Math. Ann. 235, 247—265 (1978)

On Deformations of Compactifiable Complex Manifolds

Yujiro Kawamata
Lehrstuhl für Mathematik VI, D-6800 Mannheim, Federal Republic of Germany

0. Introduction

In [11], Kodaira and Spencer developed the theory of deformations of compact manifolds. After that, Kuranishi proved the existence of a semi-universal family of deformations for any compact complex manifold [12] or [3]. Generally speaking, one presumes strongly that the theory for compact complex manifolds can be extended to a rather wide class of non-compact complex manifolds, namely, complex manifolds having a compactification. For example of such extensions see [2] or [7]. In this paper, we shall define the notion of compactifiable complex manifolds and develop the theory of their deformations.

In Section 1, we shall make some definitions for compactifiable complex manifolds. In Section 2 we shall prove a theorem of Kuranishi type for logarithmic deformations (see Definition 3), that is, deformations for a fixed compactification of given compactifiable complex manifold. We shall prove the existence of a semi-universal family of deformations for any fixed compactification of given compactifiable complex manifold (Theorem 1). As corollaries we shall prove theorems of Kodaira-Spencer type concerning the cohomology groups $H^i(\bar{X}, T_{\bar{X}}(\log \bar{D}))$, where $T_{\bar{X}}(\log \bar{D})$ is the sheaf of holomorphic tangent vectors with logarithmic zeroes along $\bar{D}$ (see Definition 4), which will play an essential role in our theory. We shall replace the T_X of the compact case by $T_{\bar{X}}(\log \bar{D})$.

In Section 3, we shall study the effect of changing the compactification of a given compactifiable complex manifold. We shall prove a "going down" theorem for logarithmic deformations (Theorem 3) and other related propositions.

In Section 4, we shall give some examples, which will illustrate deformations of compactifiable complex manifolds a concrete form. Though the theory of deformations of compactifiable complex manifolds is very similar to that of compact complex manifolds, some new phenomena will occur: 1) The parameter space of deformations may be infinite-dimensional. This is due to the absence of the minimal compactification. The latter is the main difficulty when we consider compactifiable complex manifolds of dimension greater than two. 2) The technique developed to extend the theory for compact complex manifolds to that for

0025-5831/78/0235/0247/$03.80

compactifiable ones can also be applied to the study of a singular variety S embedded in a non-singular variety A when we consider the complement $A-S$ of S in A. This observation provides a new standpoint for the study of equi-singularity. We shall develop the theory of equi-singular deformations of isolated singularity in a subsequent paper.

We note also that $H^1(\bar{X}, T_{\bar{X}}(\log \bar{D}))$ may be non-zero, even if $H^1(X, T_X)$ vanishes. For example, in case X is an affine manifold, we obtain a non-trivial deformation theory.

In the Appendix, we shall extend the theory of Albanese maps for compact complex manifolds to the theory of quasi Albanese maps for compactifiable ones, where the notion of meromorphic structures defined in Section 1 (Definition 2) is indispensable.

The author would like to express his heartly thanks to Professor S. Iitaka for his valuable advise and encouragement and to Professor M. Reid who kindly read and corrected the manuscript.

1. Definitions

Definition 1. Let X be a complex manifold and D a closed analytic subset of X. D is called an analytic subset of *simple normal crossing* if the following conditions are satisfied:

1) $D=\bigcup_{i=1}^{h} D_i$, where the $D_i (1 \leqq i \leqq h)$ are complex submanifolds of X.

2) For each $p \in X$, there exist a neighborhood U of p and a system of local coordinates $\{z_1, \ldots, z_n\}$ on U such that $D_i = \{z_{r_{i-1}} = \ldots = z_{r_i} = 0\}$ for $1 \leqq i \leqq h$, where the r_i are integers such that $-1 \leqq r_i \leqq n$ and $r_i \leqq r_j$ if $i \leqq j$, and we understand that $z_0 = 1$. Such a pair $(U; \{z_1, \ldots, z_n\})$ will be called a *logarithmic coordinate system.*

Definition 2. Let X be a complex manifold. A *non-singular compactification* $\bar{X}$ of X is a compact complex manifold such that $\bar{D} = \bar{X} - X$ is a closed analytic subset of simple normal crossing in X.

A *meromorphic structure* $\mathfrak{m}$ on a complex manifold X is a bimeromorphic equivalence class of non-singular compactifications of X. (Note that a complex manifold admits in general several meromorphic structures, see Appendix.) A *compactifiable complex manifold* $(X, \mathfrak{m})$ is a pair of a complex manifold X and a meromorphic structure $\mathfrak{m}$ on X. A *morphism* $f:(X, \mathfrak{m}) \to (X', \mathfrak{m}')$ of compactifiable complex manifolds is a morphism of the underlying complex manifolds which is compatible with the meromorphic structures, i.e., for a non-singular compactification $\bar{X}$ (resp. $\bar{X}'$) of X (resp. X') which belongs to $\mathfrak{m}$ (resp. $\mathfrak{m}'$), f induces a meromorphic map from $\bar{X}$ to $\bar{X}'$. We sometimes write simply X instead of $(X, \mathfrak{m})$ if there is no danger of confusion.

A *non-singular triple* $(X, \bar{X}, \bar{D})$ is a triple consisting of a complex manifold X, a non-singular compactification $\bar{X}$ of X and a closed analytic subset $\bar{D} = \bar{X} - X$ of simple normal crossing. For a non-singular triple $(X, \bar{X}, \bar{D})$ X is considered as a compactifiable complex manifold with the meromorphic structure $\mathfrak{m}$ defined by $\bar{X}$. We say that the non-singular triple belongs to $(X, \mathfrak{m})$.

By Hironaka [6] and Nagata [13], for a non-singular algebraic variety V defined over the complex number field $\boldsymbol{C}$, the corresponding complex manifold

$V_{an}=X$ becomes a compactifiable complex manifold with the meromorphic structure induced by the algebraic structure of V. We always assume that such an X is endowed with this meromorphic structure.

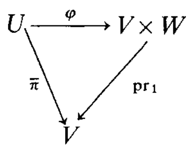
Definition 3. A *family of logarithmic deformations* of a non-singular triple $(X,\bar{X},\bar{D})$ is a 7-tuple $\mathscr{F}=(\mathscr{X},\bar{\mathscr{X}},\bar{\mathscr{D}},\bar{\pi},S,s_0,\bar{\psi})$ satisfying the following conditions:

1) $\bar{\pi}:\bar{\mathscr{X}}\to S$ is a proper smooth morphism of complex spaces $\bar{\mathscr{X}}$ and S.

2) $\bar{\mathscr{D}}$ is a closed analytic subset of $\bar{\mathscr{X}}$ and $\mathscr{X}=\bar{\mathscr{X}}-\bar{\mathscr{D}}$.

3) $\bar{\psi}:\bar{X}\to\bar{\pi}^{-1}(s_0)$ is an isomorphism such that $\psi(X)=\bar{\pi}^{-1}(s_0)\cap\mathscr{X}$.

4) $\bar{\pi}$ is locally a projection of a product space as well as the restriction of it to $\bar{\mathscr{D}}$, that is, for each $p\in\bar{\mathscr{X}}$ there exist an open neighborhood U of p and an isomorphism $\varphi:U\to V\times W$, where $V=\bar{\pi}(U)$ and $W=U\cap\bar{\pi}^{-1}(\bar{\pi}(p))$, such that the following diagram



is commutative and $\varphi(U\cap\bar{\mathscr{D}})=V\times(W\cap\bar{\mathscr{D}})$.

Remark. If $\mathscr{F}=(\mathscr{X},\bar{\mathscr{X}},\bar{\mathscr{D}},\bar{\pi},S,s_0,\bar{\psi})$ is a family of logarithmic deformations, then it is clear that $\mathscr{F}_i=(\mathscr{D}_i,\bar{\mathscr{D}}_i,\bar{\mathscr{E}}_i,\bar{\pi}_i,S,s_0,\bar{\psi}_i)$ is again a family of logarithmic deformations, where $\mathscr{D}_i=\bar{\mathscr{D}}_i-\bar{\mathscr{E}}_i$, $\bar{\mathscr{E}}_i=\bar{\mathscr{D}}_i\cap(\bar{\mathscr{D}}-\bar{\mathscr{D}}_i)$, $\bar{\pi}_i=\bar{\pi}|_{\bar{\mathscr{D}}_i}$, and $\bar{\psi}_i=\bar{\psi}|_{\bar{D}_i}$ for all i. This remark enables us to use inductive arguments on the dimension n of X.

A *family of compactifiable deformations* of a compactifiable complex manifold $(X,\mathfrak{m})$ is a 5-tuple $(\mathscr{X},\pi,S,s_0,\psi)$ obtained from a family of logarithmic deformations of a non-singular triple belonging to $(X,\mathfrak{m})$.

