Let $s \geqq k$ be the least degree of a curve D containing Z, where D does not contain C as a component. (D may be reducible.) For each $l \in \mathbb{Z}$, let*

$$c_l = h^1(\mathscr{I}_Z(l-1)) - h^1(\mathscr{I}_Z(l)).$$

Then

$$c_l = \begin{cases} 0 & \textit{for} \quad l<0 \\ l+1 & \textit{for} \quad 0 \leqq l < k \\ k & \textit{for} \quad k \leqq l < s \end{cases}$$

and

$$k=c_{s-1}>c_s>\ldots>c_q=0 \text{ for some } q; \qquad c_l=0 \quad \text{for} \quad l\geqq q.$$

Proof. By hypothesis, $C\cap D$ is a finite set of points containing Z. We take $L\subseteq \mathbb{P}^2$ a line which does not meet $C\cap D$. Then consider the exact sequence

$$0\to \mathscr{I}_Z(l-1)\to \mathscr{I}_Z(l)\to \mathcal{O}_L(l)\to 0.$$

This gives a cohomology sequence, for $l\geqq 0$

$$0\to H^0(\mathscr{I}_Z(l-1))\to H^0(\mathscr{I}_Z(l))\to H^0(\mathcal{O}_L(l))\to H^1(\mathscr{I}_Z(l-1))\to H^1(\mathscr{I}_Z(l))\to 0.$$

Note for $l<0$, from the sequence $0\to \mathscr{I}_Z(l)\to \mathcal{O}_{\mathbb{P}^2}(l)\to \mathcal{O}_Z(l)\to 0$ we get $h^1(\mathscr{I}_Z(l))=0$, so $c_l=0$ also.

Now for $l<k$, $H^0(\mathscr{I}_Z(l))=0$, so $c_l=h^0(\mathcal{O}_L(l))=l+1$. For $k\leqq l<s$, every curve containing Z consists of C plus something else, so

$$h^0(\mathscr{I}_Z(l))=h^0(\mathcal{O}_{\mathbb{P}^2}(l-k))=\tfrac{1}{2}(l-k+2)(l-k+1).$$

Thus

$$c_l=l+1-\tfrac{1}{2}(l-k+2)(l-k+1)+\tfrac{1}{2}(l-k+1)(l-k)=k.$$

In particular, $c_{s-1}=k$.

Now, for $l\geqq s$, let

$$V=\operatorname{im}(H^0(\mathscr{I}_Z(l))\to H^0(\mathcal{O}_L(l))).$$

Then V determines a linear system on L without base points, since $L\cap C\cap D=\emptyset$. And we can recover

$$c_l=\operatorname{codim}(V, H^0(\mathcal{O}_L(l))).$$

Now the last statement is a consequence of the following lemma.

Lemma 5.2 (Mumford). *Let $L=\mathbb{P}^1$, let $V\subseteq H^0(\mathcal{O}_L(l))$ be a linear system without base points, and let $V'\subseteq H^0(\mathcal{O}_L(l+1))$ be the image of $V\otimes H^0(\mathcal{O}_L(1))$. Assume $V\neq H^0(\mathcal{O}_L(l))$. Then*

$$\operatorname{codim}(V', H^0(\mathcal{O}_L(l+1))<\operatorname{codim}(V, H^0(\mathcal{O}_L(l))).$$

Proof. Let $\dim V=q$. Then we have a surjective map $V\otimes\mathcal{O}_L\to\mathcal{O}_L(l)$. Let $\mathscr{E}$ be the kernel:

$$0\to\mathscr{E}\to V\otimes\mathcal{O}_L\to\mathcal{O}_L(l)\to 0.$$

Then $\mathscr{E}$ is locally free of rank $q-1$ on L. Furthermore, since V is a subspace of $H^0(\mathcal{O}_L(l))$, we have $H^0(\mathscr{E})=0$. It follows from the classification of locally free sheaves on $\mathbb{P}^1$ [AG, V, Ex. 2.1] that $\mathscr{E}\cong\bigoplus_{i=1}^{q-1}\mathcal{O}(v_i)$, with $v_i<0$ for each i.

Now tensor with $\mathcal{O}(1)$ and take cohomology. We get

$$0\to H^0(\mathscr{E}(1))\to V\otimes H^0(\mathcal{O}_L(1))\to H^0(\mathcal{O}_L(l+1)).$$

Since each $v_i<0$, $h^0(\mathscr{E}(1))\leqq q-1$. Also, $\dim(V\otimes H^0(\mathcal{O}_L(1)))=2q$; $\dim H^0(\mathcal{O}_L(l+1)) = l+2$. So

$$a=\operatorname{codim}(V', H^0(\mathcal{O}_L(l+1))=(l+2)-2q+h^0(\mathscr{E}(1))\leqq l-q+1.$$

On the other hand

$$b=\operatorname{codim}(V, H^0(\mathcal{O}_L(l)))=l+1-q,$$

so we get $a\leqq b$. If $a=b$, then $h^0(\mathscr{E}(1))=q-1$, which implies $v_i=-1$ for each i. In that case, taking degrees, we see $\deg\mathscr{E}=-(q-1)$, so $l=q-1$, because $\deg\mathscr{E}+\deg\mathcal{O}_L(l)=\deg V\otimes\mathcal{O}_L=0$. But then $q=l+1$, so $V=H^0(\mathcal{O}_L(l))$, contrary to hypothesis. We conclude $a<b$ as required.

Next we include a key technical lemma.

Lemma 5.3 (Gieseker). *Let $b_0, b_1, \ldots, b_m$ be a sequence of nonnegative integers. Assume*

(1) *for each i, $b_i<b_{i+1}$ unless both of them are 0 or both are $\geqq m$.*

(2) $\sum_{i=0}^{m} b_i\leqq\frac{1}{2}m(m+1)$.

Then

(a) $b_0=0$

(b) *for each $0\leqq l\leqq m$, $\sum_{i=0}^{l} b_i\leqq\frac{1}{2}l(l+1)$.*

(c) *for each $0\leqq l\leqq m$, let $e_l=\frac{1}{2}l(l+1)-\sum_{0}^{l} b_i$. Then*

$$e_l\geqq\min(l, e_m).$$

Proof.

Case 1. $b_m<m$. Then we have strict inequalities, which imply $b_l<l$ for all $l>0$. Thus $b_0=0$, and for each l, $\sum_0^l b_i\leqq\sum_0^l i=\frac{1}{2}l(l+1)$.

Case 2. $b_m\geqq m$. We use induction on m, the case $m=0$ being trivial. Note $\sum_0^{m-1} b_i=\sum_0^m b_i-b_m\leqq\frac{1}{2}m(m+1)-m=\frac{1}{2}(m-1)(m)$, so the sequence $b_0, \ldots, b_{m-1}$ satisfies the hypotheses for $m-1$. So (a) and (b) follow by induction.

Proof of (c). If $e_m=0$, there is nothing to prove. If $e_m>0$, because of hypothesis (2), there must be some i with $b_i<i$. So we can pick integers $0<\alpha\leqq\beta\leqq m$ such that

$$b_i<i \quad \text{for} \quad 0<i\leqq\alpha$$
$$b_i=i \quad \text{for} \quad \alpha<i\leqq\beta$$
$$b_i>i \quad \text{for} \quad \beta<i\leqq m.$$

Then for $l\leqq\alpha$ we have $\sum_0^l b_i\leqq\frac{1}{2}l(l-1)$, so $e_l\geqq l$. For $\alpha<l\leqq\beta$, we have $e_\alpha=\ldots=e_\beta$

Then for $l>\beta$ we have $e_\beta > e_{\beta+1} > \dots > e_m$. Thus it is clear that $e_l \geqq \min(l, e_m)$ for all l.

We apply (5.3) to the c_i to get estimates for $h^1(\mathscr{I}_Z(l))$.

Proposition 5.4. *Let $Z \subseteq \mathbb{P}^2$ satisfy $(*)$. Assume $d \geqq k^2$, and choose r such that $rk \leqq d < (r+1)k$. Let $e=(r+1)k-d>0$. Then*

$$h^1(\mathscr{I}_Z(l)) = \begin{cases} d & \text{for} \quad l<0 \\ d-\frac{1}{2}(l+1)(l+2) & \text{for} \quad 0 \leqq l < k \\ d+\frac{1}{2}k(k-3)-kl & \text{for} \quad k \leqq l \leqq r-1 \end{cases}$$

and

$$h^1(\mathscr{I}_Z(l)) \leqq \begin{cases} \frac{1}{2}(r+k-l)(r+k-l-1)-e & \text{for} \quad r-1 \leqq l \leqq r+k-e-1 \\ \frac{1}{2}(r+k-l-1)(r+k-l-2) & \text{for} \quad r+k-e-1 \leqq l \leqq r+k-1 \end{cases}$$

and

$$h^1(\mathscr{I}_Z(l)) = 0 \quad \text{for} \quad l \geqq r+k-2.$$

Proof. Since $rk \leqq d$ and clearly $d \leqq sk$ since $Z \subseteq C \cap D$ (with the s of (5.1)), we have $r \leqq s$. We will apply (5.3) to the sequence $b_i = c_{r+k-i}$, with $m=k$. To check the hypotheses, note first that we have

$$k = c_{r-1} = \dots = c_{s-1} > c_s > \dots$$

so hypothesis (1) is satisfied. For hypothesis (2), note that $\sum_0^k b_i = \sum_r^\infty c_i$. On the other hand, $\sum_{-\infty}^\infty c_i = h^1(\mathscr{I}_Z(a)) - h^1(\mathscr{I}_Z(b))$ for $a \ll 0$ and $b \gg 0$, which is d. So

$$\sum_0^i b_i = d - \sum_{-\infty}^{r-1} c_i .$$

This last sum we can compute explicitly from (5.1):

$$\sum_{-\infty}^{r-1} c_i = \tfrac{1}{2}k(k+1) + (r-k)k = rk - \tfrac{1}{2}k(k-1).$$

So

$$\sum_0^k b_i = d - rk + \tfrac{1}{2}k(k-1) = \tfrac{1}{2}k(k+1) - e .$$

So hypothesis (2) is satisfied, and we see furthermore that $e=e_m$ of (5.3). Our conclusion is that for each $0 \leqq q \leqq k$,

$$\sum_0^q b_i = \tfrac{1}{2}q(q+1) - e_q ,$$

with

$$e_q \geqq \min(q, e).$$

More precisely, for $0 \leqq q \leqq e$ we have

$$\sum_0^q b_i \leqq \tfrac{1}{2}q(q+1)-q$$

and for $e \leqq q \leqq k$ we have

$$\sum_0^q b_i \leqq \tfrac{1}{2}q(q+1)-e.$$

Now for any q,

$$\sum_0^q b_i = \sum_{r+k-q}^{\infty} c_i = h^1(\mathscr{I}_Z(r+k-q-1)).$$

Making a change of variables $l=r+k-q-1$, we obtain the inequalities in the second half of the proposition.

The equalities in the first half are obtained using the formula

$$\sum_{-\infty}^{q} c_i = d - h^1(\mathscr{I}_Z(q))$$

and the values of c_i given in (5.1) for $i \leqq r-1$.

