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# Subalgebras of a Finite Algebra

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## 1. Introduction

For a finite countably decomposable von Neumann algebra $L$ we study the structures of the space consisting of all von Neumann subalgebras of $L$. Especially we study the relation $M \overset{\delta}{\subset} N$ which was previously studied by Murray and von Neumann [11] and McDuff [9].

Much of the techniques used are taken from the articles [3] and [4], but also [6] and [7] are very fundamental.

In Sect. 3 we obtain certain results about homomorphisms which are close to the identity in trace norm.

Section 4 is devoted to the study of the relation $M \overset{\delta}{\subset} N$, and we prove that if both $M$ and $N$ are factors, and one of them is finitedimensional, then for $\delta$ sufficiently small there is a unitary $u$ in $L$ such that $u^*Mu \subseteq N$. Corollary 4.3 contains a result for finitedimensional algebras, which is not dependent upon the particular dimension. At the end of Sect. 4 we prove that if two factors $M$ and $N$ satisfy $M \overset{1/8}{\subset} N$, then $M$ is isomorphic to a subalgebra of $N \otimes M_2$.

In the last paragraph we introduce the Hausdorff-metric with respect to some trace norm on the set of von Neumann subalgebras. We prove that various algebraically characterized sets have nice topological properties: The set of maximal abelian subalgebras is closed, the set of factors is a Baire space and the set of injective factors is an open and closed subset of the set of factors.

In the paper we use the notation $L^2(N, \tau)$ and $\| \|_2^\tau$ for a semifinite von Neumann algebra $N$ with a faithful normal semifinite trace $\tau$. We do not explain this notation but refer the reader to the article [13] and the book [8] especially III,7.

A special warning with respect to the word von Neumann algebra is perhaps necessary; in Sect. 4 the von Neumann algebras $M$ and $N$ always contain the identity $I$ whereas in general von Neumann algebras are not in this article assumed to contain the identity.

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## 2. Technical Lemmas

In [7] Connes proves that the polardecomposition in a semifinite factor has certain continuity properties with respect to the ultrastrong topology. ([7], Theorem 1.2.2.)

Results of this type go back to the work of Murray and von Neumann [11], and can be found in Dixmiers book [8] in Chap. III, § 7.3, pp. 273–276. Although the above mentioned result of Connes could do for our purpose we want to use the following lemma, because it is sharper in our context, and it is not too difficult to prove.

**2.1. Lemma.** *Let $N$ be a semifinite von Neumann algebra with a faithful semifinite normal trace $\tau$. Suppose $e$ is a projection in $L_2(N,\tau)$ and $h$ in $L^2(N,\tau)$ has the properties, $0 \leqq h \leqq I$ and $\|h-e\|_2 \leqq k\|e\|_2$ for some $k \in [0,1[$ then the spectral projection $E_{1-k^{1/2}}$ for $h$ corresponding to the interval $[1-k^{1/2},1]$ will satisfy*

$$\|E_{1-k^{1/2}}-e\|_2 \leqq k^{1/2}(1-k^{1/2})^{-1}\|e\|_2 .$$

*Proof.*

$$\|e\|_2^2 k^2 \geqq \|h-e\|_2^2 = \|(I-h)e\|_2^2 + \|h(I-e)\|_2^2$$

and

$$(I-h)^2 \geqq (I-E_{1-k^{1/2}})(I-h)^2 \geqq k(I-E_{1-k^{1/2}})$$

so

$$\tau(k(I-E_{1-k^{1/2}})e) \leqq \tau((I-h)^2 e) \leqq k^2\|e\|_2^2$$

or

$$\|(I-E_{1-k^{1/2}})e\|_2^2 \leqq k\|e\|_2^2 .$$

$$\|E_{1-k^{1/2}}(1-e)\|_2^2 = \tau(E_{1-k^{1/2}}(1-e)) \leqq \left(\frac{1}{1-k^{1/2}}\right)^2 \tau(h^2(1-e)).$$

Wherefore

$$\|E_{1-k^{1/2}}(1-e)\|_2^2 \leqq \left(\frac{k}{1-k^{1/2}}\right)^2 \|e\|_2^2 .$$

Finally we obtain

$$\|E_{1-k^{1/2}}-e\|_2^2 = \|(I-E_{1-k^{1/2}})e\|^2 + \|E_{1-k^{1/2}}(I-e)\|_2^2$$

and

$$\|E_{1-k^{1/2}}-e\|_2^2 \leqq k\left(1+\left(\frac{k^{1/2}}{1-k^{1/2}}\right)^2\right)\|e\|_2^2 \leqq k\left(\frac{1}{1-k^{1/2}}\right)^2 \|e\|_2^2$$

$$\|E_{1-k^{1/2}}-e\|_2 \leqq k^{1/2}(1-k^{1/2})^{-1}\|e\|_2 .$$

The lemma follows.

The next lemma has its sources within the works mentioned above, and we want just to state it for the convenience of the reader.

**2.2. Lemma.** *Let $N$ be a semifinite von Neumann algebra with a faithful normal semifinite trace $\tau$, $e \sim f$ a pair of finite equivalent projections in $N$ with*

$$\|e-f\|_2 \leqq \varepsilon \|e\|_2 .$$

*Then there is a partial isometry $v$ in $N$ such that*

$$v^*v = e\,; \quad vv^* = f\,; \quad \|v-e\|_2 \leqq 6\|e-f\|_2 \leqq 6\varepsilon\|e\|_2$$

*Proof.* See [7] Lemma 1.4, or use ([6], Lemma 1.1.4, p. 388) on the algebra $N_{(e \vee f)}$.