Definition 4. Let X be a complex manifold and D a closed analytic subset of X. The *logarithmic tangent sheaf* $T_X(\log D)$ is the subsheaf of the tangent sheaf T_X of X consisting of derivations of $\mathscr{O}_X$ which send $\mathscr{I}_D$ into itself, where $\mathscr{I}_D$ is the ideal sheaf of D in $\mathscr{O}_X$. (This definition is due to Saito [14].) To simplify the notation, we often write $T(\log D)$ etc. instead of $T_X(\log D)$, if there is no danger of confusion.

Let $(X,\bar{X},\bar{D})$ be a non-singular triple. Then $T_{\bar{X}}(\log\bar{D})$ is the sheaf of infinitesimal automorphisms of $\bar{X}$ which send $\bar{D}$ into itself. By [5], we see that $H^1(\bar{X},T(\log\bar{D}))$ is the set of infinitesimal logarithmic deformations, that is, families of logarithmic deformations over the space $\operatorname{Spec}\boldsymbol{C}[x]/(x^2)$ and that $H^2(\bar{X},T(\log\bar{D}))$ is the set of obstructions. In a usual way we have a Kodaira-Spencer map

$$\varrho_{s_0}:T_{S,s_0}\to H^1(\bar{X},T(\log\bar{D}))\,.$$

Proposition 1. *The following sequences are exact*

(1) $0\to T_X(-D)\to T_X(\log D)\to T_D\to 0$,

where T_D *is the sheaf of derivations of* $\mathscr{O}_D$.

(2) $0\to T_X(\log D)\to T_X\to N_D\to 0$,

where $N_D=\operatorname{coker}(T_D\to T_X\otimes_{\mathscr{O}_X}\mathscr{O}_D)$.

Proof. (1) is clear from the definition. (2) follows from the following diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & T_X(-D) & \longrightarrow & T_X(-D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T_X(\log D) & \longrightarrow & T_X & \longrightarrow & N_D \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T_D & \longrightarrow & T_X \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & N_D \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with the aid of the "9-lemma". Q.E.D.

In this paper we are mainly concerned with the case where D is of simple normal crossing. In this case, we easily see that $N_D = \bigoplus_{i=1}^{h} N_{D_i}$, where the N_{D_i} are the normal sheaves of the D_i in X. We also see that the sequence (1) corresponds to the Remark above.

2. A Theorem of Kuranishi Type

Definition 5. A family $\mathscr{F} = (\mathscr{X}, \bar{\mathscr{X}}, \bar{\mathscr{D}}, \bar{\pi}, S, s_0, \bar{\psi})$ of logarithmic deformations of a non-singular triple $(X, \bar{X}, \bar{D})$ is *semi-universal* if for any family $\mathscr{F}' = (\mathscr{X}', \bar{\mathscr{X}}', \bar{\mathscr{D}}', \bar{\pi}', S', s'_0, \bar{\psi}')$ of logarithmic deformations of $(X, \bar{X}, \bar{D})$ there exist an open neighborhood S'' of s'_0 in S' and a morphism $\alpha: S'' \to S$ such that the following conditions are satisfied:

1) The restriction $\mathscr{F}'|_{S''}$ of $\mathscr{F}'$ over S'' is isomorphic to the induced family $\alpha^* \mathscr{F}$.
2) For any S'_0 and α_0 satisfying the same condition as in 1), the induced tangential maps T_α and T_{α_0} from T_{S', s'_0} to T_{S, s_0} coincide.

Theorem 1. *For any non-singular triple $(X, \bar{X}, \bar{D})$ there exists a semi-universal family of logarithmic deformations of it.*

Proof. We need the following proposition ([4], § 3 or [7], III, Theorem 9.1).

Proposition 2. *Let $\bar{X}_2$ be a compact complex manifold and C a closed submanifold of it. Let $\bar{X}_1$ be the monoidal transform of $\bar{X}_2$ with center C and let D be the total transform of C. Put $X_1 = \bar{X}_1 - D$ and $X_2 = \bar{X}_2 - C$. Then for any family $\mathscr{F}_1 = (\mathscr{X}_1, \bar{\mathscr{X}}_1, \mathscr{D}, \bar{\pi}_1, S, s_0, \bar{\psi}_1)$ of logarithmic deformations of $(X_1, \bar{X}_1, D)$, there exist an open neighborhood S' of s_0 in S and a family $\mathscr{F}_2 = (\mathscr{X}_2, \bar{\mathscr{X}}_2, \mathscr{C}, \bar{\pi}_2, S', s_0, \bar{\psi}_2)$ of logarithmic deformations of $(X_2, \bar{X}_2, C)$ such that $\bar{\mathscr{X}}_1|_{S'}$ is a monoidal transform of $\bar{\mathscr{X}}_2$ with center $\mathscr{C}$ and $\mathscr{D}|_{S'}$ is the total transform of $\mathscr{C}$.*

By this proposition, we may assume that $\bar{D}$ is a divisor on $\bar{X}$. Then $T_{\bar{X}}(\log \bar{D})$ is a locally free sheaf on a compact complex manifold and the harmonic integral theory is available. Let M and N be the (union of) the underlying real analytic manifolds of $\bar{X}$ and $\bar{D}$, respectively. Let S be a complex space having a base point s_0 and let ψ be a (small) family of complex structures on M parametrized by S near the one defined by $\bar{X}$. We represent ψ by a $T_{\bar{X}}$-valued real analytic (0, 1)-differential form ω on $\bar{X}$ parametrized by S (see [3]). Then,

Lemma 1. *$N \times S$ becomes a closed analytic subset of $M \times S$ with respect to the complex structure given by ω if and only if ω is logarithmic along $\bar{D}$, that is, ω is a $T_{\bar{X}}(\log \bar{D})$-valued real analytic (0, 1)-form on $\bar{X}$.*

Proof. (only if); Let p be an arbitrary point of M and let R be a minimal equation of $\bar{D}$ around p. If $N \times S$ is analytic, then there exists a real analytic function ε around p parametrized by S such that $\varepsilon(p) \neq 0$ and $(L-\omega(L))(R\varepsilon)=0$ for any $L \in \bar{T}_{\bar{X},p}$. Since $L(R)=0$, this is equivalent to $\omega(L)(R)=\varepsilon^{-1}(L(\varepsilon)-\omega(L)(\varepsilon))R$, which proves the "only if" part.

(if); If ω is logarithmic along $\bar{D}$, the pull back η of ω to $\bar{D}$ is a $T_{\bar{D}}$-valued (0, 1)-form on $\bar{D}$, by Proposition 1 (1). Clearly $d''\eta-[\eta,\eta]=0$. Hence η defines a family of complex structures on N parametrized by S. Now, we have only to prove that the injection $N \times S \to M \times S$ is holomorphic with respect to the above defined complex structures. Let f be a holomorphic function on $M \times S$ around a point p. Then for any $L \in \bar{T}_{\bar{X},p}$, $(L-\omega(L))(f)=0$, and for any $T \in \bar{T}_S$, $T(f)=0$. Therefore, for $L \in \bar{T}_{\bar{D},p}$, $(L-\eta(L))(f)=0$, that is, the pull back of f is holomorphic on $N \times S$. Q.E.D.

As in [3], the family $\mathscr{F}$ is obtained from the subspace of $\Gamma_{\text{real analytic}}(\bar{X}, T(\log \bar{D}) \otimes \bar{T}^*)$ defined by the equations $d'\delta'\omega=0$ and $d''\omega-[\omega,\omega]=0$, where $d=d'+d''$ is the decomposition of the exterior differential operator into the operators d' and d'' of type (1,0) and (0,1), respectively, and δ' is the adjoint operator of d'' with respect to some hermitian metrics on $T(\log \bar{D})$ and $\bar{X}$. The rest of the proof of the theorem is exactly the same as [3], and we omit it. Q.E.D.

3. Changing Compactifications

Definition 6. Let X be a complex manifold and D a closed analytic subset of simple normal crossing. An *admissible center* C on X with respect to D is a closed submanifold of X of codimension at least two contained in D satisfying the following conditions: For each $p \in C$, there is an open neighborhood U of p in X and a closed analytic subset D' of simple normal crossing in U such that, if $D'=\bigcup_{i=1}^{k} D'_i$ is the irreducible decomposition of D', then $D \cap U=\bigcup_{i=1}^{h} D'_i$ and $C \cap U=\bigcap_{i=m}^{k} D'_i$, for some h and m $(1 \leqq h, m \leqq k)$, Moreover, if in addition that $h=k$, C is called a *canonical center.*

Theorem 2. *Let X be a complex manifold, D a closed analytic subset of simple normal crossing and C an admissible center on X with respect to D. Let $X^{\#}$ be the monoidal transform of X with center C and $f: X^{\#} \to X$ the natural morphism. Let*

$D^{\#}=\mathrm{red}(f^{-1}(D\cup C))$. Then, $D^{\#}$ is a closed analytic subset of simple normal crossing in $X^{\#}$ and

$$\boldsymbol{R}f_{*}T_{X^{\#}}(\log D^{\#})=T_X(\log D)(\log C),$$

where the right hand side of the equality is the intersection of $T_X(\log D)$ and $T_X(\log C)$ in T_X.