Proposition 5.5. *Let $Z \subseteq \mathbb{P}^2$ satisfy $(*)$, and suppose $d<k^2$. Then we must have $d \geqq \frac{1}{2}k(k+1)$. Choose $1 \leqq a \leqq k-1$ so that*

$$k(k-a)+\tfrac{1}{2}a(a+1) \leqq d < k(k-a+1)+\tfrac{1}{2}a(a-1),$$

and let $e=k(k-a+1)+\frac{1}{2}a(a-1)-d$, so that $e>0$. Then

$$h^1(\mathscr{I}_Z(l)) = \begin{cases} d & \text{for} \quad l<0 \\ d-\frac{1}{2}(l+1)(l+2) & \text{for} \quad 0 \leqq l < k \end{cases}$$

$$h^1(\mathscr{I}_Z(l)) \leqq \begin{cases} \frac{1}{2}(2k-a-l)(2k-a-l-1)-e & \text{for} \quad k-1 \leqq l \leqq 2k-a-e-1 \\ \frac{1}{2}(2k-a-l-1)(2k-a-l-2) & \text{for} \quad 2k-a-e-1 \leqq l \leqq 2k-a-1 \end{cases}$$

$$h^1(\mathscr{I}_Z(l))=0 \quad \text{for} \quad l \geqq 2k-a-2.$$

Proof. First note that curves of degree $k-1$ in $\mathbb{P}^2$ depend on $\frac{1}{2}(k-1)(k+2) = \frac{1}{2}k(k+1)-1$ parameters, so any set of $d<\frac{1}{2}k(k+1)$ points is contained in a curve of degree $k-1$. Since Z is not contained in any curve of degree $k-1$ by hypothesis, we have $d \geqq \frac{1}{2}k(k+1)$. Then d determines a unique a in the range $1 \leqq a \leqq k-1$ satisfying the inequality above.

The first two equalities follow directly from the values of c_i given in (5.1), as in the proof of (5.4).

For the inequalities, we will apply (5.3) to $\sum_k^\infty c_i$, taking $m=k-a$. Thus $b_i = c_{2k-a-i}$, for $i=0,\ldots,m$. To check the hypotheses of (5.3), first note by (5.1) that

$$k=c_{k-1}=\ldots=c_{s-1}>c_s>\ldots$$

so (1) is satisfied. For (2), we have

$$\sum_0^m b_i = \sum_k^\infty c_i = d - \sum_{-\infty}^{k-1} c_i.$$

But by (5.1)

$$\sum_{-\infty}^{k-1} c_i = \tfrac{1}{2}k(k+1),$$

so

$$\sum_0^m b_i = d - \tfrac{1}{2}k(k+1).$$

Substituting for e, we find

$$\begin{aligned}\sum_0^m b_i &= k(k-a+1) + \tfrac{1}{2}a(a+1) - \tfrac{1}{2}k(k+1)\\ &= \tfrac{1}{2}(k-a)(k-a+1) - e\\ &= \tfrac{1}{2}m(m+1) - e.\end{aligned}$$

So the hypothesis (2) is satisfied for $m = k-a$, and $e = e_m$.

Now the conclusion of (5.3) tells us that

$$\sum_0^q b_i \leqq \tfrac{1}{2}q(q+1) - q = \tfrac{1}{2}q(q-1) \quad \text{for} \quad 0 \leqq q \leqq e$$

and

$$\sum_0^q b_i \leqq \tfrac{1}{2}q(q+1) - e \qquad \text{for} \quad e \leqq q \leqq m.$$

On the other hand

$$\sum_0^q b_i = \sum_{2k-a-q}^{\infty} c_i = h^1(\mathscr{I}_Z(2k-a-q-1)).$$

So we make a change of variables $l = 2k-a-q-1$, which gives the statement of the proposition. In particular, for $l \geqq 2k-a-2$ we find that $h^1(\mathscr{I}_Z(l)) = 0$.

Remark 5.5.1. In fact, the proof of (5.4) works also for $r = k-1$, hence $d \geqq k(k-1)$, and gives the same answer as (5.5) in that range. Also, the proof of (5.5) works for $a = 0$, i.e. $k^2 \leqq d < k(k+1)$, and gives the same answer as (5.4) in that range.

Remark 5.5.2. I believe the inequalities of (5.4) and (5.5) are best possible, but I haven't set about checking that systematically.

§ 6. Application to Curves in $\mathbb{P}^3$

In the classification of curves in $\mathbb{P}^3$, an important question is to determine, for each degree d, the possible values of the genus g of an irreducible nonsingular curve of that degree. If the curve lies in a plane, then $g=\frac{1}{2}(d-1)(d-2)$. If the curve does not lie in a plane, then a classical result of Castelnuovo [AG, IV, 6.4] gives an upper bound for the genus, namely

$$g \leqq \begin{cases} \dfrac{d^2}{4}-d+1 & \text{if } d \text{ is even} \\ \dfrac{d^2-1}{4}-d+1 & \text{if } d \text{ is odd.} \end{cases}$$

Furthermore, equality is attained for every $d \geqq 3$, and any curve for which equality holds must lie on a quadric surface.

This suggests that a better bound should hold if the curve does not lie in a quadric surface, or more generally if it is assumed not to lie in any surface of degree $<k$ for some integer k. I made some conjectures, and Joe Harris proved the conjectured bound in the range $d>k^2$. (The best bound in the case $d \leqq k^2$ is still not known.) Harris's original proof [15] involved a subtle study of curves and their intersections with planes. The present proof, based on the techniques of § 5, resulted from Mumford's observation of parallels between Harris's proof and recent work of Gieseker on the moduli of surfaces. Recently Gruson and Peskine [14] have found another proof of this result, using the technique of liaison.

Theorem 6.1 (Harris). *Let Y be a reduced curve of degree d in $\mathbb{P}^3$, contained in an irreducible surface F of degree k, but not contained in any surface of lower degree, and assume $d>k(k-1)$. Then*

$$p_a(Y) \leqq \frac{d^2}{2k}+\tfrac{1}{2}d(k-4)+1+\tfrac{1}{2}f\left(f+1-k-\frac{f}{k}\right)$$

where $d \equiv f \pmod{k}$ and $0 \leqq f < k$.

Proof. First we choose a plane $H \subseteq \mathbb{P}^3$ such that $Z=Y \cap H$ consists of d distinct points (possible since Y is reduced) and the curve $C=F \cap H$ is irreducible. Then Z is contained in the irreducible curve C of degree k, and because of the hypothesis $d>k(k-1)$, Z cannot be contained in any curve of degree $<k$. (This simple point is the one which fails for $d \leqq k(k-1)$. Thus Z satisfies the hypotheses (*) of § 5.

Lemma 6.2. *With the above hypotheses, for each l*

$$h^0(\mathcal{O}_Y(l))-h^0(\mathcal{O}_Y(l-1)) \geqq d-h^1(\mathcal{I}_Z(l)).$$

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & H^0(\mathcal{O}_P(l)) & \longrightarrow & H^0(\mathcal{O}_H(l)) & \longrightarrow & H^1(\mathcal{O}_P(l-1))=0 \\ & & \downarrow & & \downarrow \gamma & & \downarrow \\ 0 \to H^0(\mathcal{O}_Y(l-1)) & \to & H^0(\mathcal{O}_Y(l)) & \xrightarrow{\alpha} & H^0(\mathcal{O}_Z(l)) & \xrightarrow{\beta} & H^1(\mathcal{O}_Y(l-1)) \to \dots \\ & & & & \downarrow & & \\ & & & & H^1(\mathcal{I}_Z(l)) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Now $\beta\gamma=0$ because this map factors through $H^1(\mathcal{O}_P(l-1))$ which is zero. Therefore $\operatorname{im}\gamma\subseteq\ker\beta=\operatorname{im}\alpha$. Since $h^0(\mathcal{O}_Z(l))=d$, this implies that

$$h^0(\mathcal{O}_Y(l))-h^0(\mathcal{O}_Y(l-1))\geqq d-h^1(\mathcal{I}_Z(l))$$

as required.

Proof of theorem, continued. Take $q\gg 0$. Then the Hilbert polynomial of Y gives

$$h^0(\mathcal{O}_Y(q))=dq+1-p_a.$$

Choose $r\geqq k-1$ so that $rk\leqq d<(r+1)k$. Then we can write $h^0(\mathcal{O}_Y(q))$ as

$$h^0(\mathcal{O}_Y(q))=h^0(\mathcal{O}_Y(r-1))+\sum_{l=r}^{q}(h^0(\mathcal{O}_Y(l))-h^0(\mathcal{O}_Y(l-1)).$$

Since $d\geqq rk$ and Y is contained in the irreducible surface F of degree k, any surface of degree $<r$ containing Y must contain F as as component. Therefore

$$h^0(\mathcal{O}_Y(r-1))\geqq h^0(\mathcal{O}_P(r-1))-h^0(\mathcal{O}_P(r-k-1))$$

or

$$h^0(\mathcal{O}_Y(r-1))\geqq\binom{r+2}{3}-\binom{r-k+2}{3}.$$

On the other hand, by the lemma,

$$\sum_{l=r}^{q}(h^0(\mathcal{O}_Y(l))-h^0(\mathcal{O}_Y(l-1)))\leqq d(q-r+1)-\sum_{l=r}^{q}h^1(\mathcal{I}_Z(l)).$$

Now we are in a position to apply (5.4) – cf. (5.5.1). Let $d=rk+f$, so that $e=k-f$, and substitute. We find that

$$\begin{aligned}dq+1-p_a\geqq&\binom{r+2}{3}-\binom{r-k+2}{3}+d(q-r+1)\\&-\sum_{l=r}^{r+f-1}(\tfrac{1}{2}(r+k-l)(r+k-l-1)-k+f)\\&-\sum_{l=r+f}^{r+k-1}\tfrac{1}{2}(r+k-l-1)(r+k-l-2).\end{aligned}$$

Using the formula

$$\sum_{a}^{b}\binom{l}{2}=\binom{b+1}{3}-\binom{a}{3}$$

to evaluate the sums, we find

$$\begin{aligned}p_a\leqq&1+rd-d-\binom{r+2}{3}+\binom{r+k-2}{3}+\binom{k+1}{3}\\&-\binom{k-f+1}{3}-kf+f^2+\binom{k-f}{3}.\end{aligned}$$

Then expanding and substituting $r=\frac{d-f}{k}$ we obtain

$$p_a \leqq \frac{d^2}{2k} + \tfrac{1}{2}d(k-4)+1+\tfrac{1}{2}f\left(f+1-k-\frac{f}{k}\right)$$

as required.

Remark 6.2.1. Harris actually proved more, namely that this maximum is attained, for every k, $d \geqq k^2$, by an irreducible nonsingular curve, and that the theorem remains true if instead of assuming Y is contained in a surface of degree k, we assume merely that Y is not contained in any surface of degree $<k$. For $d=rk$, with $r \geqq k$, a complete intersection Y of a surface of degree k with a surface of degree r gives equality in the theorem.

Remark 6.2.2. For $k=2$ we recover Castelnuovo's theorem. For $k=3$, if Y is contained in a nonsingular cubic surface, then one can derive this bound by an explicit study of all curves on the surface [AG, V, Ex. 4.7].

§ 7. Stable Bundles on $\mathbb{P}^2$

In this section we will apply the results of § 5 to stable rank 2 bundles on $\mathbb{P}^2$. Our purpose is to get bounds on $h^1(\mathscr{E}(l))$ for all l, which we will then use in studying stable bundles on $\mathbb{P}^3$.

Proposition 7.1. *Let $\mathscr{E}$ be a rank 2 bundle on $\mathbb{P}^2$ with $c_1=0$ (respectively, $c_1=-1$). Let $t \geqq -1$ be an integer such that $(t+1)(t+2)>c_2$ (respectively, $(t+1)^2>c_2$). Then $H^0(\mathscr{E}(t)) \neq 0$.*

Proof. The Riemann-Roch theorem for $\mathscr{E}(t)$ says that

$$\chi(\mathscr{E}(t)) = \tfrac{1}{2}c_1(c_1+2t+3)+(t+1)(t+2)-c_2 .$$

This can be obtained from the Riemann-Roch theorem for $\mathscr{E}$ (see proof of 3.2) by substituting $c_1(\mathscr{E}(t))=c_1+2t$ and $c_2(\mathscr{E}(t))=c_2+tc_1+t^2$.