## 3. Homomorphisms

**3.1. Theorem.** *Let $L$ be a finite von Neumann algebra, $\tau$ a faithful normal tracial state, $M$ a sub von Neumann algebra containing the identity $I_L$ of $L$ and $\Phi$ a star homomorphism of $M$ into $L$.*

*Suppose that for any $m$ in $M_1$ $\|\Phi(m)-m\|_2 \leqq t$, then there exist projections $q, r$ and a partial isometry $v$ in $(M \cup \Phi(M))''$ such that $v^*v = q \in \Phi(M)'$; $vv^* = r \in M'$; $\|I-v\|_2 \leqq 2t$; $\|I-r\|_2 \leqq t$; $\|I-q\|_2 \leqq t$; $\forall m \in M$; $q\Phi(m) = v^*mv$.*

*Proof.* Let $K$ be the ultraweakly closed convex hull of the set $\{u^*\Phi(u) \mid u$ unitary in $M\}$, then $K$ is also a closed convex subset of $L^2(L,\tau)$ and there will be a point $k$ in $K$ of minimal Hilbert-Schmidt norm. For $u$ unitary in $M$ the map $K \ni h \to u^*h\Phi(u)$ is a normdecreasing map (with respect to the Hilbert-Schmidt norm) of $K$ into $K$ so by definition of $k$ and the strict convexity of the unitball in Hilbert spaces we find, that for any unitary $u$ in $M$ $u^*k\Phi(u) = k$ or

$$\forall u \in M_u : k\Phi(u) = uk . \tag{1}$$

It follows easily that $\|k-I\|_2 \leqq t$ since for any unitary $u$ in $M$

$$\|\Phi(u)u^* - I\|_2 = \|\Phi(u) - u\|_2 \leqq t .$$

Let $k = vh$ be the polar decomposition of $k$, then Lemma 2.2 implies that $\|I-v\|_2 \leqq 2t$; $\|I-v^*v\|_2 \leqq t$; $\|I-vv^*\|_2 \leqq t$.

The relation (1) shows that $h \in \Phi(M)'$ and $vhv^* \in M'$, so the theorem follows.

**3.2. Corollary.** *If $M$ is a type $I_k$ factor ($k < \infty$) and $\Phi(I) = I$, then there is a unitary $u$ in $L$ such that, $\|I-u\|_2 \leqq 3t$ and $\Phi(m) = u^*Mu$.*

*Proof.* Find $v$ as in 3.1, $q = v^*v \in \Phi(M)'$; $q\Phi(m) = v^*mv$ and choose a set of matrix units $(e_{ij})$ for $M$, then the set $\{\Phi(e_{ii})(I-q), e_{ii}(I-r)\}$ consists of pairwise equivalent projections. This result follows from the Comparison Theorem ([8] III, 1.2, Theorem 2, p. 218), because for any central projection $z$ in $L$, for which $\tau(\Phi(e_{11})(I-q)z) < \tau(e_{11}(I-r)z)$ we get $(\Phi(I) = I)$ $\tau((I-q)z) < \tau((I-r)z)$ which contradicts $(I-q) \sim (I-r)$, because $q \sim r$ and the algebra is finite. Let $w$ be a partial isometry such that $w^*w = (I-q)\Phi(e_{11})$, $ww^* = (I-r)e_{11}$ and define the partial isometry $v_1$ by $v_1 = \sum_{i=1}^{k} (I-r)e_{i1}w\Phi(e_{1i})(I-q)$, then $u = v + v_1$ is unitary and

$$\|I-u\|_2 \leqq \|I-v\|_2 + \|v_1\|_2 \leqq 2t + \|I-q\|_2 \leqq 3t .$$

For any $e_{nm}$ in $M$ $u^*e_{nm}u = v^*e_{nm}v + v_1^*e_{nm}v_1 = q\Phi(e_{nm}) + (1-q)\Phi(e_{n1})\Phi(e_{1m}) = \Phi(e_{nm})$. The corollary follows. ([2] proof of Theorem 4.1).

**3.3. Corollary.** *If $M=L$, $M$ is a factor and $t\in[0,1[$, then $\Phi$ is an inner automorphism implemented by a unitary $v$ in $M$ for which $\|I-v\|_2\leqq 2t$.*

*Proof.* Find $v$, partial isometry, as in 3.1 then $vv^*\in M'\cap M$, so $vv^*=0$ or $vv^*=I$, but $\|I-vv^*\|_2\leqq t<1$, and $v$ must be a unitary.

## 4. Subalgebras

In this section we will let $L$ denote a finite von Neumann algebra of type $\mathrm{II}_1$, with a faithful tracial state $\tau$.

We want to study the relation $\overset{\delta}{\subset}$ between sub von Neumann algebras of $L$, but before we want to go into the technical details, we will fix our notation and give some explanatory remarks, which leed up to the first lemmas.

We will deal with two von Neumann subalgebras $M$ and $N$ of $L$, which do always contain the identity $I$ in $L$, whereas a homomorphism $\Phi$ of one sub von Neumann algebra into another, not necessarily needs to satisfy $\Phi(I)=I$, but always $\Phi(m^*)=\Phi(m)^*$.

Further we suppose that $L$ acts on $L^2(L,\tau)$, in the canonical way as leftmultipliers, we will let $\xi$ denote the vector $I$ in $L^2(L,\tau)$, since it is then easier to distinguish between the element $x$ in $L$ and the element $x\xi$ in $L^2(L,\tau)$, when this is necessary. ([8], I.5, I.6, [13]). Let $N$ be a sub von Neumann algebra of $L$, then there exists a faithful normal projection $\pi$ of norm one from $L$ onto $N$. The map $\pi$ is defined as a conditional expectation by

$$\forall x\in L:\pi(x)\in N \quad \text{and} \quad \forall n\in N:\tau(xn)=\tau(\pi(x)n), \tag{2}$$

([12], Proposition 4.4.23, p. 211, [15–17]).

Speaking in commutative terms the expression (2) depends upon the fact that in finite measure spaces the $L^\infty$ space is dense in the $L^1$ space, but the finiteness of the measure also implies that $L^\infty$ is a dense subset of the $L^2$ space.