Proof. The first statement is trivial. Since the required equality is local, we may assume that $X=U=\boldsymbol{D}^n$ (polydisc) with the coordinate system $\{z_1,\dots,z_n\}$. Put $H_\alpha=\{z_\alpha=0\}\subset X$. Then we may assume that $D=\bigcup_{i=1}^{h} D_i$, $D_i=\bigcap_{\alpha\in M_i} H_\alpha$ and $C=\bigcap_{i=1}^{r} H_\alpha$ where the $M_i(1\leqq i\leqq h)$ are disjoint subsets of $\{1,\dots,n\}$. Denote by H'_α the strict transform of H_α, by E the total transform of C, and set $H^{\#}=\bigcup_{\alpha=1}^{n} H'_\alpha\cup E$. Then the strict transform D'_i of D_i is equals to $\bigcap_{\alpha\in M_i} H'_\alpha$ for any $1\leqq i\leqq h$. (We may assume that $D_i\neq C$.)

First, we see that $T_{X^{\#}}(\log H^{\#})=f^{*}T_X(\log H)$ and hence $\boldsymbol{R}f_{*}T_{X^{\#}}(\log H^{\#})=T_X(\log H)\otimes_{\mathcal{O}_X}\boldsymbol{R}f_{*}\mathcal{O}_{X^{\#}}=T_X(\log H)$.

On the other hand, $f_{*}T_{X^{\#}}(\log D^{\#})=T_X(\log D)(\log C)$ is obvious. We shall show that the higher direct images vanish. We have an exact sequence

$$\begin{aligned}&0\to T_{X^{\#}}(\log H^{\#})\to T_{X^{\#}}(\log D^{\#})\\ &\to\bigoplus_{i=1}^{h}\bigoplus_{\alpha\in M_i} N_{H'_\alpha}(-D'_i)\bigoplus_{\alpha\notin M_i} N_{H'_\alpha}\longrightarrow 0.\end{aligned}$$

Since $R^pf_{*}N_{H'_\alpha}(-D'_i)=R^pf_{*}N_{H'_\alpha}=0$ for $p>0$, we have $R^pf_{*}T_{X^{\#}}(\log D^{\#})=0$ for $p>0$. Q.E.D.

In particular, if C is a canonical center with respect to D, we have

$$\boldsymbol{R}f_{*}T_{X^{\#}}(\log D^{\#})=T_X(\log D).$$

From this and Theorem 1 (and the proof of it), we deduce the following Corollaries in a usual way: We fix our non-singular triple $(X,\bar{X},\bar{D})$. Then

Corollary 1. *Let $\mathcal{F}=(\mathcal{X},\bar{\mathcal{X}},\bar{\mathcal{D}},\bar{\pi},S,s_0,\bar{\psi})$ be a family of logarithmic deformations of $(X,\bar{X},\bar{D})$. If the Kodaira-Spencer map $\varrho: T_{S,s_0}\to H^1(\bar{X},T(\log\bar{D}))$ is surjective and S is regular, then $\mathcal{F}$ is a versal family.*

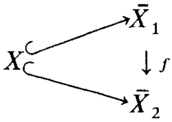
Corollary 2. *If $H^1(\bar{X},T(\log\bar{D}))=0$, then $(X,\bar{X},\bar{D})$ is rigid, i.e., every family of logarithmic deformations of $(X,\bar{X},\bar{D})$ is isomorphic to the product family near s_0.*

Corollary 3. *Let $\mathcal{F}=(\mathcal{X},\bar{\mathcal{X}},\bar{\mathcal{D}},\bar{\pi},S,s_0,\bar{\psi})$ be a family of logarithmic deformations of $(X,\bar{X},\bar{D})$. Assume that $\dim H^1(\bar{X}_t,T_t(\log\bar{D}_t))$ is constant and the Kodaira-Spencer map $\varrho_t: T_{S,t}\to H^1(\bar{X}_t,T_t(\log\bar{D}_t))$ is zero for each $t\in S$, where $\bar{X}_t=\bar{\pi}^{-1}(t)$ and $\bar{D}_t=\bar{\mathcal{D}}\cap\bar{X}_t$. Then, $\mathcal{F}$ is isomorphic to the product family near s_0.*

Corollary 4. *If $H^2(\bar{X},T(\log\bar{D}))=0$, then for any semi-universal family $\mathcal{F}=(\mathcal{X},\bar{\mathcal{X}},\bar{\mathcal{D}},\bar{\pi},S,s_0,\bar{\psi})$ of logarithmic deformations of $(X,\bar{X},\bar{D})$, S is regular at s_0.*

Proposition 3. *Let X be a complex manifold and let $(X,\bar{X}_1,\bar{D}_1)$ and $(X,\bar{X}_2,\bar{D}_2)$ be two non-singular compactifications of X. Assume that there is a morphism $f:\bar{X}_1\to\bar{X}_2$*

such that the following diagram



is commutative. (Note that $f(\bar{D}_1)=\bar{D}_2$*.) Then, there exists a functorial linear map*

$$f_*: H^i(\bar{X}_1, T_1(\log \bar{D}_1)) \to H^i(\bar{X}_2, T_2(\log \bar{D}_2))$$

for each $i \geqq 0$.

Proof. By Theorem 2, we may assume that $\bar{D}_1$ and $\bar{D}_2$ are divisors on $\bar{X}_1$ and $\bar{X}_2$, respectively. Then, by Serre duality, $H^i(\bar{X}_\varepsilon, T_\varepsilon(\log \bar{D}_\varepsilon))$ is the dual module of

$$H^{n-i}(\bar{X}_\varepsilon, \Omega^1_\varepsilon(\log \bar{D}_\varepsilon) \otimes_{\mathcal{O}_X} \Omega^n_\varepsilon)),$$

where $\varepsilon = 1$ or 2, $n = \dim X$ and $\Omega^1_\varepsilon(\log \bar{D}_\varepsilon)$ is the dual sheaf of $T_\varepsilon(\log \bar{D}_\varepsilon)$, i.e., the sheaf of holomorphic 1-forms with logarithmic poles along $\bar{D}_\varepsilon$. By the pull-back morphisms

$$\Omega^1_2(\log \bar{D}_2) \to f_* \Omega^1_1(\log \bar{D}_1) \quad \text{and} \quad \Omega^1_2 \to f_* \Omega^1_1,$$

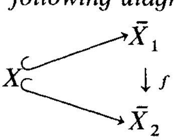
we have a functorial map

$$f^*: H^{n-i}(\bar{X}_2, \Omega^1_2(\log \bar{D}_2) \otimes \Omega^n_2)) \to H^{n-i}(\bar{X}_1, \Omega^1_1(\log \bar{D}_1) \otimes \Omega^n_1)).$$

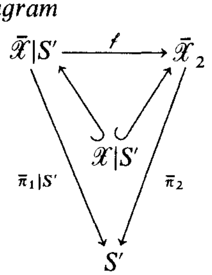
The desired f_* is obtained as the adjoint of f^*. Q.E.D.

Remark. f_* is not necessarily either injective or surjective.

Theorem 3. *Let* X *be a complex manifold,* $(X, \bar{X}_1, \bar{D}_1)$ *and* $(X, \bar{X}_2, \bar{D}_2)$ *two non-singular compactifications of* X, *and* $(\mathscr{X}, \bar{\mathscr{X}}_1, \bar{\mathscr{D}}_1, \bar{\pi}_1, S, s_0, \bar{\psi}_1)$ *a family of logarithmic deformations of* $(X, \bar{X}_1, \bar{D}_1)$. *We assume that there is a morphism* $f: \bar{X}_1 \to \bar{X}_2$ *such that the following diagram*

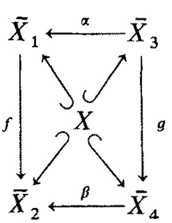


is commutative. Then, there exist a neighborhood S' *of* s_0 *in* S *and a family of logarithmic deformations* $(\mathscr{X}|_{S'}, \bar{\mathscr{X}}_2, \bar{\mathscr{D}}_2, \bar{\pi}_2, S', s_0, \bar{\psi}_2)$ *of* $(X, \bar{X}_2, \bar{D}_2)$ *satisfying the following conditions: There is a morphism* $\not{f}: \bar{\mathscr{X}}_1|S' \to \bar{\mathscr{X}}_2$ *such that the following diagram*



is commutative and $\not{f}|\bar{X}_1 = f$.

