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Lusternik-Schnirelman Theory and Non-Linear Eigenvalue Problems*

Herbert Amann

1. Introduction

Let A and B be mappings from a real infinite-dimensional Banach space X into its dual X^*. We consider an *eigenvalue problem* for the pair (A,B), namely the problem of finding an element $u \in X$, i.e. an *eigenfunction*, satisfying some normalization conditions and a real number λ, i.e. an *eigenvalue*, such that

$$A(u) = \lambda B(u). \tag{1.1}$$

In this paper we employ the ideas of Lusternik and Schnirelman [14] to establish the existence of infinitely many distinct eigenfunctions for problem (1.1).

This problem has already attracted considerable interest [3–7, 9–11, 13, 14, 18, 19]. All these investigations have been based on variants of the so-called Lusternik-Schnirelman theory.

In the early 1930's Lusternik and Schnirelman developed a theory of critical points for differentiable functions on finite-dimensional Riemannian manifolds. One of the principal tools for establishing the existence of "intermediate" critical points (i.e. of critical points not belonging to absolute maxima or minima) is the same as in the Morse theory, namely the deformation of the manifold along gradient lines. The application of this theory to infinite-dimensional eigenvalue problems of the form (1.1) which arise in connection with differential and integral equations require the generalization of the Lusternik-Schnirelman theory to infinite-dimensional manifolds. This extension has been made by Schwartz [16, 17] for Riemannian manifolds modelled on Hilbert spaces and by R. S. Palais [15] for Finsler manifolds modelled on arbitrary Banach spaces. These generalizations are based on a fundamental compactness assumption, the so-called Palais-Smale Condition.

In applying this general Lusternik-Schnirelman theory to the eigenvalue problem (1.1) one is faced with two technical difficulties. First, one

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has to impose restrictions upon the operators under consideration which guarantee the Palais-Smale Condition to be satisfied. Second, one has to impose regularity conditions upon the norm of the Banach space X (it has to have a locally Lipschitz continuous first derivative) in order to be able to construct a pseudo-gradient field defining a flow on the manifold. For a lucid discussion of these difficulties see [6].

To avoid these difficulties Browder [5–7] and Weiss [19] have employed a Galerkin approximation procedure for the eigenvalue problem (1.1) which makes no explicite use of the theory of infinite dimensional manifolds. However, in order to carry through the necessary limit arguments one has to impose definiteness restrictions upon the operator B. Assumptions of this type have been made, either explicitely or implicitely, in all of the papers which have been published on this subject. In fact, it has at least been assumed that, for all $u \neq 0$,

$$|\langle B(u), u\rangle| > 0.$$

The main purpose of this paper is to remove this condition. To do this we use an infinite dimensional argument of Lusternik-Schnirelman type which is based on two observations. First, in applying the general Lusternik-Schnirelman theory in the form of Palais to the eigenvalue problem (1.1) only manifolds have been used which are homeomorphic to the unit sphere by means of the radial projection mapping. But on manifolds of this type one can easily construct trajectories which are approximations to gradient lines of a function without integrating a differential equation. Hence, there is no need for a Finsler structure on these manifolds.

Second, the difficulty in establishing the Palais-Smale condition under reasonable assumptions upon B and its functional b stems from the fact that a weakly convergent sequence $\{u_n\}$ of normalized elements may converge to zero. This can easily be excluded if the sequence $\{b(u_n)\}$ is bounded away from zero. But the consideration of sequences of this type suffices for establishing the existence of infinitely many eigenfunctions. This means that we do not have to verify the Palais-Smale Condition "globally" but only for sequences of the above type which is a much easier task. By this way we can solve our problem under rather weak assumptions upon B which in fact are necessary too.

Applications of our general results to quasi-linear differential equations can be given very much in the same way as in Browder's work [4–7] and are omitted here. Instead, we apply our results to the Hammerstein equation $u = \lambda K F(u)$ in order to prove a general theorem on eigenfunctions which generalizes in particular recent results of Coffman [10].

2. Definitions and Statement of the Main Results

Let X be a real Banach space with dual X^* and with duality pairing $\langle\cdot,\cdot\rangle$ between X^* and X. We denote weak convergence in either X or X^* by $\rightharpoonup$ and strong convergence by $\rightarrow$.

A mapping $A: X \rightarrow X^*$ is said to be a *potential operator* with *potential* a if there exists a Gateaux-differentiable functional a: $X \rightarrow \boldsymbol{R}$ such that, for all $u, v \in X$,

$$\lim_{\tau \rightarrow 0} \tau^{-1}(a(u+\tau v)-a(u))=\langle A(u), v\rangle .$$

The potential a is uniquely determined by the requirement $a(0)=0$ *which always will be made in this paper.*

A mapping $A: X \rightarrow X^*$ is called hemicontinuous if it is continuous from line segments in X to the weak topology of X^*. It is easily seen (e.g. [3]) that a hemicontinuous mapping $A: X \rightarrow X^*$ is a potential operator if and only if, for all $u, v \in X$

$$\int_0^1(\langle A(\tau u), u\rangle-\langle A(\tau v), v\rangle) d\tau=\int_0^1\langle A(v+\tau(u-v)), u-v\rangle d\tau .$$

Moreover, its potential can be represented in the form

$$a(u)=\int_0^1\langle A(\tau u), u\rangle d\tau . \tag{2.1}$$

A mapping $A: X \rightarrow X^*$ is said to be *strongly monotone* if there exists a continuous function $\alpha:[0, \infty) \rightarrow[0, \infty)$ which is positive on $(0, \infty)$ and satisfies $\alpha(\varrho) \rightarrow \infty$ as $\varrho \rightarrow \infty$ such that, for all $u, v \in X$,

$$\langle A(u)-A(v), u-v\rangle \geqq \alpha(\|u-v\|)\|u-v\| .$$

A mapping $A: X \rightarrow X^*$ is said to satisfy *condition* $(S)_1$ if for every sequence $\{u_j\}$ in X with $u_j \rightharpoonup u$ and $A(u_j) \rightarrow v$ we have $u_j \rightarrow u$. Obviously, every strongly monotone operator satisfies condition $(S)_1$.