If we take this point of view $\pi$ becomes a densely defined linear projection from $L^2(L,\tau)$ onto a dense subset of $L^2(N,\tau)$, and moreover we find from (2) that the orthogonal projection $p$ from $L^2(L,\tau)$ onto $\overline{N\xi}$ is related to $\pi$ in the following way.

$$\forall x\in L\forall n_1,n_2\in N:(xn_1\xi\,|\,n_2\xi)=(\pi(x)n_1\xi\,|\,n_2\xi) \tag{3}$$

$$\forall x\in L:pxp=p\pi(x)p=\pi(x)p. \tag{4}$$

One should remark that since $p\xi=\xi$ (4) implies

$$\forall x\in L:px\xi=\pi(x)\xi. \tag{5}$$

Much of the later computations will take place in the algebra $(L\cup p)''$, so we want here to find out some elementary facts about this algebra. First of all we remark that $(L\cup p)'=L'\cap p'$ is isomorphic to the algebra $R_N$ of rightmultipliers with elements from $N$. This result is contained in ([1], Theorem 1.3.1, p. 154), but it follows also directly from the following simple argument.

Let $R$ be the commutant of $L$ then by ([8], I.5.2, Theorem 1, I.6.2, Theorem 2) $R$ consists of all right-multipliers so it is obvious that $R_N$ is contained in $R_L\cap p'$ $(\forall n_1,n_2\in N:pR_{n_2}n_1\xi=n_1n_2\xi=R_{n_2}pn_1\xi)$.

Suppose $R_x \in R_L \cap p'$ then from (5)

$$x\xi = R_x p\xi = pR_x\xi = px\xi = \pi(x)\xi,$$

and we find $x \in N$ since $\xi$ is separating. The relation (4) shows that $(L \cup p)''_p = N_p$, wherefore $p$ is a finite projection in $(L \cup p)''$, a fact upon which most of the following arguments depend.

It is clear that $(L \cup p)''$ is semifinite and that $(L \cup p)''$ is a factor if $N$ is a factor.

The relations between certain projections in $B(L^2(L, \tau))$ and the subalgebras of $L$ are studied in [14]. In that article C. Skau gives a characterisation of such projections which are cyclic for subalgebras of $L$.

*Definition.* Let both $M$ and $N$ be von Neumann subalgebras of $L$, $\delta \in \mathbb{R}_+ \cup \{0\}$. $M$ is said to be contained $\delta$ in $N$ if for any $m$ in the unitball $M_1$ of $M$, there is an $n$ in $N$ such that $\|m - n\|_2 \leqq \delta$. If both $M \overset{\delta}{\subset} N$ and $N \overset{\delta}{\subset} M$ we write $\|M - N\|_2 \leqq \delta$.

We will from now on assume that $M$ and $N$ are given subalgebras of $L$ for which $M \overset{\delta}{\subset} N$, if we then look at (5) once more we find that for any $m$ in $M$, $\pi(m)$ is the point in $N$ which is closest to $m$ in the Hilbert-Schmidt norm. Therefore

$$M \overset{\delta}{\subset} N \Rightarrow \forall m \in M \; \|m - \pi(m)\|_2 \leqq \delta. \tag{6}$$

Let $u$ be any unitary in $M$ then since $\pi$ is a conditional expectation

$$\pi(I - \pi(u^*)u - u^*\pi(u) + \pi(u^*)\pi(u)) = I - \pi(u^*)\pi(u).$$

The relations (6) and (2) then yield

$$M \overset{\delta}{\subset} N \Rightarrow \forall u \in M_u : \delta^2 \geqq \|u - \pi(u)\|_2^2 = \tau(I - \pi(u^*)\pi(u)). \tag{7}$$

In the articles [3] and [4] we did develop a technique, which from statements of the type obtained in (7) shows that if $\delta$ is small, the algebras are semifinite and the $\| \;\|_2$ is replaced with $\| \;\|_\infty$, then within some small distance from $\pi$ there will be a homomorphism of $M$ into $N$. It is our aim to transfer this method to the case under consideration here, and in some sence the arguments become simpler here, because the unitball in a Hilbert space has no faces other than the extreme points.

The first step in this direction was taken in the proof of Proposition 1.1 in [4], where just as here we have an algebra $L$ and a projection $p$ outside $L$ such that $(L \cup p)''$ is a semifinite algebra in which $p$ is finite projection.

Let us return to the concrete case, and in the rest of this section assume that $N$ is a factor, then there is exactly one faithful normal semifinite trace $\varphi$ on $(L \cup p)''$ for which $\varphi(p) = 1$. This trace is related to $\tau$ by the following relation

$$\forall x \in L : \varphi(xp) = \tau(\pi(x)) = \tau(x). \tag{8}$$

The first equality follows from the uniqueness of the normalised trace on $N$ and (4), the second is a consequence of (2).

Let $u$ be any unitary in $M$ then by (7)

$$\begin{aligned}\varphi((p - u^*pu)^2) &= \varphi(p + u^*pu - u^*pup - pu^*pu) \\ &= 2\varphi(p - pu^*pup) = 2\tau(I - \pi(u^*)\pi(u)) \leqq 2\delta^2.\end{aligned}$$

The norm $\| \;\|_2^\varphi$ is ultrastrongly lower semicontinuous on $(L \cup p)''$, hence for any $h$ in $\overline{co}_M(p)$ we obtain $\|h - p\|_2^\varphi \leqq 2^{1/2}\delta$.

Just as in the proof of Theorem 3.1 we find that there is a $k$ in $\overline{co}_M(p) \cap M'$, and $k$ satisfies

$$0 \leqq k \leqq I, \quad \|k-p\|_2^\varphi \leqq 2^{1/2}\delta, \quad \text{hence if} \quad \delta < 2^{-1/2},$$

Lemma 2.1 yields

$$\exists \text{ projection } q \in M' \cap (M \cup p)'': \quad \|q-p\|_2^\varphi \leqq 2^{1/4}\delta^{1/2}(1-2^{1/4}\delta^{1/2})^{-1}. \tag{9}$$

Let $\gamma$ denote $2^{1/4}\delta^{1/2}(1-2^{1/4}\delta^{1/2})^{-1}$ and remember that

$$\|q-p\|_2^2 = \|q(I-p)\|_2^2 + \|(I-q)p\|_2^2,$$

then the inequalities below are easy.

$$-\gamma^2 \leqq -\|q-p\|_2^2 \leqq -\|q(I-p)\|_2^2 = \varphi(qp-q) \leqq \varphi(p-q).$$

$$\begin{aligned} &\varphi(p-q) \leqq \varphi(p-qp) = \|(I-q)p\|_2^2 \leqq \|q-p\|_2^2 \leqq \gamma^2 \\ &\qquad |\varphi(p)-\varphi(q)| = |1-\varphi(q)| \leqq \gamma^2. \end{aligned} \tag{10}$$

In the uniform case ([3, 4]) $q$ would be equivalent to $p$ via a partial isometry close to $p$ and $q$, and the desired homomorphism is simply obtained by $M \ni m \to vmv^*$ and the identification of $(L \cup p)_p''$ with $N$. In the case of square integrable perturbations, close projections need not be equivalent, but if $q \prec p$ in the sense of Murray and von Neumann ([8], III.1.1) the following lemma can help.