Proof. We first prove that we may assume that $\bar{D}_1$ (resp. $\bar{D}_2$) is a divisor on $\bar{X}_1$ (resp. $\bar{X}_2$). By successive monoidal transformations with canonical centers, we obtain $(X, \bar{X}_4, \bar{D}_4)$ from $(X, \bar{X}_2, \bar{D}_2)$ such that $\bar{D}_4$ is a divisor on $\bar{X}_4$. Then by successive monoidal transformations with canonical centers, we can construct $(X, \bar{X}_3, \bar{D}_3)$ from $(X, \bar{X}_1, \bar{D}_1)$ satisfying the following conditions: $\bar{D}_3$ is a divisor on $\bar{X}_3$ and there exists a morphism $g: \bar{X}_3 \to \bar{X}_4$ such that the following diagram



is commutative, where α and β are natural morphisms. Blowing up $(\mathscr{X}, \bar{\mathscr{X}}_1, \bar{\mathscr{D}}_1, \bar{\pi}_1, S, s_0, \bar{\psi}_1)$ corresponding to the blowing ups of $(X, \bar{X}_1, \bar{D}_1)$, we obtain a family of logarithmic deformations $(\mathscr{X}, \bar{\mathscr{X}}_3, \bar{\mathscr{D}}_3, \bar{\pi}_3, S, s_0, \bar{\psi}_3)$ of $(X, \bar{X}_3, \bar{D}_3)$. By assumption, we get a family of logarithmic deformations $(\mathscr{X}, \bar{\mathscr{X}}_4, \bar{\mathscr{D}}_4, S', s_0, \bar{\psi}_4)(X, \bar{X}_4, \bar{D}_4)$. Finally by Proposition 2, we obtain a family of logarithmic deformations $(\mathscr{X}, \bar{\mathscr{X}}_2, \bar{\mathscr{D}}_2, \bar{\pi}_2, S'', s_0, \bar{\psi}_2)$ of $(X, \bar{X}_2, \bar{D}_2)$.

Second, we shall prove the theorem under the assumption that $\bar{D}_1$ (resp. $\bar{D}_2$) is a divisor on $\bar{X}_1$ (resp. $\bar{X}_2$). Let $\bar{D}_2 = \bigcup_{i=1}^{h} \bar{D}_{2,i}$ be the irreducible decomposition of $\bar{D}_2$, and $\bar{D}_{1,i}$ the irreducible component of $\bar{D}_1$ such that the restriction of f to $\bar{D}_{1,i}$ induces a bimeromorphic morphism to $\bar{D}_{2,i}$. By Theorem 8.1 of [7], we obtain a family of deformations $\bar{\mathscr{X}}_2 \to S'$ (resp. $\bar{\mathscr{D}}_{2,i} \to S'$) of $\bar{X}_2$ (resp. $\bar{D}_{2,i}$) and a holomorphic map $\not{f}: \bar{\mathscr{X}}_1 \to \bar{\mathscr{X}}_2$ (resp. $\not{f}: \bar{\mathscr{D}}_{1,i} \to \bar{\mathscr{D}}_{2,i}$) over S' extending f, for a small neighborhood S' of s_0 in S. Moreover, by applying Proposition 7.3 of [7] to the following diagram

$$\begin{array}{ccc} \bar{\mathscr{D}}_{1,i} & \subset & \bar{\mathscr{X}}_1 \\ {\scriptstyle \not{f}_i}\downarrow & & \downarrow{\scriptstyle \not{f}} \\ \bar{\mathscr{D}}_{2,i} & & \bar{\mathscr{X}}_2 \end{array},$$

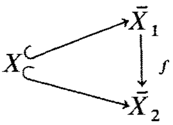
using Lemma 7.5 of [7], we find a holomorphic map $\varphi_i: \bar{\mathscr{D}}_{2,i} \to \bar{\mathscr{X}}_2$ over S' extending the inclusion map $\bar{D}_{2,i} \subset \bar{X}_2$ and making the above diagram commutative. It is easy to check that the φ_i are in fact inclusion maps for divisors of simple normal crossings and that the restriction of $\not{f}$ to the preimage of the complement of $\bar{\mathscr{D}}_2 = \bigcup_{i \in I} \bar{\mathscr{D}}_{2,i}$ in $\bar{\mathscr{X}}_2$ is an isomorphism. Q.E.D.

Remark. The morphism $\not{f}$ corresponds to the homomorphism f_* in Proposition 3. That is, the following diagram

$$\begin{array}{ccc} T_{S,s_0} & \xrightarrow{\varrho_1} & H^1(\bar{X}_1, T_1(\log \bar{D}_1)) \\ \| & & \downarrow{\scriptstyle f_*} \\ T_{S,s_0} & \xrightarrow{\varrho_2} & H^1(\bar{X}_2, T_2(\log \bar{D}_2)) \end{array}$$

is commutative, where ϱ_1 (resp. ϱ_2) is the Kodaira-Spencer map of $\bar{\mathscr{X}}_1$ (resp. $\bar{\mathscr{X}}_2$) at s_0.

Let $(X,\bar{X},\bar{D})$ be a non-singular triple, where $\bar{D}$ is a divisor on $\bar{X}$. We regard the inclusion map $i:X\subset\bar{X}$ as a special case of a toroidal embedding (cf. [10]). There corresponds a complex of simplicial cones denoted by $\Delta(X,\bar{X},\bar{D})$ or simply by $\Delta(\bar{X})$, if there is no confusion. Let $(X,\bar{X}_i,\bar{D}_i)$ $(i=1,2)$ be two non-singular compactifications of X such that the $\bar{D}_i$ are divisors on $\bar{X}_i$, respectively, and $f:\bar{X}_1\to\bar{X}_2$ a morphism such that the following diagram



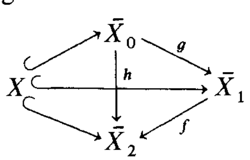
(2)

is commutative. f is called allowable if it is a morphism in the category of toroidal embeddings ([10], p. 87). In this case, $\Delta(\bar{X}_1)$ is a simplicial sub-division of $\Delta(\bar{X}_2)$. For example, a monoidal transformation with a canonical center is allowable and corresponds to a barycentric subdivision. As remarked by Iitaka, if f is allowable, then the logarithmic ramification divisor $\bar{R}_f$ vanishes.

Proposition 4. *In the situation described above, if f is allowable, then we have*

$$\boldsymbol{R}f_*T_1(\log\bar{D}_1)\cong T_2(\log\bar{D}_2)\,.$$

Proof. Since $\Delta(\bar{X}_1)$ is a simplicial subdivision of $\Delta(\bar{X}_2)$, successive barycentric subdivisions of $\Delta(\bar{X}_2)$ yields a simplicial complex Δ_0 which is a simplicial subdivision of $\Delta(\bar{X}_1)$. Corresponding to the subdivision Δ_0 of $\Delta(\bar{X}_2)$, we have a non-singular triple $(X,\bar{X}_0,\bar{D}_0)$ which is obtained by successive monoidal transformations with canonical centers from $(X,\bar{X}_2,\bar{D}_2)$. We have a commutative diagram



On the other hand, we have a morphism $f^*(T_2(\log\bar{D}_2))\to(T_1(\log\bar{D}_1))$. Indeed, for $v\in\Gamma(U,T_2(\log\bar{D}_2))$ we have $h^*v\in\Gamma(h^{-1}(U),T_0(\log\bar{D}_0))$, and hence $f^*v\in\Gamma(f^{-1}(U),T_1(\log\bar{D}_1))$. Therefore, we have a natural homomorphism

$$\varphi_{12}:T_2(\log\bar{D}_2))\to\boldsymbol{R}f_*T_1(\log\bar{D}_1))\,.$$

By Theorem 2, we have $\boldsymbol{R}h_*T_0(\log\bar{D}_0)\cong T_2(\log\bar{D}_2)$, that is, $\boldsymbol{R}f_*(\varphi_{01})\circ\varphi_{12}$ is an isomorphism. Hence, φ_{12} is injective and $\boldsymbol{R}f_*(\varphi_{01})$ is surjective. Then $\boldsymbol{R}f_*(\varphi_{01})$ is injective, and finally, φ_{12} is an isomorphism. Q.E.D.

Proposition 5. *We assume the same conditions as in Proposition 4. Then, the set of all germs of families of compactifiable deformations of X induced by the families of logarithmic deformations of $(X,\bar{X}_1,\bar{D}_1)$ coincides with that of $(X,\bar{X}_2,\bar{D}_2)$, where a germ means an equivalence class with respect to the restrictions of the base space S.*

Proof. We make use of the arguments and the notation of the proof of Proposition 4. By Theorem 3, to each family of logarithmic deformations of $(X,\bar{X}_1,\bar{D}_1)$, there corresponds a family of logarithmic deformations of $(X,\bar{X}_2,\bar{D}_2)$. On the other hand, for a family of logarithmic deformations of $(X,\bar{X}_2,\bar{D}_2)$, successive monoidal transformations yields a family of logarithmic deformations of $(X,\bar{X}_0,\bar{D}_0)$ and then, by Theorem 3, we have a family of logarithmic deformations of $(X,\bar{X}_1,\bar{D}_1)$. Q.E.D.

4. Examples

In this section, we shall handle some examples and determine the semi-universal family of compactifiable deformations of it. The main tools are the following exact sequence

$$
\begin{aligned}
0 &\to H^0(\bar{X}, T(\log \bar{D})) \to H^0(\bar{X}, T) \to \bigoplus_{i\in I} H^0(\bar{D}_i, N_i)\\
&\to H^1(\bar{X}, T(\log \bar{D})) \to H^1(\bar{X}, T) \to \bigoplus_{i\in I} H^1(\bar{D}_i, N_i)\\
&\to H^2(\bar{X}, T(\log \bar{D})) \to H^2(\bar{X}, T) \to \bigoplus_{i\in I} H^2(\bar{D}_i, N_i)
\end{aligned}
$$

deduced from Proposition 1.