Now suppose that $H^0(\mathscr{E}(t))=0$. Then by Serre duality, $H^2(\mathscr{E}(t))$ is also zero, because it is dual to $H^0(\mathscr{E}(-c_1-t-3))$ and $t \geqq -1$ implies $-c_1-t-3 \leqq t$. Thus the Riemann-Roch theorem reduces to

$$-h^1(\mathscr{E}(t)) = \tfrac{1}{2}c_1(c_1+2t+3)+(t+1)(t+2)-c_2 ,$$

which is $\leqq 0$. Substituting $c_1=0$ (respectively, $c_1=-1$) we get $(t+1)(t+2) \leqq c_2$ (respectively, $(t+1)^2 \leqq c_2$). Therefore if the opposite inequality is satisfied, $H^0(\mathscr{E}(t)) \neq 0$.

Our next objective is to get some good bounds on $h^1(\mathscr{E}(l))$ for all $l \in \mathbb{Z}$. The technique is to twist $\mathscr{E}$ by a large integer n, take a section $s \in H^0(\mathscr{E}(n))$, and let Z be the zero set of s. First we show that for n sufficiently large and s sufficiently general, Z will satisfy the hypotheses (*) of § 5. Then we can apply the results of § 5 to get bounds on $h^1(\mathscr{E}(l))$.

Proposition 7.2. *Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbb{P}^2$ with Chern classes c_1, c_2. Then for $n \gg 0$ and $s \in H^0(\mathscr{E}(n))$ sufficiently general, the zero set $Z = (s)_0$ satisfies $(*)$ of §5: namely Z consists of distinct points, and there exists a curve of least degree containing Z which is irreducible.*

Proof. For n sufficiently large, $\mathscr{E}(n-1)$ will be generated by global sections, so by (1.4), for $s \in H^0(\mathscr{E}(n))$ sufficiently general, Z will be nonsingular, i.e. consist of distinct points with multiplicity one.

Consider such a Z. Then there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(n) \to \mathscr{I}_Z(c_1 + 2n) \to 0.$$

For any integer $k > 0$, curves of degree k containing Z correspond to elements of $H^0(\mathscr{I}_Z(k))$. These can be lifted to elements of $H^0(\mathscr{E}(k - c_1 - n))$, and conversely, any element of this space which is not a multiple of s gives a curve containing Z. In particular, let t be the least integer for which $H^0(\mathscr{E}(t)) \neq 0$, and assume that n was taken $> t$. Then the least degree of a curve containing Z is $k = t + c_1 + n$, and $H^0(\mathscr{E}(t)) \cong H^0(\mathscr{I}_Z(k))$. Let $u \in H^0(\mathscr{E}(t))$ determine the curve C of least degree containing Z. We may have very little freedom in the choice of u, so we will show that if s is sufficiently general, then C is necessarily irreducible.

First we show that the curve C can be characterized as the support of the sheaf

$$\mathscr{E}(t)/(s \cdot \mathcal{O}(t-n) + u \cdot \mathcal{O}),$$

Indeed,

$$\mathscr{E}(t)/s \cdot \mathcal{O}(t-n) \cong \mathscr{I}_Z(k),$$

and the section u defines the curve C, so

$$\mathscr{E}(t)/(s \cdot \mathcal{O}(t-n) + u \cdot \mathcal{O}) \cong \mathscr{I}_Z(k)/\mathscr{I}_C(k)$$

which has support C.

Thus s and u play a symmetrical role in the definition of C. Reversing these roles, let W be the zero set of the section $u \in H^0(\mathscr{E}(t))$. Then W is a locally complete intersection zero-dimensional closed subscheme of $\mathbb{P}^2$, and there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{u} \mathscr{E}(t) \to \mathscr{I}_W(c_1 + 2t) \to 0.$$

Then the section $s \in H^0(\mathscr{E}(n)$ determines a curve of degree $k = t + c_1 + n$ containing W, which is none other than C. So to prove our result, it will be sufficient to show that for n sufficiently large and s sufficiently general, C is irreducible. Since we get all curves containing W in this way, it is enough to prove the following lemma.

Lemma 7.3. *Let W be a zero-dimensional locally complete intersection subscheme of $\mathbb{P}^2$. Then for all $m \gg 0$, there exists an irreducible curve C of degree m containing W.*

Proof. At each point $p \in W$, take local equations f, g for W. Then we can express the local intersection of the curves f, g at P as P together with some infinitely near points. So let W be represented as $P_1, \ldots, P_r$ where we include all these infinitely near points. Now blow up all the points P_i, and let $\tilde{X}$ be the blown-up surface. Let

E_i denote the total transform on $\tilde{X}$ of the exceptional curve introduced by blowing up P_i. Then the linear system of curves of degree m in $\mathbb{P}^2$ containing W corresponds to the linear system $|\pi^*(mL) - \Sigma E_i|$ on X, where L denotes a line in $\mathbb{P}^2$, and $\pi : \tilde{X} \to \mathbb{P}^2$ is the projection map (see [AG, V, §4]).

Now $\tilde{X}$ is obtained from $\mathbb{P}^2$ by a succession of blowing up points. At each step, the relative $\mathcal{O}(1)$-sheaf obtained from the **Proj** construction is just $\mathcal{O}(-E_i)$ [AG, II, §7]. Therefore, taking L as a very ample divisor on $\mathbb{P}^2$ and applying [AG, II, Ex. 7.14b] successively, we see that for all $m \gg 0$ and for suitable $r_i > 0$, the divisor $mL - \Sigma r_i E_i$ will be very ample on $\tilde{X}$. Then by Bertini's theorem [AG, II, 8.18] we can find an irreducible nonsingular curve $\tilde{C}$ in the linear system $|mL - \Sigma r_i E_i|$. Its image $C = \pi(\tilde{C})$ in $\mathbb{P}^2$ will be an irreducible curve of degree m containing W.

Theorem 7.4. *Let $\mathcal{E}$ be a rank 2 bundle on $\mathbb{P}^2$ with Chern classes c_1, c_2, and let t be the least integer such that $H^0(\mathcal{E}(t)) \neq 0$*

(a) *Assume $c_1 = 0$ and $t \geqq 0$.* Then

$$h^1(\mathcal{E}(l)) \begin{cases} = 0 & \text{for } l \leqq -c_2 + t^2 - 2 \\ \leqq c_2 - t^2 + l + 2 & \text{for } -c_2 + t^2 - 2 \leqq l \leqq -t - 3 \\ = c_2 - (l+1)(l+2) & \text{for } -t - 2 \leqq l \leqq t - 1 \\ \leqq c_2 - t_2 - l - 1 & \text{for } t - 1 \leqq l \leqq c_2 - t^2 - 1 \\ 0 & \text{for } l \geqq c_2 - t^2 - 1 . \end{cases}$$

(b) *Assume $c_1 = -1$ and $t > 0$. Then*

$$h^1(\mathcal{E}(l)) \begin{cases} = 0 & \text{for } l \leqq -c_2 + t^2 - t - 1 \\ \leqq c_2 - t^2 + t + l + 1 & \text{for } -c_2 + t^2 - t - 1 \leqq l \leqq -t - 1 \\ = c_2 - (l+1)^2 & \text{for } -t - 1 \leqq l \leqq t - 1 \\ \leqq c_2 - t^2 + t - l - 1 & \text{for } t - 1 \leqq l \leqq c_2 - t^2 + t - 1 \\ = 0 & \text{for } l \geqq c_2 - t^2 + t - 1 . \end{cases}$$

Proof. We will write only the proof of (a), since the proof of (b) is almost identical. So assume $c_1 = 0$ and $t \geqq 0$. In the first place, by Serre duality, $h^1(\mathcal{E}(l)) = h^1(\mathcal{E}(-l-3))$, so it is enough to treat the case $l \geqq -1$.

In the range $-1 \leqq l \leqq t - 1$ we have $h^0(\mathcal{E}(l)) = 0$ by definition of t, and $h^2(\mathcal{E}(l)) = h^0(\mathcal{E}(-l-3)) = 0$ by duality, so the equality follows directly from the Riemann-Roch theorem, as in the proof of (7.1).

It remains to treat the case $l \geqq t$. Pick an integer $n > 0$ and a section $s \in H^0(\mathcal{E}(n))$ by (7.2) so that the zero set $Z = (s)_0$ satisfies (*) of §5. The degree of Z is $d = c_2 + n^2$, and the least degree of a curve containing Z is $k = t + n$. Note that since $t \geqq 0$, for all $n \gg 0$, $d < k(k+1) = n^2 + n + 2nt + t$. Therefore we can apply (5.5) – cf. (5.5.1).

First we must choose a such that

$$k(k-a) + \tfrac{1}{2}a(a+1) \leqq d < k(k-a+1) + \tfrac{1}{2}a(a-1) .$$

Substituting $d = n^2 + c_2$ and $k = n + t$ this says

$$(n+t)(n+t-a) + \tfrac{1}{2}a(a+1) \leqq n^2 + c_2 \leqq (n+t)(n+t-a+1) + \tfrac{1}{2}a(a-1)$$

or

$$n(2t-a) + t^2 - at + \tfrac{1}{2}a(a+1) \leqq c_2 < n(2t-a+1) + t^2 - at + t + \tfrac{1}{2}a(a-1) .$$

So we take $a=2t$ in which case the inequalities become

$$t^2+t \leqq c_2 < n+t^2,$$

The first inequality $t^2+t \leqq c_2$ must be satisfied, because if $t(t+1) > c_2$, then by (7.1), $H^0(\mathscr{E}(t-1)) \neq 0$, which contradicts the choice of t. The second inequality is satisfied for $n \gg 0$.

Thus we are in a position to apply (5.5), with

$$d=n^2+c_2$$

$$k=n+t$$

$$a=2t$$

$$e=n+t^2-c_2.$$

Writing q as the variable instead of l, we will need the range $q \geqq k-1$. Expressing everything in terms of c_2, t, n, we get

$$h^1(\mathscr{I}_Z(q)) \begin{cases} \leqq \frac{1}{2}(2n-q)(2n-q-1)-n-t^2+c_2 & \text{for } n+t-1 \leqq q \leqq n-t^2+c_2-1 \\ \leqq \frac{1}{2}(2n-q-1)(2n-q-2) & \text{for } n-t^2+c_2-1 \leqq q \leqq 2n-1 \\ =0 & \text{for } q \geqq 2n-2. \end{cases}$$

To compute $h^1(\mathscr{E}(l))$, we use the exact sequence

$$0 \to \mathcal{O}(l-n) \to \mathscr{E}(l) \to \mathscr{I}_Z(l+n) \to 0,$$

which gives a cohomology sequence

$$0 \to H^1(\mathscr{E}(l)) \to H^1(\mathscr{I}_Z(l+n)) \to H^2(\mathcal{O}(l-n)) \to H^2(\mathscr{E}(l)).$$

For $l \geqq -1$, $h^2(\mathscr{E}(l)) = h^0(\mathscr{E}(-l-3)) = 0$. On the other hand,

$$h^2(\mathcal{O}(l-n)) = h^0(\mathcal{O}(n-l-3)) = \begin{cases} \frac{1}{2}(n-l-1)(n-l-2) & \text{for } l \leqq n-2 \\ 0 & \text{for } l \geqq n-2 \end{cases}$$

Taking $q=l+n$ above, and combining, we get

$$h^1(\mathscr{E}(l)) \begin{cases} \leqq \frac{1}{2}(n-l)(n-l-1)-n-t^2+c_2-\frac{1}{2}(n-l-1)(n-l-2) \\ \qquad \text{for } t-1 \leqq l \leqq c_2-t^2-1 \\ \leqq \frac{1}{2}(n-l-1)(n-l-2)-\frac{1}{2}(n-l-1)(n-l-2)=0 \\ \qquad \text{for } c_2-t^2-1 \leqq l \leqq n-1 \\ =0 \qquad \text{for } l \geqq n-2. \end{cases}$$

Simplifying the first expression, n drops out, and we get

$$h^1(\mathscr{E}(l)) \leqq c_2-t^2-l-1$$

as required. q.e.d

§8. Nonvanishing of $H^0(\mathscr{E}(t))$ on $\mathbb{P}^3$

In this section we prove one of the main results of this paper, which gives a specific bound on t, as a function of c_1 and c_2, so that $H^0(\mathscr{E}(t)) \neq 0$ for any rank 2 bundle on

$\mathbb{P}^3$. This gives, in principle, a method of classifying all stable bundles with given c_1 and c_2, by associating them to curves in $\mathbb{P}^3$ of bounded degree. To prove this result, we use the Riemann-Roch theorem for $\mathscr{E}$ on $\mathbb{P}^3$, plus the estimates of cohomology of the restriction of $\mathscr{E}$ to a plane which were developed in the previous section.