Condition $(S)_1$ generalizes conditions (S) and $(S)_0$ introduced by Browder [5–7]. Indeed, A is said to satisfy condition (S) if $u_j \rightharpoonup u$ and $\langle A(u_j)-A(u), u_j-u\rangle \rightarrow 0$ imply $u_j \rightarrow u$, and A is said to satisfy condition $(S)_0$ if $u_j \rightharpoonup u, A(u_j) \rightharpoonup v$ and $\langle A(u_j), u_j\rangle \rightarrow\langle v, u\rangle$ imply $u_j \rightarrow u$. It has been shown [5, Lemma 1] that (S) implies $(S)_0$ and it is trivial that $(S)_0$ implies $(S)_1$. Hence (cf. [6]), condition $(S)_1$ is satisfied for quasi-linear elliptic differential operators in generalized divergence form under weak hypotheses upon the coefficients.

A mapping $A: X \rightarrow X^*$ is said to be *odd* if, for all $u \in X$, we have $A(u)=-A(-u)$. If A is an odd hemicontinuous potential operator then by (2.1) its potential is an *even* functional, i.e. for all $u \in X$, we have $a(u)=a(-u)$.

In this paper we consider mappings satisfying the following

Assumption (A). *$A: X \to X^*$ is an odd potential operator which is uniformly continuous on bounded sets and satisfies condition $(S)_1$. For a given constant $\alpha > 0$ the level set $M_\alpha \equiv \{u \in X \mid a(u) = \alpha\}$ is bounded and each ray through the origin intersects M_α. Moreover, for every $u \neq 0$, $\langle A(u), u\rangle > 0$ and there exists a constant $\varrho_\alpha > 0$ such that $\langle A(u), u\rangle \geqq \varrho_\alpha$ on M_α.*

It is easily seen that every strongly monotone odd potential operator which is uniformly continuous on bounded sets satisfies Assumption (A).

A mapping $B: X \to X^*$ is said to be *strongly sequentially continuous* if it maps every weakly convergent sequence into a strongly convergent sequence.

A subset of X is said to be *symmetric* if it is invariant under the involution which sends u into $-u$.

Let $\mathscr{C}$ be the class of all closed symmetric subsets of X not containing the origin. For every $C \in \mathscr{C}$ we define the *genus* of C, *gen* (C), to be zero if C is empty and otherwise to be the supremum of the set of integers n such that every odd continuous map $f: C \to \mathbb{R}^{n-1}$ has a zero.

For an arbitrary symmetric subset S of $X \setminus \{0\}$ we define the *genus over compact sets* $\gamma(S)$ by

$$\gamma(S) \equiv \sup\{gen(C) \mid C \subset S, C \in \mathscr{C}, C \text{ compact}\}.$$

It is an immediate consequence of Borsuk's theorem (e.g. [17, Corollary 3.29]) that $\gamma(S) \geqq n$ if there exists an odd homeomorphism of the unit sphere in $\mathbb{R}^n$ onto a subset of S.

We are now in position to state our main result.

Theorem A. *Suppose the following hypotheses are satisfied:*

H1) *X is a real infinite dimensional uniformly convex Banach space.*

H2) *The operator $A: X \to X^*$ satisfies Assumption (A).*

H3) *The mapping $B: X \to X^*$ is a strongly sequentially continuous odd potential operator (with potential b) such that $b(u) \neq 0$ implies $B(u) \neq 0$.*

Then:

a) *The eigenvalue problem*

$$A(u) = \lambda B(u) \tag{2.2}$$

has infinitely many distinct eigenfunctions satisfying the normalization condition $a(u) = \alpha$ provided

$$\gamma\{u \in M_\alpha \mid b(u) \neq 0\} = \infty. \tag{2.3}$$

b) *For every $k \in N$ set $\mathscr{C}_k \equiv \{C \subset M_\alpha \mid C$ symmetric, compact, $gen(C) \geqq k\}$ and define β_k by*

$$\beta_k \equiv \sup_{C \subset \mathscr{C}_k} \inf_{u \in C} |b(u)|.$$

Then, if $\beta_k > 0$, *there exists an eigenfunction* $u_k \in M_\alpha$ *of* (2.2) *with*

$$|b(u_k)| = \beta_k .$$

c) *Suppose, for some* $j, m \in N$, *we have*

$$\beta_j = \beta_{j+1} = \cdots = \beta_{j+m} > 0$$

and denote by E_j *the set of all eigenfunctions on* M_α *of* (2.2) *satisfying* $|b(u)| = \beta_j$. *Then*

$$gen(E_j) \geqq m+1 .$$

d) *Condition* (2.3) *is necessary for the existence of infinitely many distinct normalized eigenfunctions of* (2.2), *i.e. there exist Banach spaces* X *and operators* A *and* B *satisfying* $H1-H3$ *but not* (2.3) *such that* (2.2) *has only finitely many distinct normalized eigenfunctions.*

Hypothesis $H3$ is obviously satisfied if B is a strongly sequentially continuous odd potential operator satisfying $|\langle B(u),u\rangle| > 0$ for $u \neq 0$. Hence, except for regularity assumptions, Theorem A generalizes all the known existence theorems for the eigenvalue problem (2.2).