In this section, X is a compactifiable complex manifold which we deform, and $(X,\bar{X},\bar{D})$ is a non-singular compactification of X.

n° 1. $\dim X = 1$.

In this case, there is only one compactification $\bar{X}$. Set $X = \bar{X} - \{p_1, \ldots, p_t\}$, and let $g = g(\bar{X})$ be the genus of $\bar{X}$. We have the following table:

g	t	$\dim H^0$	$\dim H^1$	$\bar{\kappa}(X)$	$\dim H^2$
0	0	3	0	$-\infty$	
	1	2	0	$-\infty$	
	2	1	0	0	
	$t \geq 3$	0	$t-3$	1	
1	0	1	1	0	0
	$t \geq 1$	0	t	1	
$g \geq 2$	t	0	$3g-3+t$	1	

where H^i denotes $H^i(\bar{X}, T(\log \bar{D}))$ and $\bar{\kappa}(X)$ is the logarithmic Kodaira dimension of X (cf. [8]).

n° 2. $\dim X = 2$.

Let $(X,\bar{X}_1,\bar{D}_1)$ be a non-singular triple belonging to $(X, \mathfrak{m})$. We consider the following two types of transformations of it:

i) Blowing up an ordinary double point p of $\bar{D}_1$, and

ii) blowing up a simple point p of $\bar{D}_1$.

Denote by $(X, \bar{X}_0, \bar{D}_0)$ the transform of $(X, \bar{X}_1, \bar{D}_1)$ and by f the natural morphism from $\bar{X}_0$ to $\bar{X}_1$. It is known that any two non-singular triples belonging to $(X, \mathfrak{m})$ are joined by a chain of such transformations. In Case i), f is canonical and by Theorem 2

$$H^i(\bar{X}_0, T_0(\log \bar{D}_0)) \cong H^i(\bar{X}_1, T_1(\log \bar{D}_1)).$$

In Case ii), we have the following exact sequence of sheaves

$$0 \to \boldsymbol{R}f_* T_0(\log \bar{D}_0) \to T_1(\log \bar{D}_1) \to N_{p/\bar{D}_1} \to 0.$$

Therefore, we have the following exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0(\bar{X}_0, T_0(\log \bar{D}_0)) \to H^0(\bar{X}_1, T_1(\log \bar{D}_1)) \to \boldsymbol{C} \\ &\to H^1(\bar{X}_0, T_0(\log \bar{D}_0)) \to H^1(\bar{X}_1, T_1(\log \bar{D}_1)) \to 0, \end{aligned}$$

and $H^2(\bar{X}_0, T_0(\log \bar{D}_0)) \xrightarrow{\sim} H^2(\bar{X}_1, T_1(\log \bar{D}_1))$.

We note that $\chi(T_0(\log \bar{D}_0)) = \chi(T_1(\log \bar{D}_1)) - 1$, which also follows from the following equality

$$\chi(T_{\bar{X}}(\log \bar{D})) = \frac{c_1^2 + c_2 + 3c_1\bar{c}_1 + 3\bar{c}_1^2 - 6\bar{c}_2}{6},$$

where c_i and $\bar{c}_i$ are the chern classes $c_i(T_{\bar{X}})$ and $c_i(T_{\bar{X}}(\log \bar{D}))$, respectively.

We can show easily that $(X, \mathfrak{m})$ has no minimal compactifications modulo canonical morphisms [i.e., transformations of Type i)], if and only if there exists a non-singular triple $(X, \bar{X}, \bar{D})$ belonging to $(X, \mathfrak{m})$ such that for some irreducible component E of $\bar{D}$,

i) $E = \boldsymbol{P}^1$,

ii) $E^2 = 0$, and

iii) $E \cdot \bar{D}' \leqq 1$, where $\bar{D}' = \bar{D} - E$.

In this case, $(X, \bar{X}, \bar{D})$ admits an infinite chain of blowing downs of Type ii) modulo that of Type i), and hence, $\overline{\lim} \dim H^0(\bar{X}, T(\log \bar{D})) = \infty$, where $(X, \bar{X}, \bar{D})$ varies all non-singular triples belonging to $(X, \mathfrak{m})$. In particular, $\bar{\kappa}(X) = -\infty$ (cf. [8]).

Definition 7. A *quasi-projective plane* is an algebraic variety V which is an open subvariety of the projective plane $\boldsymbol{P}^2$. We identify V with the underlying compactifiable complex manifold $(X, \mathfrak{m})$ (cf. Definition 2).

Proposition 6. *A compactifiable deformation of a quasi-projective plane is again a quasi-projective plane.*

Proof. Let $V = \boldsymbol{P}^2 - C$. By an embedded resolution of C, we obtain a non-singular triple $(X, \bar{X}, \bar{D})$. Now assume that $(X, \bar{X}, \bar{D})$ is deformed to $(X', \bar{X}', \bar{D}')$. Since the intersection numbers are topological invariants, $\bar{D}'$ is contracted to a curve C' on a compact complex surface Q, where $X' = Q - C'$. Also we have $c_1^2(Q) = c_1^2(\boldsymbol{P}^2) = 9$. Hence $Q \cong \boldsymbol{P}^2$. Q.E.D.

We fix our notation: $X = \boldsymbol{P}^2 - C$ and $(X, \bar{X}, \bar{D})$ is the non-singular triple obtained from the minimal embedded resolution of C. (Note that this is not the minimal non-singular compactification of X in Example 1 below.)

Example 1. $C = \{yz^{s-1} = x^s\}$, $s \geqq 2$.

In this case,

$$\dim H^i(\bar{X}, T(\log \bar{D})) = \begin{cases} 1 & \text{for } i=1 \\ s-3 & \text{for } i=2 \\ 0 & \text{otherwise}. \end{cases}$$

The semi-universal family of $(X, \bar{X}, \bar{D})$ is obtained from

$$C_t = \{yz^{s-1} - t_1 x^2 z^{s-2} - \dots - t_{s-3} x^{s-2} z^2 = x^s\},$$

where $t = (t_1, \dots, t_{s-3})$ is the parameter. Note that this family is not necessarily semi-universal as a family of compactifiable deformations, since X has no minimal compactification. But we can see that

$$\dim H^i(\bar{X}_t, T_t(\log \bar{D}_t)) = \begin{cases} 0 & i=0 \\ s-4 & i=1 \\ 0 & \text{otherwise}. \end{cases}$$

Hence, $X \not\cong X_t$ if $t \neq 0$. This shows that the family is semi-universal as a family of compactifiable deformations.

Example 2. $C = \{yz^{s-1} = x^s\} \cup \{x=0\}$, $s \geqq 2$.
In this case,

$$\dim H^i(\bar{X}, T(\log \bar{D})) = \begin{cases} 1 & \text{if } i=0, \\ s-2 & \text{if } i=1, \\ 0 & \text{otherwise}. \end{cases}$$

The semi-universal family of logarithmic deformations of $(X, \bar{X}, \bar{D})$ is obtained from

$$C_t = \{yz^{s-1} - t_1 x^2 z^{s-2} - \dots - t_{s-2} x^{s-1} z = x^s\} \cup \{x=0\}.$$

From this we obtain a semi-universal family of compactifiable deformations of X, since $(X, \bar{X}, \bar{D})$ is the minimal compactification of X. We have also

$$\dim H^i(\bar{X}_t, T_t(\log \bar{D}_t)) = \begin{cases} 0 & \text{if } i=0, \\ s-3 & \text{if } i=1, \\ 0 & \text{otherwise}. \end{cases}$$

Example 3. $C = \{y^q z^{p-q} = x^p\}$, $(p, q) = 1$, $p > 2q$, and $q \neq 1$.
In this case,

$$\dim H^i(\bar{X}, T(\log \bar{D})) = \begin{cases} 1 & \text{if } i=0, \\ s-1 & \text{if } i=1, \\ 0 & \text{otherwise}. \end{cases}$$

The semi-universal family of logarithmic deformations of $(X, \bar{X}, \bar{D})$ is obtained from

$$C_t = \{(yz^{s-1} - t_1 x^2 z^{s-2} - \dots - t_{s-1} x^s)^q z^r = x^p\},$$

where $p=qs+r$, $0<r<q$, and this is also semi-universal in a compactifiable sence. Furthermore,

$$\dim H^i(\bar{X}_t, T_t(\log \bar{D}_t)) = \begin{cases} 0 & \text{if } i=0, \\ s-2 & \text{if } i=1, \\ 0 & \text{otherwise}. \end{cases}$$

Example 4. $x^2y^2+y^2z^2+z^2x^2+2(x^2yz+y^2zx+z^2xy)=0$.
In this case, $H^i(\bar{X}, T(\log \bar{D}))=0$ for all i.

Thus, X is rigid.