**4.1. Lemma.** *Let $L$, $\tau$, $N$, $p$, $\varphi$ be as above, $M$ a von Neumann sub-algebra of $L$, $r$ a projection in $M' \cap (L \cup p)''$. Suppose $r \prec p$ then there is a homomorphism $\Phi$ of $M$ into $N$ such that for any $m$ in $M_1$*

$$\|\Phi(m) - \pi(m)\|_2 \leqq 26\|p-r\|_2^\varphi.$$

*Proof.* The remarks in front of (10) yield, that if we let $\alpha$ denote $\|p-r\|_2^\varphi$ then $0 \leqq (1-\varphi(r)) \leqq \alpha^2$.

Let $e$ be a projection in $(L \cup p)''$ such that $r \sim e \leqq p$, then $\|r-e\|_2^\varphi \leqq \|r-p\|_2^\varphi + \|p-e\|_2^\varphi = \alpha + \varphi(p-e)^{1/2} = \alpha + (1-\varphi(r))^{1/2} \leqq 2\alpha$. The Lemma 2.3 implies, that there exists a partial isometry $v$ in $(L \cup p)''$ such that $v^*v = e$, $vv^* = r$, and $\|e-v\|_2^\varphi \leqq 12\alpha$.

The algebra $N$ is a factor (or $p$ has central support $I$) so we get by (4) a well defined homomorphism $\Phi$ of $M$ into $N$ by

$$\forall m \in M : \Phi(m) \in N \quad \text{and} \quad \Phi(m)p = v^*mv.$$

The relations (4) and (8) show that for $m \in M_1$

$$\begin{aligned} \|\Phi(s) - \pi(s)\|_2^\tau &= \|v^*sv - psp\|_2^\varphi \leqq \|(v^*-p)sv + ps(v-p)\|_2^\varphi \\ &\leqq \|v^*-p\|_2^\varphi + \|v-p\|_2^\varphi \leqq 2(\|v-e\|_2^\varphi + \|e-p\|_2^\varphi) \leqq 26\alpha. \end{aligned}$$

The lemma follows.

The real problem left for the case $M \overset{\delta}{\subset} N$ is to find conditions under which $q$ is not a minimal projection in $M' \cap (L \cup p)''$. To this end one should remark that in the case where $M = N$, then $(N' \cap (L \cup p)'')_p$ is trivial, because $N$ is a factor, and $p$ is minimal in $N' \cap (L \cup p)''$. In the opposite direction it seems likely that if $M' \cap N$ is "big" then a $q$ which is close to $p$ ought not to be minimal in $M'\cap(L\cup p)''$. We want to show a result of this type in the following theorem.

**4.2. Theorem.** *Suppose $M\overset{\delta}{\subset}N$, $N$ is a continuous factor, $M$ is finite dimensional and $\delta\in[0,2^{-1/2}[$, then there is an isomorphism $\Phi$ of $M$ onto a von Neumann subalgebra of $N$ ($\Phi(I)=I$, here) such that*

$$\forall m\in M_1 \|\Phi(m)-m\|_2\leqq 105\gamma \quad \text{[see line following (9)].}$$

*If $\delta<2^{-1/4}$ then $\gamma<10\delta^{1/2}$.*

*Proof.* Find a projection $q$ in $M'\cap(L\cup p)''$ as above such that $\|q-p\|_2\leqq\gamma$ and $\|\varphi(q)-1\|\leqq\gamma^2$.

If $q\prec p$ then Lemma 4.1 shows that there is a homomorphism $\Psi$ of $M$ into $N$ such that

$$\forall m\in M_1: \|\Psi(m)-m\|_2\leqq\|m-\pi(m)\|_2+\|\Psi(m)-\pi(m)\|_2\leqq\delta+26\gamma.$$

The algebra $(M'\cap(L\cup p)'')_q$ is a finite continuous algebra, since $q\in M'\cap(L\cup p)''$ and the commutant of this algebra is generated by the two commuting algebras $M$ and $R_N$. The former is finite dimensional and the latter is a factor so the von Neumann algebra they generate is isomorphic to the tensorproduct and hence continuous ([8], I.2, Example 6, p. 29). Therefore if $\varphi(q)>1$ there will be a projection $r$ in $M'\cap(L\cup p)''$ such that $r\leqq q$ and $\varphi(r)=1$, furthermore we get $r\sim p$, since $(L\cup p)''$ is a factor. The distance between $p$ and $r$ is measured by $\|p-r\|_2 \leqq\|p-q\|_2+\|q-r\|_2\leqq \gamma+(\varphi(q)-1)^{1/2}\leqq 2\ \gamma$, and Lemma 4.1 shows that there exists a homomorphism $\Psi$ of $M$ into $N$ such that

$$\forall m\in M \|\Psi(m)-m\|_2\leqq\|m-\pi(m)\|_2+52\gamma\leqq\delta+52\gamma.$$

If $\Psi(I)=I$ then let us choose a projection $r$ in $(\Psi(M))'\cap N$ such that $\tau(r)<\delta$ and let $\Psi_1$ be an isomorphism of $M$ onto a sub-algebra of $Nr$ such that $\Psi_1(I)=r$. Now $\Phi$ defined by $\Phi(m)=(I-r)\Psi(m)+\Psi_1(m)$ will give the result. If $\Psi(I)\neq I$ then choose as above an isomorphism $\Psi_1$ of $M$ onto a subalgebra of $(I-\Psi(I))N(I-\Psi(I))$ and again $\Phi=\Psi+\Psi_1$ will do.

