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Lie Algebra Homology and the Macdonald-Kac Formulas

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Introduction

I.G. Macdonald's noted identities concerning Dedekind's η-function were originally obtained in [14] by means of "affine root systems". These formulas have subsequently been interpreted by V.G. Kac [11(b)] and R.V. Moody [15(c)] as the precise analogues of Weyl's denominator formula for the "Euclidean Lie algebras" (Moody's term)–certain infinite-dimensional analogues of complex semisimple Lie algebras introduced by Kac [11(a)] and Moody [15(a), (b)]. In [11(b)], Kac also sketches a new *proof* of the Macdonald identities, and in fact of a much wider class of identities–the analogues of both Weyl's character *and* denominator formulas, for a family of Lie algebras considerably more general than the Euclidean Lie algebras. These more general algebras, also introduced by Kac [11(a)] and Moody [15(a)], are the Lie algebras defined by symmetrizable (generalized) Cartan matrices (see § 2 below). The main purpose of the present paper is to generalize B. Kostant's fundamental result [12, Theorem 5.14] on the homology (or cohomology) of nilradicals of parabolic subalgebras in certain modules, from (finite-dimensional) complex semisimple Lie algebras to the Kac-Moody Lie algebras defined by symmetrizable Cartan matrices (see Theorem 8.6). We thus obtain the results in [11(b)], including the Macdonald identities, as immediate consequences of the Euler-Poincaré principle (§ 9), just as Kostant derives Weyl's character and denominator formulas in [12, § 7] from his homology theorem.

Kac's method in [11(b)] is to adapt to Lie algebras defined by symmetrizable Cartan matrices the simple proof, using Verma modules, of Weyl's character formula given by I.N. Bernstein, I.M. Gelfand and S.I. Gelfand in [1(a)]. (This proof is also presented in [6, § 7.5] and [9, § 24].) But Kac must make a certain modification: In place of the Harish-Chandra isomorphism theorem concerning the center of the universal enveloping algebra used in [1(a)] (see [6, § 7.4] or [9, § 23]), he uses a Casimir operator, which in effect plays the role of a single decisive element of the center of the universal enveloping algebra. When applied to complex semisimple Lie algebras, Kac's argument thus simplifies the proof in [1(a)] of Weyl's character formula. (Incidentally, the second named author of the present paper independently discovered this simplified proof of Weyl's character formula, but did not attempt to apply it to the infinite-dimensional Lie algebras.)

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Now in [1 (b), § 9], Bernstein, Gelfand and Gelfand construct a resolution, in terms of Verma modules, of a finite-dimensional irreducible module for a complex semisimple Lie algebra $\mathfrak{g}$, and they use it to give a simple proof of Bott's theorem [2, § 15] on the dimensions of the cohomology spaces of a maximal nilpotent subalgebra of $\mathfrak{g}$ in such a $\mathfrak{g}$-module. Their proof uses the Harish-Chandra isomorphism theorem cited above. In the present paper, we further simplify their proof by using the Casimir element in place of the Harish-Chandra theorem, we refine and extend their method so as to obtain Kostant's homology theorem [12, Theorem 5.14] in full generality (including the action of the reductive part of the parabolic subalgebra on the homology and cohomology), and we carry out the argument for Lie algebras defined by symmetrizable Cartan matrices. In particular, we considerably generalize one form of their resolution (see Theorem 8.7). Our use of the Casimir operator is somewhat similar to P. Cartier's argument in [4] (cf. also Kostant's use of the laplacian in [12]).

The main difficulty in writing this paper has been to deal with the technicalities encountered in working with infinite-dimensional Lie algebras defined by Cartan matrices. We have to contend with infinite root systems, infinite Weyl groups, infinitely long filtrations of modules, infinitely long complexes, etc. The papers [11 (a)] and [15 (a), (b)] provide excellent background, but we also need important ideas from [11 (b)], which omits many details. So in §§ 2, 3, 4 and 6 we have included among other things a detailed exposition of some of the tools used in [11 (b)], with certain modifications designed to make these tools more flexible and natural.

Specifically, we have defined "extended" Lie algebras $\mathfrak{g}^e$ (§ 2) by adjoining certain derivations to the usual Kac-Moody Lie algebras. This construction provides us with a nice framework for the set of roots (§ 2), the Weyl group (§ 2), the invariant bilinear form σ (§ 2), "weight modules" (§§ 3, 4), the generalizations of Verma modules (§ 3) and the generalizations of finite-dimensional irreducible modules (§ 6). Unlike Kac's setup in [11 (b)], ours has enough built-in flexibility so that for example when one is dealing with ordinary finite-dimensional semisimple Lie algebras, all the above concepts and related concepts reduce to the usual classical ones. This is achieved by choosing $\mathfrak{d}=0$ (i.e., $\mathfrak{g}^e=\mathfrak{g}$) in § 2. In particular, the reader who is interested in seeing a simple new proof of Kostant's clasical theorem (and hence of the Borel-Weil-Bott theorem; cf. [12]), can start with a classical Cartan matrix A arising in the usual way from a complex semisimple Lie algebra and take $\mathfrak{d}=0$ in §2, and then this paper simplifies to a series of well-known facts and easy proofs. Likewise, the reader who wants to see a short proof of Macdonald's identities (in the form [15 (c), Proposition 2]) can ignore much of this paper by taking S to be the null set (§ 3) and X to be the trivial one-dimensional module (§ 6), but of course he must work in the generality of Lie algebras defined by symmetrizable Cartan matrices. (Specializing to Euclidean Lie algebras does not really shorten our argument further.)

As was indicated above, the proof of our main results was inspired by the proof of Bott's theorem in [1 (b)]. Our argument rests on the construction of a certain natural resolution of a Lie algebra module in a general setting (Proposition 1.9; see also Proposition 7.1). This is based in turn on the complex $V(\mathfrak{b}, \mathfrak{a})$ constructed in [1 (b)] (see Proposition 1.1) and a discussion relating it directly with homology and cohomology in certain modules (§ 1), together with a useful very general Hopf algebra principle on induced modules and tensor products (see Proposition 1.7 and the subsequent Remark). The "parabolic subalgebras" of the extended Kac-Moody Lie algebras that enter into our work are the "F-parabolic subalgebras" defined in § 3. The "F" refers to the fact that the "reductive part" is finite-dimensional. F-parabolic subalgebras give rise to finite-dimensional homology in each degree (see Theorem 8.6). In place of the Verma modules used in [1(b)], we use "generalized Verma modules" – modules induced from finite-dimensional irreducible modules for F-parabolic subalgebras. Our resolution (generalizing [1(b), Theorem 9.9]) of an arbitrary quasisimple module (see § 6) is in terms of modules each of which has a finite filtration whose successive quotients are generalized Verma modules, and the highest weights of the inducing modules are related by the Weyl group in a natural way (see Theorem 8.7). This resolution will be used in subsequent work [13(c)].

In the Appendix, we present a simplified proof of our generalization of Kostant's theorem, but this simplification does not yield the resolution (Theorem 8.7).

In [13(a), (b)], certain generalized Verma modules for finite-dimensional semisimple Lie algebras are discussed from a completely different viewpoint. Hopefully, the methods of the present paper can shed some new light on the issues encountered in that work.

This paper was motivated by a desire to "understand" and generalize the first named author's explanation by means of an Euler-Poincaré principle of the Macdonald formulas for the powers $\eta^{\dim \mathfrak{a}}$ of Dedekind's η-function ($\mathfrak{a}$ a complex semisimple Lie algebra) [7]. The explanation consisted of a version of Theorem 8.6 for a special class of Euclidean Lie algebras – those of the form $\mathfrak{a}\otimes\mathbb{C}\langle x\rangle$ ($\mathfrak{a}$ as above, $\mathbb{C}\langle x\rangle$ the algebra of finite Laurent series in one variable) – and the trivial one-dimensional quasisimple module X, in the notation of Theorem 8.6. The method of proof, similar in spirit to Kostant's in [12], involved the explicit, and complicated, computation of a laplacian using a hermitian structure.

The reader is referred to the announcement [7] for a detailed discussion of what amounts to an important special case of the present paper, and to [19] for an interesting exposition of several matters concerning Macdonald's identities and Euclidean Lie algebras.

The reader is also referred to the original article [14] of I. G. Macdonald, and the paper of F. Dyson ("Missed Opportunities," Bulletin of the American Math. Soc. **78** (1972), pp. 635–653). Finally, we mention [8], which also bears some relation with the present work (see the last paragraph of § 2 in [8] and Corollary 3.6 in [7]).

We are very grateful to N. Conze-Berline for providing us with a preprint of [1(b)].[1]

§ 1. The Resolutions $V(\mathfrak{b}, \mathfrak{a}, N)$

We first recall the complexes $V(\mathfrak{b}, \mathfrak{a})$ constructed in [1(b), § 9].

Let $\mathfrak{b}$ be a Lie algebra over a field k, and $\mathfrak{a}$ a subalgebra of $\mathfrak{b}$. Let $\mathscr{A}$ and $\mathscr{B}$ be the universal enveloping algebras of $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and regard $\mathscr{A}$ as a

[1] Also, see notes added in proof at the end of the paper

subalgebra of $\mathscr{B}$. We shall identify Lie algebra modules with the corresponding universal enveloping algebra modules.

The natural action of $\mathfrak{a}$ on $\mathfrak{b}/\mathfrak{a}$ extends uniquely to an action of $\mathfrak{a}$ on the exterior algebra $\Lambda(\mathfrak{b}/\mathfrak{a})$ by derivations. Let $j \in \mathbb{Z}_+$ (the set of nonnegative integers). Then $\Lambda^j(\mathfrak{b}/\mathfrak{a})$ is an $\mathfrak{a}$-submodule of $\Lambda(\mathfrak{b}/\mathfrak{a})$, and we can form the induced $\mathfrak{b}$-module $D_j = \mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a})$ (cf. [6, Chapitre 5]). Suppose $j>0$. We define a linear map $d_j\colon D_j \to D_{j-1}$ as follows: Let $x_1, \dots, x_j \in \mathfrak{b}/\mathfrak{a}$, and choose representatives $y_1, \dots, y_j \in \mathfrak{b}$. Then for all $x \in \mathscr{B}$,

$$\begin{aligned} d_j(x \otimes x_1 \wedge \cdots \wedge x_j) &= \sum_{i=1}^{j} (-1)^{i+1} (x y_i) \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_j \\ &\quad + \sum_{1 \leq r < s \leq j} (-1)^{r+s} x \otimes \pi[y_r, y_s] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_j, \end{aligned}$$

where $\pi\colon \mathfrak{b} \to \mathfrak{b}/\mathfrak{a}$ denotes the canonical map, and $\hat{}$ signifies the omission of a symbol. It is easy to see that d_j is independent of the choice of representatives $y_1, \dots, y_j$, and that d_j is a $\mathfrak{b}$-module map.

Define $\varepsilon_0\colon D_0 \to k$ by the condition that $\varepsilon_0(b \otimes 1)$ $(b \in \mathscr{B})$ be the constant term of b. Then we have a sequence $V(\mathfrak{b}, \mathfrak{a})$ of $\mathfrak{b}$-modules and $\mathfrak{b}$-module maps

$$\cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \xrightarrow{\varepsilon_0} k \to 0.$$

The following result is obtained in [1(b), Theorem 9.1]; the proof follows standard lines and consists of a reduction to the acyclicity of the Koszul complex:

Proposition 1.1. *The sequence $V(\mathfrak{b}, \mathfrak{a})$ is exact.*

Remark. The complex $V(\mathfrak{b}, 0)$ is the standard $\mathscr{B}$-free resolution of the trivial $\mathfrak{b}$-module k.

Let $\mathfrak{c}$ be a subalgebra of $\mathfrak{b}$ such that $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{c}$ as a vector space, and let $\mathscr{C}$ be the universal enveloping algebra of $\mathfrak{c}$. The Poincaré-Birkhoff-Witt theorem immediately gives (see [1(b), Proposition 9.2]):

Proposition 1.2. *Regarded as a complex of $\mathfrak{c}$-modules, $V(\mathfrak{b}, \mathfrak{a})$ is naturally isomorphic to the standard $\mathscr{C}$-free resolution $V(\mathfrak{c}, 0)$ of the trivial $\mathfrak{c}$-module k.*

Remark. The considerations of § 1 will be applied in this paper to a generalization of the situation in which $\mathfrak{b}$ is a split semisimple Lie algebra of characteristic zero, $\mathfrak{a}$ is a parabolic subalgebra of $\mathfrak{b}$, and $\mathfrak{c}$ is the nilradical of the opposite parabolic subalgebra.

Definition. Let N be an arbitrary $\mathfrak{b}$-module. Denote by $V(\mathfrak{b}, \mathfrak{a}, N)$ the sequence of tensor product $\mathfrak{b}$-modules and $\mathfrak{b}$-module maps

$$\cdots \xrightarrow{d_2 \otimes 1} D_1 \otimes N \xrightarrow{d_1 \otimes 1} D_0 \otimes N \xrightarrow{\varepsilon_0 \otimes 1} N \to 0.$$

(See the definition of $V(\mathfrak{b}, \mathfrak{a})$ for the notation.)

Remarks. (1) The complement $\mathfrak{c}$ is not needed for the definition of $V(\mathfrak{b}, \mathfrak{a}, N)$.

(2) $V(\mathfrak{b}, \mathfrak{a}, k) \simeq V(\mathfrak{b}, \mathfrak{a})$, where k is regarded as the trivial $\mathfrak{b}$-module.

(3) In view of Proposition 1.1, $V(\mathfrak{b}, \mathfrak{a}, N)$ is an exact sequence.

(4) We shall illuminate the structure of each $D_j \otimes N$ below, showing in particular that it is $\mathscr{C}$-free. Thus $V(\mathfrak{b}, \mathfrak{a}, N)$ is a $\mathscr{C}$-free resolution of N.

We shall now relate $V(\mathfrak{b}, \mathfrak{a}, N)$ to the homology and cohomology of $\mathfrak{c}$ in certain modules.

Definition. Consider the $\mathfrak{b}$-module complex $V'(\mathfrak{b}, \mathfrak{a}, N)$ obtained from $V(\mathfrak{b}, \mathfrak{a}, N)$ by deleting the segment $\xrightarrow{\varepsilon_0 \otimes 1} N$. Denote by $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ the complex

$$\cdots \rightarrow V_1(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c}) \rightarrow V_0(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c}) \rightarrow 0$$

of vector spaces obtained by tensoring $V'(\mathfrak{b}, \mathfrak{a}, N)$ on the left, over $\mathscr{C}$, with the trivial right $\mathfrak{c}$-module k. That is, $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is the complex

$$\cdots \xrightarrow{1 \otimes (d_2 \otimes 1)} k \otimes_{\mathscr{C}} (D_1 \otimes N) \xrightarrow{1 \otimes (d_1 \otimes 1)} k \otimes_{\mathscr{C}} (D_0 \otimes N) \rightarrow 0.$$

(Each $D_j \otimes N$ is regarded as a $\mathfrak{c}$-module by restriction.)

Remark. For each $j \in \mathbb{Z}_+$, $k \otimes_{\mathscr{C}} (D_j \otimes N)$ is naturally isomorphic to the space $(D_j \otimes N)/\mathfrak{c} \cdot (D_j \otimes N)$, and the maps in $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ may be identified with the natural quotient maps coming from the maps in $V'(\mathfrak{b}, \mathfrak{a}, N)$.

Notation. Let $T: \mathscr{B} \rightarrow \mathscr{B}$ denote the transpose map of $\mathscr{B}$, i.e., the unique antiautomorphism which is -1 on $\mathfrak{b}$. Denote by N^t the right $\mathfrak{b}$-module whose space is N and on which $\mathscr{B}$ acts by the rule $n \cdot b = T(b) \cdot n$ for all $n \in N$ and $b \in \mathscr{B}$.

Proposition 1.3. *The homology of the complex $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally isomorphic as graded vector space to the homology $H_*(\mathfrak{c}, N^t)$ of $\mathfrak{c}$ in N^t, regarded as a right $\mathfrak{c}$-module by restriction.* (*See* [3, p. 282] *for the definition of Lie algebra homology in a right module.*)

Proof. By Proposition 1.2, we may replace the modules D_j and maps d_j in the definition of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ by the corresponding $\mathfrak{c}$-modules and $\mathfrak{c}$-module maps occurring in the standard resolution $V(\mathfrak{c}, 0)$. Denote these replacements by D'_j and d'_j. For each j, we have a natural linear isomorphism $\omega: k \otimes_{\mathscr{C}} (D'_j \otimes N) \rightarrow N^t \otimes_{\mathscr{C}} D'_j$, given by the condition $\omega(1 \otimes (d \otimes n)) = n \otimes d$ $(d \in D'_j, n \in N)$. Hence $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally isomorphic to the complex

$$\cdots \xrightarrow{1 \otimes d'_2} N^t \otimes_{\mathscr{C}} D'_1 \xrightarrow{1 \otimes d'_1} N^t \otimes_{\mathscr{C}} D'_0 \rightarrow 0.$$

But the homology of this complex is precisely $H_*(\mathfrak{c}, N^t)$, by [3, p. 282]. Q.E.D.

Concretely, $H_*(\mathfrak{c}, N^t)$ may be realized as the homology of the *standard homology complex*

$$\cdots \xrightarrow{\partial_3} N^t \otimes_k \Lambda^2(\mathfrak{c}) \xrightarrow{\partial_2} N^t \otimes_k \Lambda^1(\mathfrak{c}) \xrightarrow{\partial_1} N^t \otimes_k \Lambda^0(\mathfrak{c}) \rightarrow 0,$$

where for all $j > 0$, $n \in N^t$ and $c_1, \ldots, c_j \in \mathfrak{c}$,

$$\begin{aligned} &\partial_j(n \otimes c_1 \wedge \cdots \wedge c_j) = \\ &\quad = \sum_{i=1}^{j} (-1)^{i+1} (n \cdot c_i) \otimes c_1 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_j \\ &\quad + \sum_{1 \leqq r < s \leqq j} (-1)^{r+s} n \otimes [c_r, c_s] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_s \wedge \cdots \wedge c_j \end{aligned}$$

(see [3, p. 282]). Now suppose that $\mathfrak{s}$ is a subalgebra of $\mathfrak{b}$ such that $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$. The action of $\mathfrak{s}$ on $\mathfrak{c}$ extends uniquely to an action of $\mathfrak{s}$ on $\Lambda(\mathfrak{c})$ by derivations, giving each $\Lambda^j(\mathfrak{c})$ $\mathfrak{s}$-module structure. Identify the right $\mathfrak{s}$-module N^t with the (left) $\mathfrak{s}$-module N, and let $\mathfrak{s}$ act on $N^t \otimes \Lambda^j(\mathfrak{c})$ by the tensor product action on $N \otimes \Lambda^j(\mathfrak{c})$. Then it is easy to check that the action of $\mathfrak{s}$ commutes with the maps ∂_j. We thus have the *standard action* of $\mathfrak{s}$ on the standard homology complex. The resulting actions of $\mathfrak{s}$ on the homology spaces $H_j(\mathfrak{c}, N^t)$ constitute the *standard action* of $\mathfrak{s}$ on $H_*(\mathfrak{c}, N^t)$.