Example 5. $C=\{x^3y^2+y^3z^2+z^3x^2=0\}$.
In this case,

$$H^i(\bar{X}, T(\log \bar{D})) = \begin{cases} 0 & \text{if } i=0 \\ 6 & \text{if } i=1, \\ 0 & \text{otherwise}. \end{cases}$$

$$C_t=\{x^3(y+t_1z)^2+y^3(z+t_2x)^2+z^3(x+t_3y)^2=0\}$$

defines a subfamily of logarithmic deformations of $(X, \bar{X}, \bar{D})$.

Let us compute the logarithmic pluri-genera $\bar{p}_g$ and the logarithmic Kodaira dimensions $\bar{\kappa}$ of X (cf. [8]):

Example 1. $\bar{p}_m(X)=0$ for all m,
$\bar{\kappa}(X)=-\infty$.

Example 2. $\bar{p}_m(X)=1$ for all m,
$\bar{\kappa}(X)=0$.

Example 3. $\bar{p}_g(X)$ $(=\bar{p}_1(X))=0$, $\bar{p}_2(X)=1$, $\bar{p}_4(X)=2$, etc.,
$\bar{\kappa}(X)=1$.

Example 4. $\bar{p}_g(X)=0$, $\bar{p}_2(X)=3$,
$\bar{\kappa}(X)=2$.

Example 5. $\bar{p}_g(X)=3$,
$\bar{\kappa}(X)=2$.

If we compute the $\bar{p}_m$ and $\bar{\kappa}$ of small deformations of X obtained in Examples, we find that they are invariant under small deformations. Therefore, we raise the following conjecture:

Conjecture. *The logarithmic pluri-genera $\bar{p}_m$ and the logarithmic Kodaira dimensions $\bar{\kappa}$ of algebraic surfaces are invariant under global deformations*[1].

Here, we understand that the deformation means the compactifiable one, and we note that we have only to prove the invariance under small deformations.

n° 3. $\dim X=3$ $\quad X=\boldsymbol{C}\times\Sigma_m$.

We shall give here only one example for the case that $\dim X=3$. In this case, f_* in Proposition 7 need not be injective nor surjective and the parameter space of the family of compactifiable deformations may be infinite dimensional.

Let Σ_m be the Hirzebruch surface of degree m $(m\geqq 0)$ and $X=\boldsymbol{C}\times\Sigma_m$. X admits a non-singular compactification $\Sigma_{n,m}=\Sigma_n\times_{\boldsymbol{C}^*}\Sigma_m$, where $n\in\boldsymbol{Z}$, and Σ_n is considered

[1] The invariance of $\bar{\kappa}$ under deformation is proved recently by the author

as a fiber bundle with the structure group $\boldsymbol{C}^{\times}$. By a chain of elementary transformations for Σ_n:

$$\begin{array}{ccccccccc} \cdots & & & & Y_n & & & & Y_{n+1} & & & & \cdots \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & \Sigma_{n-1} & & & & \Sigma_n & & & & \Sigma_{n+1} & & \end{array}$$

we obtain a chain of non-singular compactifications of X:

$$\begin{array}{ccccccccc} \cdots & & & & Y_n \times_{\boldsymbol{C}^{\times}} \Sigma_m & & & & Y_{n+1} \times_{\boldsymbol{C}^{\times}} \Sigma_m & & & & \cdots \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & \Sigma_{n-1,m} & & & & \Sigma_{n,m} & & & & \Sigma_{n+1,m} & & \end{array}$$

for which f_*'s in Proposition 7 are not injective nor surjective and $\dim \varinjlim H^1(\bar{X}, T(\log \bar{D})) = \infty$, where the direct limit is taken for that chain with respect to f_*. In fact, X has too many deformations: $\Sigma_{-n,m}$ $(n>0)$ is covered by coordinate neighborhoods

$$\{U_{ij}; (z_i, t_j, \zeta_{ij})\} \quad \text{for} \quad i, j = 1, 2.$$

The relations are

$$\zeta_{11} = z_2^m \zeta_{21} = t_2^n \zeta_{12} = z_2^m t_2^n \zeta_{22}, \qquad z_1 = z_2^{-1}, \quad \text{and} \quad t_1 = t_2^{-1}.$$

For polynomials $P_h(t) = \sum_{l=0}^{n} a_{hl} t^l$, $h = 1, \ldots, m-1$, of degree n in t, we set

$$\begin{cases} \zeta_{11} = z_2^m \zeta_{21} + \sum_{h=1}^{m-1} P_h(t_1) z_2^h = t_2^{-n} \zeta_{12} \\ \quad = t_2^{-n} z_2^m \zeta_{22} + t_2^{-m} \sum_{h=1}^{m-1} t_2^n P_h(t_1) z_2^h, \\ z_1 = z_2^{-1}, \qquad t_1 = t_2^{-1}. \end{cases}$$

Then this defines a family of logarithmic deformations of $(X, \Sigma_{-n,m}, D_{-n})$ which is effective as a family of compactifiable deformations of X. Note that the dimension of the parameter space is $n(m-1)$, which is equals to that of the image of $H^1(\Sigma_{-n,m}, T_{\Sigma_{-n,m}}(\log D_{-n}))$ in $\varinjlim H^1(\bar{X}, T(\log \bar{D}))$, though

$$\dim H^1(\Sigma_{-n,m}, T_{\Sigma_{-n,m}}(\log D_{-n})) = 2mn - 2.$$

n° **4.** Semi-complex tori (see Appendix).

Proposition 8. *A small compactifiable deformation of a semi-complex torus is again a semi-complex torus.*

Proof. Let $(X, \bar{X}_1, \bar{D}_1)$ be the logarithmic deformation inducing the compactifiable deformation in question. Since the deformation is small, $\dim H^q(\bar{X}_1, \Omega_1^p(\log \bar{D}_1))$ is equals to that of the fiber at the origin by [15] and [2]. (Note that a semi-complex torus admits a Kählerian manifold as a non-singular compactification and hence "théorie de Hodge mixed" is available.) Let $\alpha: X \to \mathscr{A}$ be the quasi-Albanese map of

X. We have $\dim X = \dim \mathscr{A}$ by Proposition B of the Appendix. Replacing $\bar{X}_1$ by a non-singular compactification $\bar{X}$ of X dominating $\bar{X}_1$, we have a morphism $\bar{\alpha}: \bar{X} \to \bar{\mathscr{A}} = \mathscr{A} \times_{\mathscr{T}} \boldsymbol{P}^n$, which is an extension of α. Note that

$$H^q(\bar{X}, \Omega^p(\log \bar{D})) \cong H^q(\bar{X}_1, \Omega^p(\log \bar{D}_1)).$$

First, we prove that $\bar{\alpha}$ is surjective. By definition, $\alpha^*: H^1(\mathscr{A}, \boldsymbol{C}) \to H^1(X, \boldsymbol{C})$ is an isomorphism. On the other hand, since the family of logarithmic deformations is topologically trivial, we have $H^*(X, \boldsymbol{C}) = \Lambda^* H^1(X, \boldsymbol{C})$. Hence, if $\bar{\alpha}$ degenerates, we have a contradiction. Set $\bar{D} = \bar{X} - X$ and $\bar{L} = \bar{\mathscr{A}} - \mathscr{A} = \bigcup_{i=0}^{h} L_i$. By the topological triviality of the family of logarithmic deformations, we have $K_{\bar{X}} + \bar{D} \equiv \bar{D}_0$ in $H^2(\bar{X}, \boldsymbol{C})$ for some effective divisor $\bar{D}_0 \subset \bar{D}$. (The left hand side of the equality is equals to $\bar{R} = \bar{R}_\alpha$, the logarithmic ramification divisor of α, by definition.) Hence, $\bar{D}_0 - \bar{R} \sim \alpha^* M$ for some M in $H^1(\bar{\mathscr{A}}, \mathcal{O}) = H^1(A, \mathcal{O})$, where $\sim$ denotes rational equivalence and A is the base space of the $\mathscr{T}$-bundle $\mathscr{A}$. Then $\bar{\alpha}_* \bar{D}_0 - \bar{\alpha}_* \bar{R} \sim dM$, where d is the degree of $\bar{\alpha}$. It is easy to see that $\bar{\alpha}_* \bar{D}_0 \subset \bar{L}$. Taking the intersection with the zero section s, we have $\bar{\alpha}_* \bar{D}_0 \cdot s = 0$ in $H^{2n+2}(\bar{\mathscr{A}}, \boldsymbol{C})$. Hence, $\bar{\alpha}_* \bar{R} \cdot s = 0$ in $H^{2n+2}(\bar{\mathscr{A}}, \boldsymbol{C})$. Both $\bar{D}_0$ and $\bar{R}$ are effective, this shows that $\bar{\alpha}_* \bar{D}_0 \cdot s = \bar{\alpha}_* \bar{R} \cdot s = 0$. Therefore, $M \cdot s = 0$, and $M = 0$. Thus, if $\bar{D}_0 \neq \bar{R}$, then we have $H^0(\bar{X}, K_{\bar{X}} + \bar{D}) \geqq 2$, which is a contradiction. Q.E.D.