Instead of $\gamma$ we want to introduce the rough estimates $\gamma\leqq 3/2\delta^{1/2}$ and $105\delta<150\delta^{1/2}$ when $\delta<10^{-6}$.

**4.3. Corollary.** *Suppose $M$ is a type $I_k$ factor ($k\in N$), $\delta<10^{-6}$, then there is a unitary $u$ in $L$ such that $\|I-u\|_2\leqq 450\delta^{1/2}$ and $u^*Mu\subseteqq N$.*

*Proof.* Follows from 3.2 and 4.2.

The remarkable thing in 4.2 and 4.3 is of course that the results are not dependent of the dimension of $M$, but only upon the finiteness of the dimension of $M$ and the continuity of $N$.

Corollary 4.3 can be generalised to arbitrary finite dimensional algebras $M$ if $L$ is a factor.

Let us now turn to the case where $N$ is a finite dimensional factor of type $I_k$, and let $e$ be a minimal projection in $N$ then $ep$ is minimal in $N_p=(L\cup p)''_p$, so we will have $\varphi(ep)=1/k$ and for any projection $r$ in $(L\cup p)''$ either $\varphi(r)=\infty$ or $\varphi(r)\in\{n/k\,|\,n\in\mathbb{N}\}$.

Suppose $M$ is a von Neumann subalgebra, $\delta < 2^{-1/2}$ and $M \overset{\delta}{\subset} N$, then there is a projection $q$ in $M' \cap (L \cup p)''$ such that $\|p-q\|_2^\varphi \leqq 2^{1/4}\delta^{1/2}(1-2^{1/4}\delta^{1/2})^{-1}$. (9) and $|1-\varphi(q)| < 2^{1/2}\delta(1-2^{1/4}\delta^{1/2})^{-2}$, (10).

These remarks yield very easy the following.

**4.4. Theorem.** *Suppose $N$ is a type $I_k$ factor $\delta < 10^{-2}k^{-1}$ and $M \overset{\delta}{\subset} N$ then there is an identity preserving homomorphism $\Phi$ of $M$ into $N$, such that $\forall m \in M_1 : \|\Phi(m)-m\|_2 \leqq 39\delta^{1/2}$.*

*Proof.* For $\delta < 10^{-2}k^{-1}$, $2^{1/2}\delta(1-2^{1/4}\delta^{1/2})^{-2} < k^{-1}$ so $\varphi(q)=1$ and $q \sim p$. Since $\delta < 10^{-2}$ we get $\|p-q\|_2^\varphi \leqq 2^{1/4}\delta^{1/2}(1-2^{1/4}\delta^{1/2})^{-1} < 3/2\delta^{1/2} - \delta/26$ so by Lemma 4.1, we find that there is a homomorphism $\Phi$ of $M$ into $N$ such that $\forall m \in M_1 : \|\Phi(m)-m\|_1 \leqq 39\delta^{1/2}$. Moreover the proof of 4.1 yields that when $q$ is equivalent to $p$, then $\Phi(I)=I$.

**4.5. Corollary.** *If $M$ is a type $I_k$ factor, $N$ is an arbitrary factor, and $\|M-N\|_2 < 10^{-2}k^{-1}$, then $N$ is a type $I_k$ factor.*

*Proof.* 4.4 gives that there is a homomorphism $\Phi$ of $N$ into $M$, but $N$ is a finite factor, wherefore ([8], III, 5.2, Proposition 2 or 3, p. 257) either $\Phi$ vanish or $\Phi$ is faithful, $\Phi(I)=I$ however and $N$ must be a type $I_n$ algebra, where $n$ is a divisor in $k$. Theorem 4.4 can then be used once more upon $M \overset{\delta}{\subset} N$ because $\delta < 10^{-2}k^{-1} < 10^{-2}n^{-1}$. The corollary follows.

Let us turn to the situation where $M \overset{\delta}{\subset} N$ and $M$ is continuous.

Again if $q \prec p$ we get by 4.1 a homomorphism $\Phi$ of $M$ into $N$ such that for any $m$ in $M_1$

$$\|\Phi(m)-m\|_2 \leqq \delta + 26\gamma \leqq 40\delta^{1/2} \quad \text{when} \quad \delta < 10^{-6}.$$

We will then assume that $q \succ p$. The continuity assumptions on $M$ implies, that there is a projection $e$ in $M$ such that $\tau(e)=\varphi(p)/\varphi(q)$ and $p \sim qe$. If the ratio $\varphi(p)/\varphi(q)$ is rational the result is clear, if not choose projections $e_n$ in $M$, $e_n \leqq e_{n+1}$ such that $\tau(e_n)$ is rational $\varphi(qe_n)=\varphi(q)\tau(e_n)$ $(q \in M')$ and $\sup\tau(e_n)=\varphi(p)/\varphi(q)$, then $e=\sup e_n$ will do. Now $eq \sim p$ and $\|eq-p\|_2^\varphi \leqq \|p-q\|_2^\varphi + \|q-eq\|_2^\varphi = \|p-q\|_2^\varphi + (\varphi(q)-\varphi(p))^{1/2}$.

By (9) and (10) we obtain $\|eq-p\|_2^\varphi \leqq 2\gamma$ and Lemma 4.1 applied to the algebra $eMe + \mathbb{C}(I-e)$ yields a homomorphism $\Phi$ from $eMe$ into $N$ such that for any $m$ in $M_1$

$$\|\Phi(m)-m\|_2 \leqq \delta + 52\gamma \leqq 80\delta^{1/2}.$$

We have now proved the following theorem.

**4.6. Theorem.** *Suppose $M$ is a continuous von Neumann subalgebra of $L$, $N$ is a subfactor $0 < \delta < 10^{-6}$ and $M \overset{\delta}{\subset} N$, then there exists a projection $e$ in $M$ and a homomorphism $\Phi$ of $M_e$ into $N$ such that*

$$\|I-e\|_2 < 2\delta^{1/2}$$

*and*

$$\|\Phi(eme)-m\|_2 \leqq 80\delta^{1/2} \quad \textit{for any } m \textit{ in } M_1.$$

*We close this section by a result which have no conditions of continuity or discreteness.*

**4.7. Theorem.** *Suppose $M$ is an arbitrary von Neumann subalgebra in $L$, $N$ is a factor and $M \overset{1/8}{\subset} N$ then there exists a nontrivial homomorphism $\Phi$ of $M$ into $N \otimes M_2$.*

*Proof.* Instead of $L$ and $\tau$ let us consider $L_0 = L \otimes M_2$ and $\tau_0 = \tau \otimes \mathrm{tr}$.