A mapping $B: X \to X^*$ is called *homogenous of degree* $\beta \geqq 0$ if, for every $u \in X$ and every $\tau > 0$,

$$B(\tau u) = \tau^\beta B(u) .$$

Let $B: X \to X^*$ be a homogenous potential operator. Then by (2.1), $b(u) = \dfrac{1}{\beta+1} \langle B(u),u\rangle$. Hence, *every homogenous potential operator has the property that* $b(u) \neq 0$ *implies* $B(u) \neq 0$.

Let X be an arbitrary real Banach space. A linear operator $K: X \to X^*$ is said to be *non-negative* if, for all $u, v \in X$,

$$\langle Ku,v\rangle = \langle Kv,u\rangle \quad \text{and} \quad \langle Ku,u\rangle \geqq 0 .$$

It is easily seen (e.g. [1, Lemma 2.1]) that a non-negative linear operator is continuous.

Let $K: X \to X^*$ be an arbitrary bounded linear operator with null-space $N(K)$ and range $R(K)$. Then K induces an injective bounded linear operator $\hat{K}$ on the factor space $X/N(K)$ with $R(\hat{K}) = R(K)$. Hence $\hat{K}$ has an (in general unbounded) inverse $L: R(K) \to X/N(K)$ which will be called *the generalized inverse of* K.

Let $K: X \to X^*$ be a linear operator and let $F: X^* \to X$ be a non-linear mapping. The non-linear operator equation

$$u = KF(u)$$

in X^* is called the *Hammerstein equation* since it is the abstract analogue of a non-linear integral equation of Hammerstein type.

The following theorem on the existence of eigenfunctions for the Hammerstein equation generalizes results of Vainberg [18] and Coffman [10]. Vainberg considers the case where X is a Hilbert space and K has infinitely many eigenvalues and in [10] it is supposed that for all $u \neq 0$, $\langle Ku, u\rangle > 0$ and that F is monotone.

Theorem B. *Let X be an arbitrary real Banach space. Suppose $K: X \to X^*$ is a non-negative compact linear operator. Suppose $F: \overline{R(K)} \to X$ is an odd continuous potential operator (with potential f). Finally suppose that, for all non-zero $u \in \overline{R(K)}$, $f(u) \neq 0$ and $KF(u) \neq 0$.*

Then, for every $\alpha > 0$, the eigenvalue problem for the Hammerstein equation

$$u = \lambda K F(u) \tag{2.4}$$

has at least dim $R(K)$ many distinct pairs of eigenfunctions $(u, -u)$ satisfying the normalization condition

$$\langle u, Lu\rangle = \alpha$$

where L denotes the generalized inverse of K.

Remark. By the definition of a potential operator, F is a mapping from $Z \equiv \overline{R(K)}$ into Z^*. The canonical imbedding of X into X^{**} and the fact that $Z \subset X^*$ imply that $X \subset Z^*$. Hence the assumption $F: \overline{R(K)} \to X$, i.e. $R(F) \subset X$, is meaningful.

3. Some Auxiliary Results

Let X be an arbitrary real Banach space. It is easily seen that a mapping $A: X \to X^*$ which is uniformly continuous on bounded sets is *bounded*, i.e. maps bounded sets into bounded sets (e.g. [18, p. 18]).

Let $A: X \to X^*$ be an operator satisfying Assumption (A). We define a mapping $p: X \backslash \{0\} \to (0, \infty)$ by

$$a(p(u)u) = \alpha.$$

This mapping has the following properties.

Lemma 3.1. *p is a well-defined even functional which is bounded on sets bounded away from zero. It has a Fréchet derivative $p': X \to X^*$ which is odd and uniformly continuous on bounded sets which are bounded away from zero. Moreover, for all $u \in M_\alpha$ and all $v \in X$ with $\langle A(u), v\rangle = 0$, we have $\langle p'(u), v\rangle = 0$.*

Proof. By the assumptions upon M_α, for every $u \neq 0$ there exists at least one $p(u) > 0$ such that $a(p(u)u) = \alpha$. Suppose there is a ray through the origin intersecting M_α twice. Then, for some $u \neq 0$ and some $\tau > 1$, we have $a(u) = a(\tau u) = \alpha$ and hence, by Assumption (A),

$$0 = a(\tau u) - a(u) = \int_1^\tau \langle A(\sigma u), \sigma u \rangle \frac{d\sigma}{\sigma} > 0$$

which is impossible. Hence p is well-defined. By Assumption (A), M_α is symmetric, bounded, and bounded away from zero. This implies that p is even and bounded on sets which are bounded away from zero.

