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Characters of the Nullcone

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1. Introduction

Let G be a semi-simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. The nullcone N of $\mathfrak{g}$ consists of the nilpotent elements of $\mathfrak{g}$. Its co-ordinate ring $A(N)$ is a graded ring $\sum_{n \geqq 0} A_n(N)$. We fix a maximal torus T of G and a dominant chamber C in the character group P of T. If $\lambda \in C$ let E_λ denote the irreducible G-module with highest weight λ. Let $d_n(\lambda)$ be the multiplicity of E_λ in the G-module $A_n(N)$.

Let R be the root system. Let R_+ be the set of the positive roots. Put $\varrho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. If $\chi \in P$ let $p_n(\chi)$ be the number of maps $f : R_+ \to \{0\} \cup \mathbb{N}$ such that $n = \sum f(\alpha)$ and $\chi = \sum f(\alpha)\alpha$. Let W be the Weyl group. If $w \in W$, put $R(w) := R_+ \cap -wR_+$ and $n(w) := \# R(w)$ and $\varepsilon(w) := (-1)^{n(w)}$. We prove the following theorem.

Theorem. *If* $\lambda \in C$ *then* $d_n(\lambda) = \sum_{w \in W} \varepsilon(w)\, p_n(w(\lambda + \varrho) - \varrho)$.

Remarks. The sequence $d_*(\lambda)$ is equivalent to the sequence of generalized exponents $m_*(\lambda)$ introduced by B. Kostant in [5]. In fact we have

$$d_n(\lambda) = \# \{i \mid m_i(\lambda) = n\} \quad \text{and} \quad m_i(\lambda) = \min\left\{m \mid i \leqq \sum_{n=0}^{m} d_n(\lambda)\right\}.$$

March 1976 I obtained the Theorem using the cohomological methods of [4]. After a conversation with T. A. Springer it became clear that the cohomology could be eliminated from the proof, see below. September 1978 I learned that D. Peterson had obtained the same Theorem, independently and by different methods.

2. We may assume that G is simply connected. The Grothendieck group $R(T)$ of the (finite dimensional) T-modules is identified with the group ring $\mathbb{Z}[P]$. If E is a T-module its class in $R(T)$ is denoted by $\mathrm{ch}(E)$. Let V be an affine T-variety. Assume that the co-ordinate ring $A(V)$ is graded in such a way that the

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homogeneous parts $A_n(V)$ are T-modules. Then we define the character of V to be the formal power series

$$\mathrm{CH}(V,z):=\sum_{n=0}^{\infty}\mathrm{ch}(A_n(V))z^n.$$

Let E be a T-module with $\mathrm{ch}(E)=\sum m(i)e^{\chi(i)}$, so $m(i)\in\mathbb{Z}$ and $\chi(i)\in P$. Since the coordinate ring of E is the symmetric algebra $S(E^*)$ on the dual E^* we obtain

$$\mathrm{CH}(E,z)=\prod(1-e^{-\chi(i)}z)^{-m(i)}.$$

3. Lemma. *If we put* $W(z):=\sum_{w\in W}z^{n(w)}$ *the nullcone* N *satisfies*

$$\mathrm{CH}(N,z)=W(z)\prod_{\alpha\in R}(1-e^{\alpha}z)^{-1}.$$

Proof. Put $r=\mathrm{rank}(\mathfrak{g})$. By Sect. 2 we have

$$\mathrm{CH}(\mathfrak{g},z)=(1-z)^{-r}\prod_{\alpha\in R}(1-e^{\alpha}z)^{-1}.$$

B. Kostant has shown that $A(\mathfrak{g})=H\otimes A(\mathfrak{g})^G$ where $H=\sum H_n$ and the G-module H_n is isomorphic to $A_n(N)$ for every n, cf. [5] Theorem 11. Let $m_1,\ldots,m_r$ be the exponents of the root system R. Put $d(i)=m_i+1$. The ring of invariants $A(\mathfrak{g})^G$ is generated by algebraically independent homogeneous polynomials $f_1,\ldots,f_r$ of degrees $d(1),\ldots,d(r)$. This implies

$$\sum\mathrm{ch}(A_n(\mathfrak{g})^G)z^n=\prod(1-z^{d(i)})^{-1}.$$

It follows that

$$\mathrm{CH}(N,z)=(1-z)^{-r}\prod_{i=1}^{r}(1-z^{d(i)})\prod_{\alpha\in R}(1-e^{\alpha}z)^{-1}.$$

So the lemma follows from the identity

$$W(z)=(1-z)^{-r}\prod_{i=1}^{r}(1-z^{d(i)}),\ \text{cf. [6] 2.6}.$$

4. The Weyl group action on P is extended to the ring $R(T)[[z]]$ in such a way that z is W-invariant. Let J be the endomorphism of $R(T)[[z]]$ given by $J=\sum\varepsilon(w)w$.