It is not difficult to see, that if we let $\tilde{M}$ and $\tilde{N}$ denote $M \otimes \mathbb{C}$ and $N \otimes \mathbb{C}$ in $L \otimes M_2$, then $\tilde{M} \overset{1/8}{\subset} \tilde{N}$ and we may proceed exactly as above and find projections $p, q$ and a trace $\varphi$ on $(L_0 \cup p)''$ such that $pK = \overline{\tilde{N}\xi}$, $\varphi(p) = 1$, $\|q - p\|_2^\varphi < 1$ and $q \in \tilde{M}' \cap (L_0 \cup p)''$.

Let $e_{ij}$ $i, j = 1, 2$ be matrix units for $\mathbb{C} \otimes M_2$ then the rangeprojections $r_{ij}$ of $e_{ij}p$ are mutually orthogonal and equivalent to $p$. The orthogonality is seen by the following computation. Let $n_1, n_2$ be operators in $N$ then

$$(e_{12}(n_1 \otimes I)\xi \mid e_{11}(n_2 \otimes I)\xi) = \tau(n_2^* n_1)\,\mathrm{tr}(e_{12}) = 0 \quad \text{etc.}$$

The equivalence $r_{ij} \sim p$ follows by a similar argument. Let $n_1, n_2$ be in $N$ then

$$((e_{11} - e_{22})(n_1 \otimes I)\xi \mid (n_2 \otimes I)\xi) = \tfrac{1}{2}\tau(n_2^* n_1) - \tfrac{1}{2}\tau(n_2^* n_1) = 0,$$

and we can conclude that $p(e_{11} - e_{22})p = 0$ so

$$pe_{11}p = \tfrac{1}{2}p + \tfrac{1}{2}p(e_{11} - e_{22})p = \tfrac{1}{2}p.$$

If we then define $r$ as the projection onto $\overline{N \otimes M_2 \xi}$ then $r = r_{11} + r_{12} + r_{21} + r_{22}$ so $\varphi(r) = 4$ and $r \in (L_0 \cup p)''$. On the other hand $p$ equals the rangeprojection of $e_{11}r + e_{21}r$ so $(L_0 \cup r)'' = (L_0 \cup p)''$.

The relation (4) is now applicable again and we find that $(L_0 \cup r)_r''$ is isomorphic to $N \otimes M_2$. The relation (10) becomes for $\delta = 1/8$

$$|1 - \varphi(q)| \leqq \gamma^2 = (2^{5/4} - 1)^{-2} < 1$$

and $0 < \varphi(q) < 2$.

Therefore $q$ is equivalent to a subprojection of $r$ inside $(L_0 \cup r)''$ and we get a non trivial homomorphism of $M$ into $N \otimes M_2$.

## 5. Topological Properties

We want to maintain the notation from Section 4, $L$ is a countably decomposable type $\mathrm{II}_1$ von Neumann algebra with a faithful normal tracial state $\tau$. $\mathscr{S}$ will denote the set of all von Neumann subalgebras of $L$ (not necessarily containing $I$), and we equip this space with the topology generated by the metric $d(M, N) = \|M - N\|_2$. We do start by showing that $\mathscr{S}$ is a complete metric space with respect to $d$. This is followed by a result which shows that the relative commutant operation is continuous on $\mathscr{S}$. This in turn shows that certain subsets of $\mathscr{S}$ are closed.

Section 4 has some topological consequences in the space $\mathscr{S}$, in particular we mention that the set of injective subfactors is open and closed in the set of all factors in $\mathscr{S}$.

**5.1. Theorem.** *$(\mathcal{S}, d)$ is a complete metric space.*

*Proof.* Let $(M_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{S}, d)$ and $(\pi_n)_{n\in\mathbb{N}}$ the corresponding set of conditional expectations of $L$ onto the algebras $M_n$.

Just as in Sect. 4 the $\pi_n$'s are chosen such that any of them is the restriction of the orthogonal projection $P_n$ onto the closure of $M_n$ in $L^2(L,\tau)$. Suppose $x$ is an operator in the unitball $L_1$ of $L$, then for any pair of natural numbers $m, n$, we get by (7) of Sect. 4 that

$$\begin{aligned}\|\pi_m(x)-\pi_n(x)\|_2^2 &= (\pi_n(x)|\pi_n(x)-\pi_m(x)) + (\pi_m(x)|\pi_m(x)-\pi_n(x))\\ &= (\pi_n(x)|x-\pi_m(x)) + (\pi_m(x)|x-\pi_n(x))\\ &= ((\pi_n(x)-\pi_m(\pi_n(x)))|x-\pi_m(x)) + ((\pi_m(x)-\pi_n(\pi_m(x)))|x-\pi_n(x))\\ &\leqq 4d(M_n, M_m). \end{aligned} \tag{*}$$

The computation shows that for any $x$ in $L$ the sequence $(\pi_n(x))_{n\in\mathbb{N}}$ is a $L^2$ Cauchy sequence consisting of elements from the closed ball in $L$ (operator norm) with radius $\|x\|$. This ball is complete with respect to the $L^2$ norm, therefore we may define a linear map $\pi$ of norm one from $L$ into $L$ by

$$\forall x \in L : \pi(x) = \lim_{n\to\infty} \pi_n(x).$$

The limit is taken in the ultrastrong topology since this topology coincides with the $L^2$ topology on bounded sets. It is obvious that $\pi$ is positive, and not difficult to see that $\pi$ is a projection. In fact let $x \in L$ then for any $\varepsilon > 0$ there exists $n_0$ such that $\|\pi(x)-\pi_n(x)\|_2 < \varepsilon$ for $n > n_0$. Therefore if $n > n_0$ we obtain

$$\|\pi(x)-\pi_n(\pi(x))\|_2 \leqq \|\pi(x)-\pi_n(x)\| + \|\pi_n(\pi_n-\pi)(x)\|_2 \leqq 2\varepsilon$$

so $\pi^2(x) = \pi(x)$.