Proposition 1.4. *Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{a}$ such that $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$. Then $\mathfrak{s}$ acts in a natural way on the complex $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, and this action is naturally equivalent to the standard action of $\mathfrak{s}$ on the standard homology complex for computing $H_*(\mathfrak{c}, N^t)$. In particular, the natural action of $\mathfrak{s}$ on the homology of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally equivalent to the standard action of $\mathfrak{s}$ on $H_*(\mathfrak{c}, N^t)$.*

Proof. In the notation of the definition of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, the natural action of $\mathfrak{s}$ is given as follows:

Let $j \in \mathbb{Z}_+$, $d \in D_j$, $n \in N$ and $s \in \mathfrak{s}$. Then $s \cdot (1 \otimes (d \otimes n)) = 1 \otimes s \cdot (d \otimes n)$. (See the Remark following the definition of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$.) Now $D_j = \mathscr{B} \otimes_{\mathscr{A}} \Lambda^j(\mathfrak{b}/\mathfrak{a})$. We may take d of the form $c \otimes x$, where $c \in \mathscr{C}$ and $x \in \Lambda^j(\mathfrak{b}/\mathfrak{a})$. Then $s \cdot d = sc \otimes x = [s, c] \otimes x + c \otimes s \cdot x$, because $\mathfrak{s} \subset \mathfrak{a}$. In the notation of the proof of Proposition 1.3, $D'_j = \mathscr{C} \otimes_k \Lambda^j(\mathfrak{c})$. Identifying $k \otimes_{\mathscr{C}} (D_j \otimes N)$ with $k \otimes_{\mathscr{C}} (\mathscr{C} \otimes_k \Lambda^j(\mathfrak{c}) \otimes_k N)$ as in the first step of that proof, we see that $s \in \mathfrak{s}$ acts by the rule

$$s \cdot (1 \otimes (c \otimes y \otimes n)) = 1 \otimes ([s, c] \otimes y \otimes n) + 1 \otimes (c \otimes s \cdot y \otimes n) + 1 \otimes (c \otimes y \otimes s \cdot n),$$

where $c \in \mathscr{C}$, $y \in \Lambda^j(\mathfrak{c})$, $n \in N$ and $s \cdot y$ denotes the natural action. Applying the isomorphism ω in the above proof, we see that s acts on $N^t \otimes_{\mathscr{C}} D'_j$ by the condition

$$s \cdot (n \otimes (1 \otimes y)) = -n \cdot s \otimes (1 \otimes y) + n \otimes (1 \otimes s \cdot y)$$

(same notation), where on the right-hand side, s acts on the right on n. Finally, identifying $N^t \otimes_{\mathscr{C}} D'_j$ with $N \otimes_k \Lambda^j(\mathfrak{c})$, we recover the standard action of s on the standard complex. Thus $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally $\mathfrak{s}$-module isomorphic to the standard complex, regarded as an $\mathfrak{s}$-module complex. Passing to homology gives the last assertion. Q.E.D.

Here is the corresponding picture for cohomology:

Definition. Denote by $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ the complex of vector spaces

$$\cdots \leftarrow V^1(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c}) \leftarrow V^0(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c}) \leftarrow 0$$

obtained by dualizing $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$. That is, $V^j(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is the dual space of $V_j(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ for each j, and the maps in $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ are the contragredients of the maps in $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$.

Remark. For each $j \in \mathbb{Z}_+$, $V^j(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally isomorphic to the space $\mathrm{Hom}_{\mathfrak{c}}(D_j \otimes N, k)$, where k is regarded as the trivial $\mathfrak{c}$-module. $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is obtained by taking the complex of $\mathfrak{c}$-invariants in the dual complex to $V'(\mathfrak{b}, \mathfrak{a}, N)$ (using the notation of the definition of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$).

Proposition 1.5. *The homology of the complex $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally isomorphic as graded vector space to the cohomology $H^*(\mathfrak{c}, N^*)$ of $\mathfrak{c}$ in the contragredient $\mathfrak{b}$-module N^*, regarded as a $\mathfrak{c}$-module by restriction. (See [3, p. 282] for the definition of this cohomology.) This in turn is naturally isomorphic to the graded vector space of dual spaces $H_j(\mathfrak{c}, N^t)^*$.*

Proof. By definition of $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, its j-th homology is naturally isomorphic to the dual space of the j-th homology of $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, for each j. By Proposition 1.3, this is naturally isomorphic to the dual space of $H_j(\mathfrak{c}, N^t)$. But this dual is naturally isomorphic to $H^j(\mathfrak{c}, N^*)$. In fact, $H_j(\mathfrak{c}, N^t)$ is the j-th homology of the complex

$$\cdots \xrightarrow{1 \otimes d_2'} N^t \otimes_{\mathscr{C}} D_1' \xrightarrow{1 \otimes d_1'} N^t \otimes_{\mathscr{C}} D_0' \to 0,$$

in the notation of the proof of Proposition 1.3, and $H^j(\mathfrak{c}, N^*)$ is the j-th homology of the dual complex, namely,

$$\cdots \xleftarrow{\mathrm{Hom}(d_2', 1)} \mathrm{Hom}_{\mathfrak{c}}(D_1', N^*) \xleftarrow{\mathrm{Hom}(d_1', 1)} \mathrm{Hom}_{\mathfrak{c}}(D_0', N^*) \leftarrow 0$$

(see [3, p. 282]). Q.E.D.

$H^*(\mathfrak{c}, N^*)$ may be realized as the homology of the *standard cohomology complex*

$$\cdots \xleftarrow{\delta_3} \mathrm{Hom}_k(\Lambda^2(\mathfrak{c}), N^*) \xleftarrow{\delta_2} \mathrm{Hom}_k(\Lambda^1(\mathfrak{c}), N^*) \xleftarrow{\delta_1} \mathrm{Hom}_k(\Lambda^0(\mathfrak{c}), N^*) \leftarrow 0,$$

where for all $j \in \mathbb{Z}_+$, $f \in \mathrm{Hom}_k(\Lambda^j(\mathfrak{c}), N^*)$ and $c_1, \ldots, c_{j+1} \in \mathfrak{c}$,

$$\begin{aligned}(\delta_{j+1} f)(c_1 \wedge \cdots \wedge c_{j+1}) &= \sum_{i=1}^{j+1} (-1)^{i+1} c_i \cdot f(c_1 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_{j+1}) \\ &+ \sum_{i \leqq r < s \leqq j+1} (-1)^{r+s} f([c_r, c_s] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_s \wedge \cdots \wedge c_{j+1})\end{aligned}$$

(see [3, p. 282]). Identifying $\mathrm{Hom}_k(\Lambda^j(\mathfrak{c}), N^*)$ with $(N \otimes \Lambda^j(\mathfrak{c}))^*$ in the natural way for each j, we see easily that the standard cohomology complex is the dual complex to the standard homology complex (see above). If $\mathfrak{s}$ is a subalgebra of $\mathfrak{b}$ such that $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$, then $\mathfrak{s}$ acts in the obvious way on $\mathrm{Hom}(\Lambda^j(\mathfrak{c}), N^*)$ for each j, and the maps δ_j commute with this action, as is easily seen. In fact, the resulting *standard action* of $\mathfrak{s}$ on the standard cohomology complex is the contragredient of the standard action of $\mathfrak{s}$ on the standard homology complex. The induced action of $\mathfrak{s}$ on homology is called the *standard action* of $\mathfrak{s}$ on $H^*(\mathfrak{c}, N^*)$, and it is clear that for each j, the standard action of $\mathfrak{s}$ on $H^j(\mathfrak{c}, N^*)$ is the contragredient of the standard action of $\mathfrak{s}$ on $H_j(\mathfrak{c}, N^t)$.

From Proposition 1.4 and the above, we now have:

Proposition 1.6. *Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{a}$ such that $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$. Then $\mathfrak{s}$ acts in a natural way on the complex $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, and this action is naturally equivalent to the standard action of $\mathfrak{s}$ on the standard cohomology complex for computing $H^*(\mathfrak{c}, N^*)$. In particular, the natural action of $\mathfrak{s}$ on the homology of $V^*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$ is naturally equivalent to the standard action of $\mathfrak{s}$ on $H^*(\mathfrak{c}, N^*)$. For each $j \in \mathbb{Z}_+$, $H^j(\mathfrak{c}, N^*)$ is naturally isomorphic to the $\mathfrak{s}$-module contragredient to $H_j(\mathfrak{c}, N^t)$, provided with the standard action of $\mathfrak{s}$.*

We shall now illuminate the structure of $V(\mathfrak{b}, \mathfrak{a}, N)$, and in particular, its terms $D_j \otimes N$ (see Remark (4) following the definition of $V(\mathfrak{b}, \mathfrak{a}, N)$), by deriving the following useful general principle, not requiring the existence of $\mathfrak{c}$, concerning inducing and tensor products:

Proposition 1.7. *Let M be an $\mathfrak{a}$-module and N a $\mathfrak{b}$-module. Then there is a natural isomorphism of $\mathfrak{b}$-modules (indicated in the proof below)*

$$(\mathscr{B} \otimes_{\mathscr{A}} M) \otimes_k N \simeq \mathscr{B} \otimes_{\mathscr{A}} (M \otimes_k N).$$

(The left-hand side is the tensor product of $\mathfrak{b}$-modules. $M \otimes_k N$ on the right is the tensor product of $\mathfrak{a}$-modules, with N regarded as an $\mathfrak{a}$-module by restriction.)

Proof. Let $\Delta: \mathscr{B} \to \mathscr{B} \otimes_k \mathscr{B}$ be the diagonal map, that is, the unique algebra homomorphism such that $\Delta(b) = b \otimes 1 + 1 \otimes b$ for all $b \in \mathfrak{b}$. We shall first construct a map from $\mathscr{B} \otimes_{\mathscr{A}} (M \otimes N)$ to $(\mathscr{B} \otimes_{\mathscr{A}} M) \otimes N$. Fix $b \in \mathscr{B}$, let $\Delta(b) = \sum_i b_{1i} \otimes b_{2i}$ $(b_{li} \in \mathscr{B})$, and define

$$\varphi_b: M \otimes N \to (\mathscr{B} \otimes_{\mathscr{A}} M) \otimes N$$

by $\varphi_b(m \otimes n) = \sum_i (b_{1i} \otimes m) \otimes b_{2i} \cdot n$. Now define

$$\varphi: \mathscr{B} \otimes_{\mathscr{A}} (M \otimes N) \to (\mathscr{B} \otimes_{\mathscr{A}} M) \otimes N$$

by $\varphi(b \otimes x) = \varphi_b(x)$ $(b \in \mathscr{B}, x \in M \otimes N)$. This map is well defined because if $a \in \mathscr{A}$ and $\Delta(a) = \sum_j a_{1j} \otimes a_{2j}$ $(a_{lj} \in \mathscr{A})$, then

$$\Delta(ba) = \Delta(b)\Delta(a) = \sum_{i,j} b_{1i} a_{1j} \otimes b_{2i} a_{2j},$$

and so

$$\begin{aligned}\varphi_{ba}(m \otimes n) &= \sum_{i,j} (b_{1j} a_{1j} \otimes m) \otimes b_{2i} a_{2j} \cdot n = \sum_{i,j} (b_{1i} \otimes a_{1j} \cdot m) \otimes b_{2i} a_{2j} \cdot n \\ &= \varphi_b(a \cdot (m \otimes n)).\end{aligned}$$

It is clear that φ is a $\mathfrak{b}$-module map.

Recall that $T: \mathscr{B} \to \mathscr{B}$ is the transpose antiautomorphism. To define a map from $(\mathscr{B} \otimes_{\mathscr{A}} M) \otimes N$ to $\mathscr{B} \otimes_{\mathscr{A}} (M \otimes N)$, first fix $n \in N$ and let

$$\psi_n: \mathscr{B} \otimes_{\mathscr{A}} M \to \mathscr{B} \otimes_{\mathscr{A}} (M \otimes N)$$

be the map $\psi_n(b \otimes m) = \sum_i b_{1i} \otimes (m \otimes T(b_{2i}) \cdot n)$ (with b and b_{li} as above). This map is well defined. In fact, let $a \in \mathscr{A}$ with $\Delta(a) = \sum_j a_{1j} \otimes a_{2j}$ as above. Then

$$\begin{aligned}\psi_n(ba \otimes m) &= \sum_{i,j} b_{1i} a_{1j} \otimes (m \otimes T(b_{2i} a_{2j}) \cdot n) \\ &= \sum_{i,j} b_{1i} \otimes a_{1j} \cdot (m \otimes T(a_{2j}) T(b_{2i}) \cdot n).\end{aligned}$$

Let $\Delta(a_{2j}) = \sum_s a_{2j1s} \otimes a_{2j2s}$, where the a's are in $\mathscr{A}$. Since

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta: \mathscr{B} \to \mathscr{B} \otimes \mathscr{B} \otimes \mathscr{B},$$

we have

$$\psi_n(b\,a \otimes m) = \sum_{i,j,s} b_{1i} \otimes (a_{1j} \cdot m \otimes a_{2j1s} T(a_{2j2s}) T(b_{2i}) \cdot n).$$

But

$$\sum_s a_{2j1s} T(a_{2j2s}) = \varepsilon(a_{2j}) 1,$$

where $\varepsilon: \mathscr{A} \to k$ is the augmentation map, and 1 is the identity element of $\mathscr{A}$ (see [18, p. 73]). Hence

$$\begin{aligned}\psi_n(b\,a \otimes m) &= \sum_{i,j} b_{1i} \otimes (a_{1j} \cdot m \otimes \varepsilon(a_{2j}) T(b_{2i}) \cdot n) \\ &= \sum_i b_{1i} \otimes (a \cdot m \otimes T(b_{2i}) \cdot n) = \psi_n(b \otimes a \cdot m).\end{aligned}$$

Thus ψ_n is well defined. Now define

$$\psi: (\mathscr{B} \otimes_{\mathscr{A}} M) \otimes N \to \mathscr{B} \otimes_{\mathscr{A}} (M \otimes N)$$

by $\psi(x \otimes n) = \psi_n(x)$ $(x \in \mathscr{B} \otimes_{\mathscr{A}} M, n \in N)$. Computations similar to the above show that ψ is a $\mathfrak{b}$-module map, and that ψ is a left and right inverse of φ. Q.E.D.

Remark. Proposition 1.7 generalizes [6, Proposition 5.1.15], the case $\dim N = 1$. Proposition 1.7 also holds more generally for an arbitrary Hopf algebra $\mathscr{B}$ and Hopf subalgebra $\mathscr{A}$ in place of the universal enveloping algebras of $\mathfrak{b}$ and $\mathfrak{a}$.[2] In fact, our proof uses only the definition of Hopf algebra (see [18, Chapter IV]) and formulas 1 on p. 74 and (iii), (iv) on p. 79 of [18]. Applied to group algebras of finite groups, Proposition 1.7 becomes an important result of Frobenius (see for example [5, p. 268, Theorem (38.5)]).

As a sample of the usefulness of Proposition 1.7, we quickly show the following:

Corollary 1.8. *The tensor product of a free $\mathscr{B}$-module with an arbitrary $\mathscr{B}$-module is free.*

Proof. The most general free $\mathscr{B}$-module is of the form $\mathscr{B} \otimes_k M$, where M is a vector space – that is, the module is a $\mathfrak{b}$-module induced from some module for the zero subalgebra of $\mathfrak{b}$. Applying Proposition 1.7 to the case $\mathfrak{a} = 0$ shows that the tensor product of the $\mathscr{B}$-module $\mathscr{B} \otimes_k M$ with a $\mathscr{B}$-module N is naturally isomorphic to the free $\mathscr{B}$-module $\mathscr{B} \otimes_k (M \otimes_k N)$ (where $\mathscr{B}$ acts by left multiplication on the first factor). Q.E.D.

Applied to $\mathscr{C}$ in place of $\mathscr{B}$, this corollary implies what we claimed above – that each $D_j \otimes N$ is free as a $\mathscr{C}$-module. But Proposition 1.7 immediately implies the following more precise result (see also Propositions 1.4 and 1.6):

Proposition 1.9. *For each $j \in \mathbb{Z}_+$, let D_j^N be the $\mathfrak{b}$-module induced by the tensor product $\Lambda^j(\mathfrak{b}/\mathfrak{a}) \otimes_k N$ of $\mathfrak{a}$-modules (where N is regarded as an $\mathfrak{a}$-module by restriction). That is,*

$$D_j^N = \mathscr{B} \otimes_{\mathscr{A}} (\Lambda^j(\mathfrak{b}/\mathfrak{a}) \otimes N).$$

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2 J. Humphreys has informed us that he and M. Sweedler are also aware of this fact, at least in the case when $\mathscr{A} = k$

Then there are $\mathfrak{b}$*-module maps* d_j^N $(j>0)$ *and* ε_0^N *such that the* $\mathfrak{b}$*-module complex* $V(\mathfrak{b}, \mathfrak{a}, N)$ *is naturally isomorphic to the exact sequence of* $\mathfrak{b}$*-modules*

$$\cdots \xrightarrow{d_2^N} D_1^N \xrightarrow{d_1^N} D_0^N \xrightarrow{\varepsilon_0^N} N \to 0.$$

For each $j \in \mathbb{Z}_+$, D_j^N *is free as a* $\mathscr{C}$*-module; and in fact as a* $\mathscr{C}$*-module,*

$$D_j^N \simeq \mathscr{C} \otimes_k (\Lambda^j(\mathfrak{c}) \otimes N),$$

with $\mathscr{C}$ *acting by left multiplication on the first factor. Moreover, let* $\mathfrak{s}$ *be a subalgebra of* $\mathfrak{a}$ *such that* $[\mathfrak{s}, \mathfrak{c}] \subset \mathfrak{c}$. *Then the complex*

$$\cdots \xrightarrow{1 \otimes d_2^N} k \otimes_{\mathscr{C}} D_1^N \xrightarrow{1 \otimes d_1^N} k \otimes_{\mathscr{C}} D_0^N \to 0, \qquad (*)$$

where k *is regarded as the trivial right* $\mathfrak{c}$*-module, has a natural* $\mathfrak{s}$*-module structure, and it is naturally isomorphic, as a complex of* $\mathfrak{s}$*-modules, to* $V_*(\mathfrak{b}, \mathfrak{a}, N, \mathfrak{c})$, *and hence to the standard homology complex for computing* $H_*(\mathfrak{c}, N^t)$, *provided with the standard* $\mathfrak{s}$*-module action. The resulting action of* $\mathfrak{s}$ *on the homology of the complex* $(*)$ *is naturally* $\mathfrak{s}$*-module equivalent to the standard action of* $\mathfrak{s}$ *on* $H_*(\mathfrak{c}, N^t)$. *Finally, for each* $j \in \mathbb{Z}_+$, *the* $\mathfrak{s}$*-module contragredient to* $H_j(\mathfrak{c}, N^t)$ *is naturally isomorphic to* $H^j(\mathfrak{c}, N^*)$, *provided with the standard action of* $\mathfrak{s}$.

Remarks. (1) It is of course possible to describe the maps d_j^N and ε_0^N precisely, using the proof of Proposition 1.7 together with the above description of the maps $d_j\colon D_j \to D_{j-1}$, but we shall not have to do this. The information stated in Proposition 1.9 is all that we shall need to know about $V(\mathfrak{b}, \mathfrak{a}, N)$.

(2) For each $j \in \mathbb{Z}_+$, the term $k \otimes D_j^N$ in the complex $(*)$ may be identified with the tensor product $\mathfrak{s}$-module $\Lambda^j(\mathfrak{c}) \otimes N$.