Lemma 8.1 (Riemann-Roch). *Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbb{P}^3$ with Chern classes c_1, c_2.*

(a) *If $c_1=0$ then*

$$\chi(\mathscr{E}(l))=\tfrac{1}{3}(l+1)(l+2)(l+3)-c_2(l+2)$$

for any $l\in\mathbb{Z}$.

(b) *If $c_1=-1$, then*

$$\chi(\mathscr{E}(l))=\tfrac{1}{6}(l+1)(l+2)(2l+3)-\tfrac{1}{2}c_2(2l+3)$$

for any $l\in\mathbb{Z}$.

Proof. This is another special case of the general Riemann-Roch theorem, and the particular formulas can be computed as in the proof of (3.2).

Theorem 8.2. *Let $\mathscr{E}$ be a rank 2 bundle on $\mathbb{P}^3$ with Chern classes c_1, c_2.*

(a) *Assume $c_1=0$, and let $t\geqq 0$ be an integer such that either*

(1) *$t\geqq c_2-2$ and $(t+1)(t+3)>3c_2$, or*

(2) *$t\leqq c_2-2$ and $(t+1)(t+2)(2t+3)>3c_2(c_2+1)$.*

Then $H^0(\mathscr{E}(t))\neq 0$.

(b) *Assume $c_1=-1$, and let $t\geqq 0$ be such that either*

(1) *$t\geqq c_2-2$ and $(t+1)(t+2)>3c_2$, or*

(2) *$t\leqq c_2-2$ and $2t(t+1)(t+2)>3c_2^2$.*

Then $H^0(\mathscr{E}(t))\neq 0$.

Proof. We will only write the proof of (a), since the proof of (b) is almost identical. So assume $c_1=0$. If $\mathscr{E}$ is not stable, then $H^0(\mathscr{E})\neq 0$ (3.0.1) so there is nothing to prove. So we may assume that $\mathscr{E}$ is stable. Then by (3.3) there is a plane $H\subseteq\mathbb{P}^3$ such that $\mathscr{E}|_H$ is semistable (and in the case $c_1=-1$ it is even stable). Let r be the least integer for which $H^0(\mathscr{E}|_H(r))\neq 0$. Then $r\geqq 0$.

Now suppose for some $t\geqq 0$ that $H^0(\mathscr{E}(t))=0$. Then also $H^3(\mathscr{E}(t))=0$ since it is dual to $H^0(\mathscr{E}(-t-4))$. Thus the Riemann-Roch theorem for $\mathscr{E}(t)$ says

$$-h^1(\mathscr{E}(t))+h^2(\mathscr{E}(t))=\tfrac{1}{3}(t+1)(t+2)(t+3)-c_2(t+2).$$

By duality $h^2(\mathscr{E}(t))=h^1(\mathscr{E}(-t-4))$. We estimate this latter dimension by comparing with $\mathscr{E}|_H$. For any $l<0$, $H^0(\mathscr{E}|_H(l-1))=0$, so there is an exact sequence

$$0\to H^1(\mathscr{E}(l-1))\to H^1(\mathscr{E}(l))\to H^1(\mathscr{E}|_H(l))\to\dots$$

If $t\geqq c_2-2$, then $-t-4\leqq -c_2-2$, so by (7.4), for any $l\leqq -t-4$, $h^1(\mathscr{E}|_H(l))=0$. Since in any case $H^1(\mathscr{E}(l))=0$ for $l\ll 0$, we conclude that $H^1(\mathscr{E}(-t-4))=0$. Then since $h^1(\mathscr{E}(t))\geqq 0$, the Riemann-Roch theorem above gives

$$\tfrac{1}{3}(t+1)(t+2)(t+3)-c_2(t+2)\leqq 0.$$

Dividing out $t+2$ we find

$$(t+1)(t+3)\leqq 3c_2.$$

So if the opposite inequality is satisfied, we must have $H^0(\mathscr{E}(t))\neq 0$. This is Case 1.

Now suppose on the other hand that $t\leqq c_2-2$. Then $-t-4\geqq -c_2-2$, so by (7.4), noting $r\geqq 0$, we have

$$h^1(\mathscr{E}|_H(-t-4))\leqq c_2-t-2.$$

From the exact sequence above, we see that

$$h^1(\mathscr{E}(-t-4))\leqq \sum_{l=-\infty}^{-t-4} h^1(\mathscr{E}|_H(l)).$$

Since the estimate of (7.4) for $h^1(\mathscr{E}|_H(l))$ increases by 1 each time, we obtain

$$h^1(\mathscr{E}(-t-4))\leqq \tfrac{1}{2}(c_2-t-2)(c_2-t-1).$$

Using this estimate for $h^2(\mathscr{E}(t))$, and using $h^1(\mathscr{E}(t))\geqq 0$, the Riemann-Roch theorem gives

$$\tfrac{1}{3}(t+1)(t+2)(t+3)-c_2(t+2)\leqq \tfrac{1}{2}(c_2-t-2)(c_2-t-1).$$

Simplifying gives

$$(t+1)(t+2)(2t+3)\leqq 3c_2(c_2+1).$$

So if the opposite inequality is satisfied, we must have $H^0(\mathscr{E}(t))\neq 0$. This is Case 2.

Remark 8.2.1. Using Barth's theorem [6] over $\mathbb{C}$ that if $\mathscr{E}$ is stable on $\mathbb{P}^3$ with $c_1=0$, $c_2\geqq 2$, then $\mathscr{E}|_H$ is stable for almost all H, we can improve the estimate of (a)(2) in (8.2). It is enough to assume

$$(t+2)(t+3)(2t-1)>3c_2(c_2-1).$$

Remark and Conjecture 8.2.2. In both (a) and (b), Case 1 applies for $c_2=1,2,3,4,5$, and Case 2 applies for $c_2\geqq 5$. Here is a table of the least t satisfying the hypotheses of the theorem for small c_2.

c_2	$\leqq 0$	1	2	3	4	5	6	7	8	9	10	11	12
$c_1=\ \ 0: t$	0	1	1	2	2	3	3	3	4	4	5	5	5
$c_1=-1: t$	0		2		3		3		4		5		6

Note that in Case 1, $t\sim\sqrt{3c_2}$, whereas in Case 2 $t\sim(\tfrac{3}{2}c_2^2)^{1/3}$. We *conjecture* that the quadratic bound of Case 1 applies for all c_2, e.g. for $c_1=0$, that $t>\sqrt{3c_2+1}-2$ implies $H^0(\mathscr{E}(t))\neq 0$.

Remark 8.2.3. Atiyah has observed that this conjecture is true for *instanton bundles*, i.e., those stable bundles $\mathscr{E}$ with $c_1=0$ for which $H^1(\mathscr{E}(-2))=0$. In that case, the bundle $\mathscr{E}$ can be recovered as the homology $\ker\beta/\operatorname{im}\alpha$ of a *monad*

$$\mathcal{O}(-1)^d \xrightarrow{\alpha} \mathcal{O}^{2d+2} \xrightarrow{\beta} \mathcal{O}(1)^d,$$

where $d=c_2(\mathscr{E})$, α is injective, $\operatorname{coker}\alpha$ is locally free, and β is surjective (see [3] or [7a]). Therefore, letting $\mathscr{F}=\ker\beta$, there are exact sequences

$$0\to\mathscr{F}\to\mathcal{O}^{2d+2}\xrightarrow{\beta}\mathcal{O}(1)^d\to 0$$

and

$$0\to\mathcal{O}(-1)^d\to\mathscr{F}\to\mathscr{E}\to 0.$$

Twisting by an integer t and taking cohomology gives

$$0\to H^0(\mathcal{O}(t-1)^d)\to H^0(\mathscr{F}(t))\to H^0(\mathscr{E}(t))\to 0$$

and

$$0\to H^0(\mathscr{F}(t))\to H^0(\mathcal{O}(t)^{2d+2})\to H^0(\mathcal{O}(t+1)^d)\to \dots .$$

From this it follows that for $t\geqq 0$,

$$h^0(\mathscr{E}(t))\geqq(2d+2)\binom{t+3}{3}-d\binom{t+4}{3}-d\binom{t+2}{3},$$

which simplifies to

$$h^0(\mathscr{E}(t))\geqq\tfrac{1}{3}(t+2)(t^2+4t+3-3d).$$

The condition that $t^2+4t+3-3d>0$ is precisely $t>\sqrt{3d+1}-2$. We conclude that if $t>\sqrt{3c_2+1}-2$, then $H^0(\mathscr{E}(t))\neq 0$, as required.

It seems reasonable to expect that for a sufficiently general instanton bundle, this bound on t is also the best possible, but we have no proof.

Corollary 8.3.[4] *The set of stable rank 2 bundles on* $\mathbb{P}^3$ *with given Chern classes* c_1, c_2 *forms a bounded family.*

Proof. We may assume $c_1=0$ or $c_1=-1$. Then according to the theorem, $H^0(\mathscr{E}(t))\neq 0$, t depending on c_2. Therefore for some $0<l\leqq t$, there is a section $s\in H^0(\mathscr{E}(l))$ whose zero set is a curve in $\mathbb{P}^3$. According to (1.1) $\mathscr{E}$ is determined by this curve Y and an isomorphism ξ of ω_Y with $\mathcal{O}_Y(m)$ for a certain m. Since the degree and arithmetic genus of Y are determined, these curves Y form a bounded family parametrized by part of the Hilbert scheme; the choice of isomorphism ξ is again a finite-dimensional choice. Thus the family of stable $\mathscr{E}$ with given c_1 and c_2 is bounded.

Corollary 8.4. *The possible values of* c_1, c_2, α *for a normalized stable rank 2 bundle on* $\mathbb{P}^3$ *are*

$c_1=0,\quad \alpha=0,\quad c_2\geqq 1;$

$c_1=0,\quad \alpha=1,\quad c_2\geqq 3;$

$c_1=-1,\quad c_2$ *even* $\geqq 2.$

In particular, for any stable rank 2 bundle, $c_1^2-4c_2<0$.

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4 Maruyama [30, p. 92] has announced this result for stable rank 2 bundles on any nonsingular variety. This is an independent proof in this case

Proof. Indeed, we have seen that all these values are possible (3.1.1), (3.1.2), (3.1.3). Conversely, $c_1 c_2 \equiv 0 \pmod 2$ by (2.2) and if $\mathscr{E}$ is semistable, then $c_1^2 - 4c_2 \leqq 0$ by (3.5). Thus we have only to eliminate the cases $c_1 = c_2 = 0$ and $c_1 = 0, \alpha = 1, c_2 = 1, 2$.

If $c_1 = c_2 = 0$, then by (8.2), $H^0(\mathscr{E}) \neq 0$, so $\mathscr{E}$ is not stable.

If $c_1 = 0$, $c_2 = 1$, then Wever (8.4.1) shows that $\mathscr{E}(1)$ corresponds to two skew lines as in (3.1.1), so $\alpha = 0$.

If $c_1 = 0$, $c_2 = 2$, then $H^1(\mathscr{E}(-2)) = 0$ by (9.4), so $\alpha = 0$.