Since A is a continuous potential operator its potential a is Fréchet differentiable. Moreover, for every $u \in X \setminus \{0\}$ and every $\tau > 0$,

$$\frac{d}{d\tau} a(\tau u) = \langle A(\tau u), u \rangle = \tau^{-1} \langle A(\tau u), \tau u \rangle > 0.$$

Hence, by the implicit function theorem, p is continuously differentiable and from

$$0 = \big(a(p(u)u)\big)' = \langle A(p(u)u), u \rangle \, p'(u) + p(u) \, A(p(u)u)$$

we obtain the explicit representation

$$p'(u) = -\frac{p^2(u)}{\langle A(p(u)u), p(u)u \rangle} A(p(u)u). \tag{3.1}$$

Therefore, since A is bounded and since p is bounded on sets which are bounded away from zero, p' is bounded on bounded sets which are bounded away from zero. Hence, by means of the relation

$$p(u) - p(v) = \int_0^1 \langle p'(v + \tau(u-v)), u - v \rangle \, d\tau$$

which is true for all $u, v \in X$ with $0 \notin \{v + \tau(u-v) \mid 0 \leqq \tau \leqq 1\}$, it follows easily that p is uniformly continuous on bounded sets which are bounded away from zero. Hence, by (3.1), p' has the stated continuity property. The last statement follows from (3.1) since $p(u) = 1$ if $u \in M_\alpha$. Hence, since p' is obviously odd, the lemma has been proved.

For completeness we include a proof of the following result (compare [18]).

Lemma 3.2. *Let X be reflexive and let $B: X \to X^*$ be a strongly continuous potential operator (with potential b). Then B is uniformly continuous on bounded sets and b is weakly sequentially continuous.*

Proof. Suppose there is some bounded set $M \subset X$ such that B is not uniformly continuous on M. Then, there exists an $\varepsilon > 0$ and sequences $\{u_j\}$, $\{v_j\} \subset M$ such that, for all $j \in \mathbb{N}$,

$$\|u_j - v_j\| \leqq \frac{1}{j} \quad \text{and} \quad \|B(u_j) - B(v_j)\| \geqq \varepsilon. \tag{3.2}$$

Since M is weakly sequentially compact, by taking suitable subsequences, we may assume that $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$. Hence, by the first relation of (3.2), $u = v$, and since B is strongly continuous, $B(u_j) - B(v_j) \to 0$ which contradicts the second inequality of (3.2). Hence B is uniformly continuous on bounded sets.

Suppose $u_n \rightharpoonup u$. Then

$$b(u_n) = \int_0^1 \langle B(\sigma u_n), u_n \rangle \, d\sigma$$

and, for every $\sigma \in [0, 1]$, $\langle B(\sigma u_n), u_n \rangle \to \langle B(\sigma u), u \rangle$. Since B is uniformly continuous it is bounded. Hence, by the Theorem on Dominated Convergence, $b(u_n) \to b(u)$, i.e. b is weakly sequentially continuous. q.e.d.

Let X be a uniformly convex (hence reflexive) Banach space. Then, for every $u \in X^*$, there is exactly one $J(\mathrm{u}) \in X$ such that,

$$\langle u, J(u) \rangle = \|u\|^2 \quad \text{and} \quad \|J(u)\| = \|u\|.$$

Hence, the *duality mapping* $J : X^* \to X$ is well-defined and it is easily seen [12, Lemma 1.2] that J is uniformly continuous on bounded sets.

Let X be a uniformly convex Banach space. Let $A : X \to X^*$ satisfy Assumption (A) and let $B : X \to X^*$ be an odd mapping which is uniformly continuous on bounded sets. For every $u \in M_\alpha$ we define

$$D(u) \equiv B(u) - \frac{\langle B(u), u \rangle}{\langle A(u), u \rangle} A(u)$$

and

$$T(u) \equiv J(D(u)) - \frac{\langle A(u), J(D(u)) \rangle}{\langle A(u), u \rangle} u.$$

Then T is an odd mapping from M_α into X which is uniformly continuous, hence bounded. Therefore there exist constants $\tau_0, \gamma_0 > 0$ such that, for all $\tau \in [-\tau_0, \tau_0]$ and all $u \in M_\alpha$,

$$\|u + \tau T(u)\| \geqq \gamma_0.$$

Hence, by Lemma 3.1, the mapping $H : M_\alpha \times [-\tau_0, \tau_0] \to M_\alpha$ given by

$$H(u, \tau) \equiv p(u + \tau T(u))(u + \tau T(u))$$

is well-defined, odd and uniformly continuous. Moreover, for every $u \in M_\alpha$,

$$H(u,0) = u.$$

The following lemma will play an important role in the proof of the main result.

Lemma 3.3. *Let X be a uniformly convex real Banach space. Let $A: X \to X^*$ satisfy Assumption (A) and let $B: X \to X^*$ be an odd potential operator (with potential b) which is uniformly continuous on bounded sets. Then, there exists a mapping $r: M_\alpha \times [-\tau_0, \tau_0] \to \mathbf{R}$ with $r(u,\tau) \to 0$ as $\tau \to 0$, uniformly in $u \in M_\alpha$, such that, for every $u \in M_\alpha$ and every $\tau \in [-\tau_0, \tau_0]$,*

$$b(H(u,\tau)) = b(u) + \int_0^\tau [\|D(u)\|^2 + r(u,\sigma)]\, d\sigma. \tag{3.3}$$

Proof. For every $(u,\tau) \in M_\alpha \times [-\tau_0, \tau_0]$ we have

$$b(H(u,\tau)) = b(u) + \int_0^\tau \left\langle B(H(u,\sigma)), \frac{\partial}{\partial\sigma} H(u,\sigma) \right\rangle d\sigma.$$