The next result, which follows easily from the standard elementary properties of induced modules (cf. [6, Chapitre 5, pp. 163–164]), will be helpful in studying the modules D_j^N occurring in Proposition 1.9.

Propositon 1.10. *Any* $\mathfrak{a}$*-module filtration* $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ *of an* $\mathfrak{a}$*-module* M *such that* $M = \bigcup M_i$ *naturally defines a* $\mathfrak{b}$*-module filtration*

$$0 = \mathscr{B} \otimes_{\mathscr{A}} M_0 \subset \mathscr{B} \otimes_{\mathscr{A}} M_1 \subset \mathscr{B} \otimes_{\mathscr{A}} M_2 \subset \cdots$$

of the induced $\mathfrak{b}$*-module* $\mathscr{B} \otimes_{\mathscr{A}} M$ *such that* $\mathscr{B} \otimes_{\mathscr{A}} M = \bigcup (\mathscr{B} \otimes_{\mathscr{A}} M_i)$. *Moreover, we have a natural* $\mathfrak{b}$*-module isomorphism*

$$\mathscr{B} \otimes_{\mathscr{A}} (M_{i+1}/M_i) \simeq (\mathscr{B} \otimes_{\mathscr{A}} M_{i+1})/(\mathscr{B} \otimes_{\mathscr{A}} M_i)$$

for each $i \in \mathbb{Z}_+$.

§ 2. The Setting

In this section, we shall mostly review, with some modification, some basic definitions and results from the papers of Kac and Moody – especially [11(a), (b)] and [15(a)]. We shall also introduce notation to be used thoughout this paper.

Let $l \in \mathbb{Z}_+$, and let $A = (A_{ij})_{i,j \in \{1,\ldots,l\}}$ be an $l \times l$ *Cartan matrix*. That is, $A_{ij} \in \mathbb{Z}$ (the set of integers) for all i and j, $A_{ii} = 2$ for all i, $A_{ij} \leqq 0$ whenever $i \neq j$ and $A_{ji} = 0$

whenever $A_{ij}=0$. (Later in this section we shall also impose a symmetrizability condition on A.) Assume that k is a field of characteristic zero. Consider the (possibly infinite-dimensional) Lie algebra $\mathfrak{g}_1=\mathfrak{g}_1(A)$ over k defined by $3l$ generators h_i, e_i, f_i $(i=1, \ldots, l)$ with the relations $[h_i, h_j]=0$, $[e_i, f_j]=\delta_{ij}h_i$, $[h_i, e_j]=A_{ij}e_j$, $[h_i, f_j]=-A_{ij}f_j$ for all $i, j=1, \ldots, l$, and $(ad\, e_i)^{-A_{ij}+1}e_j=0=(ad f_i)^{-A_{ij}+1}f_j$ whenever $i \neq j$. Let $\mathfrak{h}$ be the abelian aubalgebra of $\mathfrak{g}_1$ spanned by $h_1, \ldots, h_l$. For each l-tuple $(n_1, \ldots, n_l)$ of nonnegative (resp., nonpositive) integers not all zero, define $\mathfrak{g}_1(n_1, \ldots, n_l)$ to be the subspace of $\mathfrak{g}_1$ spanned by the elements

$$[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{r-1}}, e_{i_r}] \ldots]]$$

(resp.,

$$[f_{i_1}, [f_{i_2}, \ldots, [f_{i_{r-1}}, f_{i_r}] \ldots]]),$$

where e_j (resp., f_j) occurs $|n_j|$ times. Also define $\mathfrak{g}_1(0, \ldots, 0)=\mathfrak{h}$ and $\mathfrak{g}_1(n_1, \ldots, n_l)=0$ for any other l-tuple of integers. Then

$$\mathfrak{g}_1=\coprod_{(n_1, \ldots, n_l)\in \mathbb{Z}^l} \mathfrak{g}_1(n_1, \ldots, n_l),$$

this is a Lie algebra gradiation of $\mathfrak{g}_1$, and the elements $h_1, \ldots, h_l, e_1, \ldots, e_l, f_1, \ldots, f_l$ are linearly independent in $\mathfrak{g}_1$ (see [11(a)], [15(a)]). In particular, $\dim \mathfrak{h}=l$. The space $\mathfrak{g}_1(0, \ldots, 0, 1, 0, \ldots, 0)$ (resp., $\mathfrak{g}_1(0, \ldots, 0, -1, 0, \ldots, 0)$) is nonzero and is spanned by e_i (resp., f_i); here ± 1 is in the i-th position. Also, each space $\mathfrak{g}_1(n_1, \ldots, n_l)$ is finite-dimensional. There is clearly a Lie algebra involution η of $\mathfrak{g}_1$ interchanging e_i and f_i and taking h_i to $-h_i$ for all $i=1, \ldots, l$, and η takes each space $\mathfrak{g}_1(n_1, \ldots, n_l)$ onto $\mathfrak{g}_1(-n_1, \ldots, -n_l)$.

The $\mathbb{Z}^l$-graded Lie algebra $\mathfrak{g}_1$ contains a unique graded ideal $\mathfrak{r}_1$ maximal among those graded ideals not intersecting the span of h_i, e_i and f_i $(1 \leqq i \leqq l)$ (cf. [11(a)], [15(a)]). Let $\mathfrak{g}=\mathfrak{g}(A)$ be the $\mathbb{Z}^l$-graded Lie algebra $\mathfrak{g}_1(A)/\mathfrak{r}_1$. The images in $\mathfrak{g}$ of $h_i, e_i, f_i, \mathfrak{h}$ and $\mathfrak{g}_1(n_1, \ldots, n_l)$ shall be denoted $h_i, e_i, f_i, \mathfrak{h}$ and $\mathfrak{g}(n_1, \ldots, n_l)$, respectively. The involution η of $\mathfrak{g}_1$ clearly induces an involution of $\mathfrak{g}$.

Remark. If A is a classical Cartan matrix of finite type (i.e., one arising in the usual way from a finite-dimensional split semisimple Lie algebra), then $\mathfrak{r}_1=0$ and $\mathfrak{g}(A)$ is the split semisimple Lie algebra whose Cartan matrix is A, by Serre's theorem on generators and relations for such a Lie algebra [17, Chapitre VI, p. 19]. If A is a Euclidean matrix, in the sense of [15(a)], then $\mathfrak{r}_1=0$, and $\mathfrak{g}(A)$ is a Euclidean Lie algebra (see [11(a)] and [15(b)]). More generally, $\mathfrak{r}_1=0$ whenever A is symmetrizable, in the sense defined below; this assertion is the second part of the Corollary in [11(b)]. We do not know whether $\mathfrak{r}_1=0$ in general; this is conjectured in [11(a), Chapter II, § 7].

Under the adjoint action, $\mathfrak{h}$ acts as scalars on the spaces $\mathfrak{g}(n_1, \ldots, n_l)$, giving rise to linear functionals on $\mathfrak{h}$. However, if the matrix A is singular, the resulting functionals span only a proper subspace of $\mathfrak{h}^*$ (* denotes dual). In order to remedy this defect, we shall enlarge $\mathfrak{g}$ by adjoining some derivations.

Let D_i $(1 \leqq i \leqq l)$ be the i-th degree derivation of $\mathfrak{g}$, that is, the derivation which acts on $\mathfrak{g}(n_1, \ldots, n_l)$ as scalar multiplication by n_i. Then $D_1, \ldots, D_l$ span an l-dimensional space $\mathfrak{d}_0$ of commuting derivations of $\mathfrak{g}$.

Let $\mathfrak{d}$ be a subspace of $\mathfrak{d}_0$. Since $\mathfrak{d}$ may be regarded as an abelian Lie algebra acting on the $\mathfrak{d}$-module $\mathfrak{g}$ by derivations, we may form the semidirect product

Lie algebra $\mathfrak{g}^e = \mathfrak{d} \times \mathfrak{g}$ (e for "extended") with respect to this action. Then $\mathfrak{h}^e = \mathfrak{d} \oplus \mathfrak{h}$ is an abelian Lie subalgebra of $\mathfrak{g}^e$ which acts via scalar multiplication on each space $\mathfrak{g}(n_1, \ldots, n_l)$. Define $\alpha_1, \ldots, \alpha_l \in (\mathfrak{h}^e)^*$ by the conditions $[h, e_i] = \alpha_i(h) e_i$ for all $h \in \mathfrak{h}^e$ and all $i = 1, \ldots, l$. Note that $\alpha_j(h_i) = A_{ij}$ for all $i, j = 1, \ldots, l$.

We now make the basic assumption that $\mathfrak{d}$ is chosen so that $\alpha_1, \ldots, \alpha_l$ are linearly independent. This is always possible, because this condition holds for $\mathfrak{d} = \mathfrak{d}_0$. (In this case, we have $\alpha_i(D_j) = \delta_{ij}$ for all $i, j = 1, \ldots, l$.) But we may wish to choose $\mathfrak{d}$ smaller than $\mathfrak{d}_0$, and in fact if A is nonsingular, then $\mathfrak{d} = 0$ is a natural choice.

For all $\varphi \in (\mathfrak{h}^e)^*$, define

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} \mid [h, x] = \varphi(h) x \text{ for all } h \in \mathfrak{h}^e\}.$$

Note that $[\mathfrak{g}^\varphi, \mathfrak{g}^\psi] \subset \mathfrak{g}^{\varphi + \psi}$ for all $\varphi, \psi \in (\mathfrak{h}^e)^*$. It is clear that $e_i \in \mathfrak{g}^{\alpha_i}$ and $f_i \in \mathfrak{g}^{-\alpha_i}$ for each $i = 1, \ldots, l$, and that for all $(n_1, \ldots, n_l) \in \mathbb{Z}^l$,

$$\mathfrak{g}(n_1, \ldots, n_l) \subset \mathfrak{g}^{n_1 \alpha_1 + \ldots + n_l \alpha_l}.$$

Since $\alpha_1, \ldots, \alpha_l$ are linearly independent, this inclusion is an equality, and the decomposition

$$\mathfrak{g} = \coprod_{(n_1, \ldots, n_l) \in \mathbb{Z}^l} \mathfrak{g}(n_1, \ldots, n_l)$$

coincides with the decomposition

$$\mathfrak{g} = \coprod_{\varphi \in (\mathfrak{h}^e)^*} \mathfrak{g}^\varphi.$$

Define the *roots* of $\mathfrak{g}$ (with respect to $\mathfrak{h}^e$) to be the nonzero elements φ of $(\mathfrak{h}^e)^*$ such that $\mathfrak{g}^\varphi \neq 0$. Let Δ be the set of roots, Δ_+ (the set of *positive* roots) the set of roots which are nonnegative integral linear combinations of $\alpha_1, \ldots, \alpha_l$, and $\Delta_- = -\Delta_+$ (the set of *negative* roots). Then $\Delta = \Delta_+ \cup \Delta_-$, $\mathfrak{g}^0 = \mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\varphi \in \Delta_+} \mathfrak{g}^\varphi \oplus \coprod_{\varphi \in \Delta_-} \mathfrak{g}^\varphi,$$

and $\dim \mathfrak{g}^{-\varphi} = \dim \mathfrak{g}^\varphi$ for all $\varphi \in \Delta$.

Let R be the linear subspace of $(\mathfrak{h}^e)^*$ with basis $\alpha_1, \ldots, \alpha_l$, so that R is also the span of Δ. Then the restriction map $R \to \mathfrak{h}^*$ is an isomorphism if and only if the Cartan matrix A is nonsingular.

For each $i = 1, \ldots, l$, define the linear transformation $r_i \colon R \to R$ by the conditions $r_i \alpha_j = \alpha_j - A_{ij} \alpha_i$ for all $j = 1, \ldots, l$. Equivalently, $r_i \varphi = \varphi - \varphi(h_i) \alpha_i$ for all $\varphi \in R$. We have $r_i \alpha_i = -\alpha_i$, and r_i acts as the identity on the $(l-1)$-dimensional space of $\varphi \in R$ such that $\varphi(h_i) = 0$. Let W (the *Weyl group*) be the group of linear automorphisms of R generated by the reflections r_i.

For each $i = 1, \ldots, l$, let $\mathfrak{u}_i$ be the Lie algebra spanned by h_i, e_i and f_i, so that $\mathfrak{u}_i \simeq \mathfrak{sl}(2, k)$. The defining relations for $\mathfrak{g}$ imply that for each $j = 1, \ldots, l$, e_j is contained in a finite-dimensional irreducible $\mathfrak{u}_i$-module, and so is f_j. Since $\mathfrak{g}$ is generated by the e_j and f_j, $\mathfrak{g}$ and in fact every $\mathfrak{u}_i$-submodule of $\mathfrak{g}$ is a sum, and hence a direct sum, of finite-dimensional irreducible $\mathfrak{u}_i$-submodules. Applying this to the $\mathfrak{u}_i$-module $\coprod_{n \in \mathbb{Z}} \mathfrak{g}^{\varphi + n\alpha_i}$ $(\varphi \in \Delta \cup \{0\})$ shows easily that $r_i \Delta = \Delta$ and $\dim \mathfrak{g}^\varphi = \dim \mathfrak{g}^{r_i \varphi}$ for all $\varphi \in \Delta$. Hence also $W\Delta = \Delta$ and $\dim \mathfrak{g}^\varphi = \dim \mathfrak{g}^{w\varphi}$ for all $w \in W$ and

$\varphi\in\Delta$. Moreover, for each $i=1,\ldots,l$, $n\alpha_i\in\Delta(n\in\mathbb{Z})$ implies $n=\pm 1$. Furthermore, for each α_i and $\varphi\in\Delta\cup\{0\}$, the "$\alpha_i$-root string" $\{\varphi+n\alpha_i\in\Delta\cup\{0\}\mid n\in\mathbb{Z}\}$ is finite and is an unbroken string of the form $\varphi-p\alpha_i$, $\varphi-(p-1)\alpha_i,\ldots,\varphi+q\alpha_i$, where $p,q\in\mathbb{Z}_+$.

The following elementary fact is basic:

Proposition 2.1. *For all $i=1,\ldots,l$, r_i permutes the elements of $\Delta_+-\{\alpha_i\}$.*

Proof. Let $\varphi\in\Delta_+-\{\alpha_i\}$. Then $\varphi=\sum_{j=1}^{l} n_j\alpha_j$ with each $n_j\in\mathbb{Z}_+$ and some $n_{j_0}>0$ with $j_0\neq i$. Then

$$r_i\varphi=\varphi-\varphi(h_i)\alpha_i=\Big(\sum_{j\neq i} n_j\alpha_j\Big)+(n_i-\varphi(h_i))\alpha_i,$$

where $\varphi(h_i)\in\mathbb{Z}$. Since $r_i\varphi\in\Delta$ and its j_0-coefficient is positive, $r_i\varphi\in\Delta_+$. Q.E.D.

Define the set Δ_R of *real* roots to be the set of Weyl group transforms of $\alpha_1,\ldots,\alpha_l$, and define the set Δ_I of *imaginary* roots to be $\Delta-\Delta_R$. Then $\dim\mathfrak{g}^\varphi=1$ for all $\varphi\in\Delta_R$, since $\dim\mathfrak{g}^{\alpha_i}=1$ for each i. If $\varphi\in\Delta_I$, then $\dim\mathfrak{g}^\varphi$ need not be 1 (see [11(a)], [15(b)]). Clearly, $W\Delta_R=\Delta_R$, $W\Delta_I=\Delta_I$, $\Delta_R=-\Delta_R$ and $\Delta_I=-\Delta_I$. We also have $W(\Delta_I\cap\Delta_+)=\Delta_I\cap\Delta_+$, by Proposition 2.1.

For all $w\in W$, define $\Phi_w=\Delta_+\cap w\Delta_-=\{\varphi\in\Delta_+\mid w^{-1}\varphi\in\Delta_-\}$. Clearly, $\Phi_w\subset\Delta_R\cap\Delta_+$, Φ_1 is empty and $\Phi_{r_i}=\{\alpha_i\}$ for all $i=1,\ldots,l$. Let $n(w)$ be the number of elements in Φ_w. (We shall see presently that $n(w)<\infty$.) Let $l(w)$ be the *length* of w, that is, the smallest nonnegative integer j such that w can be writtens as $r_{i_1}r_{i_2}\ldots r_{i_j}(1\leqq i_m\leqq l)$. The following is established in [16, §2] (see also [10]):

Proposition 2.2. *Let $w\in W$ and $i\in\{1,\ldots,l\}$.* Then

(1) $n(w)=l(w)$.

(2) If $\alpha_i\notin\Phi_w$, then $\Phi_{r_iw}=r_i\Phi_w\cup\{\alpha_i\}$ (disjoint union), and $l(r_iw)=l(w)+1$.

(3) If $\alpha_i\in\Phi_w$, then $\Phi_{r_iw}=r_i(\Phi_w-\{\alpha_i\})$, and $l(r_iw)=l(w)-1$.

For every finite subset Φ of Δ, define $\langle\Phi\rangle=\sum_{\varphi\in\Phi}\varphi\in R$. Note that by Proposition 2.2(1), Φ_w is finite for all $w\in W$, so that $\langle\Phi_w\rangle$ is defined. Proposition 2.2(2), (3) immediately imply:

Corollary 2.3. *For all $w\in W$ and $i\in\{1,\ldots,l\}$, $\langle\Phi_{r_iw}\rangle=r_i\langle\Phi_w\rangle+\langle\Phi_{r_i}\rangle$.*

Proposition 2.4. *Let $w\in W$, Φ a finite subset of Δ_+, and $\gamma\in R$ a finite sum of not necessarily distinct positive imaginary roots. If $\langle\Phi_w\rangle=\langle\Phi\rangle+\gamma$, then $\gamma=0$ and $\Phi=\Phi_w$. In particular, Φ consists of real roots.*

Proof (A. Feingold). Define a partial ordering on R as follows: If $\varphi,\psi\in R$, we say that $\varphi\leqq\psi$ if $\psi-\varphi$ is a nonnegative integral linear combination of $\alpha_1,\ldots,\alpha_l$. Let $\beta_1,\ldots,\beta_m$ be the distinct elements of Φ, and $\gamma_1,\ldots,\gamma_n$ the distinct elements of Φ_w. If $n=0$, the result is clear, so assume that $n>0$. Then $w^{-1}\langle\Phi_w\rangle<0$, and $w^{-1}\gamma\geqq 0$ because w^{-1} preserves $\Delta_I\cap\Delta_+$. Thus $w^{-1}\langle\Phi\rangle<0$, so that $w^{-1}\beta_i<0$ for some $i=1,\ldots,m$. Hence $\beta_i\in\Phi_w$, and so $\beta_i=\gamma_j$ for some $j=1,\ldots,n$. Then $\langle\Phi_w-\{\gamma_j\}\rangle=\langle\Phi-\{\beta_i\}\rangle+\gamma$. If $n=1$, we are clearly done, so assume that $n>1$. As above, $w^{-1}\langle\Phi-\{\beta_i\}\rangle<0$. Hence $m>1$, and there exists $i'\neq i$ such that

$w^{-1}\beta_{i'}<0$, i.e., $\beta_{i'}\in\Phi_w$. Since $\beta_{i'}\neq\beta_i$, we have $\beta_{i'}=\gamma_{j'}$ for some $j'\neq j$. A continuation of this process proves the result. Q.E.D.