Lemma. *Let* V *be a subset of* R_+. *Put* $W(V)=\{w\in W\mid R(w)\subset V\}$. *Then we have*

$$J\Big(e^{\varrho}\prod_{\alpha\in V}(1-e^{-\alpha}z)\Big)=J(e^{\varrho})\sum_{w\in W(V)}z^{n(w)}.$$

Proof. If $A\subset R_+$ we put $|A|=\sum_{\alpha\in A}\alpha$. We have

$$J\Big(e^{\varrho}\prod_{\alpha\in V}(1-e^{-\alpha}z)\Big)=\sum_{A\subset V}(-z)^{\# A}J(e^{\varrho-|A|}).$$

In [6], p. 166, I.G. Macdonald proved that $J(e^{\varrho-|A|})\neq 0$ if and only if $A=R(w)$ for some $w\in W$. Moreover the element w is necessarily unique. It satisfies $n(w)=\#A$ and $J(e^{\varrho-|A|})=\varepsilon(w)J(e^{\varrho})$.

5. Proposition. $\mathrm{CH}(N,z)J(e^{\varrho})=J\left(e^{\varrho}\prod_{\alpha\in R_+}(1-e^{\alpha}z)^{-1}\right)$.

Proof. Put $x=\prod_{\alpha\in R}(1-e^{\alpha}z)^{-1}$ and $y=e^{\varrho}\prod_{\alpha\in R_+}(1-e^{-\alpha}z)$. Since x is W-invariant the righthand side of our formula is equal to $J(xy)=xJ(y)$. Now the equality follows from the Lemmas 3 and 4.

Remark. This proof is a simplification of an idea used in [4] p. 251.

6. *Proof of the Theorem.* The numbers $p_n(\chi)$ are determined by

$$\prod_{\alpha\in R_+}(1-e^{\alpha}z)^{-1}=\sum_{\chi\in P,n\geqq 0}p_n(\chi)e^{\chi}z^n.$$

The proposition implies

$$\begin{aligned}\mathrm{ch}(A_n(N))J(e^{\varrho})&=\sum_{\chi\in P}p_n(\chi)J(e^{\varrho+\chi})\\&=\sum_{\lambda\in C}\sum_{w\in W}\varepsilon(w)p_n(w(\lambda+\varrho)-\varrho)J(e^{\lambda+\varrho}).\end{aligned}$$

Since G-modules are characterized by their formal character, the Theorem follows by Weyl's character formula

$$\mathrm{ch}(E_\lambda)J(e^{\varrho})=J(e^{\lambda+\varrho}),$$

cf. [1], Chap. 8, Sect. 9.

7. *Remark.* Let U be a maximal unipotent subgroup of G, normalized by T, whose weights are the positive roots. The ring of invariants $A(N)^U$ is a graded T-module with character

$$f(z)=\sum\mathrm{ch}(A_n(N)^U)z^n=\sum_{\lambda\in C,n\geqq 0}d_n(\lambda)e^{\lambda}z^n.$$

By the Theorem of Hadziev-Grosshans, cf. [3] and [2], the ring $A(N)^U$ is finitely generated. It follows that $f(z)$ is an element of the quotient field of the polynomial ring $R(T)[z]$.

Problem. Write $f(z)$ as a quotient $g(z)/h(z)$ with $g(z)$ and $h(z)$ in $R(T)[z]$ or rather in $\mathbb{Z}[C][z]$.

Remark. Let us define $D_n(\lambda):=\sum_{w\in W}\varepsilon(w)p_n(w(\lambda+\varrho)-\varrho)$ for all $\lambda\in P$, so that $D_n(\lambda)=d_n(\lambda)$ if $\lambda\in C$. The formal power series

$$F(z):=\sum_{\lambda\in P,n\geqq 0}D_n(\lambda)e^{\lambda}z^n$$

satisfies

$$F(z)=e^{-\varrho}J\left(e^{\varrho}\prod_{\alpha\in R_+}(1-e^{\alpha}z)^{-1}\right).$$

Example 1. Let G be of type A_2, with simple roots α and β. A weight $\lambda = x\alpha + y\beta$ satisfies $d_n(\lambda) = 1$ if and only if $d_n(\lambda) \neq 0$ if and only if x and y are integers with

$$\max\{2x - y, 2y - x\} \leqq n \leqq x + y.$$

It follows that

$$f(z) = \frac{1 + e^{\alpha+\beta} z^2 + e^{2\alpha+2\beta} z^4}{(1 - e^{\alpha+\beta} z)(1 - e^{2\alpha+\beta} z^3)(1 - e^{\alpha+2\beta} z^3)}.$$

Generators and relations for $A(N)^U$ are easily obtained.

Example 2. Let G be of type B_2, with simple roots α (short) and β (long). A rather tedious calculation shows that

$$f(z) = \frac{1 + e^{3\alpha+2\beta} z^4}{(1 - e^{\alpha+\beta} z^2)(1 - e^{2\alpha+2\beta} z^2)(1 - e^{2\alpha+\beta} z)(1 - e^{2\alpha+\beta} z^3)}.$$

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