By Lemma 3.1, and since obviously $\langle A(u), T(u) \rangle = 0$, we find

$$\begin{aligned} \frac{\partial}{\partial\sigma} H(u,\sigma) &= \langle p'(u + \sigma T(u)), T(u) \rangle (u + \sigma T(u)) + p(u + \sigma T(u)) T(u) \\ &= T(u) + R(u,\sigma) \end{aligned}$$

with

$$\begin{aligned} R(u,\sigma) \equiv\ & (p(u + \sigma T(u)) - p(u)) T(u) \\ & + \langle p'(u + \sigma T(u)) - p'(u), T(u) \rangle (u + \sigma T(u)). \end{aligned}$$

Hence $R(u,\sigma) \to 0$ as $\sigma \to 0$, uniformly in $u \in M_\alpha$. Using these results we find

$$b(H(u,\tau)) = b(u) + \int_0^1 [\langle B(u), T(u) \rangle + r(u,\sigma)]\, d\sigma$$

with

$$r(u,\sigma) \equiv \langle B(u), R(u,\sigma) \rangle + \left\langle B(H(u,\sigma)) - B(u), \frac{\partial}{\partial\tau} H(u,\sigma) \right\rangle$$

which shows that

$$\lim_{\sigma \to 0} r(u,\sigma) = 0,$$

uniformly in $u \in M_\alpha$. Finally,

$$\langle B(u), T(u)\rangle = \langle D(u), J(D(u))\rangle = \|D(u)\|^2$$

which proves the lemma.

In the following, for every $\beta > 0$, we denote by N_β the set

$$N_\beta \equiv \{u \in M_\alpha \mid |b(u)| \geqq \beta\}.$$

Lemma 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Let $\beta > 0$ be fixed and suppose there exist an open set $U \subset M_\alpha$ and positive constants δ, ϱ with $\varrho < \beta$ such that*

$$\|D(u)\| \geqq \delta \text{ if } u \in V_\varrho \equiv \{u \in M_\alpha \mid u \notin U, \, ||b(u)| - \beta| \leqq \varrho\}.$$

Then, there exists an $\varepsilon > 0$ and an odd continuous operator H_ε mapping $N_{\beta-\varepsilon} \backslash U$ into $N_{\beta+\varepsilon}$.

Proof. Choose $\tau_1 \in (0, \tau_0]$ such that, for all $u \in M_\alpha$ and $\sigma \in [-\tau_1, \tau_1]$, we have $|r(u,\sigma)| \leqq \delta^2/2$. Set $t(u,\tau) \equiv \tau \operatorname{sgn} b(u)$. Then, according to Lemma 3.3, for all $u \in V_\varrho$ and all $\tau \in [0, \tau_1]$,

$$|b(H(u, t(u,\tau)))| \geqq |b(u)| + \frac{1}{2}\delta^2 \tau.$$

Set $\varepsilon = \min(\varrho, \frac{1}{4}\delta^2 \tau_1)$. Then, for every $u \in V_\varrho \cap N_{\beta-\varepsilon}$,

$$|b(H(u, t(u,\tau_1)))| \geqq |b(u)| + \frac{1}{2}\delta^2 \tau_1 \geqq |b(u)| + 2\varepsilon \geqq \beta + \varepsilon.$$

By (3.3), for every $u \in V_\varrho$, $|b(H(u, t(u, \cdot)))|$ is strictly increasing in some interval $[0, \sigma)$ containing $[0, \tau_1]$. Hence, for every $u \in V_\varepsilon$, the functional

$$t_\varepsilon(u) \equiv \min\{\tau \geqq 0 \mid |b(H(u, t(u,\tau)))| = \beta + \varepsilon\} \operatorname{sgn} b(u)$$

is uniquely defined, continuous in $u \in V_\varepsilon$ and satisfies $0 \leqq |t_\varepsilon(u)| \leqq \tau_1$. Therefore the mapping $H_\varepsilon : N_{\beta-\varepsilon} \backslash U \to N_{\beta+\varepsilon}$ defined by

$$H_\varepsilon(u) = \begin{cases} H(u, t_\varepsilon(u)) & \text{if } u \in V_\varepsilon, \\ u & \text{if } u \in N_{\beta-\varepsilon} \backslash (U \cup V_\varepsilon) \end{cases}$$

has the desired properties. q.e.d.

Finally we need the following lemma which establishes a "local Palais-Smale Condition" (compare e.g. [6]).

Lemma 3.5. *Let the hypotheses of Lemma 3.3 be satisfied and suppose $b(u) \neq 0$ implies $B(u) \neq 0$. Let $\beta > 0$ be fixed and suppose there exists a* *sequence* $\{u_j\} \subset M_\alpha$ *with* $|b(u_j)| \geqq \beta$ *and* $D(u_j) \to 0$. *Then* $\{u_j\}$ *has a strongly convergent subsequence converging to an eigenfunction* $u \in M_\alpha$ *of* (2.2).

Proof. Since M_α is bounded and X is reflexive, by passing to a suitable subsequence if necessary, we may assume that $u_j \rightharpoonup u$ and

$$\mu_j \equiv \frac{\langle B(u_j), u_j\rangle}{\langle A(u_j), u_j\rangle} \to \mu .$$

Since b is weakly continuous, $|b(u_j)| \to |b(u)| \geqq \beta$. This implies $u \neq 0$ and $B(u) \neq 0$. Since $D(u_j) \to 0$, we find

$$\mu_j A(u_j) = B(u_j) - D(u_j) \to B(u) \neq 0,$$

hence, in particular, $\mu \neq 0$. This implies $A(u_j) \to \mu^{-1} B(u)$ and therefore, by Condition $(S)_1$, $u_j \to u$. Hence $A(u) = \lambda B(u)$ with
$\lambda = \mu^{-1} = \langle A(u), u\rangle / \langle B(u), u\rangle$. q.e.d.