We want to extend the action of the Weyl group W from R to all of $(\mathfrak{h}^e)^*$. To do this, define the linear automorphism $r_i'(1\leqq i\leqq l)$ of $(\mathfrak{h}^e)^*$ by the condition $r_i'\varphi=\varphi-\varphi(h_i)\alpha_i$ for all $\varphi\in(\mathfrak{h}^e)^*$. Then $r_i'\,|\,R=r_i$, and the restriction map to R gives a homomorphism from the group W' generated by $r_1',\dots,r_l'$ onto W.

It is known that W is a Coxeter group (cf. [15(a), p. 216, Proposition 5] and [16, § 2]). Specifically, for all $i,j=1,\dots,l$ with $i\neq j$, define m_{ij} to be 2, 3, 4, 6 or ∞ according as $A_{ij}A_{ji}$ is 0, 1, 2, 3 or $\geqq 4$, respectively. Then W is the group generated by $r_1,\dots,r_l$ subject to the relations $r_i^2=1$ for all $i=1,\dots,l$ and $(r_i r_j)^{m_{ij}}=1$ for all $i,j=1,\dots,l$ with $i\neq j$. (Here $(r_i r_j)^\infty=1$ is interpreted to be the vacuous relation.)

Now for each $i=1,\dots,l$, $(r_i')^2=1$. Also, the argument used in [15(a), Proposition 5] to show that $(r_i r_j)^{m_{ij}}=1$ whenever $i\neq j$ also proves that $(r_i' r_j')^{m_{ij}}=1$ whenever $i\neq j$. Thus there exists a group homomorphism from W to W' taking r_i to r_i' for each i. Clearly, this is a left and right inverse of the restriction homomorphism $W'\to W$ indicated above, and so this restriction map is an isomorphism. Hence we may now write r_i for r_i', and we may identify W with the group of linear automorphisms of $(\mathfrak{h}^e)^*$ generated by the automorphisms $r_i(1\leqq i\leqq l)$ defined by $r_i\varphi=\varphi-\varphi(h_i)\alpha_i$ for all $\varphi\in(\mathfrak{h}^e)^*$.

Define $\rho\in(\mathfrak{h}^e)^*$ to be any fixed element satisfying the conditions $\rho(h_i)=1$ for all $i=1,\dots,l$. (If the Cartan matrix A is nonsingular and $\mathfrak{d}=0$, then ρ is determined uniquely and $\rho\in R=(\mathfrak{h}^e)^*$.)

Proposition 2.5. *For all $i\in\{1,\dots,l\}$, $r_i\rho=\rho-\alpha_i$. For all $w\in W$, $\langle\Phi_w\rangle=\rho-w\rho$.*

Proof. The first assertion is immediate, and establishes the second for all Weyl group elements of length 1. Use induction on the length of w, and assume the second assertion is true for $w'\in W$ with $l(w')<l(w)$. Let $w=r_{i_1}r_{i_2}\dots r_{i_j}$ be a minimal expression for w, and set $w'=r_{i_2}\dots r_{i_j}$. This is a minimal expression for w', so that $l(w')=l(w)-1$. Then

$$\rho-w\rho=\rho-r_{i_1}w'\rho=\rho-r_{i_1}\rho+r_{i_1}(\rho-w'\rho)=\langle\Phi_{r_{i_1}}\rangle+r_{i_1}\langle\Phi_{w'}\rangle,$$

by the first assertion and the induction hypothesis. But this equals $\langle\Phi_w\rangle$, by Corollary 2.3. Q.E.D.

Corollary 2.6. *The only Weyl group element which fixes ρ is the identity. Equivalently, if $w_1\rho=w_2\rho(w_1,w_2\in W)$, then $w_1=w_2$.*

Proof. If $w\rho=\rho(w\in W)$, then $\langle\Phi_w\rangle=0$ by Proposition 2.5, and so $w=1$ by Proposition 2.2(1). Q.E.D.

Corollary 2.7. *If $w_1, w_2\in W$ and $\Phi_{w_1}=\Phi_{w_2}$, or even if $\langle\Phi_{w_1}\rangle=\langle\Phi_{w_2}\rangle$, then $w_1=w_2$.*

Proof. Simply apply Proposition 2.5 and Corollary 2.6. Q.E.D.

Proposition 2.8. *Let T be the subset of $(\mathfrak{h}^e)^*$ consisting of the elements of the form $-\langle\Phi\rangle-\gamma$, where Φ is a finite subset of Δ_+ and γ is a finite sum of not necessarily distinct positive imaginary roots. Then $\rho+T$ is W-invariant.*

Proof. It is sufficient to show that $r_i(\rho-\langle\Phi\rangle-\gamma)-\rho\in T$ for all $i=1,\dots,l$. If $\alpha_i\in\Phi$, then $\Phi=\Phi'\cup\{\alpha_i\}$, where $\Phi'=\Phi-\{\alpha_i\}$. We have

$$r_i(\rho-\langle\Phi\rangle-\gamma)-\rho=-\alpha_i-r_i\langle\Phi'\rangle+\alpha_i-r_i\gamma=-r_i\langle\Phi'\rangle-r_i\gamma\in T$$

by Proposition 2.1 and the fact that r_i preserves $\Delta_I\cap\Delta_+$. If $\alpha_i\notin\Phi$, then

$$r_i(\rho-\langle\Phi\rangle-\gamma)-\rho=-\alpha_i-r_i\langle\Phi\rangle-r_i\gamma\in T$$

for the same reasons. Q.E.D.

Definitions. Let $\lambda\in(\mathfrak{h}^e)^*$. Call λ *integral* if $\lambda(h_i)\in\mathbb{Z}$ for all $i=1,\dots,l$, and call λ *dominant integral* if $\lambda(h_i)\in\mathbb{Z}_+$ for all $i=1,\dots,l$. Let $P\subset(\mathfrak{h}^e)^*$ be the set of dominant integral elements.

Remarks. (1) Every root is integral.

(2) W preserves the set of integral elements.

(3) ρ is dominant integral.

Proposition 2.9. *Let $\lambda\in(\mathfrak{h}^e)^*$ be integral, and let U be a W-invariant subset of $(\mathfrak{h}^e)^*$ all of whose elements are of the form $\lambda-\sum_{i=1}^{l}n_i\alpha_i$ where each $n_i\in\mathbb{Z}_+$. Then every element of U is W-conjugate to a dominant integral element of U.*

Proof. Let $\mu\in U$. Choose $w\in W$ so that in the expression $w\mu=\lambda-\sum_{i=1}^{l}n_i\alpha_i(n_i\in\mathbb{Z}_+)$, the sum $\sum_{i=1}^{l}n_i$ is minimal. Then $w\mu$ is dominant integral. Indeed, $w\mu\in U$, and hence is integral. If $m=(w\mu)(h_i)<0$ for some i, then $r_i w\mu=w\mu-m\alpha_i\in U$ has an expression $\lambda-\sum_{i=1}^{l}m_i\alpha_i$ in which $\sum_{i=1}^{l}m_i<\sum_{i=1}^{l}n_i$, a contradiction. Thus $w\mu\in P$. Q.E.D.

Assume that the Cartan matrix A is *symmetrizable*, i.e., that there are positive rational numbers $q_1,\dots,q_l$ such that $\operatorname{diag}(q_1,\dots,q_l)A$ is a symmetric matrix. Define a symmetric bilinear form σ on R by the condition $\sigma(\alpha_i,\alpha_j)=q_iA_{ij}$ for all $i,j=1,\dots,l$. Note that $q_i=\sigma(\alpha_i,\alpha_i)/2$ for each i. Set $x_{\alpha_i}=q_ih_i=\sigma(\alpha_i,\alpha_i)h_i/2$ in $\mathfrak{h}$ for all $i=1,\dots,l$, and for all $\varphi\in R$, with $\varphi=\sum_{i=1}^{l}a_i\alpha_i(a_i\in k)$, define $x_\varphi=\sum_{i=1}^{l}a_ix_{\alpha_i}$ in $\mathfrak{h}$. Transfer σ to the symmetric bilinear form τ_0 on $\mathfrak{h}$ determined by the condition $\tau_0(x_{\alpha_i},x_{\alpha_j})=\sigma(\alpha_i,\alpha_j)$ for all $i,j=1,\dots,l$. Then $\tau_0(x_\varphi,x_\psi)=\sigma(\varphi,\psi)$ for all $\varphi,\psi\in R$. Now $\sigma(\alpha_i,\alpha_j)=\alpha_j(x_{\alpha_i})$ for all $i,j=1,\dots,l$, so that $\tau_0(x_\varphi,x_\psi)=\sigma(\varphi,\psi)=\psi(x_\varphi)=\varphi(x_\psi)$ for all $\varphi,\psi\in R$. The form τ_0 extends to a symmetric $\mathfrak{g}$-invariant bilinear form τ on $\mathfrak{g}$ such that $[a,b]=\tau(a,b)x_\varphi$ for all $\varphi\in\Delta$, $a\in\mathfrak{g}^\varphi$ and $b\in\mathfrak{g}^{-\varphi}$ (see [11(a)], [15(a)]). In particular, $\tau(e_i,f_i)=2/\sigma(\alpha_i,\alpha_i)$ for all $i=1,\dots,l$. For all $\varphi\in\Delta$ and $\psi\in\Delta\cup\{0\}$, $\tau(\mathfrak{g}^\varphi,\mathfrak{g}^\psi)=0$ unless $\psi=-\varphi$, and τ induces a nonsingular pairing between $\mathfrak{g}^\varphi$ and $\mathfrak{g}^{-\varphi}$ (see [11(a), 15(a)]).

It is clearly possible to extend the symmetric form σ on R to a symmetric form σ on $(\mathfrak{h}^e)^*$ satisfying the following condition: For all $\varphi\in R$ and $\lambda\in(\mathfrak{h}^e)^*$, $\sigma(\lambda,\varphi)=\lambda(x_\varphi)$. Fix such a form σ on $(\mathfrak{h}^e)^*$.

Proposition 2.10. *The form σ on $(\mathfrak{h}^e)^*$ is W-invariant.*

Proof. Let $\lambda, \mu \in (\mathfrak{h}^e)^*$, and let $i \in \{1, \ldots, l\}$. Then

$$\begin{aligned}\sigma(r_i \lambda, r_i \mu) &= \sigma(\lambda - \lambda(h_i)\alpha_i, \mu - \mu(h_i)\alpha_i) \\ &= \sigma(\lambda, \mu) - \lambda(h_i)\mu(x_{\alpha_i}) - \mu(h_i)\lambda(x_{\alpha_i}) + \sigma(\alpha_i, \alpha_i)\lambda(h_i)\mu(h_i) \\ &= \sigma(\lambda, \mu) - \tfrac{1}{2}\sigma(\alpha_i, \alpha_i)\lambda(h_i)\mu(h_i) - \tfrac{1}{2}\sigma(\alpha_i, \alpha_i)\mu(h_i)\lambda(h_i) \\ &\quad + \sigma(\alpha_i, \alpha_i)\lambda(h_i)\mu(h_i) = \sigma(\lambda, \mu). \quad \text{Q.E.D.}\end{aligned}$$

The following result is immediate from the definitions:

Proposition 2.11. *If $\mu \in (\mathfrak{h}^e)^*$ is dominant integral and $i \in \{1, \ldots, l\}$, then $\sigma(\mu, \alpha_i)$ is a nonnegative rational number, and $\sigma(\mu + \rho, \alpha_i)$ is a positive rational number.*

Proposition 2.12. *Let $\mu \in (\mathfrak{h}^e)^*$ be dominant integral, let $\nu = \mu - \sum_{i=1}^{l} n_i \alpha_i$, where each $n_i \in \mathbb{Z}_+$, and suppose that $\nu + \rho$ is dominant integral. Then*

$$\sigma(\mu + \rho, \mu + \rho) - \sigma(\nu + \rho, \nu + \rho) \geqq 0$$

(i.e., this is a nonnegative rational number), and we have equality if and only if $\mu = \nu$.

Proof. We have

$$\begin{aligned}\sigma(\mu + \rho, \mu + \rho) - \sigma(\nu + \rho, \nu + \rho) &= \sigma(\mu + \rho, \mu + \rho) - \sigma(\mu + \rho - \textstyle\sum n_i \alpha_i, \mu + \rho - \sum n_i \alpha_i) \\ &= \sigma(\mu + \rho, \textstyle\sum n_i \alpha_i) + \sigma(\sum n_i \alpha_i, \nu + \rho) \\ &\geqq \sigma(\mu + \rho, \textstyle\sum n_i \alpha_i) \geqq 0,\end{aligned}$$

by Proposition 2.11. That result also implies that equality holds only if each $n_i = 0$. Q.E.D.

Proposition 2.13. *Let $T \subset (\mathfrak{h}^e)^*$ be as in Proposition 2.8, let $\mu \in (\mathfrak{h}^e)^*$ be dominant integral, and let T' be a W-invariant subset of $(\mathfrak{h}^e)^*$ all of whose elements are of the form $\mu - \sum_{i=1}^{l} n_i \alpha_i$ where each $n_i \in \mathbb{Z}_+$. Suppose $\lambda = \tau + \tau'$, with $\tau \in T$ and $\tau' \in T'$. Then*

$$\sigma(\mu + \rho, \mu + \rho) - \sigma(\lambda + \rho, \lambda + \rho) \geqq 0$$

(i.e., this is a nonnegative rational number), and equality holds if and only if there exists $w \in W$ such that $\lambda + \rho = w(\mu + \rho)$, or equivalently, such that $\tau = w\rho - \rho = -\langle \Phi_w \rangle$ (see Proposition 2.5) and $\tau' = w\mu$. In case of equality, λ determines w and hence also τ and τ'.

Proof. By Proposition 2.8, $\rho + T + T'$ is W-invariant. Also, every element of $\rho + T + T'$ is of the form $\mu + \rho - \sum_{i=1}^{l} n_i \alpha_i$ with $n_i \in \mathbb{Z}_+$. Thus by Proposition 2.9, there exists $w \in W$ such that $w^{-1}(\lambda + \rho)$ is a dominant integral element of $\rho + T + T'$. Then we may write $w^{-1}(\lambda + \rho) - \rho = \mu - \sum_{i=1}^{l} n_i \alpha_i$ with $n_i \in \mathbb{Z}_+$. Applying Proposition 2.12 to $\nu = w^{-1}(\lambda + \rho) - \rho$, and using the W-invariance of σ (Proposition

2.10), we have

$$\sigma(\mu+\rho,\mu+\rho)-\sigma(\lambda+\rho,\lambda+\rho)\geqq 0,$$

and equality holds if and only if $w^{-1}(\lambda+\rho)=\mu+\rho$. This last condition is equivalent to $w^{-1}(\tau+\rho)-\rho+w^{-1}\tau'=\mu$. By the definitions and the W-invariance of $\rho+T$ and of T', this condition is in turn equivalent to the pair of conditions $w^{-1}(\tau+\rho)-\rho=0$ and $w^{-1}\tau'=\mu$. This proves all but the uniqueness of w. To prove this uniqueness, suppose $\lambda+\rho=w'(\mu+\rho)$ for some $w'\in W$, and let $w_0=w^{-1}w'$. Then $w_0(\mu+\rho)=\mu+\rho$, and so $w_0\rho-\rho+w_0\mu=\mu$. But $\mu=w^{-1}\tau'\in T'$ since T' is W-invariant, and $w_0\rho-\rho\in T$, so that again by the definitions of T and T', we must have $w_0\rho-\rho=0$ and $w_0\mu=\mu$. By Corollary 2.6, $w_0=1$. Q.E.D.

§ 3. Modules Induced from an F-Parabolic Subalgebra

Every subset S of $\{1,\dots,l\}$ defines in the obvious way a square submatrix B of the Cartan matrix A which is also a Cartan matrix. We shall say that S is *of finite type* if B is a classical Cartan matrix of finite type. $S=\emptyset$ is an example.

Assume that S is of finite type. It is clear that there is a natural injection $\mathfrak{g}(B)\hookrightarrow\mathfrak{g}(A)$ (using the notation of § 2). We may identify the finite-dimensional split semisimple Lie algebra $\mathfrak{g}(B)$ with the subalgebra $\mathfrak{g}_S$ of $\mathfrak{g}=\mathfrak{g}(A)$ generated by $\{h_i,e_i,f_i\}_{i\in S}$. Let $\mathfrak{h}_S$ be the span of $\{h_i\}_{i\in S}$, $\Delta^S=\Delta\cap\coprod_{i\in S}\mathbb{Z}\alpha_i$, $\Delta^S_+=\Delta_+\cap\Delta^S$ and $\Delta^S_-=\Delta_-\cap\Delta^S$. Then

$$\mathfrak{g}_S=\mathfrak{h}_S\oplus\coprod_{\varphi\in\Delta^S_+}\mathfrak{g}^\varphi\oplus\coprod_{\varphi\in\Delta^S_-}\mathfrak{g}^\varphi.$$

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}=\coprod_{\varphi\in\Delta_+}\mathfrak{g}^\varphi;\quad \mathfrak{n}^-=\coprod_{\varphi\in\Delta_-}\mathfrak{g}^\varphi;\quad \mathfrak{n}_S=\coprod_{\varphi\in\Delta^S_+}\mathfrak{g}^\varphi;\ \ \mathfrak{n}_S^-=\coprod_{\varphi\in\Delta^S_-}\mathfrak{g}^\varphi;$$

$$\mathfrak{u}=\coprod_{\varphi\in\Delta_+-\Delta^S_+}\mathfrak{g}^\varphi;\quad \mathfrak{u}^-=\coprod_{\varphi\in\Delta_--\Delta^S_-}\mathfrak{g}^\varphi;$$

$\mathfrak{r}=\mathfrak{g}_S+\mathfrak{h}$ and $\mathfrak{p}=\mathfrak{r}\oplus\mathfrak{u}$. This last is a subalgebra because $[\mathfrak{r},\mathfrak{u}]\subset\mathfrak{u}$. (It is also clear that $[\mathfrak{r},\mathfrak{u}^-]\subset\mathfrak{u}^-$.) We have the following relations: $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$; $\mathfrak{g}_S=\mathfrak{n}_S^-\oplus\mathfrak{h}_S\oplus\mathfrak{n}_S$; $\mathfrak{n}=\mathfrak{n}_S\oplus\mathfrak{u}$; $\mathfrak{n}^-=\mathfrak{n}_S^-\oplus\mathfrak{u}^-$; $\mathfrak{r}=\mathfrak{n}_S^-\oplus\mathfrak{h}\oplus\mathfrak{n}_S$ and $\mathfrak{g}=\mathfrak{u}^-\oplus\mathfrak{p}$.

We call $\mathfrak{p}$ the *F-parabolic subalgebra* of $\mathfrak{g}$ defined by B. (The "F" refers to the finite-dimensionality of $\mathfrak{g}_S$.) Note that if $S=\emptyset$, the associated F-parabolic subalgebra is $\mathfrak{h}\oplus\mathfrak{n}$. If A is classical of finite type, the F-parabolic subalgebras are the parabolic subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h}\oplus\mathfrak{n}$.

Since B is nonsingular, the restrictions to $\mathfrak{h}_S$ of the α_i with $i\in S$ form a linearly independent family of elements of $\mathfrak{h}_S^*$. Let $\mathfrak{h}^S=\bigcap_{i\in S}\operatorname{Ker}(\alpha_i|\mathfrak{h})\subset\mathfrak{h}$. Then $\mathfrak{h}=\mathfrak{h}_S\oplus\mathfrak{h}^S$, $\mathfrak{r}=\mathfrak{g}_S\oplus\mathfrak{h}^S$, and $\mathfrak{r}$ is a (finite-dimensional) reductive Lie algebra with commutator subalgebra $\mathfrak{g}_S$ and center $\mathfrak{h}^S$.