By the above proposition, we can construct a local universal family of compactifiable deformations of a semi-complex torus X. That is, ng-dimensional family of deformations of the fiber space structures over g^2-dimensional family of deformations of the base complex tori. Actually, this family is obtained by the logarithmic deformations of the standard compactification $X \subset \bar{X} = X \times_{\mathscr{T}} P^n$. In fact,

$$\dim H^i(\bar{X}, T(\log \bar{D})) = \binom{g}{i}(g+n),$$

and the space S in Theorem 1 is the whole $H^1(\bar{X}, T(\log \bar{D}))$.

Appendix

Quasi-Albanese Maps for Compactifiable Complex Manifolds

Iitaka defined the quasi-Albanese maps for algebraic varieties over the complex number field $\boldsymbol{C}$ which are not necessarily complete (cf. [9]). We shall extend it for compactifiable complex manifolds.

Proposition A. *Let $(X, \bar{X}, \bar{D})$ be a non-singular triple where $\bar{D}$ is a divisor on $\bar{X}$ and $\Omega^p_{\bar{X}}(\log \bar{D})$ the sheaf of p-forms on $\bar{X}$ with logarithmic poles along $\bar{D}$ (cf.* [1] *or* [2]*). Then we have a spectral sequence*

$$E_1^{pq} = H^q(\bar{X}, \Omega^p_{\bar{X}}(\log \bar{D})) \Rightarrow H^n(X, \boldsymbol{C}).$$

Proof. Let $j: X \hookrightarrow \bar{X}$ be the inclusion map. Consider the following diagram

$$j^* \Omega^{\cdot}_{\bar{X}}(\log \bar{D}) \xrightarrow{\alpha} \Omega^{\cdot}_X \xleftarrow{\beta} \boldsymbol{C}_X,$$

where α is an identity and β is a quasi-isomorphism obtained from the Poincaré lemma. By adjunction, we get a morphism

$$\Omega^{\cdot}_{\bar{X}}(\log \bar{D}) \longrightarrow \boldsymbol{R}j_*\boldsymbol{C}_X$$

in the derived category. We can easily check that it is an isomorphism and the proposition is proved. Q.E.D.

From this proposition and the Rieman-Roch theorem we can easily show that $c_n(T_{\bar{X}}(\log \bar{D})) = e(X) =$ the Euler number of X.

Proposition B. *Let $(X, \mathfrak{m})$ be a compactifiable complex manifold, $(X, \bar{X}_i, \bar{D}_i)$ $(i=1,2)$ two non-singular compactifications belonging to $(X, \mathfrak{m})$ where the $\bar{D}_i$ are divisors on $\bar{X}_i$, and $f: \bar{X}_1 \to \bar{X}_2$ a morphism such that the following diagram*

$$\begin{array}{ccc} & & \bar{X}_1 \\ & \nearrow & \\ X & & \downarrow f \\ & \searrow & \\ & & \bar{X}_2 \end{array}$$

is commutative. Then we have

$$\boldsymbol{R}f_*\Omega^p_1(\log \bar{D}_1) = \Omega^p_2(\log \bar{D}_2),$$

where the $\Omega^p_i(\log \bar{D}_i)$ are the sheaves of p-forms with logarithmic poles along the $\bar{D}_i$.

Proof. First, we assume that f is a monoidal transformation with a canonical center. Since $\Omega^p_1(\log \bar{D}_1) = f^*\Omega^p_2(\log \bar{D}_2)$, the assertion holds.

Next, we assume that f is a monoidal transformation with an admissible center C. Fix a point $p \in \bar{X}_2$ and pick a divisor $\bar{D}'_2 = \sum_{j \in J} \bar{D}'_{2,j}$ around p such that C is canonical with respect to $\bar{D}_2 + \bar{D}'_2$. We prove the assertion by induction on $\operatorname{Card} J$ and $\dim X$. Pick $j_0 \in J$ and put $J' = J - \{j_0\}$. Set $\bar{D}''_2 = \bar{D}'_2 - \bar{D}'_{2,j_0}$. Let $\bar{D}'_1$ (resp. $\bar{D}''_1$, and $\bar{D}'_{1,j}$) be the strict transform of $\bar{D}'_2$ (resp. $\bar{D}''_2$, and $\bar{D}'_{2,j}$). Then we have an exact sequence of sheaves on $\bar{X}_1$

$$0 \longrightarrow \Omega^p_1(\log \bar{D}_1) \longrightarrow \Omega^p_1(\log(\bar{D}_1 + \bar{D}'_{1,j_0})) \xrightarrow{\text{res}} \Omega^{p-1}_{j_0}(\log(\bar{D}_1 \cap \bar{D}'_{1,j_0})) \longrightarrow 0, \tag{4}$$

where res denotes the residue map along $\bar{D}'_{1,j_0}$. By the induction hypothesis, we have

$$R^i f_* \Omega^p_1(\log(\bar{D}_1 + \bar{D}'_{1,j_0}) = 0,$$

and

$$R^i f_* \Omega^{p-1}_{j_0}(\log(\bar{D}_1 \cap \bar{D}'_{1,j_0})) = 0, \quad \text{for} \quad i > 0.$$

On the other hand,

$$f_* \Omega^p_1(\log(\bar{D}_1 + \bar{D}'_{1,j_0})) = \Omega^p_2(\log(\bar{D}_2 + \bar{D}'_{2,j_0})),$$

$$f_* \Omega^{p-1}_{j_0}(\log(\bar{D}_1 \cap \bar{D}'_{1,j_0})) = \Omega^{p-1}_{j_0}(\log(\bar{D}_2 \cap \bar{D}'_{2,j_0})),$$

and the map

$$\operatorname{res}: \Omega_2^p(\log(\bar{D}_2+\bar{D}'_{2,j_0})\to\Omega_{j_0}^{p-1}(\log(\bar{D}_2\cap\bar{D}'_{2,j_0}))$$

is clearly surjective. In view of the exact sequence of sheaves on $\bar{X}_2$ induced by the exact sequence (4), we get the desired result.

Finally, we consider the general case. There exists a pull-back homomorphism $f^*\Omega_2^p(\log\bar{D}_2)\to\Omega_1^p(\log\bar{D}_1)$. The rest of the proof is analogous to the arguments given in the proof of Proposition 4. Q.E.D.

Definition A. A *semi-complex torus* of type (g,n) is a compactifiable complex manifold $(X,\mathfrak{m})$ of the following type:

1) The underlying complex manifold X is isomorphic to a quotient space $V/\bar{\Delta}$, where $V\cong \boldsymbol{C}^{\bar{g}}$, $\bar{g}=g+n$ and $\bar{\Delta}=\boldsymbol{Z}^{2g+n}$ is a subgroup of V generating a $(2g+n)$-dimensional $\boldsymbol{R}$-subspace (resp. the whole V) over $\boldsymbol{R}$ (resp. over $\boldsymbol{C}$).

2) There exists a submodule Δ_0 of $\bar{\Delta}$ of rank n generating an n-dimensional $\boldsymbol{C}$-linear subspace H of V over $\boldsymbol{C}$ and the image Δ of $\bar{\Delta}$ in V/H generates the whole V/H over $\boldsymbol{R}$. By this, X becomes the total space of a $\mathcal{T}$-bundle over a complex torus $A=V/(H+\bar{\Delta})$, where $\mathcal{T}=(\boldsymbol{C}^\times)^n$. Then, $\mathfrak{m}$ is the meromorphic structure on X determined by the non-singular compactification $\bar{X}=X\times_{\mathcal{T}}\boldsymbol{P}^n$.

Note that a semi-complex torus has a Lie group structure induced by that of V.

Remark. Such a complex manifold X admits several meromorphic structures. For example, let $\bar{\Delta}$ be a discrete subgroup of $\boldsymbol{C}^2$ generated by e_1, e_2 and $\alpha e_1+\beta e_2$, where $\{e_1,e_2\}$ is a $\boldsymbol{C}$-basis of $\boldsymbol{C}^2$, and α, β are general complex numbers. Then, two submodules $\Delta_0^1=\langle e_1\rangle$, and $\Delta_0^2=\langle e_2\rangle$ of $\bar{\Delta}$ determine distinct meromorphic structures on $\boldsymbol{C}^2/\bar{\Delta}$.

Let X be a $\mathcal{T}$-bundle over a complex torus A. Then X admits a natural non-singular compactification $\bar{X}=X\times_{\mathcal{T}}\mathbb{P}^n$. We always regard X as a compactifiable complex manifold with such a meromorphic structure, if not stated otherwise.

Definition B. A *quasi-Albanese map* $(\alpha,\mathcal{A}(X,\mathfrak{m}))$ of a compactifiable complex manifold $(X,\mathfrak{m})$ is a pair consisting of a semi-complex torus $\mathcal{A}(X,\mathfrak{m})$ and a compactifiable morphism $\alpha:X\to\mathcal{A}(X,\mathfrak{m})$ satisfying the following universality condition: If $g:X\to\mathcal{B}$ is a compactifiable morphism into a semi-complex torus $\mathcal{B}$, then there exists a unique compactifiable morphism $h:\mathcal{B}\to\mathcal{A}(X,\mathfrak{m})$, which is a composition of a Lie group homomorphism and a translation, such that $g=h\circ\alpha$.