Example 8.4.1. Let $\mathscr{E}$ be a stable rank 2 bundle on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 1$. Then $H^0(\mathscr{E}) = 0$ because $\mathscr{E}$ is stable, and by (8.2), $H^0(\mathscr{E}(1)) \neq 0$. Therefore if $s \in H^0(\mathscr{E}(1))$ is a nonzero section, its zero scheme Y must be of codimension 2. The curve Y will be of degree 2, and its dualizing sheaf ω_Y will be isomorphic to $\mathcal{O}_Y(-2)$. Thus Y must be a union of two skew lines, or a certain multiplicity 2 structure on a single line. It could not be a conic or two lines meeting because in those cases $\omega_Y \cong \mathcal{O}_Y(-1)$.

Knowing the structure of Y, one can analyze the structure of $\mathscr{E}$ and describe all such bundles up to isomorphism. This is done in the thesis of Pete Wever [46], and the main results are these: (1) Any such bundle has a section $s \in H^0(\mathscr{E}(1))$ whose zero scheme is two skew lines. (2) Any two such bundles differ by an automorphism of $\mathbb{P}^3$. (3) The set of all such bundles has a fine moduli space isomorphic to $\mathbb{P}^5 - G(1,3)$, where $G(1,3)$ is the Grassmann variety of lines in $\mathbb{P}^3$.

This classification of stable bundles with $c_1 = 0$, $c_2 = 1$ has been obtained independently by Barth [6] (at least over $\mathbb{C}$), by another method. He calls them null-correlation bundles.

§9. Stable Rank 2 Bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 2$

In this section we study the structure of stable rank 2 bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 2$. Our first main result (9.7) is that such a bundle $\mathscr{E}$ determines a unique nonsingular quadric surface $Q \subseteq \mathbb{P}^3$ and a certain linear system on Q, and that conversely this data determines $\mathscr{E}$. From this we can show that the variety of moduli M of these bundles is irreducible and nonsingular of dimension 13. From this we can also see that up to automorphisms of $\mathbb{P}^3$, there is a 1-parameter family of inequivalent bundles of this type. Then we study the restriction of $\mathscr{E}$ to planes and lines in $\mathbb{P}^3$. One main result (9.13) is that $\mathscr{E}$ is uniquely determined by its divisor of jumping lines (with one exception in characteristic 3). It is not known how generally the divisor of jumping lines determines a stable bundle on $\mathbb{P}^3$. Another result is an explicit construction of the divisor of jumping lines in terms of the quadric surface Q and the linear system mentioned above. We were not able to decide whether the variety of moduli M is a rational variety, although it appears likely that it is.

Throughout this section, $\mathscr{E}$ will denote a stable bundle on $\mathbb{P}^3$ with Chern classes $c_1 = 0$ and $c_2 = 2$. Since $\mathscr{E}$ is stable, $H^0(\mathscr{E}) = 0$, but according to (8.2), $H^0(\mathscr{E}(1)) \neq 0$. Let $s \in H^0(\mathscr{E}(1))$ be a nonzero section. Then the zero set $Y = (s)_0$ of s will be a curve (meaning a locally complete intersection closed subscheme of $\mathbb{P}^3$) of degree 3 such that $\omega_Y \cong \mathcal{O}_Y(-2)$. Our first task is to classify all such curves Y in $\mathbb{P}^3$.

Proposition 9.1. *Let Y be a curve of degree 3 in $\mathbb{P}^3$ with $\omega_Y \cong \mathcal{O}_Y(-2)$. Then Y is either*

(a) *the union of three nonintersecting lines; or*

(b) *the union of a line with another line not meeting it, where the second has a multiplicity 2 scheme structure given by a homogeneous ideal of the form $(x^2, xy, y^2, fx+gy)$, where f and g are linearly independent linear forms in the remaining variables z, w, and where we have taken $x=y=0$ as the equations of the reduced line; or*

(c) *a single line with a multiplicity 3 scheme structure given by a homogeneous ideal of the form $(x^3, x^2y, xy^2, y^3, fx+gy+ax^2+bxy+cy^2)$, where f and g are linearly independent linear forms in the remaining variables z, w, and a, b, c are constants.*

Proof. From the hypotheses, one sees immediately that the arithmetic genus p_a of Y must be -2. Thus Y cannot be an integral curve. If it is reduced, each connected component Z must be of degree $\leqq 3$ and have $\omega_Z \cong \mathcal{O}_Z(-2)$. The only connected reduced curve with this property is a line $\mathbb{P}^1$. This gives case (a), which we have studied earlier (3.1.1). The curve Y cannot have a conic or two intersecting lines in its support, so the only remaining possibilities are two lines, one with a multiplicity 2 structure, or one line with a multiplicity 3 structure. It remains to see which multiplicity 2 or 3 structures Z on a line (say $x=y=0$) have the property that $\omega_Z \cong \mathcal{O}_Z(-2)$.

First let Z be a multiplicity 2 structure on the line X given by $x=y=0$, such that $\omega_Z \cong \mathcal{O}_Z(-2)$. Then according to Ferrand's theorem (1.5), taking $m=2$, the ideal sheaf $\mathcal{I}_Z$ is obtained as the kernel of a surjective map $u: \mathcal{I}_X \to \mathcal{O}_X$. Hence there is an exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_X^2 \to \mathcal{I}_X/\mathcal{I}_X^2 \xrightarrow{u} \mathcal{O}_X \to 0.$$

Since $\mathcal{I}_X/\mathcal{I}_X^2 \cong \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)$, with generators x and y, the map u is given by two linearly independent forms f, g in the homogeneous variables z, w along the line X. Therefore $\mathcal{I}_Z/\mathcal{I}_X^2$ is generated by $fx+gy$, and the homogeneous ideal of Z is of the form $(x^2, xy, y^2, fx+gy)$, as required.

Now suppose Y is a multiplicity 3 structure on the line X, with $\omega_Y \cong \mathcal{O}_Y(-2)$. Since we do not have a general classification of multiplicity 3 scheme structures on a curve analogous to Ferrand's theorem, we will use an *ad hoc* argument. Take an affine 3-space $\mathbb{A}^3 \subseteq \mathbb{P}^3$ with affine coordinates x, y, t, and pass to the ring $k(t)[x, y]$. Then Z corresponds to a multiplicity 3 structure on the point $x=y=0$ in $\mathbb{A}^2_{k(t)}$. Therefore its ideal can be written in the form $(x^3, x^2y, xy^2, y^3, fx+gy+ax^2+bxy+cy^2)$, where f, g, a, b, c are polynomials in t, and f and g are not both zero. Going back to $\mathbb{P}^3$, this shows that the homogeneous ideal I_Y of Y contains a homogeneous polynomial of the form $h=fx+gy+ax^2+bxy+cy^2$, where f, g, a, b, c are polynomials in z and w, with f, g of some degree r and a, b, c of degree $r-1$. Since Y is locally complete intersection, we see that

$$I_Y = (x^3, x^2y, xy^2, y^3, fx+gy+ax^2+bxy+cy^2)$$

and furthermore that f and g can have no common zeros along X. Now the arithmetic genus of such a curve is easily computed as a function of $r=\deg f$.

Indeed, there is an exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^3}(-r-1)/(x^2,xy,y^2)\xrightarrow{h}\mathcal{O}_{\mathbb{P}^3}/(x^3,x^2y,xy^2,y^3)\to\mathcal{O}_Y\to 0$$

from which a short calculation gives $p_a=1-3r$. In order to have $p_a=-2$ we must have $r=1$. Thus f and g are linearly independent linear forms, and a,b,c, are constants, as required.

Lemma 9.2. *With the same hypotheses as* (9.1), *the curve* Y *is contained in a unique quadric surface* $Q\subseteq\mathbb{P}^3$, *necessarily nonsingular.*

Proof. In case (a) this is classical. In case (b), it is clear from the ideal that the double line is contained in a 3-parameter family of quadrics. Thus we can find at least one quadric Q containing Y. This quadric must be irreducible, for otherwise the double line would be contained in a plane, which is impossible as we see from its ideal. Furthermore Q must be nonsingular, because any two lines on a quadric cone meet. Then it is clear that Y is a divisor of type $(3,0)$ on Q (recall [AG, II, 6.6.1] that the divisor class group of Q is $\mathbb{Z}\oplus\mathbb{Z}$, generated by a line in each of the two rulings; by the *type* we mean the class in $\mathbb{Z}\oplus\mathbb{Z}$). Therefore Q is unique, because Y cannot be a subset of a divisor $Q\cap Q'$, which is of type $(2,2)$.

In case (c) it is clear from the ideal that Y is contained in a unique quadric surface Q. It must be irreducible since Y is not contained in a plane. It must be nonsingular, because the triple line scheme on a quadric cone fails to be locally complete intersection at the vertex.

Lemma 9.3. *Given* $\mathscr{E}$ *as above, and a section* $s\in H^0(\mathscr{E}(1))$ *with zero set* Y, *the nonsingular quadric surface* Q *of* (9.2) *containing* Y *depends only on* $\mathscr{E}$, *and not on the choice of* s. *Furthermore,* $\dim H^0(\mathscr{E}(1))=2$, *and as* $s\in H^0(\mathscr{E}(1))$ *varies,* Y *describes a linear system of dimension* 1 *and type* $(3,0)$ *on* Q.

Proof. First we show that $\dim H^0(\mathscr{E}(1))=2$. Indeed, we have an exact sequence

$$0\to\mathcal{O}\xrightarrow{s}\mathscr{E}(1)\to\mathscr{I}_Y(2)\to 0,$$

and (9.2) implies that $\dim H^0(\mathscr{I}_Y(2))=1$, from which it follows immediately.

Now let $t\in H^0(\mathscr{E}(1))$ be a section linearly independent from s. Then the image of t in $H^0(\mathscr{I}_Y(2))$ gives an equation for Q. Therefore

$$\mathscr{E}(1)/(s,t)\cong\mathscr{I}_{Y,Q}(2),$$

where $\mathscr{I}_{Y,Q}$ denotes the ideal sheaf of Y on Q. Since s and t form a basis of $H^0(\mathscr{E}(1))$, the left-hand side of this isomorphism depends only on $\mathscr{E}$. We can recover Q as the support of the coherent sheaf on the right-hand side, so Q depends only on $\mathscr{E}$.

It is easy to see that there is a linear map $H^0(\mathscr{E}(1))\to H^0(\mathcal{O}_Q(3,0))$ whereby the section s goes to the section cutting out Y on Q. So as s varies, Y cuts out a linear system of type $(3,0)$ and of dimension 1.

Lemma 9.4. *The dimension of the intermediate cohomology of* $\mathscr{E}$ *is as follows:*

$$h^1(\mathscr{E}(l))=\begin{cases}2 & \textit{if}\quad l=0,-1\\ 0 & \textit{otherwise}\end{cases}$$

$$h^2(\mathscr{E}(l))=\begin{cases}2 & \textit{if}\quad l=-3,-4\\ 0 & \textit{otherwise}\end{cases}$$

Proof. From the exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E}(1) \to \mathcal{I}_Y(2) \to 0$$

we have $H^1(\mathcal{E}(l)) \cong H^1(\mathcal{I}_Y(l+1))$. From the exact sequence

$$0 \to \mathcal{I}_Q \to \mathcal{I}_Y \to \mathcal{I}_{Y,Q} \to 0$$

we have $H^1(\mathcal{I}_Y(l+1)) \cong H^1(\mathcal{I}_{Y,Q}(l+1))$. Now since $\mathcal{I}_{Y,Q}$ is an invertible sheaf on Q, its cohomology depends only on the linear equivalence class of Y. Since Y is linearly equivalent on Q to a union of three disjoint lines, we may assume, for the purposes of this calculation, that Y is a union of three disjoint lines. Then from the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0$$

it is easy to compute that

$$h^1(\mathcal{I}_Y(m)) = \begin{cases} 2 & \text{if } m = 0,1 \\ 0 & \text{otherwise}. \end{cases}$$

This gives the result for h^1, and the statement for h^2 follows by Serre duality.