For every $\mathfrak{d}$-invariant subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, denote by $\mathfrak{t}^e$ the subalgebra $\mathfrak{d}\oplus\mathfrak{t}$ of $\mathfrak{g}^e$. (This notation is consistent with the notation $\mathfrak{g}^e$ and $\mathfrak{h}^e$ (§ 2).)

Call $\mathfrak{p}^e=\mathfrak{r}^e\oplus\mathfrak{u}$ the *F-parabolic subalgebra* of $\mathfrak{g}^e$ defined by B. The $\mathfrak{g}^e$-modules induced by finite-dimensional irreducible modules for an F-parabolic subalgebra will be of central importance to us. We now proceed to discuss these induced modules.

Definition. Let

$$P_S = \{\lambda \in (\mathfrak{h}^e)^* \mid \lambda(h_i) \in \mathbb{Z}_+ \text{ for all } i \in S\}.$$

Proposition 3.1. *There is a natural bijection, denoted $\lambda \mapsto M(\lambda)$, between P_S and the set of (isomorphism classes of) finite-dimensional irreducible $\mathfrak{r}^e$-modules which are irreducible as $\mathfrak{g}_S$-modules. The correspondence is described as follows: The highest weight space (relative to $\mathfrak{h}_S$ and $\mathfrak{n}_S$) of the $\mathfrak{g}_S$-module $M(\lambda)$ is $\mathfrak{h}^e$-stable, and λ is the resulting weight for the action of $\mathfrak{h}^e$.*

The proof is straightforward and may be omitted. Note that $\mathfrak{h}^S$ acts via multiplication by scalars on $M(\lambda)$. Also, $M(\lambda)$ is the direct sum of its weight spaces for $\mathfrak{h}^e$ (in the obvious sense), and the weights for $\mathfrak{h}^e$ in $M(\lambda)$ are all of the form $\lambda - \sum_{i \in S} n_i \alpha_i$ with $n_i \in \mathbb{Z}_+$.

We define the $\mathfrak{g}^e$-module $V^{M(\lambda)}$ ($\lambda \in P_S$) to be the $\mathfrak{g}^e$-module induced by the irreducible $\mathfrak{p}^e$-module which is $M(\lambda)$ as an $\mathfrak{r}^e$-module and which is annihilated by $\mathfrak{u}$. That is, let $\mathscr{G}^e$ and $\mathscr{P}^e$ (regarded as a subalgebra of $\mathscr{G}^e$) denote the universal enveloping algebras of $\mathfrak{g}^e$ and $\mathfrak{p}^e$, respectively. Then $V^{M(\lambda)}$ is the $\mathfrak{g}^e$-module $\mathscr{G}^e \otimes_{\mathscr{P}^e} M(\lambda)$.

Remark. If the Cartan matrix A is classical of finite type, if $\mathfrak{d} = 0$ (see § 2), and if $S = \emptyset$ (so that $\mathfrak{p}^e$ is a classical Borel subalgebra), then the modules $V^{M(\lambda)}$ are just the Verma modules (cf. [6, Chapitre 7]).

$V^{M(\lambda)}$ satisfies the universal property that any $\mathfrak{r}^e$-module map of $M(\lambda)$ into the $\mathfrak{u}$-invariant subspace of a $\mathfrak{g}^e$-module X extends uniquely to a $\mathfrak{g}^e$-module map of $V^{M(\lambda)}$ into X. Let $\mathscr{U}^-$ be the universal enveloping algebra of $\mathfrak{u}^-$. Then since $\mathfrak{g}^e = \mathfrak{u}^- \oplus \mathfrak{p}^e$, the multiplication map in $\mathscr{G}^e$ induces a linear isomorphism $\mathscr{G}^e \simeq \mathscr{U}^- \otimes \mathscr{P}^e$, by the Poincaré-Birkhoff-Witt theorem. Thus we have a natural linear isomorphism $V^{M(\lambda)} \simeq \mathscr{U}^- \otimes M(\lambda)$.

We digress to clarify some general terminology, some of which we have already used. Let X be an $\mathfrak{h}^e$-module (for example, a $\mathfrak{g}^e$-module or a $\mathfrak{p}^e$-module regarded as an $\mathfrak{h}^e$-module by restriction), and let $\nu \in (\mathfrak{h}^e)^*$. Define the *weight space* $X_\nu \subset X$ corresponding to ν to be $\{x \in X \mid h \cdot x = \nu(h)x \text{ for all } h \in \mathfrak{h}^e\}$. Call ν a *weight* of X if $X_\nu \neq 0$, and call the nonzero elements of X_ν *weight vectors* with weight ν. Call X a *weight module* if it is a (direct) sum of its weight spaces.

Clearly, $\mathfrak{g}^e$ is itself a weight module under the adjoint action, and for all $\varphi \in (\mathfrak{h}^e)^* - \{0\}$ (in particular, for all $\varphi \in \Delta$), the weight space $(\mathfrak{g}^e)_\varphi$ is just $\mathfrak{g}^\varphi$. Also, $(\mathfrak{g}^e)_0 = \mathfrak{h}^e$. Thus the set of weights of $\mathfrak{g}^e$ is just $\Delta \cup \{0\}$, and lies in the subspace R of $(\mathfrak{h}^e)^*$. If X is a $\mathfrak{g}^e$-module, $\varphi \in R$ and $\nu \in (\mathfrak{h}^e)^*$, then $\mathfrak{g}^\varphi \cdot X_\nu \subset X_{\nu+\varphi}$.

If X and Y are $\mathfrak{h}^e$-modules, and $\varphi, \psi \in (\mathfrak{h}^e)^*$, then $X_\varphi \otimes Y_\psi \subset (X \otimes Y)_{\varphi+\psi}$ in the tensor product module. It follows that the tensor product of weight modules is a weight module, and the weights of the tensor product are the sums of the weights of the factors. In particular, if $\mathfrak{t}$ is an $\mathfrak{h}^e$-invariant subalgebra of $\mathfrak{g}^e$, then the tensor algebra of $\mathfrak{t}$, and any of its quotients, such as the universal enveloping algebra of $\mathfrak{t}$, are weight modules with weights in $R \subset (\mathfrak{h}^e)^*$. For example, $\mathscr{G}^e$ is a weight module with weights in R. If X is a $\mathfrak{g}^e$-module, $\varphi \in R$ and $\nu \in (\mathfrak{h}^e)^*$, then $\mathscr{G}^e_\varphi \cdot X_\nu \subset X_{\nu+\varphi}$.

A $\mathfrak{g}^e$-module X is called a *highest weight module* if it is generated by an $\mathfrak{n}$-invariant weight vector x. The *highest weight vector* x is uniquely determined up to a nonzero scalar, and its weight is called the *highest weight* of X. The weight space of x is the *highest weight space* of X. The highest weight space of X is one-dimensional, X is a weight module with finite-dimensional weight spaces, and all the weights of X are of the form $\nu - \sum_{i=1}^{l} n_i \alpha_i$ $(n_i \in \mathbb{Z}_+)$, where $\nu \in (\mathfrak{h}^e)^*$ is the highest weight. These facts follow easily from the decomposition $\mathfrak{g}^e = \mathfrak{n}^- \oplus \mathfrak{h}^e \oplus \mathfrak{n}$.

Returning to the modules $V^{M(\lambda)}$, we note the following obvious fact:

Proposition 3.2. *For all $\lambda \in P_S$, the $\mathfrak{g}^e$-module $V^{M(\lambda)}$ is a highest weight module with highest weight λ. The highest weight space (relative to $\mathfrak{h}_S$ and $\mathfrak{n}_S$) of the $\mathfrak{g}_S$-module $M(\lambda)$ is the highest weight space of $V^{M(\lambda)}$ (identifying $M(\lambda)$ with the $\mathfrak{p}^e$-submodule $1 \otimes M(\lambda)$ of $V^{M(\lambda)}$).*

§ 4. The Casimir Operator; the Category $\mathscr{C}$

For $\nu \in (\mathfrak{h}^e)^*$, define $D(\nu) = \left\{ \nu - \sum_{i=1}^{l} n_i \alpha_i \mid n_i \in \mathbb{Z}_+ \right\} \subset (\mathfrak{h}^e)^*$. Let $\mathscr{C}$ be the category of $\mathfrak{g}^e$-modules X such that X is a weight module whose weight spaces are finite-dimensional and whose set of weights lies in a finite union of sets of the form $D(\nu)$ $(\nu \in (\mathfrak{h}^e)^*)$. Every highest weight module lies in $\mathscr{C}$ (see § 3). Also, $\mathscr{C}$ is stable under the operations of submodules and quotient modules. Following [11(b)], we shall define an operator Γ_X, to be called the *Casimir operator*, on $X \in \mathscr{C}$.

Recall that the Cartan matrix A is symmetrizable, and that this gives rise to a $\mathfrak{g}$-invariant symmetric bilinear form τ on $\mathfrak{g}$ such that $[a,b] = \tau(a,b) x_\varphi$ for all $\varphi \in \Delta$, $a \in \mathfrak{g}^\varphi$ and $b \in \mathfrak{g}^{-\varphi}$; $\tau(e_i, f_i) = 2/\sigma(\alpha_i, \alpha_i)$ for each $i = 1, \dots, l$; τ induces a nonsingular pairing between $\mathfrak{g}^\varphi$ and $\mathfrak{g}^{-\varphi}$; and $\tau(\mathfrak{g}^\psi, \mathfrak{g}^\eta) = 0$ for all $\psi, \eta \in R$ such that $\psi + \eta \neq 0$ (see § 2). Let $s_1, \dots, s_m$ be a basis of $\mathfrak{g}^\varphi$ $(\varphi \in \Delta)$. Then there is a unique dual basis $t_1, \dots, t_m$ of $\mathfrak{g}^{-\varphi}$, relative to τ. Set $\omega_\varphi = \sum_{i=1}^{m} t_i s_i \in \mathscr{G}^e$. Then ω_φ is independent of the basis $s_1, \dots, s_m$. Indeed, let $\iota_\varphi \in (\mathfrak{g}^\varphi)^* \otimes \mathfrak{g}^\varphi \simeq \operatorname{End} \mathfrak{g}^\varphi$ be the unique element corresponding to $1 \in \operatorname{End} \mathfrak{g}^\varphi$, let $\kappa_\varphi \colon (\mathfrak{g}^\varphi)^* \to \mathfrak{g}^{-\varphi}$ be the linear isomorphism defined by $\tau \mid \mathfrak{g}^\varphi \times \mathfrak{g}^{-\varphi}$, and let $f \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathscr{G}^e$ be the map induced by multiplication. Then $\omega_\varphi = f \circ (\kappa_\varphi \otimes 1)(\iota_\varphi)$.

Since $X \in \mathscr{C}$, it is clear that the (possibly infinite) sum $2 \sum_{\varphi \in \Delta_+} \omega_\varphi$ acts as a well-defined operator on X. Denote this operator by Γ_1. Note that $\omega_{\alpha_i} = \frac{1}{2} \sigma(\alpha_i, \alpha_i) f_i e_i$ for all $i = 1, \dots, l$.

Define $\rho \in (\mathfrak{h}^e)^*$ as in § 2, so that $\rho(h_i) = 1$ for all $i = 1, \dots, l$. Note that $\rho(x_{\alpha_i}) = \frac{1}{2} \sigma(\alpha_i, \alpha_i)$ for each i.

Now define the operator Γ_2 on X as follows: Recall that σ is a symmetric bilinear form on $(\mathfrak{h}^e)^*$ such that for all $\varphi \in R$ and $\lambda \in (\mathfrak{h}^e)^*$, we have $\sigma(\lambda, \varphi) = \lambda(x_\varphi)$ (see § 2). Let $\nu \in (\mathfrak{h}^e)^*$. Then Γ_2 acts on the weight space X_ν as scalar multiplication by $\sigma(\nu + \rho, \nu + \rho)$.

Definition. For the $\mathfrak{g}^e$-module $X \in \mathscr{C}$, define the *Casimir operator* $\Gamma_X \in \operatorname{End} X$ to be $\Gamma_1 + \Gamma_2$ (see above).

The following result, which is clear, says essentially that $X \mapsto \Gamma_X$ defines a functor from $\mathscr{C}$ to the category of morphisms of $\mathscr{C}$:

Proposition 4.1. *Let $X, Y \in \mathscr{C}$ and $f: X \to Y$ a $\mathfrak{g}^e$-module map. Then $f \circ \Gamma_X = \Gamma_Y \circ f$: $X \to Y$.*

Here is the crucial property of Γ_X (cf. [11(b)]):

Proposition 4.2. *For $X \in C$, the Casimir operator Γ_X commutes with the action of $\mathfrak{g}^e$ on X.*

Proof. It is clear that Γ_X commutes with the action of $\mathfrak{h}^e$. It is sufficient to show that for each $i = 1, \ldots, l$, $e_i \Gamma_X = \Gamma_X e_i$ and $f_i \Gamma_X = \Gamma_X f_i$ as operators on X.

Fix $i = 1, \ldots, l$. Let $\Phi \subset \Delta_+$ be a finite union of α_i-root strings (see § 2). Then $M = \coprod_{\varphi \in \Phi} \mathfrak{g}^\varphi$ and $N = \coprod_{\varphi \in -\Phi} \mathfrak{g}^\varphi$ are $\mathfrak{u}_i$-modules (in the notation of § 2) which are contragredient under the ($\mathfrak{g}$-invariant) form τ. Let $\iota_M \in M^* \otimes M \simeq \operatorname{End} M$ correspond to $1 \in \operatorname{End} M$, and let κ_M: $M^* \to N$ be the linear isomorphism defined by $\tau | M \times N$, so that κ_M is a $\mathfrak{u}_i$-module map. Let f: $\mathfrak{g} \otimes \mathfrak{g} \to \mathscr{G}^e$ be the map induced by multiplication. Then it is clear that $f \circ (\kappa_M \otimes 1)(\iota_M) \in \mathscr{G}^e$ is $\mathfrak{u}_i$-invariant (i.e., it commutes with $\mathfrak{u}_i$) and that this element is precisely $\sum_{\varphi \in \Phi} \omega_\varphi$.

Let $\nu \in (\mathfrak{h}^e)^*$ and let $x \in X_\nu$. Since $\mathfrak{g}^\varphi \cdot x = \mathfrak{g}^\varphi \cdot (e_i \cdot x) = \mathfrak{g}^\varphi \cdot (f_i \cdot x) = 0$ for all but finitely many $\varphi \in \Delta_+$, we may choose a finite subset $\Phi \subset \Delta_+$ which is a finite union of α_i-root strings and such that

$$\Gamma_1 \cdot x = 2\omega_{\alpha_i} \cdot x + 2 \sum_{\varphi \in \Phi} \omega_\varphi \cdot x = \sigma(\alpha_i, \alpha_i) f_i e_i \cdot x + 2 \sum_{\varphi \in \Phi} \omega_\varphi \cdot x,$$

$$\Gamma_1 \cdot (e_i \cdot x) = \sigma(\alpha_i, \alpha_i) f_i e_i^2 \cdot x + 2 \sum_{\varphi \in \Phi} \omega_\varphi e_i \cdot x$$

and

$$\Gamma_1 \cdot (f_i \cdot x) = \sigma(\alpha_i, \alpha_i) f_i e_i f_i \cdot x + 2 \sum_{\varphi \in \Phi} \omega_\varphi f_i \cdot x.$$

Thus

$$\begin{aligned} e_i \cdot (\Gamma_1 \cdot x) - \Gamma_1 \cdot (e_i \cdot x) &= \sigma(\alpha_i, \alpha_i)(e_i f_i e_i - f_i e_i^2) \cdot x \\ &= \sigma(\alpha_i, \alpha_i) h_i e_i \cdot x = \sigma(\alpha_i, \alpha_i)(\nu(h_i) + 2) e_i \cdot x \end{aligned}$$

and

$$\begin{aligned} f_i \cdot (\Gamma_1 \cdot x) - \Gamma_1 \cdot (f_i \cdot x) &= \sigma(\alpha_i, \alpha_i)(f_i^2 e_i - f_i e_i f_i) \cdot x \\ &= -\sigma(\alpha_i, \alpha_i) f_i h_i \cdot x = -\sigma(\alpha_i, \alpha_i) \nu(h_i) f_i \cdot x. \end{aligned}$$

On the other hand,

$$\begin{aligned} e_i \cdot (\Gamma_2 \cdot x) - \Gamma_2 \cdot (e_i \cdot x) &= (\sigma(\nu + \rho, \nu + \rho) - \sigma(\nu + \rho + \alpha_i, \nu + \rho + \alpha_i)) e_i \cdot x \\ &= (-2\sigma(\nu + \rho, \alpha_i) - \sigma(\alpha_i, \alpha_i)) e_i \cdot x \\ &= (-2\nu(x_{\alpha_i}) - 2\rho(x_{\alpha_i}) - \sigma(\alpha_i, \alpha_i)) e_i \cdot x \\ &= -\sigma(\alpha_i, \alpha_i)(\nu(h_i) + 2) e_i \cdot x \end{aligned}$$

and

$$f_i \cdot (\Gamma_2 \cdot x) - \Gamma_2 \cdot (f_i \cdot x) = (\sigma(v+\rho, v+\rho) - \sigma(v+\rho-\alpha_i, v+\rho-\alpha_i)) f_i \cdot x$$
$$= (2\sigma(v+\rho, \alpha_i) - \sigma(\alpha_i, \alpha_i)) f_i \cdot x = \sigma(\alpha_i, \alpha_i) v(h_i) f_i \cdot x.$$

Hence $e_i \cdot (\Gamma_X \cdot x) = \Gamma_X \cdot (e_i \cdot x)$ and $f_i \cdot (\Gamma_X \cdot x) = \Gamma_X \cdot (f_i \cdot x)$ for all $x \in X_v$ and hence for all $x \in X$. Q.E.D.

Corollary 4.3. *Let X be a highest weight module for $\mathfrak{g}^e$, with highest weight $\lambda \in (\mathfrak{h}^e)^*$ – for example, the module $V^{M(\lambda)}$, in the notation of Proposition 3.2. Then X lies in $\mathscr{C}$, and the Casimir operator of X acts on X as scalar multiplication by $\sigma(\lambda+\rho, \lambda+\rho)$.*

Proof. This follows immediately from Proposition 4.2 and the fact that Γ_X multiplies the highest weight vector of X by $\sigma(\lambda+\rho, \lambda+\rho)$. Q.E.D.

Lemma 4.4. *Let $X \in \mathscr{C}$. Then X has a (possibly finite) $\mathfrak{g}^e$-module filtration $0 = X_0 \subset X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup X_i$ and each $\mathfrak{g}^e$-module X_{i+1}/X_i $(i \geqq 0)$ is a highest weight module. In particular, if $X \neq 0$, then X contains a highest weight vector.*

Proof. Let $v_1, \ldots, v_r \in (\mathfrak{h}^e)^*$ be such that the set of weights of X lies in $D(v_1) \cup \cdots \cup D(v_r)$. Call $\lambda, \mu \in (\mathfrak{h}^e)^*$ *compatible* if $\lambda - \mu \in \sum_{i=1}^{l} \mathbb{Z}\alpha_i$. If λ and μ are compatible, then clearly there exists $v \in (\mathfrak{h}^e)^*$ such that $D(\lambda) \cup D(\mu) \subset D(v)$. Hence we may assume that $v_1, \ldots, v_r$ are mutually incompatible.