4. Proof of Theorem I

Let X be an arbitrary real Banach space and recall that $\mathscr{C}$ denotes the set of all closed symmetric subsets of X not containing the origin. In the following lemma we state the relevant properties of the genus. For short proofs of these properties see [9].

Lemma 4.1. *Let* $C, C_1, C_2 \in \mathscr{C}$ *be arbitrary.*

i) *If there exists an odd continuous mapping* $F: C_1 \to C_2$, *in particular if* $C_1 \subset C_2$, *then* $gen(C_1) \leqq gen(C_2)$;

ii) $gen(C_1 \cup C_2) \leqq gen(C_1) + gen(C_2)$;

iii) *If* C *is compact then* $gen(C) < \infty$ *and* C *has an open symmetric neighborhood* U *with* $\bar{U} \in \mathscr{C}$ *and* $gen(\bar{U}) = gen(C)$;

iv) *If there exists an odd homeomorphism of the unit sphere in* $\mathbb{R}^n$ *onto* C *then* $gen(C) = n$.

Let Assumption (A) be satisfied and recall that, for every $k \in N$,

$$\mathscr{C}_k = \{C \subset M_\alpha | C \text{ symmetric, compact, } gen(C) \geqq k\}.$$

Since the radial projection of M_α onto the unit sphere of X obviously is an odd homeomorphism, for every $k \in N$, the set $\mathscr{C}_k$ is non-void.

We begin with the proof of part b of Theorem A.

Proposition 1. *Let the hypotheses of Theorem* A *be satisfied and suppose that, for some* $k \in N$,

$$\beta_k = \sup_{C \in \mathscr{C}_k} \min_{u \in C} |b(u)| > 0. \tag{4.1}$$

Then there exists an eigenfunction $u_k \in M_\alpha$ *of* (2.2) *satisfying* $|b(u_k)| = \beta_k$.

Proof. Our assumptions imply the existence of a sequence $\{u_n\} \subset M_\alpha$ with $|b(u_n)| \to \beta_k$ and $D(u_n) \to 0$. Indeed, otherwise we could find positive constants δ and ϱ such that, for all

$$u \in V_\varrho \equiv \{u \in M_\alpha \,\big|\, |b(u)| - \beta_k| \leqq \varrho\},$$

we have $\|D(u)\| \geqq \delta$. Without loss of generality we may assume that $\varrho < \beta_k$. Hence, by Lemma 3.4 (with $U = \phi$), there exists an $\varepsilon > 0$ and an odd continuous map H_ε such that $H_\varepsilon(N_{\beta_k - \varepsilon}) \subset N_{\beta_k + \varepsilon}$. By definition of β_k there exists a $C_\varepsilon \in \mathscr{C}_k$ such that $|b(u)| \geqq \beta_k - \varepsilon$ on C_ε, i.e. $C_\varepsilon \subset N_{\beta_k - \varepsilon}$. Hence $|b(u)| \geqq \beta_k + \varepsilon$ on $H_\varepsilon(C_\varepsilon)$. But Lemma 4.1 implies $H_\varepsilon(C_\varepsilon) \in \mathscr{C}_k$ which contradicts the definition of β_k. This establishes the existence of a sequence $\{u_n\}$ with the properties stated above.

By Lemma 3.5 we can find a convergent subsequence converging to an eigenfunction $u \in M_\alpha$. Since, obviously, $|b(u)| = \beta_k$, the proposition is proved.

In the following we denote by E the set of all eigenfunctions of (2.2) with $b(u) \neq 0$, i.e.

$$E \equiv \{u \in M_\alpha \,|\, b(u) \neq 0, D(u) = 0\},$$

and we recall that, for every $k \in N$, E_k denotes the set of all eigenfunctions "on the level β_k", i.e.

$$E_k \equiv \{u \in M_\alpha \,|\, D(u) = 0, |b(u)| = \beta_k\}.$$

Now we prove part c of Theorem A.

Proposition 2. *Let the hypotheses of Theorem A be satisfied. Suppose, for some $j, m \in N$, we have*

$$\beta_j = \beta_{j+1} = \cdots = \beta_{j+m} > 0.$$

Then gen $(E_j) \geqq m + 1$.

Proof. By Lemma 3.5, E_j is compact and, by Proposition 1, E_j is not void. Hence there exists an open symmetric neighborhood U_j of E_j in M_α such that $gen(\bar{U}_j) = gen(E_j) < \infty$.

Next we prove the existence of an open symmetric neighborhood U of E_j in M_α and of positive constants ϱ and δ such that

$$gen(\bar{U}) = gen(E_j) \tag{4.2}$$

and

$$u \in \{u \in M_\alpha \backslash U \,\big|\, |b(u)| - \beta_j| \leqq \varrho\} \text{ implies } \|D(u)\| \geqq \delta. \tag{4.3}$$

To do this we observe that there exists a positive ϱ such that

$$F \equiv E \cap \{u \in M_\alpha \,|\, |b(u) - \beta_j| \leqq \varrho\} \subset U_j.$$

Indeed, otherwise we could find a sequence $\{u_n\} \in M_\alpha \backslash U_j$ with $|b(u_n)| \to \beta_j$ and $\{u_n\} \subset E$. Then, by Lemma 3.5, there exists a convergent subsequence $u_n \to u$ such that $u \in E \backslash U_j$ which is impossible.