Remark 9.4.1. Since $H^0(\mathcal{E}(l)) = 0$ for $l \leqq 0$, by Serre duality $H^3(\mathcal{E}(l)) = 0$ for $l \geqq -4$. So we can apply the theorem of Castelnuovo [32, p. 99] to conclude that $\mathcal{E}(2)$ is generated by global sections. (Of course $\mathcal{E}(1)$ is not generated by global sections: its global sections generate the stalks only at points not on Q.) It follows from (1.4), at least in characteristic 0, that $\mathcal{E}(2)$ will have a section whose zero set is an irreducible nonsingular elliptic curve of degree 6. This shows that in characteristic 0 the construction of (4.3.3) actually produces *all* stable bundles with $c_1 = 0$ and $c_2 = 2$.

Lemma 9.5. *With $\mathcal{E}, Q$ as in* (9.3), *the linear system of curves Y on Q induced by varying $s \in H^0(\mathcal{E}(1))$ is a linear system without base points.*

Proof. The support of Y is always a union of lines, so it will be sufficient to show that for any line L, there exists a section $s \in H^0(\mathcal{E}(1))$ not vanishing along L. Considering $\mathcal{I}_L \otimes \mathcal{E}(1)$ as a subsheaf of $\mathcal{E}(1)$, since $\dim H^0(\mathcal{E}(1)) = 2$, it will be sufficient to show that $\dim H^0(\mathcal{I}_L \otimes \mathcal{E}(1)) \leqq 1$.

Let h_1 and h_2 be linear forms defining the line L. Then the exact sequence

$$0 \to \mathcal{O}(-2) \xrightarrow{h_2, -h_1} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{h_1, h_2} \mathcal{I}_L \to 0,$$

tensored with $\mathcal{E}(1)$, gives an exact sequence

$$0 = H^0(\mathcal{E})^2 \to H^0(\mathcal{I}_L \otimes \mathcal{E}(1)) \to H^1(\mathcal{E}(-1)) \xrightarrow{h_2, -h_1} H^1(\mathcal{E})^2.$$

I claim in fact that the map

$$h_1 : H^1(\mathcal{E}(-1)) \to H^1(\mathcal{E})$$

has kernel of dimension $\leqq 1$, which will prove our result. Let H be the plane defined by h_1. From the exact sequence

$$0 \to \mathcal{E}(-1) \xrightarrow{h_1} \mathcal{E} \to \mathcal{E}|_H \to 0$$

we get

$$0=H^0(\mathscr{E})\to H^0(\mathscr{E}|_H)\to H^1(\mathscr{E}(-1))\xrightarrow{h_1} H^1(\mathscr{E}).$$

Since $H^1(\mathscr{E}(-2))=0$, the same sequence shows that $H^0(\mathscr{E}(-1)|_H)=0$. Therefore if $H^0(\mathscr{E}|_H)\neq 0$, a section $t\in H^0(\mathscr{E}|_H)$ must have a zero-set Z of codimension 2 in $H=\mathbb{P}^2$, which will consist of two points, since $c_2(\mathscr{E}|_H)=2$. So we will have an exact sequence

$$0\to \mathcal{O}_H\to\mathscr{E}|_H\to\mathscr{I}_{Z,H}\to 0,$$

from which it is clear that $\dim H^0(\mathscr{E}|_H)=1$. This completes the proof.

Corollary 9.6. *If* char. $k\neq 3$, *then there exists a section* $s\in H^0(\mathscr{E}(1))$ *whose zero set is three disjoint lines. Thus the construction of* (3.1.1) *gives all stable bundles with* $c_1=0$ *and* $c_2=2$.

Proof. The linear system of the curves Y is of type $(3,0)$ on Q. Regarding Q as $\mathbb{P}^1\times\mathbb{P}^1$, this linear system is induced by a linear system g_3^1 of degree 3 and dimension 1 on one of the factors $\mathbb{P}^1$. Since the g_3^1 has no base points, it induces a morphism of degree 3 from $\mathbb{P}^1$ to $\mathbb{P}^1$. If char. $k\neq 3$, this morphism will be separable, so the general member of the g_3^1 will consist of three distinct points. The corresponding curve Y on Q will be three disjoint lines.

Remark 9.6.1. On the other hand, if char. $k=3$, the Frobenius morphism of $\mathbb{P}^1$ to $\mathbb{P}^1$ corresponds to a g_3^1 consisting of $\{3P|P\in\mathbb{P}^1\}$. From (9.7) below it follows that for any choice of Q and a choice of one factor in $Q=\mathbb{P}^1\times\mathbb{P}^1$, there exists a bundle $\mathscr{E}$ giving rise to the given Q and g_3^1. Thus in characteristic 3 there are stable bundles $\mathscr{E}$ with $c_1=0$ and $c_2=2$ such that for every $s\in H^0(\mathscr{E}(1))$, the zero set $Y=(s)_0$ is a single line with a multiplicity 3 scheme structure.

Theorem 9.7. *Each stable bundle* $\mathscr{E}$ *on* $\mathbb{P}^3$ *with* $c_1=0$ *and* $c_2=2$ *determines*

(a) *a nonsingular quadric surface* $Q\subseteq\mathbb{P}^3$,

(b) *a choice of one of the two factors in the isomorphism* $Q\cong\mathbb{P}^1\times\mathbb{P}^1$, *and*

(c) *a linear system* g_3^1 *of degree 3 and dimension 1 on* $\mathbb{P}^1$, *without base points.*

Conversely, any such data (a), (b), (c) *arise from a unique such bundle* $\mathscr{E}$.

Proof. We have seen in the previous lemmas that as $s\in H^0(\mathscr{E}(1))$ varies, the corresponding curves Y sweep out a linear system of type $(3,0)$ without base points on a uniquely determined nonsingular quadric surface Q. This picks out one of the two families of rulings on Q and gives a g_3^1 without base points on $\mathbb{P}^1$.

To prove the converse we show, using a technique similar to the proof of (1.1), that $\mathscr{E}(1)$ can be recovered as an extension of coherent sheaves. As in (9.3) pick two linearly independent sections $s,t\in H^0(\mathscr{E}(1))$ and consider the exact sequence

$$0\to\mathcal{O}\oplus\mathcal{O}\xrightarrow{s,t}\mathscr{E}(1)\to\mathscr{I}_{Y,Q}(2)\to 0.$$

Now $\mathscr{I}_{Y,Q}\cong\mathcal{O}_Q(-3,0)$, where we denote the two generators of Pic Q by $\mathcal{O}_Q(1,0)$ and $\mathcal{O}_Q(0,1)$. With this notation, $\mathcal{O}_Q\otimes\mathcal{O}_{\mathbb{P}^3}(1)=\mathcal{O}_Q(1,1)$, so $\mathscr{I}_{Y,Q}(2)\cong\mathcal{O}_Q(-1,2)$. The exact sequence above determines an element $\xi\in\operatorname{Ext}^1_{\mathbb{P}^3}(\mathcal{O}_Q(-1,2),\mathcal{O}\oplus\mathcal{O})$. This Ext^1 is the global sections of the corresponding sheaf $\mathscr{E}xt^1$; using the isomorphism

$\omega_Q \cong \mathscr{E}xt^1(\mathcal{O}_Q, \omega_{\mathbb{P}^3})$, we can interpret ξ as an element of $H^0(\mathcal{O}_Q(3,0))^2$, or as two elements $\xi_1, \xi_2 \in H^0(\mathcal{O}_Q(3,0))$. These of course are the generators of the g_3^1.

Conversely, given the data (a), (b), (c), let $\xi_1, \xi_2 \in H^0(\mathcal{O}_Q(3,0))$ correspond to divisors generating the g_3^1. They determine an element $\xi \in H^0(\mathcal{O}_Q(3,0))^2$, which by the above reasoning is isomorphic to $\mathrm{Ext}^1_{\mathbb{P}^3}(\mathcal{O}_Q(-1,2), \mathcal{O}\oplus\mathcal{O})$. This ξ determines an extension

$$0 \to \mathcal{O}\oplus\mathcal{O} \to \mathscr{E}(1) \to \mathcal{O}_Q(-1,2) \to 0$$

for some coherent sheaf $\mathscr{E}$ on $\mathbb{P}^3$. We must show that $\mathscr{E}$ is locally free, stable, and has Chern classes $c_1=0$ and $c_2=2$.

To show that $\mathscr{E}(1)$ is locally free, apply the functor $\mathscr{H}om(\cdot, \mathcal{O})$ to the above sequence. We obtain (in part)

$$\ldots \to \mathscr{H}om(\mathcal{O}^2, \mathcal{O}) \xrightarrow{\delta} \mathscr{E}xt^1(\mathcal{O}_Q(-1,2), \mathcal{O}) \to \mathscr{E}xt^1(\mathscr{E}(1), \mathcal{O}) \to \mathscr{E}xt^1(\mathcal{O}^2, \mathcal{O}) = 0.$$
$$\wr\|$$
$$\mathcal{O}_Q(3,0)$$

The two global sections $(1,0)$ and $(0,1)$ of $\mathscr{H}om(\mathcal{O}^2, \mathcal{O})$ go by δ to $\xi_1, \xi_2 \in H^0(\mathcal{O}_Q(3,0))$ by construction. Since these generate a linear system without base points, δ is a surjective map of sheaves. Therefore $\mathscr{E}xt^1(\mathscr{E}(1), \mathcal{O}) = 0$. On the other hand, clearly $\mathscr{E}(1)$ has projective dimension 1 over each local ring, so by a lemma of Serre [42, lemme 9, p. 2–08], $\mathscr{E}(1)$ is locally free.

It is clear that $\mathscr{E}$ has Chern classes $c_1=0$ and $c_2=2$. Twisting the above sequence by -1 it is immediate that $H^0(\mathscr{E})=0$, so $\mathscr{E}$ is stable.

Corollary 9.8. *The variety of moduli M of these bundles is an irreducible nonsingular variety of dimension 13.*

Proof. The nonsingular quadric surfaces $Q \subsetneq \mathbb{P}^3$ are parametrized by $\mathbb{P}^9 - \Delta$, where Δ is the discriminant locus of a quadratic form in 4 variables. The moduli variety M is fibred over $\mathbb{P}^9 - \Delta$ by two copies of a variety U, where $U \subsetneq G(1,3)$ is the open subset of the Grassmann variety of 2-dimensional subvector spaces of the 4-dimensional vector space $H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ corresponding to linear systems without base points. Thus M is nonsingular of dimension 13. It is connected, because by varying Q one can connect data (a), (b), (c) corresponding to the two different choices in (b).

Proposition 9.9. *Up to automorphisms of $\mathbb{P}^3$, there is a 1-parameter family of nonisomorphic bundles $\mathscr{E}$. More precisely, the quotient space $M/$*PGL(3) *of the variety of moduli M by the group* PGL(3) *of automorphisms of $\mathbb{P}^3$ is isomorphic to the quotient space $U/$*PGL(1) *of the space of g_3^1 without base points on $\mathbb{P}^1$ by the group* PGL(1) *of automorphisms of $\mathbb{P}^1$.*

Proof. For any choice of Q and of one factor in the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, there is a morphism $\varphi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ with image Q, sending the first factor onto the chosen factor. For any such φ, the inverse image $\varphi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ is isomorphic to $p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. Therefore according to the general theory of morphisms to $\mathbb{P}^n$ [AG, II, §7], any two such morphisms differ by an automorphism of $\mathbb{P}^3$. Thus

PGL(3) acts transitively on the data (a), (b) of (9.7). Next, note that if $\sigma \in$PGL(3) leaves Q and the choice of factors fixed, then σ induces an element of PGL(1) on each factor. Conversely, any pair of elements $\tau_1, \tau_2 \in$PGL(1) induces an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which extends to an automorphism of $\mathbb{P}^3$. Therefore the quotient space M/PGL(3) is isomorphic to the space U of g_3^1 without base points on $\mathbb{P}^1$, modulo the action of PGL(1). Since U is an open subset of $G(1,3)$, this is clearly a 1-parameter family.