Now let $M$ denote the image $\pi(L)$, then for any pair $a, b$ in $M$

$$\pi(ab) = \lim_{n\to\infty} \pi_n(ab)$$

$$ab = \lim_{n\to\infty} \pi_n(a)\pi_n(b) = \lim_{n\to\infty} \pi_n(a\pi_n(b))$$

and

$$ab - \pi(ab) = \lim_{n\to\infty} \pi_n(a(b-\pi_n(b))) = 0.$$

In order to show that $M$ belongs to $\mathcal{S}$ we just have to show that $M$ is weakly closed.

Let $x$ belong to $\bar{M}$, by Kaplansky's Density Theorem we can to $\varepsilon > 0$ find $m$ in $M$ such that $\|x-m\|_2 \leqq \varepsilon$ hence

$$\|x-\pi(x)\|_2 = \|x-m+\pi(m)-\pi(x)\|_2 \leqq 2\|x-m\|_2 \leqq 2\varepsilon,$$

and $x = \pi(x)$ so $M \in \mathcal{S}$.

The relation (*) shows that if $n_0$ is chosen in $\mathbb{N}$ such that $d(M_m, M_n) < \varepsilon^2/4$ for all $m, n > n_0$ then for any $x$ in $M_1$ $\|x-\pi_n(x)\|_2 < \varepsilon$ for $n > n_0$ and for any $y$ in $(M_n)_1$ $\|y-\pi(y)\|_2 < \varepsilon$ for $n > n_0$ so $d(M_n, M) < \varepsilon$ for $n > n_0$ and $\mathcal{S}$ is complete.

*5.2. Definition.* For any $M$ in $\mathcal{S}$, $C(M)$ denotes the relative commutant of $M$ i.e. $C(M) = M' \cap L$.

We want now to show that $C$ is a continuous map of $\mathcal{S}$ into $\mathcal{S}$.

**5.3. Theorem.** *For any $M, N$ in $\mathscr{S}$ $\|C(M)-C(N)\|_2 \leqq 2\|M-N\|_2$.*

*Proof.* Put $\delta = \|M-N\|_2$ and let $x \in (C(M))_1$, then we find exactly as in the proof of Theorem 3.1 that $\overline{co}_N(x)$ meets $(C(N))_1$, hence

$$d(x, C(N)_1) \leqq \sup\{\|uxu^* - x\|_2 \mid u \text{ unitary in } N\}.$$

To any unitary $u$ in $N$ we find $m$ in $M_1$ such that $\|u-m\|_2 \leqq \delta$. Therefore

$$\|uxu^* - x\|_2 = \|ux - xu\|_2 = \|(u-m)x - x(u-m)\|_2 \leqq 2\delta.$$

**5.4. Corollary.** $\|C(M) \cap M - C(N) \cap N\|_2 \leqq 5\|M-N\|_2$.

*Proof.* Put $\delta = \|M-N\|_2$ to $m$ in $(C(M) \cap M)_1$ find $k$ in $C(N)_1$ and $n$ in $N_1$ such that $\|m-n\|_2 \leqq \delta$, $\|m-k\|_2 \leqq 2\delta$. Choose $r$ in $\overline{co}_N(n) \cap N'$ then $r \in C(N) \cap N$ and
$\|m-r\|_2 \leqq \|m-k\|_2 + \|k-r\|_2 \leqq 2\delta + \|k-r\|_2 \leqq 2\delta + \|k-n\|_2 \leqq 2\delta$
$+ \|k-m\|_2 + \|m-n\|_2 \leqq 5\delta$.

**5.5. Proposition.** *The subset $\mathscr{M}\mathscr{A}$ of $\mathscr{S}$ consisting of maximal abelian von Neumann subalgebras is closed.*

*Proof.* Suppose $M \in \overline{\mathscr{M}\mathscr{A}}$ and let $\delta > 0$, then for any $A$ in $\mathscr{S}$ with $\|M-A\| \leqq \delta$ we get because $A = C(A)$

$$\|M - C(M)\|_2 \leqq \|M-A\|_2 + \|C(A) - C(M)\|_2 \leqq 3\delta.$$

Since $\delta$ is arbitrary $M = C(M)$ and $M$ is maximal abelian.

5.6. *Definition.* A von Neumann algebra $N$ in $\mathscr{S}$ is said to be normal if $N = C(C(N))$.

**5.7. Proposition.** *The set $\mathscr{N}$ of normal elements in $\mathscr{S}$ is closed.*

*Proof.* Let $M \in \overline{\mathscr{N}}$ and let $\delta > 0$, choose $N$ in $\mathscr{N}$ such that $\|M-N\|_2 \leqq \delta$ then

$$\|M - C(C(M))\|_2 \leqq \|M-N\|_2 + \|C(C(N)) - C(C(M))\|_2 \leqq 5\delta.$$

Again $\delta$ is arbitrary and the proposition follows.

**5.8. Proposition.** *The set $\mathscr{A}$ of abelian algebras in $\mathscr{S}$ is closed.*

*Proof.* Suppose $M \in \overline{\mathscr{A}}$ and $m, n \in M$. To any $\delta$ find $A$ in $\mathscr{A}$ such that $\|M-A\|_2 \leqq \delta$. Find $a, b$ in $A_1$ such that $\|m-a\|_2 \leqq \delta$ and $\|n-b\|_2 \leqq \delta$, then

$$\|mn - nm\|_2 \leqq \|m(n-b)\|_2 + \|(m-a)b\|_2 + \|b(a-m)\|_2 + \|(n-b)m\|_2 \leqq 4\delta.$$

**5.9. Proposition.** *Let $A$ be an abelian element in $\mathscr{S}$ and $\mathscr{S}_A$ the algebras in $\mathscr{S}$ with center $A$, then $\mathscr{S}_A$ is closed.*

*Proof.* Let $M \in \overline{\mathscr{S}_A}$ and $\delta > 0$, choose $N \in \mathscr{S}_A$ such that $\|M-N\|_2 < \delta$ then by 5.4 $\|M \cap M^c - A\|_2 \leqq 5\delta$. The result follows.