Proposition C. *For a compactifiable complex manifold* $(X,\mathfrak{m})$, *there exists a quasi-Albanese map* $(\alpha,\mathcal{A}(X,\mathfrak{m}))$ *uniquely up to isomorphisms.*

Proof. Let $\bar{X}$ be a non-singular compactification of X belonging to $\mathfrak{m}$. The cohomology groups $H^q(\bar{X},\Omega^p(\log\bar{D}))$ are invariants of $(X,\mathfrak{m})$, by Proposition B. Let $\bar{F}$ (resp. F) be the subspace of $H^0(\bar{X},\Omega^1(\log\bar{D}))$ [resp. $H^0(\bar{X},\Omega^1)$] consisting of d-closed forms, and let $\bar{F}^*=\operatorname{Hom}_{\boldsymbol{C}}(\bar{F},\boldsymbol{C})$ [resp. $F^*=\operatorname{Hom}_{\boldsymbol{C}}(F,\boldsymbol{C})$] be the dual space of $\bar{F}$ (resp. F), let $\bar{\Delta}$ (resp. Δ) be the image of $H_1(X,\boldsymbol{Z})$ [resp. $H_1(\bar{X},\boldsymbol{Z})$] in $\bar{F}^*$ (resp. F^*), with $p:\bar{F}^*\to F^*$ the projection, and let $H=\ker p$, $\Delta_0=\bar{\Delta}\cap H$, $\bar{\Delta}_c$ (resp. Δ_c) the smallest closed Lie subgroup of $\bar{F}^*$ (resp. F^*) containing $\bar{\Delta}$ (resp. Δ) such that the connected component of $\bar{\Delta}_c$ (resp. Δ_c) is a $\boldsymbol{C}$-linear subspace of $\bar{F}^*$ (resp. F^*). By

Proposition A, we have $\dim H = \dim \bar{F} - \dim F = \dim H^1(\bar{X}, C) - \dim H^1(X, C) \geqq \operatorname{rank} \Delta_0$. On the other hand, $\bar{\Delta}$ generates the whole of $\bar{F}^*$ over C by the duality of $H^1(X, C)$ and $H_1(X, C)$. Hence $\dim H = \operatorname{rank} \Delta_0$. We note that F^*/Δ_c is an Albanese torus $A(\bar{X})$ of $\bar{X}$ (cf. [16], p. 102). Therefore, we get a semi-complex torus $\mathscr{A}(X, \mathfrak{m}) = \bar{F}^*/\bar{\Delta}$ of type (g, n), where $g = \dim F^*/\Delta_c \leqq \dim H^1(\bar{X}, \mathcal{O}_X)$ and $n = \dim H^1(X, C) - H^1(\bar{X}, C)$, by the submodule Δ_0 of $\bar{\Delta}$. Fix an arbitrary point $x_0 \in X$ and define the morphism $\alpha: X \to \mathscr{A}(X, \mathfrak{m})$ by the following integral:

$$\alpha(x)(\omega) = \int_{x_0}^{x} \omega$$

for all $x \in X$ and for all $\omega \in \bar{F}$. α is clearly compactifiable. The construction $(X, \mathfrak{m}) \to (\alpha, \mathscr{A}(X, \mathfrak{m}))$ is functorial, i.e., if $f: (X_1, \mathfrak{m}_1) \to (X_2, \mathfrak{m}_2)$ is a morphism of compactifiable complex manifolds, then there exists a morphism $\mathscr{A}(f): \mathscr{A}(X_1, m_1) \to \mathscr{A}(X_2, m_2)$ of compactifiable complex manifolds which is a Lie group homomorphism and satisfies the condition that $\mathscr{A}(f \circ g) = \mathscr{A}(f) \circ \mathscr{A}(g)$. If $(X, \mathfrak{m})$ is a semi-complex torus, then $\mathscr{A}(X, \mathfrak{m})$ is isomorphic to $(X, \mathfrak{m})$, hence the universality. Q.E.D.

Corollary. *$\mathscr{A}(X, \mathfrak{m})$ is a $\mathscr{T}$-bundle over the Albanese torus $A(\bar{X})$, where $\mathscr{T} = (C^\times)^n$, $n = \dim H^1(X, C) - \dim H^1(\bar{X}, C)$.*

Proof. See the proof of the proposition.

Lemma A. *Let $\bar{D} = \bigcup_{i \in I} \bar{D}_i$ be the irreducible decomposition of $\bar{D}$ and $\mathrm{cl}: C^I \to H^2(\bar{X}, C)$ the linear map defined by the following formula: $\mathrm{cl}(\ldots, \alpha_i, \ldots) =$ the cohomology class of $\sum_{i \in I} \alpha_i \bar{D}_i$ in X. Then we have*

$$\dim H^1(X, C) - \dim H^1(\bar{X}, C) = \dim \ker(\mathrm{cl}).$$

Proof. We have a local cohomology sequence:

$$0 \to H^1(\bar{X}, C) \to H^1(X, C) \xrightarrow{b} H^2_{\bar{D}}(\bar{X}, C) \xrightarrow{c} H^2(\bar{X}, C).$$

Note that $H^2_{\bar{D}}(\bar{X}, C) \cong \bigoplus_{i \in I} H^2_{\bar{D}_i}(\bar{X}, C) = \bigoplus_{i \in I} C$, b is induced by the residue map, and that c is exactly cl. Q.E.D.

Remark. Note that $\bar{D}$ need not be a divisor on $\bar{X}$ in the above lemma.

Lemma B. *Let $\pi: X \to A$ be a $\mathscr{T}$-bundle over a complex torus A of dimension g, $\bar{X} = X \times_{\mathscr{T}} P^n$ the canonical compactification of X, $L = \bar{X} - X = \bigcup_{i=1}^{n} L_i$ the irreducible decomposition of L. Then the following conditions are equivalent:*

1) *X is a semi-complex torus,*
2) *π is a topologically trivial bundle,*
3) *$\mathrm{cl}(L_0) = \ldots = \mathrm{cl}(L_n)$ in $H^2(\bar{X}, C)$,*
4) *X is diffeomorphic to $(S^1 \times S^1)^g \times (S^1 \times C)^n$.*

Proof. 1)⇒4) is clear. By Lemma A, 4)⇒3). To prove 3)⇒2), we may assume that $n = 1$. Since $\bar{\pi}^*(\pi) = \mathcal{O}_{\bar{X}}(L_0 - L_1)$, we have $\mathrm{cl}(\pi) = (L_0 - L_1)|L_0$, which proves 3)⇒2). 2)⇒4) is clear and 3)⇒1) is Lemma A. Q.E.D.

References

1. Deligne,P.: Equations différentielles à points singuliers réguliers. Lecture Notes in Mathematics 163. Berlin, Heidelberg, New York: Springer 1970
2. Deligne,P.: Théorie de Hodge. II. Publ. Math. I.H.E.S. **40**, 5—58 (1971)
3. Douady, A.: Le problème des modules pour les variétés analytiques complexes. Sém. Bourbaki n° 277, 1964/65
4. Fujiki, A., Nakano, S.: Supplement to "On the inverse of monoidal transform". Publ. RIMS, Kyoto Univ. **7**, 637—644 (1971/72)
5. Grothendieck, A.: A general theory of fiber spaces with structure sheaf. Univ. of Kansas Report n° 4, 2nd ed. 1958
6. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. of Math. **79**, 109—326 (1964)
7. Horikawa, E.: On deformations of holomorphic maps. I. J. Math. Soc. Japan **25**, 372—396 (1973); II. J. Math. Soc. Japan **26**, 647—667 (1974); III. Math. Ann. **222**, 275—282 (1976)
8. Litaka,S.: On logarithmic Kodaira dimension of algebraic varieties. To appear in: Complex analysis and algebraic geometry. Tokyo: Iwanami
9. Iitaka,S.: Logarithmic forms of algebraic varieties. J. Fac. Sci., Univ. Tokyo, Sec. 1A, **23**, 525—544 (1976)
10. Kempf,G. et al.: Toroidal embeddings. I. Lecture Notes in Mathematics 339. Berlin, Heidelberg, New York: Springer 1973
11. Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures. I, II. Ann. of Math. **67**, 328—466 (1958); III. **71**, 43—76 (1960)
12. Kuranishi, M.: Deformations of compact complex manifolds. Montréal, Canada: Les Presses de l'Université 1971
13. Nagata, M.: Imbedding of an abstract variety in a complete variety. J. Math. Kyoto Univ. **2—1**, 1—10 (1962)
14. Saito, K.: On the uniformization of complements of discriminant loci. A.M.S. Summer Institute 1975
15. Schneider, M.: Halbstetigkeitssätze für relativ analytische Räume. Inv. Math. **16**, 161—176 (1972)
16. Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. Lecture Notes in Mathematics 439. Berlin, Heidelberg, New York: Springer 1975

Received August 8, 1977