Every weight μ of X lies in exactly one set $D(v_j)$ $(1 \leqq j \leqq r)$. Write $v_j - \mu = \sum_{i=1}^{l} n_i \alpha_i$ $(n_i \in \mathbb{Z}_+)$, and define $N(\mu) = \sum_{i=1}^{l} n_i \in \mathbb{Z}_+$. For each $n \in \mathbb{Z}_+$, define $X(n)$ to be the direct sum of all the weight spaces X_μ of X such that $N(\mu) = n$. Then $\dim X(n) < \infty$ for each $n \in \mathbb{Z}_+$, since the weight spaces of X are finite-dimensional. It is clear that if Y is a submodule or a quotient module of X, then the function $n \mapsto \dim Y(n)$ may be defined exactly as for X, and that $\dim Y(n) \leqq \dim X(n)$ for every $n \in \mathbb{Z}_+$.

Let n_X be the nonnegative integer which is minimal such that $X(n_X) \neq 0$, let μ be a weight of X such that $N(\mu) = n_X$, and let x be a weight vector in X_μ. Then x is clearly $\mathfrak{n}$-invariant, so that the $\mathfrak{g}^e$-submodule X_1 of X generated by x is a highest weight module. For the quotient module X/X_1, we have $n_{X/X_1} \geqq n_X$, and if $n_{X/X_1} = n_X$, then $\dim (X/X_1)(n_X) < \dim X(n_X)$. Applying the above procedure to X/X_1, we find a submodule $X_2 \supset X_1$ of X such that X_2/X_1 is a highest weight module, $n_{X/X_2} \geqq n_{X/X_1}$, and if these are equal, then $\dim (X/X_2)(n_{X/X_1}) < \dim (X/X_1)(n_{X/X_1})$. Continuing inductively, we get a filtration of X with the desired properties. Q.E.D.

Definitions. For all $X \in \mathscr{C}$, let $\Theta(X) = \{c \in k \mid \Gamma_X x = cx$ for some $x \in X,\ x \neq 0\}$. For all $c \in k$, let

$$X_{(c)} = \{x \in X \mid (\Gamma_X - c)^n x = 0 \text{ for some } n > 0\}.$$

It is clear that $\Theta(X) = \{c \in k \mid X_{(c)} \neq 0\}$.

Remark. If X is a highest weight module with highest weight $\lambda \in (\mathfrak{h}^e)^*$, then $\Theta(X) = \{\sigma(\lambda+\rho, \lambda+\rho)\}$ (see Corollary 4.3).

Proposition 4.5. *For all* $X \in \mathscr{C}$,

$$X = \coprod_{c \in \Theta(X)} X_{(c)}.$$

Let $0 = X_0 \subset X_1 \subset X_2 \subset \dots$ *be any filtration of* X *with the properties in Lemma* 4.4, *and let* $\lambda_i \in (\mathfrak{h}^e)^*$ *be the highest weight of* X_{i+1}/X_i, *for each i. Then*

$$\Theta(X) = \{\sigma(\lambda_i + \rho, \lambda_i + \rho)\}_i.$$

Proof. For a subspace Y of X, let $[Y]$ denote the smallest Γ_X-invariant subspace of X containing Y. Let $c_1, c_2, \dots \in k$ be the distinct elements of the (possibly finite) set $\{\sigma(\lambda_i + \rho, \lambda_i + \rho)\}_i$. Since every finite-dimensional subspace Y of X is contained in some X_i, Corollary 4.3 implies that $[Y]$ is a finite-dimensional subspace of X_i and is annihilated by a product of powers of a finite number of operators of the form $\Gamma_X - c_j$. By standard finite-dimensional linear algebra, $[Y] = \coprod_j ([Y] \cap X_{(c_j)})$. Taking for Y the members of a filtration $0 = Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of X such that each Y_p is finite-dimensional and $X = \cup Y_p$, we see that $X = \coprod_j X_{(c_j)}$, and that if $c \in k$ is such that $X_{(c)} \neq 0$, then $c = c_j$ for some j. (We may construct such a filtration by taking $Y_p = \coprod_{n<p} X(n)$, where $X(n)$ is the space defined in the proof of Lemma 4.4.)

For each c_j, $X_{(c_j)} \neq 0$. Indeed, choose i so that $c_j = \sigma(\lambda_i + \rho, \lambda_i + \rho)$, and choose a vector $x \in X_{i+1}$, such that $x \notin X_i$. Then no product of powers of operators of the form $\Gamma_X - c$, $c \in k$, $c \neq c_j$, can send x into X_i, let alone to zero. Hence $X_{(c_j)} \neq 0$.

Since $\Theta(X) = \{c \in k \mid X_{(c)} \neq 0\}$, we have $\Theta(X) = \{c_1, c_2, \dots\}$. Q.E.D.

The following is clear from Propositions 4.1, 4.2 and 4.5:

Proposition 4.6. *Let* $c \in k$. *Then* $X \mapsto X_{(c)}$ *is a functor from* $\mathscr{C}$ *to* $\mathscr{C}$ *which transforms exact sequences to exact sequences. In particular, if* $Y \in \mathscr{C}$ *and* X *is a* $\mathfrak{g}^e$-*submodule of* Y, *then* $(Y/X)_{(c)} = Y_{(c)}/X_{(c)}$.

Proof. Proposition 4.2 implies that $X_{(c)}$ is a $\mathfrak{g}^e$-submodule of X, and Proposition 4.1 shows that if $X, Y \in \mathscr{C}$ and $f: X \to Y$ is a $\mathfrak{g}^e$-module map, then $f: X_{(c)} \to Y_{(c)}$. The exactness of the resulting functor follows from the first assertion of Proposition 4.5. Q.E.D.

Proposition 4.7. *Let* $X \in \mathscr{C}$, *let* $0 = X_0 \subset X_1 \subset X_2 \subset \dots$ *be any filtration of* X *with the properties stated in Lemma* 4.4, *and let* $\lambda_i \in (\mathfrak{h}^e)^*$ *be the highest weight of* X_{i+1}/X_i, *for each i. Let* $c \in k$. *Then* $X_{(c)}$ *has a (possibly finite)* $\mathfrak{g}^e$-*module filtration* $0 = Y_0 \subset Y_1 \subset Y_2 \subset \dots$ *such that* $X_{(c)} = \cup Y_j$, *and the family of* $\mathfrak{g}^e$-*modules* Y_{j+1}/Y_j *coincides (up to isomorphism) with the family of* $\mathfrak{g}^e$-*modules* X_{i+1}/X_i *for which*

$$\sigma(\lambda_i + \rho, \lambda_i + \rho) = c.$$

Proof. Just apply the functor $Z \mapsto Z_{(c)}$ to the given filtration, and use the above results. Q.E.D.

§ 5. The $\mathfrak{p}^e$-Modules $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e)$

Recall from § 3 that $S \subset \{1, \dots, l\}$ is a subset of finite type; that $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{u} = \mathfrak{g}_S \oplus \mathfrak{h}^S \oplus \mathfrak{u}$ is the corresponding F-parabolic subalgebra of $\mathfrak{g}$; that $\mathfrak{p}^e = \mathfrak{r}^e \oplus \mathfrak{u}$ is the corresponding F-parabolic subalgebra of $\mathfrak{g}^e$; and that $\mathfrak{g}_S$ is a finite-dimensional split semisimple Lie algebra. Also, $\mathfrak{g}_S = \mathfrak{n}_S^- \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S$ and $\mathfrak{g}^e = \mathfrak{u}^- \oplus \mathfrak{p}^e$.

Proposition 5.1. *Every $\mathfrak{g}_S$-invariant subspace of $\mathfrak{g}^e$ (under the adjoint action) is a direct sum of finite-dimensional irreducible $\mathfrak{g}_S$-modules.*

Proof. For each $i \in \{1, \ldots, l\}$, e_i and f_i are each contained in a finite-dimensional $\mathfrak{g}_S$-module. Indeed, if $i \in S$, then $e_i, f_i \in \mathfrak{g}_S$. Suppose that $i \notin S$. Then by the defining relations for $\mathfrak{g}$ (§ 2), f_i is an $\mathfrak{n}_S$-invariant weight vector for $\mathfrak{h}_S$, and for each $j \in S$, some power of $\operatorname{ad} f_j$ annihilates f_i. A well-known principle [6, Lemme 7.2.4] now implies that the $\mathfrak{g}_S$-module generated by f_i is finite-dimensional (and irreducible). Similarly, e_i is contained in a finite-dimensional $\mathfrak{g}_S$-module.

Also, $\mathfrak{h}^e$ is contained in the finite-dimensional $\mathfrak{g}_S$-module $\mathfrak{r}^e$.

Since $\mathfrak{h}^e$ and $\{e_i, f_i\}_{i=1,\ldots,l}$ generate $\mathfrak{g}^e$, every element of $\mathfrak{g}^e$ is thus contained in a finite-dimensional $\mathfrak{g}_S$-module. Hence by the complete reducibility theorem for the finite-dimensional semisimple Lie algebra $\mathfrak{g}_S$, $\mathfrak{g}^e$ is a sum of finite-dimensional irreducible $\mathfrak{g}_S$-modules. The proposition follows. Q.E.D.

Lemma 5.2. *Let X be an $\mathfrak{r}^e$-module which is a weight module with weights all of the form $\nu + \sum_{i \in S} c_i \alpha_i (c_i \in k)$ for some fixed $\nu \in (\mathfrak{h}^e)^*$. Then the weight spaces of X (for $\mathfrak{h}^e$) are precisely the weight spaces for the action of $\mathfrak{h}_S$ (the Cartan subalgebra of $\mathfrak{g}_S$) on X. In particular, every finite-dimensional irreducible $\mathfrak{g}_S$-submodule of X is $\mathfrak{r}^e$-stable and is of the form $M(\lambda)$ for some $\lambda \in P_S$ (see Proposition* 3.1).

Proof. This result follows immediately from the fact that the restrictions to $\mathfrak{h}_S$ of the α_i with $i \in S$ form a linearly independent family of elements of $\mathfrak{h}_S^*$ (see § 3). Q.E.D.

Lemma 5.3. *Let X be an $\mathfrak{r}^e$-module which is a weight module with weights all of the form $\nu - \sum_{i \in S} n_i \alpha_i (n_i \in \mathbb{Z}_+)$ for some fixed $\nu \in (\mathfrak{h}^e)^*$. Assume that X is a (direct) sum of finite-dimensional irreducible $\mathfrak{g}_S$-modules. Then X has only finitely many weights. In particular, if each weight space of X is finite-dimensional, then X is finite-dimensional.*

Proof. In view of the first assertion of Lemma 5.2, it is sufficient to show that X has only finitely many weights for $\mathfrak{h}_S$. We shall in fact show that only finitely many inequivalent irreducible $\mathfrak{g}_S$-modules can occur in X. The restrictions $\bar{\alpha}_i$ to $\mathfrak{h}_S$ of the α_i with $i \in S$ form a system of simple roots in $\mathfrak{h}_S^*$, and the $\mathfrak{h}_S$-weights of X are all of the form $\mathscr{X} - \sum_{i \in S} n_i \bar{\alpha}_i (n_i \in \mathbb{Z}_+)$ for a certain $\mathscr{X} \in \mathfrak{h}_S^*$. It is sufficient to show that only finitely many linear forms of this type in $\mathfrak{h}_S^*$ can be dominant integral with respect to $\{\bar{\alpha}_i\}$. But this follows easily from the standard fact that any dominant integral linear form in $\mathfrak{h}_S^*$ is a nonnegative rational linear combination of the $\bar{\alpha}_i$, $i \in S$ (cf. for example [9, p. 72, Exercises 7 and 8]). Q.E.D.

Definition. Let X be an $\mathfrak{h}^e$-module and let $\nu \in (\mathfrak{h}^e)^*$. Define $X\{\nu\}$ to be the (direct) sum of all the weight spaces in X with weights of the form $\nu - \sum_{i \in S} n_i \alpha_i$ where each $n_i \in \mathbb{Z}_+$.

For each $j \in \mathbb{Z}_+$, the j-th exterior power $\Lambda^j(\mathfrak{u}^-)$ is an $\mathfrak{r}^e$-module by natural extension by derivations of the adjoint action of $\mathfrak{r}^e$ on $\mathfrak{u}^-$.

Proposition 5.4. *Let* $S' = \{1, \ldots, l\} - S$. *For each* $j \in \mathbb{Z}_+$,

$$\Lambda^j(\mathfrak{u}^-) = \coprod \Lambda^j(\mathfrak{u}^-)\{-\sum_{i\in S'} m_i\alpha_i\},$$

where the direct sum is over all sequences $(m_i)_{i\in S'}$ *of nonnegative integers. Each direct summand is finite-dimensional, and is a direct sum of finitely many irreducible* $\mathfrak{r}^e$*-modules of the form* $M(\lambda)$ *(see Proposition* 3.1), *where* $\lambda \in R \subset (\mathfrak{h}^e)^*$ *is a weight for the action of* $\mathfrak{h}^e$ *on* $\Lambda^j(\mathfrak{u}^-)$, *and* $\lambda \in P_S$.

Proof. Clearly, $\mathfrak{u}^-$ is a weight module with weights all of the form $-\sum_{i=1}^{l} n_i\alpha_i (n_i \in \mathbb{Z}_+)$, and the same is true of each $\Lambda^j(\mathfrak{u}^-)$. This proves the first assertion. Also, each space $\Lambda^j(\mathfrak{u}^-)\{-\sum_{i\in S'} m_i\alpha_i\}$ is the sum of all the weight spaces in $\Lambda^j(\mathfrak{u}^-)$ with weights of the form $-\sum_{i\in S'} m_i\alpha_i + \sum_{i\in S} n_i\alpha_i$ with $n_i \in \mathbb{Z}$, and so each such space is an $\mathfrak{r}^e$-module. By Proposition 5.1, every $\mathfrak{g}_S$-invariant subspace of $\Lambda^j(\mathfrak{u}^-)$ is a direct sum of finite-dimensional irreducible $\mathfrak{g}_S$-modules. Also, since each root space in $\mathfrak{u}^-$ is finite-dimensional, it is clear that each weight space in $\Lambda^j(\mathfrak{u}^-)$ is finite-dimensional. Thus all the hypotheses of Lemmas 5.2 and 5.3 hold for each space $\Lambda^j(\mathfrak{u}^-)\{-\sum_{i\in S'} m_i\alpha_i\}$, and the proposition follows. Q.E.D.

Proposition 5.5. *For each* $j \in \mathbb{Z}_+$, $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e)$ *has a filtration* $0 = M_0 \subset M_1 \subset M_2 \subset \ldots$ *of* $\mathfrak{p}^e$*-modules such that* $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) = \cup M_i$, *each nonzero quotient* $M_{i+1}/M_i (i \in \mathbb{Z}_+)$ *is a trivial* $\mathfrak{u}$*-module and is isomorphic as an* $\mathfrak{r}^e$*-module to a module* $M(\lambda)$ *with* λ *as in Proposition* 5.4, *and* $\coprod_{i\in\mathbb{Z}_+} M_{i+1}/M_i$ *is isomorphic as an* $\mathfrak{r}^e$*-module to* $\Lambda^j(\mathfrak{u}^-)$.

Proof. As $\mathfrak{r}^e$-modules, $\mathfrak{g}^e/\mathfrak{p}^e \simeq \mathfrak{u}^-$ and $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) \simeq \Lambda^j(\mathfrak{u}^-)$. For each $n \in \mathbb{Z}_+$, let N_n be the direct sum of all the $\mathfrak{r}^e$-submodules of $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e)$ of the form $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e)\{-\sum_{i\in S'} m_i\alpha_i\}$, where $(m_i)_{i\in S'}$ is a sequence of nonnegative integers whose sum is less than n. By Proposition 5.4, we have $0 = N_0 \subset N_1 \subset N_2 \subset \ldots$; $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) = \cup N_n$; $\mathfrak{u} \cdot N_{n+1} \subset N_n$ for each $n \in \mathbb{Z}_+$; N_{n+1}/N_n is a finite-dimensional $\mathfrak{r}^e$-module which is a direct sum of finitely many $\mathfrak{r}^e$-modules $M(\lambda)$ with λ as in Proposition 5.4; and $\coprod_{n\in\mathbb{Z}_+} N_{n+1}/N_n$ is isomorphic as an $\mathfrak{r}^e$-module to $\Lambda^j(\mathfrak{u}^-)$. An obvious refinement of the sequence of N_n's gives the desired sequence of M_i's. Q.E.D.

§ 6. Quasisimple $\mathfrak{g}^e$-Modules

Definition. Let X be a $\mathfrak{g}^e$-module. X is *quasisimple* (cf. [11(b)]) if X is a highest weight module with a highest weight vector x such that there exists $n \in \mathbb{Z}_+$ with $f_i^n \cdot x = 0$ for all $i = 1, \ldots, l$.

Proposition 6.1. *The set of weights of a quasisimple* $\mathfrak{g}^e$*-module* X *is stable under the Weyl group* W *acting on* $(\mathfrak{h}^e)^*$ *(see* §2). *For every weight* μ *of* X *and every* $w \in W$, $\dim X_\mu = \dim X_{w\mu}$. *In particular, if* μ *is the highest weight of* X, *then* $\dim X_{w\mu} = 1$ *for all* $w \in W$.

Proof. As in §2, let $\mathfrak{u}_i$ be the three-dimensional simple Lie algebra spanned by h_i, e_i, $f_i (1 \leq i \leq l)$. Since X is generated by a highest weight vector contained in a finite-dimensional irreducible $\mathfrak{u}_i$-module, and since $\mathfrak{g}$ is a sum of such $\mathfrak{u}_i$-modules (see §2), so is X. Let μ be a weight of X. The fact that $\coprod_{n \in \mathbb{Z}} X_{\mu + n\alpha_i}$ is $\mathfrak{u}_i$-stable shows easily that $\dim X_\mu = \dim X_{r_i \mu}$, and the rest is easy. Q.E.D.

Proposition 6.2. *Let $P \subset (\mathfrak{h}^e)^*$ be the set of dominant integral elements, as in §2. The highest weight of a quasisimple $\mathfrak{g}^e$-module lies in P. Conversely, let $\mu \in P$. Then there exist quasisimple $\mathfrak{g}^e$-modules $X^\mu_{\max}$ and $X^\mu_{\min}$ with highest weight μ, universal in the sense that if X is any quasisimple $\mathfrak{g}^e$-module with highest weight μ, then there are $\mathfrak{g}^e$-module surjections $X^\mu_{\max} \to X \to X^\mu_{\min}$. $X^\mu_{\min}$ is also characterized as the unique irreducible highest weight module with highest weight μ.*

Proof. The first assertion follows easily from the standard representation theory of the three-dimensional simple Lie algebras $\mathfrak{u}_i (1 \leq i \leq l)$ spanned by h_i, e_i and f_i.

Apply the theory of §3 to the set $S = \emptyset$. Then the F-parabolic subalgebra is $\mathfrak{h} \oplus \mathfrak{n}$, $\mathfrak{g}_S = 0$, $\mathfrak{r}^e = \mathfrak{h}^e$, and for all $\mu \in (\mathfrak{h}^e)^*$, the $\mathfrak{h}^e$-module $M(\mu)$ is a one-dimensional weight space with weight μ. The induced $\mathfrak{g}^e$-module $V^{M(\mu)}$ is a highest weight module with highest weight μ. Let $v_0 \in V^{M(\mu)}$ be a highest weight vector. Then if X is any highest weight module with highest weight μ and highest weight vector x, there is a unique surjection $V^{M(\mu)} \to X$ taking v_0 to x.