Obviously, we may assume that $\varrho < \beta_j$. Then, again by Lemma 3.5, F is compact. Hence, it has a positive distance σ from the closed set $M_\alpha \backslash U_j$. Denote by U the $\sigma/2$-neighborhood of F. Then, again by Lemma 3.5, it is easily seen that there exists a $\delta > 0$ such that U and δ satisfy (4.3). Finally, since $E_j \subset U \subset \bar{U} \subset U_j$ we have

$$gen(E_j) \leqq gen(\bar{U}) \leqq gen(\bar{U}_j) = gen(E_j)$$

which proves (4.2).

Therefore, by Lemma 3.4, there exists an $\varepsilon > 0$ and an odd continuous mapping H_ε with $\overline{H_\varepsilon(N_{\beta_j-\varepsilon} \backslash U)} \subset N_{\beta_j+\varepsilon}$.
By definition of β_{j+m} there exists a $C_\varepsilon \in \mathscr{C}_{j+m}$ with

$$|b(u)| \geqq \beta_{j+m} - \varepsilon = \beta_j - \varepsilon$$

on C_ε.

It is easily seen that the genus over compact sets γ has on $\mathscr{C}$ the same monotonicity and subadditivity properties as the genus itself. Hence, by

$$gen(E_j) = \gamma(E_j) \leqq \gamma(\bar{U}) \leqq gen(\bar{U}) = gen(E_j),$$

we find

$$gen\,(E_j) = \gamma(\bar{U}).$$

Suppose $gen\ (E_j) \leqq m$. Then, by the above mentioned properties of γ,

$$\begin{aligned} \gamma(N_{\beta_j+\varepsilon}) &\geqq \gamma(\overline{H_\varepsilon(N_{\beta_j-\varepsilon} \backslash U)}) \geqq \gamma(N_{\beta_j-\varepsilon} \backslash U) \\ &\geqq \gamma(N_{\beta_j-\varepsilon} \cup \bar{U}) - \gamma(\bar{U}) \geqq \gamma(N_{\beta_j-\varepsilon}) - gen(E_j). \end{aligned}$$

Hence

$$gen(C_\varepsilon) = \gamma(C_\varepsilon) \leqq \gamma(N_{\beta_j-\varepsilon}) \leqq \gamma(N_{\beta_j+\varepsilon}) + m.$$

By definition of β_j we find

$$\gamma(N_{\beta_j+\varepsilon}) < j,$$

hence

$$gen(C_\varepsilon) < j + m.$$

On the other hand, $C_\varepsilon \in \mathscr{C}_{j+m}$, hence

$$gen(C_\varepsilon) \geqq j + m,$$

which gives a contradiction. Hence $gen(E_j) \geqq m + 1$. q.e.d.

The next proposition finally proves part a of Theorem A.

Proposition 3. *Let the hypotheses of Theorem* A *be satisfied and suppose* (2.3) *holds. Then the eigenvalue problem* (2.2) *has infinitely many distinct eigenfunctions on* M_α.

Proof. Condition (2.3) guaranties that, for every $k \in N$, $\beta_k > 0$. Hence, by Proposition 1, the set E is not empty. Suppose E is a finite set. Then, since obviously $\beta_1 \geqq \beta_2 \geqq \cdots$, there exists a $j \in N$ such that $\beta_k = \beta_j$ for all $k \geqq j$. By Proposition 2, this implies $gen(E_j) = \infty$ which, by Lemma 4.1, gives a contradiction to the finiteness of E. Hence the statement follows. q.e.d.

Proof of Part d *of Theorem* A. Let X be a real infinite dimensional Hilbert space and set $X^* = X$. We identify A with the identity mapping on X and we denote by B a linear symmetric compact operator on X with only finitely many distinct positive eigenvalues $\mu_1, \ldots, \mu_m$ and corresponding normalized eigenfunctions $v_1, \ldots, v_m$. Hence

$$Bu = \sum_{i=1}^{m} \mu_i \langle v_i, u \rangle v_i$$

and

$$b(u) = \frac{1}{2} \sum_{j=1}^{m} \mu_i \langle v_i, u \rangle^2 .$$

Therefore, for every $\alpha > 0, u \in \{u \in M_\alpha | b(u) \neq 0\}$ if and only if

$$\|u\| = \alpha \quad \text{and} \quad u \notin [span(v_1, \ldots, v_m)]^\perp .$$

We define an odd continuous mapping $f: X \to \boldsymbol{R}^m$ by

$$f(u) = (\langle v_1, u \rangle, \ldots, \langle v_m, u \rangle).$$

Since this map has no zero if $b(u) \neq 0$ it follows that

$$\gamma\{u \in M_\alpha | b(u) \neq 0\} \leqq m .$$

Hence all the assumptions of part d of Theorem A are fulfilled but the "eigenvalue" problem

$$u = \lambda B(u)$$

has only the m normalized pairs of eigenfunctions $\pm \alpha v_1, \ldots, \pm \alpha v_m$. Hence Theorem A is completely proved.