Remark 9.9.1. It seems an interesting problem in elementary invariant theory to describe the quotient space U/PGL(1) more completely. An equivalent problem is to classify degree 3 morphisms of $\mathbb{P}^1$ to itself up to automorphisms of $\mathbb{P}^1$.

Next we will study the restriction of our bundle $\mathscr{E}$ to planes and lines in $\mathbb{P}^3$.

Proposition 9.10. *Let $\mathscr{E}$ be a bundle as above, and let Q be the quadric surface associated to $\mathscr{E}$. Let H be a plane in $\mathbb{P}^3$. If H is transversal to Q, then $\mathscr{E}|_H$ is stable on H. If H is tangent to Q, then $\mathscr{E}|_H$ is semistable but not stable.*

Proof. If H is transversal to Q, let $s \in H^0(\mathscr{E}(1))$ be a section and let Y be its zero set. From the exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(1) \to \mathscr{I}_Y(2) \to 0,$$

tensoring with $\mathcal{O}_H$ gives an exact sequence

$$0 \to \mathcal{O}_H \xrightarrow{\bar{s}} \mathscr{E}(1)|_H \to \mathscr{I}_Z(2) \to 0,$$

where $Z = Y \cap H$ is a 3-point scheme in H. Those three points Z (distinct or coincindent) lie on the irreducible conic $C = Q \cap H$, so the scheme Z is not contained in any line. Therefore $H^0(\mathscr{I}_Z(1)) = 0$, and so $H^0(\mathscr{E}|_H) = 0$, which implies that $\mathscr{E}|_H$ is stable.

On the other hand, if H is tangent to Q, then H contains one line in each of the two rulings on Q. Therefore there exists a section $s \in H^0(\mathscr{E}(1))$ vanishing along one of those lines. The image $\bar{s}$ of this section in $H^0(\mathscr{E}(1)|_H)$ vanishes on a set of codimension 1 in H, so $H^0(\mathscr{E}|_H) \neq 0$. This shows that $\mathscr{E}|_H$ is not stable. But we have already seen, in the proof of (9.5), that $H^0(\mathscr{E}(-1)|_H) = 0$. Therefore $\mathscr{E}|_H$ is semistable.

Remark 9.10.1. In particular, $\mathscr{E}|_H$ is semistable for all planes $H \subsetneqq \mathbb{P}^3$. This strengthens (3.3) in this case. On the other hand, the fact that $\mathscr{E}|_H$ is stable for almost all H illustrates a general theorem of Barth (see footnote to (3.3)).

Proposition 9.11. *Let $\mathscr{E}, Q$ be as above, and let L be a line in $\mathbb{P}^3$. We study the restriction of $\mathscr{E}$ to L.*

(a) *If L is in the 2nd family of lines on Q (not those which form the curves Y) then $\mathscr{E}|_L \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(2)$.*

(b) *If L is in the 1st family of lines on Q, and is a double or triple line of the curve Y of the g_3^1 containing it, then $\mathscr{E}|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$.*

(c) *If $L \not\subseteq Q$, but L meets some divisor Y of the g_3^1 in two points, then $\mathscr{E}|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$.*

(d) *If L is a line of the 1st family on Q, which is a simple line of the divisor Y containing it, then* $\mathscr{E}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L$.

(e) *If* $L \not\subseteq Q$, *and L meets each divisor Y of the* g_3^1 *in at most one point, then* $\mathscr{E}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L$.

Proof. We know from the structure of vector bundles on $\mathbb{P}^1$ and the fact that $c_1(\mathscr{E})=0$ that for any line L, $\mathscr{E}|_L \cong \mathcal{O}(-a) \oplus \mathcal{O}(a)$ for some integer a. Pick a section $s \in H^0(\mathscr{E}(1))$ and let

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(1) \to \mathscr{I}_Y(2) \to 0$$

be the corresponding exact sequence. If Y meets L in a finite number of points $P_1, \ldots, P_r$, $r=0,1,2,3$, then tensoring with L gives an exact sequence

$$0 \to \mathcal{O}_L \to \mathscr{E}(1)|_L \to \mathcal{O}_L(2 - \sum P_i) \oplus \sum k_{P_i} \to 0,$$

where k_{P_i} is the constant sheaf k at P_i. If $r=0$, then $H^0(\mathscr{E}(-2)|_L)=0$ so we must have $\mathscr{E}|_L \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. If $r=1$ one sees easily the only possibility for $\mathscr{E}|_L$ is $\mathcal{O} \oplus \mathcal{O}$. Similarly if $r=2$, $\mathscr{E}|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$, and if $r=3$, $\mathscr{E}|_L \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(2)$.

If L is in the 2nd family of lines on Q, then L meets every Y in three points, which proves (a). Conversely, if L is not one of those lines, then there exists a Y in the g_3^1 with $L \cap Y = \emptyset$, so $\mathscr{E}|_L$ must be either $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Cases (c) and (e) follow similarly from this discussion. In case (d), since $\mathscr{E}(1)|_Y \cong (\mathscr{I}_Y/\mathscr{I}_Y^2)^\vee$, it follows that $\mathscr{E}(1)|_L$ is the normal bundle of L, which is $\mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$. Hence $\mathscr{E}|_L = \mathcal{O}_L \oplus \mathcal{O}_L$.

The only remaining case is (b). Such a line is a limiting position of a family of lines of type (c), so by semicontinuity, $\mathscr{E}|_L \cong \mathcal{O}(-a) \oplus \mathcal{O}(a)$ with $a \geqq 1$. On the other hand we have seen that $a<2$ except in case (a), so we must have $a=1$.

Remark 9.11.1. This result illustrates the theorem of Grauert and Mülich (see Barth [6]) that if $\mathscr{E}$ is stable with $c_1=0$ then $\mathscr{E}|_L \cong \mathcal{O}_L \oplus \mathcal{O}_L$ for almost all lines L. The lines for which this does not hold are called *jumping lines* for $\mathscr{E}$, and they form a closed, codimension one subset of the Grassmann variety of lines on $\mathbb{P}^3$. We will call lines of type (a) *double jumping lines.*

Our next objective is to show that $\mathscr{E}$ is determined by its divisor of jumping lines. First we need a preliminary result of plane projective geometry.

Proposition 9.12. *Let C be an irreducible conic in* $\mathbb{P}_k^2$, *and let a* g_3^1 *without base points be given on C. For each divisor* $D = P_1 + P_2 + P_3 \in g_3^1$, *consider the lines* L_{ij} *joining* P_i *and* P_j, *for* $1 \leqq i < j \leqq 3$. *If* $P_i = P_j$, *take* L_{ij} *to be the tangent to C at* P_i. *Then*

(a) *the totality of the lines* L_{ij}, *as D varies in the* g_3^1, *forms an irreducible line conic* Γ^* *in the dual projective plane* $\mathbb{P}^2$;

(b) *the* g_3^1 *is uniquely determined by* Γ^*; *and*

(c) *keeping C fixed, let U denote the set of all* g_3^1 *without base points on C, considered as an open subset of the Grassmann variety* $G(1,3)$ *in a certain projective space* $\mathbb{P}_1^5$; *furthermore let* $\mathbb{P}_2^5$ *denote the parameter space of all line conics in* $\mathbb{P}^{2*}$. *Then there is a linear transformation of* $\mathbb{P}_1^5$ *to* $\mathbb{P}_2^5$ *which for each* $g_3^1 \in U$ *gives the point corresponding to the line conic* Γ^*. *Thus the set of* Γ^* *which arise in this way is an open subset of a quadric hypersurface in* $\mathbb{P}_2^5$.

Proof. Any two conics in $\mathbb{P}^2$ are projectively equivalent, so it is sufficient to prove these statements for any given conic C. We will use affine coordinates for simplicity. So let C be the conic $y=x^2$. To describe a g_3^1 on C, we use the projection of C onto the x-axis, and give a g_3^1 on the x-axis by specifying two cubic polynomials

$$f=a_0x^3+a_1x^2+a_2x+a_3$$
$$g=b_0x^3+b_1x^2+b_2x+b_3.$$

Then a general member of the g_3^1 is given by the three roots x_1, x_2, x_3 of the polynomial $f+tg$ depending on a parameter t. Since the question is symmetric in the three roots, we consider only the points $P_1=(x_1, x_1^2)$ and $P_2=(x_2, x_2^2)$ on C corresponding to x_1 and x_2. The line $L=L_{12}$ joining P_1 and P_2 has equation

$$y=(x_1+x_2)x-x_1x_2,$$

and this equation also gives the tangent line at P_1 if $P_1=P_2$. If we write a general line in the form $y=ux-v$, then this line has (affine) line coordinates $(u,v)=(x_1+x_2, x_1x_2)$. We wish to find the locus of the point $(u,v)\in\mathbb{P}^{2*}$ as t varies.

For this purpose we express x_1, x_2, x_3 in terms of the given a_i, b_i which define the g_3^1:

$$x_1+x_2+x_3=-\frac{a_1+tb_1}{a_0+tb_0}$$

$$x_1x_2+x_1x_3+x_2x_3=\frac{a_2+tb_2}{a_0+tb_0}$$

$$x_1x_2x_3=-\frac{a_3+tb_3}{a_0+tb_0}.$$

We combine these equations with

$$u=x_1+x_2$$
$$v=x_1x_2,$$

and then eliminate x_1, x_2, x_3, t so as to leave an expression involving only u, v, a_i, b_i. The result of this computation depends only on u, v and the quantities $p_{ij}=a_ib_j-a_jb_i$, $0\leqq i<j\leqq 3$, which are the Plücker coordinates of the point of $U\subseteq\mathbb{P}_1^5$ representing the given g_3^1: the result is

$$p_{03}u^2+p_{02}uv+p_{01}v^2+p_{13}u+(p_{12}-p_{03})v+p_{23}=0.$$

This is manifestly a conic Γ^* in $\mathbb{P}^{2*}$. Furthermore its coefficients are 6 linearly independent linear forms in the p_{ij}, so Γ^* is determined by a linear transformation of $\mathbb{P}_1^5$ to $\mathbb{P}_2^5$, as claimed in (c).

The fact that Γ^* is irreducible follows easily from the construction: a reducible line conic would consist of two pencils of lines; for a triangle of lines to belong to two pencils, one of the vertices of the triangle must be the axis of one of the pencils. Then this axis would lie on C, which is impossible since the g_3^1 is supposed to be without base points.

Finally, it is clear that the g_3^1 is determined by Γ^*; to find the divisor $D \in g_3^1$ containing any given point P, merely take the lines of Γ^* passing through P and intersect them with C.

Remark 9.12.1. The line conic $\Gamma^* \subseteq \mathbb{P}^{2*}$ consists of the set of tangent lines of a point conic $\Gamma \subseteq \mathbb{P}^2$. Thus each divisor D of the g_3^1 on C determines a triangle which is inscribed in C and circumscribed about Γ (Fig. 1).

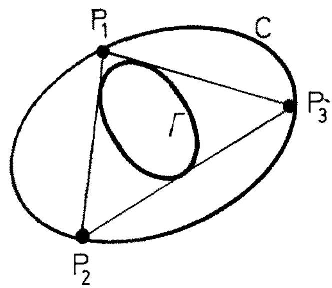

Fig. 1

This is closely related to the classical "porism" of Poncelet which says in particular that if C and Γ are two conics in the plane, and if there exists one triangle inscribed in C and circumscribed about Γ, then in fact every point of C is a vertex of such a triangle. If that happens, then the triples of points on C forming the vertices of these triangles are parametrized by the points of C, three times over, so by Lüroth's theorem those triples of points form a g_3^1 without base points on C. Our result is a converse to this statement. Similarly, of course, the points of tangency of the sides of the triangles on Γ form a g_3^1 on Γ.

Remark 9.12.2. Dualizing the result of (9.12) and interpreting it as in (9.12.1), we see that if a g_3^1 is given on a conic Γ, and if for each divisor $D = Q_1 + Q_2 + Q_3$ of points in the g_3^1 we form the tangents to Γ at the points Q_i, and let these tangents intersect at three points P_1, P_2, P_3, then as D varies, the points P_i will move on another conic C.