Let $\mu \in P$, and let $n_i = \mu(h_i)$ for $i = 1, \ldots, l$, so that each $n_i \in \mathbb{Z}_+$. For each $i = 1, \ldots, l$, the representation theory of $\mathfrak{u}_i$ implies that $e_i \cdot (f_i^{n_i+1} \cdot v_0) = 0$, and the fact that $[e_j, f_i] = 0$ for $j \neq i$ implies that $f_i^{n_i+1} \cdot v_0$ is $\mathfrak{n}$-invariant. Let Y be the $\mathfrak{g}^e$-module quotient of $V^{M(\mu)}$ by the submodule generated by $\{f_i^{n_i+1} \cdot v_0\}_{i=1,\ldots,l}$. Then Y is quasisimple with highest weight μ. Y clearly has a largest submodule not intersecting the highest weight space of Y, and the corresponding quotient $\mathfrak{g}^e$-module Z is irreducible and quasisimple with highest weight μ. Let X be quasisimple with highest weight μ, and let $x \in X$ be a highest weight vector. The representation theory of $\mathfrak{u}_i$ shows that $f_i^{n_i+1} \cdot x = 0$ for each $i = 1, \ldots, l$. Thus we have surjections $Y \to X \to Z$. Hence we may take $X^\mu_{\max} = Y$ and $X^\mu_{\min} = Z$. To prove the last assertion, let X be an irreducible highest weight module with highest weight μ, and let $x \in X$ be a highest weight vector. Arguing as above, we see that for each $i = 1, \ldots, l$, $f_i^{n_i+1} \cdot x$ is $\mathfrak{n}$-invariant. Thus this vector generates a proper $\mathfrak{g}^e$-submodule of X, and so must be zero. Hence X is quasisimple and must be isomorphic to $X^\mu_{\min}$. Q.E.D.

Remark. We shall see in §9 that in fact $X^\mu_{\max} = X^\mu_{\min}$, so that every quasisimple $\mathfrak{g}^e$-module is irreducible (cf. [11(b), Corollary]), and P indexes the quasisimple $\mathfrak{g}^e$-modules.

Return again to the situation in which $S \subset \{1, \ldots, l\}$ is an arbitrary subset of finite type (§3).

Proposition 6.3. *Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$ (see Proposition 6.2), let $j \in \mathbb{Z}_+$, and consider the $\mathfrak{r}^e$-module $Y = \Lambda^j(\mathfrak{u}^-) \otimes X$ (cf. §5). Then*

$$Y = \coprod Y\{\mu - \sum_{i \in S'} m_i \alpha_i\},$$

where the notations $Y\{.\}$ and S' are as in Proposition 5.4, and the direct sum is over all sequences $(m_i)_{i\in S'}$ of nonnegative integers. Each direct summand is itself a direct sum of finitely many irreducible $\mathfrak{r}^e$-modules of the form $M(\lambda)$ (see Proposition 3.1), where λ is a weight of Y and $\lambda \in P_S$.

Proof. Both X and $\Lambda^j(\mathfrak{u}^-)$ are weight modules with finite-dimensional weight spaces. Since the weights of X are of the form $\mu - \sum_{i=1}^{l} n_i \alpha_i (n_i \in \mathbb{Z}_+)$ and the weights of $\Lambda^j(\mathfrak{u}^-)$ are of the form $-\sum_{i=1}^{l} n_i \alpha_i (n_i \in \mathbb{Z}_+)$, we see that Y is a weight module with finite-dimensional weight spaces, the first assertion of the proposition holds, and each direct summand $Y\{\mu - \sum_{i\in S'} m_i \alpha_i\}$ is $\mathfrak{r}^e$-stable.

Let x be a highest weight vector of X. By [6, Lemme 7.2.4], the $\mathfrak{g}_S$-module generated by x is a finite-dimensional irreducible $\mathfrak{g}_S$-module (cf. the proof of Proposition 5.1). Since $X = \mathscr{G} \cdot x$ ($\mathscr{G}$ the universal enveloping algebra of $\mathfrak{g}$), Proposition 5.1 implies that X, and hence every $\mathfrak{g}_S$-invariant subspace of X, is a direct sum of finite-dimensional irreducible $\mathfrak{g}_S$-modules. This is also true of $\Lambda^j(\mathfrak{u}^-)$, by Proposition 5.1, and so it is true of Y. Our result now follows from Lemmas 5.2 and 5.3, just as in the proof of Proposition 5.4. Q.E.D.

Imitating the proof of Proposition 5.5, we have:

Proposition 6.4. *Let X be a quasisimple $\mathfrak{g}^e$-module, let $j \in \mathbb{Z}_+$, and let Z be the $\mathfrak{p}^e$-module $\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) \otimes X$. Then Z has a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots$ of $\mathfrak{p}^e$-modules such that $Z = \cup M_i$, each nonzero quotient $M_{i+1}/M_i (i \in \mathbb{Z}_+)$ is a trivial $\mathfrak{u}$-module and is $\mathfrak{r}^e$-isomorphic to a module $M(\lambda)$ with λ as in Proposition 6.3, and $\coprod_{i\in\mathbb{Z}_+} M_{i+1}/M_i$ is $\mathfrak{r}^e$-isomorphic to $\Lambda^j(\mathfrak{u}^-) \otimes X$.*

Remark. Since the trivial one-dimensional $\mathfrak{g}^e$-module is quasisimple, Propositions 6.3 and 6.4 include Propositions 5.4 and 5.5 as special cases.

§ 7. The $\mathfrak{r}^e$-Module Complex $C_*(X)$

Recall that $\mathscr{G}^e$, $\mathscr{P}^e$ and $\mathscr{U}^-$ are the universal enveloping algebras of $\mathfrak{g}^e$, $\mathfrak{p}^e$ and $\mathfrak{u}^-$, respectively. Proposition 1.9 immediately gives us a $\mathscr{U}^-$-free resolution of a quasisimple $\mathfrak{g}^e$-module X:

Proposition 7.1. *Let X be a quasisimple $\mathfrak{g}^e$-module. For each $j \in \mathbb{Z}_+$, let D_j^X be the $\mathfrak{g}^e$-module $\mathscr{G}^e \otimes_{\mathscr{P}^e} (\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) \otimes X)$, where X is regarded as a $\mathfrak{p}^e$-module by restriction. Then there is an exact sequence of $\mathfrak{g}^e$-modules and $\mathfrak{g}^e$-module maps*

$$\cdots \xrightarrow{d_2^X} D_1^X \xrightarrow{d_1^X} D_0^X \xrightarrow{\varepsilon_0^X} X \to 0,$$

with this complex naturally isomorphic to $V(\mathfrak{g}^e, \mathfrak{p}^e, X)$ (see §1). Each D_j^X is $\mathscr{U}^-$-free. More precisely, D_j^X is isomorphic as $\mathscr{U}^-$-module and as $\mathfrak{r}^e$-module to

$$\mathscr{U}^- \otimes_k (\Lambda^j(\mathfrak{u}^-) \otimes X),$$

with $\mathcal{U}^-$ acting by left multiplication on the first factor, and $\mathfrak{r}^e$ acting via tensor product action on the tensor product of the three $\mathfrak{r}^e$-modules.

The above resolution of X gives rise to the following chain complex:

$$\cdots \xrightarrow{1\otimes d_2^X} k\otimes_{\mathcal{U}^-} D_1^X \xrightarrow{1\otimes d_1^X} k\otimes_{\mathcal{U}^-} D_0^X \rightarrow 0,$$

where k is regarded as the trivial right $\mathfrak{u}^-$-module. Call this complex $C_*(X)$, and its homology $H_*(X)$.

The following is clear from Proposition 1.9:

Proposition 7.2. *$C_*(X)$ is in a natural way an $\mathfrak{r}^e$-module complex, and is naturally isomorphic to the standard complex for computing $H_*(\mathfrak{u}^-, X^t)$, provided with the standard action of $\mathfrak{r}^e$. The complex $C_*(X)$ is an $\mathfrak{r}^e$-module complex of the following form:*

$$\cdots \rightarrow \Lambda^1(\mathfrak{u}^-)\otimes_k X \rightarrow \Lambda^0(\mathfrak{u}^-)\otimes_k X \rightarrow 0,$$

where X is regarded as an $\mathfrak{r}^e$-module by restriction. $H_*(X)$ is naturally $\mathfrak{r}^e$-module isomorphic to $H_*(\mathfrak{u}^-, X^t)$, provided with the standard action of $\mathfrak{r}^e$ (see §1).

We shall compute $H_*(X)=H_*(\mathfrak{u}^-, X^t)$.

Definition. Let $\Psi=(\lambda_1, \lambda_2, \ldots)$ be a sequence, possibly finite, of elements of P_S (see Proposition 3.1). A $\mathfrak{g}^e$-module V is said to be *of type Ψ* if V has a strictly increasing (possibly finite) $\mathfrak{g}^e$-module filtration $0=V_0\subset V_1\subset V_2\subset \ldots$ such that $V=\cup V_i$ and such that the sequence of $\mathfrak{g}^e$-modules V_1/V_0, V_2/V_1, ... coincides up to rearrangement with the sequence of induced modules $V^{M(\lambda_1)}$, $V^{M(\lambda_2)}$, ... (in the notation of §3).

By Propositions 1.10 (applied to $\mathfrak{p}^e$ and $\mathfrak{g}^e$ in place of $\mathfrak{a}$ and $\mathfrak{b}$), and 6.4, we have:

Lemma 7.3. *Let X be a quasisimple $\mathfrak{g}^e$-module, let $j\in\mathbb{Z}_+$, and let $\Psi^j=(\lambda_1, \lambda_2, \ldots)$ be the (possibly finite) sequence of elements of P_S such that $\Lambda^j(\mathfrak{u}^-)\otimes X\simeq\coprod M(\lambda_i)$ as an $\mathfrak{r}^e$-module (see Proposition 6.3). Then the $\mathfrak{g}^e$-module D_j^X (see Proposition 7.1) is of type Ψ^j.*

Remark. The sequence Ψ^j is uniquely determined up to order, and by Proposition 6.3, each λ_i occurs with only finite multiplicity in Ψ^j.

It is easy to see from the last assertion of Proposition 7.1 that the $\mathfrak{g}^e$-module D_j^X lies in the category $\mathscr{C}$ of §4. Thus for each $c\in k$, we may apply the functor $Y\mapsto Y_{(c)}$ (see Proposition 4.6) to the resolution displayed in Proposition 7.1, and by Proposition 4.6, the result is an exact sequence. Let $\Theta=(\bigcup_{j\in\mathbb{Z}_+}\Theta(D_j^X))\cup\Theta(X)$, in the notation of Proposition 4.5, and let $\Theta'=\Theta-\Theta(X)$. Let μ be the highest weight of X, so that $\Theta(X)=\{\sigma(\mu+\rho, \mu+\rho)\}$ (see the Remark preceding Proposition 4.5). Then by Proposition 4.5, the complex in Proposition 7.1 is the direct sum of the exact $\mathfrak{g}^e$-module complex

$$\cdots \rightarrow (D_1^X)_{(\sigma(\mu+\rho,\, \mu+\rho))} \rightarrow (D_0^X)_{(\sigma(\mu+\rho,\, \mu+\rho))} \rightarrow X \rightarrow 0$$

and the exact $\mathfrak{g}^e$-module complexes

$$\cdots \to (D_1^X)_{(c)} \to (D_0^X)_{(c)} \to 0,$$

as c varies through Θ'. (We need not specify the maps.) Hence by Proposition 4.7 and Lemma 7.3, we have:

Lemma 7.4. *Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$ (see Proposition 6.2). For each $j \in \mathbb{Z}_+$, let Ψ_μ^j be the subsequence of Ψ^j (see Lemma 7.3) consisting of those λ_i in Ψ^j such that $\sigma(\lambda_i+\rho, \lambda_i+\rho)=\sigma(\mu+\rho, \mu+\rho)$. Then the resolution displayed in Proposition 7.1 is a direct sum of an exact $\mathfrak{g}^e$-module complex*

$$\cdots \to E_1 \to E_0 \to X \to 0$$

and a denumerable family of exact $\mathfrak{g}^e$-module complexes

$$\cdots \to E_1^{(n)} \to E_0^{(n)} \to 0$$

such that for each $j \in \mathbb{Z}_+$, E_j is of type Ψ_μ^j.

The following result is clear:

Lemma 7.5. *In the notation of Proposition 7.2 and Lemma 7.4, the $\mathfrak{r}^e$-module complex $C_*(X)$ is the direct sum of the $\mathfrak{r}^e$-module complex $B_*(X)$*

$$\cdots \to k \otimes_{\mathscr{U}^-} E_1 \to k \otimes_{\mathscr{U}^-} E_0 \to 0$$

and the $\mathfrak{r}^e$-module complexes $B_*^{(n)}(X)$

$$\cdots \to k \otimes_{\mathscr{U}^-} E_1^{(n)} \to k \otimes_{\mathscr{U}^-} E_0^{(n)} \to 0,$$

where k is regarded as the trivial right $\mathfrak{u}^-$-module. In particular, $H_*(X)$ is naturally $\mathfrak{r}^e$-module isomorphic to the direct sum of the homologies of $B_*(X)$ and of the $B_*^{(n)}(X)$.

Lemma 7.6. *In the notation of the last lemma, the homology of $B_*^{(n)}(X)$ is zero for each n. In particular, $H_*(X)$ is naturally $\mathfrak{r}^e$-module isomorphic to the homology of $B_*(X)$.*

Proof. Fix n. For each $j \in \mathbb{Z}_+$, D_j^X is free as a $\mathscr{U}^-$-module (Proposition 7.1), and $E_j^{(n)}$ is a $\mathscr{U}^-$-module direct summand of D_j^X. Hence $E_j^{(n)}$ is projective as a $\mathscr{U}^-$-module (see [3, p. 6]). Denote the $\mathscr{U}^-$-module map from $E_{j+1}^{(n)}$ to $E_j^{(n)}$ by ∂_{j+1}. Since $E_0^{(n)}$ is projective, there is a $\mathscr{U}^-$-map $f_1: E_0^{(n)} \to E_1^{(n)}$ such that $\partial_1 \circ f_1 = \text{identity}$. Thus $E_1^{(n)} = (\operatorname{Ker} \partial_1) \oplus (\operatorname{Im} f_1) = (\operatorname{Im} \partial_2) \oplus (\operatorname{Im} f_1)$, and $\partial_1: \operatorname{Im} f_1 \to E_0^{(n)}$ is a $\mathscr{U}^-$-isomorphism. Since $E_1^{(n)}$ is projective, so is $\operatorname{Im} \partial_2$, and thus there is a $\mathscr{U}^-$-map $f_2: \operatorname{Im} \partial_2 \to E_2^{(n)}$ such that $\partial_2 \circ f_2 = \text{identity on } \operatorname{Im} \partial_2$. Hence $E_2^{(n)} = (\operatorname{Im} \partial_3) \oplus (\operatorname{Im} f_2)$, and $\partial_2: \operatorname{Im} f_2 \to \operatorname{Im} \partial_2$ is a $\mathscr{U}^-$-isomorphism. Continuing in this way and then tensoring over $\mathscr{U}^-$ with k, we see that the homology of $B_*^{(n)}(X)$ is zero. Q.E.D.

Remark. This proof may be shortened by quoting the fact that $\operatorname{Tor}_*^{\mathfrak{u}^-}(k, 0)=0$ and that $\operatorname{Tor}_*^{\mathfrak{u}^-}(k, 0)$ may be computed using any $\mathscr{U}^-$-projective resolution of the zero module 0 (see [3]).

Lemma 7.7. *Let $\lambda \in P_S$ (see § 3). Then $V^{M(\lambda)}$ is canonically isomorphic as an $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module to the module induced from the $\mathfrak{r}^e$-module $M(\lambda)$. Linearly, $V^{M(\lambda)} \simeq$*

$\mathcal{U}^- \otimes_k M(\lambda)$, $\mathfrak{u}^-$ acts by left multiplication on the first factor, and $\mathfrak{r}^e$ acts by the tensor product of the adjoint action on $\mathcal{U}^-$ with the natural action on $M(\lambda)$.

Proof. This is all straightforward, using the Poincaré-Birkhoff-Witt theorem. The first assertion is a special case of [6, Proposition 5.1.14]. Q.E.D.

Lemma 7.8. *Let $\Psi=(\lambda_1, \lambda_2, \ldots)$ be a sequence, possibly finite, of elements of P_S, and let V be a $\mathfrak{g}^e$-module of type Ψ. Assume also that V is a (direct) sum of $\mathfrak{r}^e$-modules of the form $M(\lambda)$ with $\lambda \in P_S$. Then as an $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module, V is isomorphic to the module induced from the $\mathfrak{r}^e$-module $\coprod_i M(\lambda_i)$.*

Proof. By the last lemma, V has a strictly increasing $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module filtration $0=V_0 \subset V_1 \subset V_2 \subset \cdots$ such that $V=\bigcup V_i$ and such that the sequence of $\mathfrak{r}^e \oplus \mathfrak{u}^-$-modules $V_1/V_0, V_2/V_1, \ldots$ coincides up to rearrangement with the sequence of $\mathfrak{r}^e \oplus \mathfrak{u}^-$-modules induced from $M(\lambda_1), M(\lambda_2), \ldots$. We may assume for convenience that no rearrangement is necessary. Consider the exact sequence of $\mathfrak{r}^e \oplus \mathfrak{u}^-$-modules

$$0 \to V_1 \to V_2 \xrightarrow{f} V_2/V_1 \to 0.$$

The hypothesis that V, and hence V_2, is a sum of $M(\lambda)$'s, and the fact that V_2/V_1 contains a copy Y of $M(\lambda_2)$, imply that V_2 contains a copy of $M(\lambda_2)$ disjoint from $V_1 \subset V_2$. Thus there is an $\mathfrak{r}^e$-module map $g\colon Y \to V_2$ such that $f \circ g$ is the identity on Y. By the universal property of the induced $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module V_2/V_1, g extends to an $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module map $g\colon V_2/V_1 \to V_2$, and $f \circ g$ must be the identity on V_2/V_1. Hence V_2 is isomorphic to the $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module induced from the $\mathfrak{r}^e$-module $M(\lambda_1) \oplus M(\lambda_2)$. Repeating this argument for the exact sequences of $\mathfrak{r}^e \oplus \mathfrak{u}^-$-modules

$$0 \to V_2 \to V_3 \to V_3/V_2 \to 0,$$
$$0 \to V_3 \to V_4 \to V_4/V_3 \to 0, \ldots,$$

we see that V is $\mathfrak{r}^e \oplus \mathfrak{u}^-$-isomorphic to the direct sum of modules induced from the $\mathfrak{r}^e$-modules $M(\lambda_1), M(\lambda_2), \ldots$. Q.E.D.

Remark. Each of the $\mathfrak{g}^e$-modules E_j appearing in Lemma 7.4 satisfies the hypotheses of Lemma 7.8. Indeed, each module D_j^X in Proposition 7.1 is a direct sum of $\mathfrak{r}^e$-modules of the form $M(\lambda)$, by the last assertion of Proposition 7.1, and the argument used to prove Proposition 6.3.