**5.10. Corollary.** *The set of factors in $\mathscr{S}$ containing $I$ is a Baire space in the metric $d$.*

**5.11. Theorem.** *The set $\mathscr{F}$ of subfactors in $\mathscr{S}$ (not necessarily containing the identity $I$) is a Baire space in the metric $d$.*

*Proof.* Let $M \in \overline{\mathscr{F}}$ and $n \in \mathbb{N}$. Choose $N_n$ in $\mathscr{F}$ such that $\|M - N_n\|_2 \leqq 1/n$ then $\|M \cap M^c - \mathbb{C} I_{N_n}\|_2 \leqq 5/n$ so $\mathbb{C} I_{N_n}$ converges to $M \cap M^c$.

Let $p_n$ denote the orthogonal projection onto $\mathbb{C} I_{N_n} \xi$ then for any $m$ in $M \cap M^c$ $p_n m\xi \to m\xi$ and it is possible to see that $(M \cap M^c)\xi$ is one dimensional, and consequently $M$ is a factor.

We want now to investigate the topological consequences of the results in Sect. 4.

*5.12. Definition.* Let $M$ and $N$ be von Neumann algebras, then $M$ is said to be quasicontained in $N$ if there exists an $n$ in $\mathbb{N}$ such that $M$ is isomorphic to a subalgebra of $N \otimes M_n$.

If $M$ is quasicontained in $N$ we write $M \prec N$. If $M \prec N$ and $N \prec M$ we say that $M$ and $N$ are quasiequivalent and we write $M \sim N$.

**5.13. Theorem.** *Let $M, N \in \mathscr{S}$. If $\|M - N\|_2 < \frac{1}{8}$ and $I_L$ belongs to both $M$ and $N$, then $M \sim N$.*

*Proof.* The result is an immediate consequence of Theorem 4.7.

**5.14. Corollary.** *The set of all injective subfactors in $\mathscr{S}_{\mathbb{C}}$ is open and closed in $\mathscr{S}_{\mathbb{C}}$.*

*Proof.* It is obvious that any von Neumann subalgebra of $M \otimes M_2$ is injective when $M$ is injective and finite, and the corollary follows.

**5.15. Corollary.** *If $L^2(L, \tau)$ is separable, then the set of hyperfinite subfactors in $\mathscr{S}_{\mathbb{C}}$ is open and closed in $\mathscr{S}_{\mathbb{C}}$.*

*Proof.* Follows from 5.14 and the result in Sect. 5 of [7].

**5.16. Theorem.** (a) *The set of continuous factors in $\mathscr{S}_{\mathbb{C}}$ is open and closed.*

(b) *For each $n \in \mathbb{N}$ the set of factors in $\mathscr{S}_{\mathbb{C}}$ of type $I_n$ is an open and closed subset of $\mathscr{S}_{\mathbb{C}}$.*

*Proof.* The part (a) follows from Theorem 4.7, whereas (b) follows from Corollary 4.5.

*5.17. Remark.* The reason why we did introduce the relation $\sim$ was the hope that the set of equivalence-classes should perform a tractable invariant.

If we take equivalence-classes of continuous factors only, 5.14 shows that the invariant for the hyperfinite factor consists of only one point, but on the other hand by Connes result mentioned in 5.15 it is clear that a factor with separable predual, and only one equivalence-class, must be the hyperfinite ($\mathrm{II}_1$) factor. Let us recall that a finite $\mathrm{II}_1$ factor $M$ has property $\Gamma$ if and only if for any $m_1, \ldots, m_k$ in $M$ and any $\varepsilon > 0$ there exists a projection $e$ in $M$ of trace $\frac{1}{2}$ such that $\|[e, m_i]\|_2 \leqq \varepsilon$ for all $i \in \{1 \ldots k\}$. (See [7], Theorem 2.1 proof of b$\Rightarrow$a).

**5.18. Proposition.** *The set $\mathscr{G}$ of factors with property $\Gamma$ is closed.*

*Proof.* Let $M \in \overline{\mathscr{G}}$, suppose $m_1, \ldots, m_k$ in $M$ and $\varepsilon > 0$ are given. Choose $\delta > 0$ such that $3\delta + \left(\max_{1 \leqq i \leqq k} \|n_i\|\right)\gamma < \varepsilon$, $\gamma = 2\delta^{1/2}(1 - (2\delta)^{1/2})^{-1}$, find $N$ in $\mathscr{G}$, $n_1, \ldots, n_k$ in $N$ and a projection $e$ in $N$ with $\tau(e) = \frac{1}{2}$ such that

$$\|M - N\|_2 < \delta, \quad \|[n_i, e]\|_2 < \delta, \quad \|m_i - n_i\|_2 < \delta \quad \text{for } i = 1 \ldots k.$$

Let $h$ be a selfadjoint operator in $M$ of norm less than one such that $\|h-e\|_2<\delta$ then $0\leqq h^2\leqq I$ and $\|h^2-e\|_2\leqq\|h(h-e)\|_2+\|(h-e)e\|_2<2\delta$. By Lemma 2.1 there exists a spectral projection $g$ for $h^2$ such that $\|g-e\|_2\leqq(2\delta)^{1/2}(1-(2\delta)^{1/2})^{-1}\cdot 2^{-1/2}$. By 5.16 we know that $M$ is continuous, hence we can find a projection $f$ in $M$ such that $\tau(f)=\frac{1}{2}$ and $\|f-g\|_2\leqq|\frac{1}{2}-\tau(g)|^{1/2}\leqq\|e-g\|_2$. The $f$ is then close to $e$ in fact

$$\|f-e\|_2\leqq\|f-g\|_2+\|e-g\|_2\leqq 2\|e-g\|\leqq\gamma.$$

Finally we obtain

$$\|[f,m_i]\|_2\leqq 2\|m_i-n_i\|_2+2\|f-e\|_2\|n_i\|_\infty+\|[e,n_i]\|_2\leqq\varepsilon,$$

and $M$ has property $\Gamma$.

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