Theorem 9.13. *Assume* char. $k \neq 3$. *Then the bundle* $\mathscr{E}$ *is uniquely determined by the set* $Z \subseteq G(1,3)$ *of its jumping lines.*

Proof. First consider a plane $H \subseteq \mathbb{P}^3$. If H is transversal to the quadric surface Q, then H contains only lines of types (c) and (e) in the classification of (9.11). In particular, the jumping lines of $\mathscr{E}$ in H are all of type (c). More precisely, the g_3^1 of curves Y on Q cuts out a g_3^1 on the irreducible conic $C = Q \cap H$ in H, and the jumping lines in H are precisely the lines of the line conic Γ^* in H^* associated to this g_3^1 as in (9.12).

On the other hand, suppose H is tangent to Q. Then H contains a line L of the first family on Q and a line M of the second family. Let Y be the divisor of the g_3^1 on Q which contains L, and let $R, S \in M$ be the two points where the two lines of Y besides L meet the plane H. Then any line in H passing through R or S will meet L, hence will be a jumping line of type (c), except for M itself, which is a double jumping line of type (a). Thus the set of jumping lines in H consists of the two pencils of lines with axes R and S.

Summing up, for any plane $H \subsetneq \mathbb{P}^3$, the set of jumping lines of $\mathscr{E}$ contained in H is a line conic in H^*, which is irreducible if H is transversal to Q, and reducible if H is tangent to Q. This characterizes the set of planes tangent to Q in terms of the set Z of jumping lines, and so Q itself is uniquely determined by Z. Furthermore the choice of the two families of lines on Q is determined, because the line M of the second family in a tangent plane H is characterized as the line joining the axes R and S of the two pencils of jumping lines. (At this point we need to know that at least for one such H, the points R and S will be distinct. Since char. $k \neq 3$, this follows from (9.6).)

Now take any plane H transversal to Q. Since Q is already determined by Z, the conic $C = Q \cap H$ is also determined. As we saw above, the jumping lines in H form the line conic Γ^* of (9.12), and this in turn uniquely determines the g_3^1 on C. Since we also have determined the choice of line family on Q, we recover the g_3^1 on Q. Therefore by (9.7), $\mathscr{E}$ is uniquely determined.

Remark 9.13.1. The same argument works also in the case char. $k = 3$, unless the g_3^1 of (9.7) corresponds to the Frobenius isomorphism of $\mathbb{P}^1$ to $\mathbb{P}^1$. In that case the Q and g_3^1 of (9.7) are uniquely determined by Z, but we cannot determine the choice of family of rulings on Q, unless we specify in addition which lines of Z are double jumping lines. Indeed, in this case Z is simply the set of all lines tangent to Q.

We conclude this section with a geometric description of the divisor of jumping lines, which is useful in studying the existence of real jumping lines over $\mathbb{C}$ [20].

Theorem 9.14. *Assume* char. $k \neq 3$. *Given a bundle* $\mathscr{E}$ *as above, denote by* $G = G(1,3) \subsetneq \mathbb{P}^5$ *the Grassmann variety of lines in* $\mathbb{P}^3$. *The set of lines in the first family on the quadric surface* Q *associated with* $\mathscr{E}$ *forms a conic* $\gamma \subsetneq G$. *Let* π *be the unique plane in* $\mathbb{P}^5$ *containing* γ. *The* g_3^1 *on* Q *induces a* g_3^1 *on* γ. *For each divisor* $D = P_1 + P_2 + P_3$ *of the* g_3^1, *let the tangent lines to* γ *at the points* P_i *intersect in* 3 *further points* X_1, X_2, X_3. *As* D *varies, by* (9.12.2), *the points* X_i *lie on another conic* $\Gamma \subsetneq \pi$. *Let* $\pi^* \subsetneq \mathbb{P}^5$ *be the plane dual to* π *with respect to the quadric surface* Q. *Let* W *be the cone with vertex* π^* *over the conic* $\Gamma \subsetneq \pi$. *Then* W *is a quadric hypersurface in* $\mathbb{P}^5$, *and* $W \cap G$ *is the divisor* Z *of jumping lines of* $\mathscr{E}$.

Proof. A jumping line $L \subsetneq \mathbb{P}^3$ can be characterized by the property $H^0(\mathscr{E}(-1)|_L) \neq 0$. From the exact sequence

$$0 \to \mathscr{I}_L \to \mathcal{O} \to \mathcal{O}_L \to 0, \tag{1}$$

tensoring with $\mathscr{E}(-1)$, we get

$$0 \to H^0(\mathscr{E}(-1)|_L) \to H^1(\mathscr{I}_L \otimes \mathscr{E}(-1)) \xrightarrow{\alpha} H^1(\mathscr{E}(-1)).$$

Thus L is a jumping line if and only if $\ker \alpha \neq 0$. Let h_1 and h_2 be linear forms defining L. Then from the exact sequence

$$0 \to \mathcal{O}(-2) \xrightarrow{h_2, -h_1} \mathcal{O}(-1)^2 \xrightarrow{h_1, h_2} \mathscr{I}_L \to 0, \tag{2}$$

tensoring with $\mathscr{E}(-1)$ and using (9.4), we get an isomorphism

$$\delta_L : H^1(\mathscr{I}_L \otimes \mathscr{E}(-1)) \xrightarrow{\approx} H^2(\mathscr{E}(-3)).$$

Being the coboundary map of the exact sequence (2), we might think of δ_L as multiplication by $1/(h_1 \wedge h_2)$. In particular, the map

$$\alpha' = \alpha \circ \delta_L^{-1} : H^2(\mathscr{E}(-3)) \to H^1(\mathscr{E}(-1))$$

is linear in the Plücker coordinates of the line L. The jumping lines are described by the equation $\det \alpha' = 0$. Since these are both 2-dimensional vector spaces, by (9.4), we see that the variety $W \subseteq \mathbb{P}^5$ defined by $\det \alpha' = 0$ is a quadric hypersurface, and that $Z = G \cap W$.

To compute W more precisely, we now choose a section $s \in H^0(\mathscr{E}(1))$ whose zero set Y consists of three distinct lines Y_1, Y_2, Y_3. Furthermore fix two linear equations defining each line Y_i. Use the exact sequences

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{E}(1) \to \mathscr{I}_Y(2) \to 0 \tag{3}$$

and

$$0 \to \mathscr{I}_Y \to \mathcal{O} \to \mathcal{O}_Y \to 0 \tag{4}$$

to form the rows of the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}}) & \longrightarrow & H^0(\mathcal{O}_Y) & \xrightarrow{\delta_Y} & H^1(\mathscr{I}_Y) & \longrightarrow & 0 \\
 & & & & \approx \Big\downarrow \delta_L & & \Big\uparrow \approx & & \\
 & & & & & & H^1(\mathscr{E}(-1)) & & \\
 & & & & & & \Big\uparrow \alpha & & \\
 & & & & H^1(\mathcal{O}_Y(-2)) & & H^1(\mathscr{I}_L \otimes \mathscr{E}(-1)) & & \\
 & & & & \delta_s \approx \Big\downarrow \delta_Y & & \approx \Big\downarrow \delta_L & & \\
0 & \longleftarrow & H^3(\mathcal{O}(-4)) & \longleftarrow & H^2(\mathscr{I}_Y(-2)) & \longleftarrow & H^2(\mathscr{E}(-3)) & \longleftarrow & 0
\end{array}$$

The right-hand column consists of the maps α and δ_L described above, plus the isomorphism $H^1(\mathscr{E}(-1)) \xrightarrow{\approx} H^1(\mathscr{I}_Y)$ coming from exact sequence (3). The map δ_Y in the middle column comes from (4). The map δ_L in the middle column is a coboundary map obtained from (2) by tensoring with $\mathcal{O}_Y$, and assuming that $Y \cap L = \emptyset$, so that $\mathscr{I}_L \otimes \mathcal{O}_Y \cong \mathcal{O}_Y$. The diagram clearly commutes.

We identify $H^0(\mathcal{O}_{\mathbb{P}})$ with k and $H^0(\mathcal{O}_Y)$ with k^3, and the map between them sends 1 to (1, 1, 1). Using the linear forms defining each Y_i we can get a specific isomorphism of $H^1(\mathcal{O}_Y(-2))$ with k^3, and using the coordinates of $\mathbb{P}^3$ we have a specific isomorphism of $H^3(\mathcal{O}(-4))$ with k. Furthermore fix an isomorphism $\varphi : \wedge^2 \mathscr{E} \xrightarrow{\approx} \mathcal{O}$. Then according to (1.1) the exact sequence (3) determines elements $\xi_1, \xi_2, \xi_3 \in k$ such that with these identifications, the map $\delta_s \circ \delta_Y : H^1(\mathcal{O}_Y(-2)) = k^3 \to H^3(\mathcal{O}(-4)) = k$ is given by (ξ_1, ξ_2, ξ_3).

Finally, we must identify the map δ_L of the middle column. The key point is that if L, M are disjoint lines in $\mathbb{P}^3$, with Plücker coordinates l and m, and if we use the coordinates m to get a natural isomorphism $H^1(\mathcal{O}_M(-2))=k$, then the corresponding map

$$\delta_L^{-1}: H^1(\mathcal{O}_M(-2))=k \to H^0(\mathcal{O}_M)=k,$$

where δ_L is obtained from (2) as above, is simply multiplication by $\langle l,m\rangle$, where $\langle\ ,\ \rangle$ is the bilinear form on the 6-dimensional vector space of Plücker coordinates p_{ij} corresponding to the quadratic form

$$p_{01}p_{23}-p_{02}p_{13}+p_{03}p_{12}$$

which defines G in $\mathbb{P}^5$.

Applying this in our case, let P_1, P_2, P_3 be the Plücker coordinates of Y_1, Y_2, Y_3, and let y_i be the linear form $\langle\ ,P_i\rangle$ on $\mathbb{P}^5$. Then the map

$$\delta_L^{-1}: H^1(\mathcal{O}_Y(-2))=k^3 \to H^0(\mathcal{O}_Y)=k^3$$

is just multiplication by y_i in the ith place.

Now we can rewrite our big diagram explicitly as follows:

$$\begin{array}{ccccccc} 0 \to k & \xrightarrow{(1,1,1)} & k^3 & \longrightarrow & V & \to 0 \\ & & \uparrow & {\scriptstyle (y_1,y_2,y_3)} & \uparrow {\scriptstyle \alpha'} & \\ 0 \leftarrow k & \xleftarrow{(\xi_1,\xi_2,\xi_3)} & k^3 & \longleftarrow & W & \leftarrow 0 \end{array}$$

From this diagram it is easy to compute $\det\alpha'$, which is uniquely determined up to a scalar multiple. It is

$$\xi_3 y_1 y_2 + \xi_2 y_1 y_3 + \xi_1 y_2 y_3 .$$

This then is the equation of W, where the y_i are linear forms on $\mathbb{P}^5$, and the ξ_i are constants.

Since y_i is the linear form $\langle\ ,P_i\rangle$, the hyperplane $y_i=0$ in $\mathbb{P}^5$ is just the tangent hyperplane to G at P_i. Therefore the linear space $y_1=y_2=y_3=0$ is the intersection of these hyperplanes, which is just the plane π^* dual to the plane π containing P_1, P_2, P_3. On the other hand, the line $y_i=0$ in π is the tangent to γ at P_i. It is clear from the equation of W that $W\cap\pi$ contains the 3 points $y_1=y_2=0$, $y_1=y_3=0$, $y_2=y_3=0$ where these tangents meet. Now, since W is independent of the choice of $s\in H^0(\mathscr{E}(1))$ and hence of the curve Y on Q, this holds for any choice of Y, so we see that $W\cap\pi$ is the conic Γ determined by γ and its g_3^1 according to (9.12.2). Now from the equation of W it is clear that it is the cone with vertex π^* over Γ, as required.

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