In case of the general non-linear eigenvalue problem (1.1) there is no hope for a "completeness result" for the system of eigenfunctions. Hence, since it was our aim to prove "only" the existence of infinitely many distinct eigenfunctions it was possible to consider the variational problem (4.1). In the finite dimensional case however, this is not the appropriate problem to be studied since we do not obtain those eigenfunctions with $b(u)=0$. In the finite dimensional case one has to investigate the variational principle

$$\sup_{C \in \mathscr{C}_k} \min_{u \in C} b(u) \tag{4.4}$$

which can be done without difficulties. By this way we obtain the following

Theorem 1. *Let X be a finite dimensional real Banach space. Let $A: X \to X^*$ be an odd continuous potential operator with $\langle A(u), u \rangle > 0$ if $u \neq 0$ and choose $\alpha > 0$, such that every ray through the origin intersects M_α. Let $B: X \to X^*$ be a continuous odd potential operator such that $B(u) \neq 0$ on M_α. Then, the non-linear eigenvalue problem*

$$A(u) = \lambda B(u)$$

has at least dim X many distinct pairs $(u, -u)$ of eigenfunctions on M_α.

Proof. Since X is finite dimensional we can assume that X is uniformly convex. Using the variational problem (4.4) instead of (4.1) the proof follows in exactly the same way as the proof of Theorem 1. Since X is finite dimensional, every bounded sequence has a convergent subsequence and all the difficulties of the infinite dimensional case disappear. q.e.d.

5. Eigenvalue Problems for Hammerstein Equations

The essential tool for applying the general theory of the previous paragraphs to the Hammerstein equation is the following "splitting lemma".

Lemma 5.1. *Let X be an arbitrary real Banach space and let $K: X \to X^*$ be a compact, non-negative, linear operator. Then, there exists a Hilbert space Y and a linear operator $S: X \to Y$ with the following properties:*

i) *S is compact, $R(S)$ is dense in Y, and the dual operator $S^*: Y \to X^*$ is injective.*

ii) *$K = S^* S$.*

iii) $R(K) \subset R(S^*) \subset \overline{R(K)}$.

Proof. The existence of Y and S with the properties stated in i) and ii) (except the compactness) follows from a more general result of Browder and Gupta [8, Theorem 4]. The compactness of S is proved in [2, Lemma 1]. $R(K) \subset R(S^*)$ follows from ii) whereas $R(S^*) \subset \overline{R(K)}$ follows from the fact that $R(S)$ is dense in Y. q.e.d.

The fundamental idea in constructing the Hilbert space Y consists in introducing on the factor space $X/N(K)$ the inner product

$$(u,v) \equiv \langle Ku, v\rangle$$

and taking the completion. Hence, by means of the generalized inverse $L: R(K) \to X/N(K)$, for every $u \in R(K)$, we obtain

$$\|Lu\|_Y^2 = \langle u, Lu\rangle. \tag{5.1}$$

Proof of Theorem B. By Lemma 5.1 the eigenvalue problem (2.4) takes the form

$$u = \lambda S^* S F(u). \tag{5.2}$$

Hence, every eigenfunction necessarily has the form $u = S^* v$ with some $v \in Y$. Therefore, since S^* is injective, (5.2) is equivalent to

$$v = \lambda S F(S^* v).$$

It is easily seen that $B \equiv S \circ F \circ S^*$ is an odd potential operator with potential $b \equiv f \circ S^*$. Moreover, S being compact, S^* is compact too and therefore B is strongly continuous. Finally, $B(v) = 0$ if and only if $KF(S^* v) = 0$. Therefore, by our hypotheses, for all non-zero $v \in Y$, $B(v) \neq 0$ and $b(v) \neq 0$.

Since S^* is injective it follows from part iii) of Lemma 5.1 that dim $Y = \dim R(K)$. Therefore, by applying Theorem A or Theorem 1, respectively, to the problem

$$v = \lambda B(v) \tag{5.3}$$

in the Hilbert space Y we find that on every sphere $\{v \in Y \mid \|v\|_Y^2 = \alpha\}$ there exist at least dim $R(K)$ many distinct pairs of eigenfunctions of (5.3). To every eigenfunction $v \in Y$ of (5.3) there corresponds an eigenfunction $u = S^* v$ of (2.4) and vice versa. Since every eigenfunction u of (2.4) obviously belongs to $R(K)$ there exists a $w \in X$ with $v = Sw$. Hence $\alpha = \|v\|_Y^2 = (Sw, Sw) = \langle Kw, w\rangle = \langle u, Lu\rangle$ and the theorem follows. q.e.d.

In Theorem B the Hammerstein equation is considered as an equation in the dual space of a Banach space. But, since it is only assumed that F is defined on the closed linear subspace $\overline{R(K)}$, we may identify X

with the dual of a Banach space Z, i.e. $X = Z^*$ and suppose that

$$K : X \to Z \subset Z^{**} = X^*.$$

By this way we obtain the following

Corollary. *Let X be an arbitrary real Banach space and suppose $K : X^* \to X$ is a nonnegative compact linear operator. Suppose $F : X \to X^*$ is an odd continuous potential operator such that, for all non-zero $u \in X$, $f(u) \neq 0$ and $KF(u) \neq 0$.*

Then, for every $\alpha > 0$, the eigenvalue problem

$$u = \lambda K F(u)$$

in X has at least dim $R(K)$ many distinct pairs $(u, -u)$ of eigenfunctions satisfying the normalization condition

$$\langle Lu, u \rangle = \alpha.$$

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H. Amann
Institut für Mathematik
Ruhr-Universität Bochum
D-4630 Bochum
Federal Republic of Germany

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