Proposition 7.9. *Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$ (see Proposition 6.2). For each $j \in \mathbb{Z}_+$, define $C_j(X)$ to be the $\mathfrak{r}^e$-module $\Lambda^j(\mathfrak{u}^-) \otimes X$, so that the standard $\mathfrak{r}^e$-module complex for computing the standard action of $\mathfrak{r}^e$ on $H_*(\mathfrak{u}^-, X^t)$ has the form*

$$\cdots \to C_1(X) \to C_0(X) \to 0.$$

Then $H_*(\mathfrak{u}^-, X^t)$ is naturally $\mathfrak{r}^e$-module isomorphic to the homology of a certain subcomplex $B_*(X)$ of $C_*(X)$, where for each $j \in \mathbb{Z}_+$, $B_j(X)$ is the sum of all the $\mathfrak{r}^e$-submodules of $C_j(X)=\Lambda^j(\mathfrak{u}^-) \otimes X$ isomorphic to $M(\lambda)$ (see Proposition 3.1) where $\lambda \in P_S$ and $\sigma(\lambda+\rho, \lambda+\rho)=\sigma(\mu+\rho, \mu+\rho)$.

Proof. In view of Proposition 7.2, the $\mathfrak{r}^e$-module complex $C_*(X)$ defined in Proposition 7.9 may be identified with the one defined before Proposition 7.2. For each $j \in \mathbb{Z}_+$, E_j is of type Ψ_μ^j (see Lemma 7.4). By Lemma 7.8 and the subsequent Remark, E_j is isomorphic as an $\mathfrak{r}^e \oplus \mathfrak{u}^-$-module to the module induced by $\coprod M(\lambda_i)$ where λ_i ranges through the sequence Ψ_μ^j. Defining the subcomplex $B_*(X)$ of $C_*(X)$ as in Lemma 7.5, we see that $B_j(X)$ is an $\mathfrak{r}^e$-submodule of $C_j(X)$, $\mathfrak{r}^e$-isomorphic to $\coprod M(\lambda_i)$ (λ_i ranging through Ψ_μ^j). But by the Remark following Lemma 7.3, and the definition of Ψ_μ^j, the only $\mathfrak{r}^e$-submodule of $C_j(X)$ $\mathfrak{r}^e$-isomorphic to $\coprod M(\lambda_i)$ (λ_i ranging through Ψ_μ^j) is the sum of all the $\mathfrak{r}^e$-submodules of $C_j(X)$ isomorphic to $M(\lambda)$, where $\lambda \in P_S$ and $\sigma(\lambda+\rho, \lambda+\rho) = \sigma(\mu+\rho, \mu+\rho)$. By Lemma 7.6, $H_*(\mathfrak{u}^-, X^t)$ is naturally $\mathfrak{r}^e$-isomorphic to the homology of $B_*(X)$. Q.E.D.

§ 8. The Homology and the Resolution

Using the results of § 2, we shall complete the computation of $H_*(\mathfrak{u}^-, X^t)$ (see § 7).

Recall from § 3 that Δ_+^S is the set of positive roots which are nonnegative integral linear combinations of the α_i with $i \in S$. Let $\Delta_+(S)$ be the complement of Δ_+^S in Δ_+. Then $\mathfrak{u}^- = \coprod_{\varphi \in -\Delta_+(S)} \mathfrak{g}^\varphi$ (see § 3).

Definition. Let W_S^1 be the subset of the Weyl group W consisting of those $w \in W$ such that $\Phi_w \subset \Delta_+(S)$ (see § 2 for the notation).

Proposition 8.1. $W_S^1 = \{w \in W \mid w^{-1} \Delta_+^S \subset \Delta_+\}$.

Proof. If $w \in W$, then

$$w \in W_S^1 \Leftrightarrow \Delta_+^S \cap w\Delta_- = \emptyset \Leftrightarrow -\Delta_+^S \subset w\Delta_- \Leftrightarrow \Delta_+^S \subset w\Delta_+. \quad \text{Q.E.D.}$$

Remark. It is not hard to show, as in [10] (cf. also [12]) that if W_S is defined to be the subgroup of W generated by $\{r_i\}_{i \in S}$, then every element of W can be expressed uniquely in the form $w_1 w^1$, where $w_1 \in W_S$ and $w^1 \in W_S^1$, $l(w_1 w^1) = l(w_1) + l(w^1)$ (see § 2 for the notation), and W_S^1 can be characterized as the set of elements of minimal length in the cosets $W_S w$ ($w \in W$); each such coset contains a unique element of minimal length. We shall not need these facts.

Since $\mathfrak{u}^-$ is a weight module (for $\mathfrak{h}^e$; recall the terminology of § 3), it is clear that $\Lambda^j(\mathfrak{u}^-)$ is a weight module for all $j \in \mathbb{Z}_+$.

Proposition 8.2. *For each $j \in \mathbb{Z}_+$, the weights of $\Lambda^j(\mathfrak{u}^-)$ lie in the subset T of $(\mathfrak{h}^e)^*$ defined in Proposition 2.8. Let $w \in W_S^1$, and suppose that $l(w) = j$. Then $-\langle \Phi_w \rangle$ (see § 2) is a weight of $\Lambda^j(\mathfrak{u}^-)$, and the corresponding weight space is one-dimensional. If $w \in W$ and either $w \notin W_S^1$ or $l(w) \neq j$, then $-\langle \Phi_w \rangle$ is not a weight of $\Lambda^j(\mathfrak{u}^-)$.*

Proof. Recall from § 3 that $\mathfrak{n}^- = \coprod_{\varphi \in \Delta_-} \mathfrak{g}^\varphi$. Choose a basis $\{b_i\}_{i \in I}$ of $\mathfrak{n}^-$ such that for each $i \in I$, b_i lies in the root space $\mathfrak{g}^{-\varphi_i}$ ($\varphi_i \in \Delta_+$), and let I' be the subset of I such that $\{b_i\}_{i \in I'}$ is a basis of $\mathfrak{u}^- \subset \mathfrak{n}^-$. Assume that the index set I is linearly ordered. Then $\{b_{i_1} \wedge \cdots \wedge b_{i_j}\}_{i_1 < \cdots < i_j}$ ($i_m \in I$) is a basis of $\Lambda^j(\mathfrak{n}^-)$, and $\{b_{i_1} \wedge \cdots \wedge b_{i_j}\}_{i_1 < \cdots < i_j}$ ($i_m \in I'$) is a basis of $\Lambda^j(\mathfrak{u}^-)$. If $j \in \mathbb{Z}_+$ varies in these expressions, we get bases of $\Lambda(\mathfrak{n}^-)$ and $\Lambda(\mathfrak{u}^-)$.

For each sequence $i_1 < \cdots < i_j$ ($i_m \in I$), $b_{i_1} \wedge \cdots \wedge b_{i_j}$ is a weight vector with weight $- \sum_{m=1}^{j} \varphi_{i_m}$. The fact that $\dim \mathfrak{g}^{-\varphi} = 1$ for each real root $\varphi \in \Delta_+$ (see § 2) yields the first assertion of the proposition. Let $w \in W$ with $l(w) = j$. Then by Proposition 2.2(1), Φ_w is of the form $\{\varphi_{i_1}, \ldots, \varphi_{i_j}\}$, with $i_1 < \cdots < i_j$, and $-\langle \Phi_w \rangle$ is the weight of $b_{i_1} \wedge \cdots \wedge b_{i_j}$. Proposition 2.4 and the one-dimensionality of the real root spaces imply that every other basis vector of $\Lambda(\mathfrak{n}^-)$ has a weight different from $-\langle \Phi_w \rangle$, so that the weight space $\Lambda(\mathfrak{n}^-)_{-\langle \Phi_w \rangle}$ is one-dimensional and lies in $\Lambda^j(\mathfrak{n}^-)$. If $w \in W_S^1$, then $b_{i_1}, \ldots, b_{i_j} \in \mathfrak{u}^-$, so that $\Lambda(\mathfrak{n}^-)_{-\langle \Phi_w \rangle} \subset \Lambda^j(\mathfrak{u}^-)$. If $w \notin W_S^1$, then $\varphi_{i_m} \notin \Delta_+(S)$ for some $m = 1, \ldots, j$, and so $b_{i_1} \wedge \cdots \wedge b_{i_j} \notin \Lambda(\mathfrak{u}^-)$. Hence $-\langle \Phi_w \rangle$ is not a weight of $\Lambda(\mathfrak{u}^-)$ in this case. Q.E.D.

Proposition 8.3. *Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$ (see Proposition 6.2). For all $j \in \mathbb{Z}_+$, define $\mathscr{W}^j \subset (\mathfrak{h}^e)^*$ to be the set of weights λ of $\Lambda^j(\mathfrak{u}^-) \otimes X$ such that $\sigma(\lambda+\rho, \lambda+\rho) = \sigma(\mu+\rho, \mu+\rho)$, and set $\mathscr{W} = \bigcup_{j \in \mathbb{Z}_+} \mathscr{W}^j$, so that $\mathscr{W}$ is the set of weights λ of $\Lambda(\mathfrak{u}^-) \otimes X$ such that $\sigma(\lambda+\rho, \lambda+\rho) = \sigma(\mu+\rho, \mu+\rho)$. Then there is a natural bijection between $\mathscr{W}$ and W_S^1, and for each $j \in \mathbb{Z}_+$, $\mathscr{W}^j$ corresponds bijectively to $\{w \in W_S^1 \mid l(w) = j\}$. In particular, the sets $\mathscr{W}^j$ are disjoint. The correspondence is given as follows: If $w \in W_S^1$ with $l(w) = j$, then $\lambda = -\langle \Phi_w \rangle + w\mu = w(\mu+\rho) - \rho$ is the associated element of $\mathscr{W}^j$. The corresponding weight space $(\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda$ is one-dimensional, and is the tensor product $\Lambda^j(\mathfrak{u}^-)_{-\langle \Phi_w \rangle} \otimes X_{w\mu}$ of one-dimensional weight spaces.*

Proof. Let $w \in W_S^1$ with $l(w) = j$. Setting $\lambda = -\langle \Phi_w \rangle + w\mu$, we have $\lambda = w(\mu+\rho) - \rho$ by Proposition 2.5. By Proposition 8.2, $\Lambda^j(\mathfrak{u}^-)_{-\langle \Phi_w \rangle}$ is one-dimensional, and by Proposition 6.1, so is $X_{w\mu}$. Then $\Lambda^j(\mathfrak{u}^-)_{-\langle \Phi_w \rangle} \otimes X_{w\mu} \subset (\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda$, and it is clear that $\sigma(\lambda+\rho, \lambda+\rho) = \sigma(\mu+\rho, \mu+\rho)$. Thus $\lambda \in \mathscr{W}^j$, and we have a map from W_S^1 to $\mathscr{W}$ such that the elements of length j map into $\mathscr{W}^j$.

Conversely, by Proposition 8.2, the weights of $\Lambda(\mathfrak{u}^-)$ lie in the set T of Proposition 2.13, and by Propositions 6.1 and 6.2, the set of weights of X is a set of the type T' in Proposition 2.13. Let $\lambda \in \mathscr{W}$. Clearly, the weight space $(\Lambda(\mathfrak{u}^-) \otimes X)_\lambda$ is a finite direct sum of tensor products $\Lambda(\mathfrak{u}^-)_\tau \otimes X_{\tau'}$, where $\tau \in T$ is a weight of $\Lambda(\mathfrak{u}^-)$, $\tau' \in T'$, and $\lambda = \tau + \tau'$. By Proposition 2.13, there exists a unique element $w \in W$ such that $\lambda = -\langle \Phi_w \rangle + w\mu = w(\mu+\rho) - \rho$, and we must have $\tau = -\langle \Phi_w \rangle$ and $\tau' = w\mu$. Since τ is a weight of $\Lambda\mathfrak{u}^-$, $w \in W_S^1$ by Proposition 8.2. This establishes the bijection $W_S^1 \leftrightarrow \mathscr{W}$.

Repeating the argument in the last paragraph with $\Lambda^j(\mathfrak{u}^-)$ in place of $\Lambda(\mathfrak{u}^-)$, and $\lambda \in \mathscr{W}^j$ in place of $\lambda \in \mathscr{W}$, and using the same notation otherwise, we see that since $\tau = -\langle \Phi_w \rangle$ is a weight of $\Lambda^j(\mathfrak{u}^-)$, we must have $l(w) = j$ by Proposition 8.2. This shows that the above bijection $W_S^1 \leftrightarrow \mathscr{W}$ restricts to bijections

$$\{w \in W_S^1 \mid l(w) = j\} \leftrightarrow \mathscr{W}^j$$

for each $j \in \mathbb{Z}_+$. In particular, the sets $\mathscr{W}^j$ are disjoint.

Finally, in the context of the last paragraph, we have

$$(\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda = \Lambda^j(\mathfrak{u}^-)_{-\langle \Phi_w \rangle} \otimes X_{w\mu},$$

and the two factors on the right are one-dimensional by Propositions 8.2 and 6.1, respectively. Q.E.D.

Proposition 8.4. *In the notation of the last sentence of Proposition* 8.3, $(\Lambda^j(\mathfrak{u}^-)\otimes X)_\lambda$ *is annihilated by* e_i *for all* $i\in S$ *(regarding* $\Lambda^j(\mathfrak{u}^-)\otimes X$ *as the tensor product of modules for the Lie algebra* $\mathfrak{g}_S$*; see* § 3).

Proof. It suffices to show that for all $i\in S$, $\lambda+\alpha_i$ is not a weight of $\Lambda^j(\mathfrak{u}^-)\otimes X$. Suppose that it is a weight. Then $\lambda+\alpha_i=\tau+\tau'$, where τ is a weight of $\Lambda^j(\mathfrak{u}^-)$ and τ' is a weight of X. Applying the inequality in Proposition 2.13 (with T' the set of weights of X) to $\tau+\tau'$, we get

$$\sigma(\mu+\rho,\mu+\rho)-\sigma(\lambda+\alpha_i+\rho,\lambda+\alpha_i+\rho)\geqq 0.$$

But by the definition of λ in Proposition 8.3, this becomes

$$-2\sigma(w(\mu+\rho),\alpha_i)-\sigma(\alpha_i,\alpha_i)\geqq 0,$$

i.e.,

$$2\sigma(\mu+\rho,w^{-1}\alpha_i)+\sigma(\alpha_i,\alpha_i)\leqq 0.$$

Since $w\in W_S^1$, Proposition 8.1 implies that $w^{-1}\alpha_i\in\Delta_+$. By Proposition 2.11 and the fact that $\sigma(\alpha_i,\alpha_i)>0$, the left-hand side of this inequality is positive, a contradiction. Q.E.D.

Since $\Lambda^j(\mathfrak{u}^-)\otimes X$ is a direct sum of finite-dimensional $\mathfrak{g}_S$-modules by Proposition 6.3, it is clear from Proposition 8.4 that the $\mathfrak{g}_S$-submodule Y of $\Lambda^j(\mathfrak{u}^-)\otimes X$ generated by the one-dimensional space $(\Lambda^j(\mathfrak{u}^-)\otimes X)_\lambda$ is finite-dimensional and irreducible. Since $\mathfrak{h}^e$ acts according to the weight λ on the highest weight space of Y (with respect to $\mathfrak{g}_S$), Y must be isomorphic to the $\mathfrak{r}^e$-module $M(\lambda)$, and $\lambda\in P_S$ (see Proposition 3.1). Hence by Proposition 8.3, we have:

Theorem 8.5. *Let* X *be a quasisimple* $\mathfrak{g}^e$*-module with highest weight* $\mu\in P$ *(see Proposition* 6.2*), assume that* $S\subset\{1,\dots,l\}$ *is a subset of finite type (see* § 3*), and let* $j\in\mathbb{Z}_+$. *The correspondence* $w\mapsto w(\mu+\rho)-\rho$ *is a bijection from the (finite) set* $\{w\in W_S^1\,|\,l(w)=j\}$ *onto the set of all weights* λ *of* $\Lambda^j(\mathfrak{u}^-)\otimes X$ *such that* $\sigma(\lambda+\rho,\lambda+\rho)=\sigma(\mu+\rho,\mu+\rho)$ *(see* § 2 *for the notation). Each such weight* $w(\mu+\rho)-\rho$ *occurs with multiplicity one in* $\Lambda^j(\mathfrak{u}^-)\otimes X$, *with weight space* $\Lambda^j(\mathfrak{u}^-)_{-\langle\Phi_w\rangle}\otimes X_{w\mu}$; $w(\mu+\rho)-\rho\in P_S$; *and the weight space of* $w(\mu+\rho)-\rho$ *generates a copy of the irreducible* $\mathfrak{r}^e$*-module* $M(w(\mu+\rho)-\rho)$, *which occurs with multiplicity one in the* $\mathfrak{r}^e$*-module* $\Lambda^j(\mathfrak{u}^-)\otimes X$ *(see* §§ 2, 3 *for the notation). In particular, in any direct sum decomposition of the* $\mathfrak{r}^e$*-module* $\Lambda^j(\mathfrak{u}^-)\otimes X$ *as* $\coprod M(\lambda_i)$ *where* $\lambda_i\in P_S$ *(see Lemma* 7.3*), those* λ_i *for which* $\sigma(\lambda_i+\rho,\lambda_i+\rho)=\sigma(\mu+\rho,\mu+\rho)$ *must coincide (up to rearrangement) with the* $w(\mu+\rho)-\rho$ *as* w *ranges through* $\{w\in W_S^1\,|\,l(w)=j\}$, *each such* $M(\lambda_i)$ *appearing exactly once in the decomposition* $\coprod M(\lambda_i)$. *Finally, the correspondence* $w\mapsto w(\mu+\rho)-\rho$ *is a bijection from* W_S^1 *onto the set of all weights* λ *of* $\Lambda(\mathfrak{u}^-)\otimes X$ *such that* $\sigma(\lambda+\rho,\lambda+\rho)=\sigma(\mu+\rho,\mu+\rho)$, *so that in particular, the irreducible* $\mathfrak{r}^e$*-modules* $M(w(\mu+\rho)-\rho)$ *are all inequivalent as* w *ranges through* W_S^1.

Proposition 7.9 now immediately gives (cf. also Lemmas 7.5 and 7.6):

Theorem 8.6. *Let X, μ, S and j be as in Theorem 8.5. The j-th homology space $H_j(\mathfrak{u}^-, X^t)$ is finite-dimensional, and when provided with the standard $\mathfrak{r}^e$-module action (see § 1), it is naturally $\mathfrak{r}^e$-module isomorphic to the direct sum*

$$\coprod M(w(\mu+\rho)-\rho)$$

of inequivalent irreducible $\mathfrak{r}^e$-modules, as w ranges through the finite set of elements W_S^1 of length j (see Proposition 3.1). The j-th cohomology space $H^j(\mathfrak{u}^-, X^*)$, with the standard $\mathfrak{r}^e$-module action, is finite-dimensional and is naturally isomorphic to the contragredient $\mathfrak{r}^e$-module $H_j(\mathfrak{u}^-, X^t)^*$ (see Proposition 1.6). Let $C_j(X)$ be the $\mathfrak{r}^e$-module $\Lambda^j(\mathfrak{u}^-)\otimes X$, so that the standard $\mathfrak{r}^e$-module complex for computing the standard action of $\mathfrak{r}^e$ on $H_*(\mathfrak{u}^-, X^t)$ has the form

$$\cdots \to C_1(X) \to C_0(X) \to 0.$$

Then $H_j(\mathfrak{u}^-, X^t)$ may be naturally identified with the $\mathfrak{r}^e$-submodule $B_j(X)$ of $C_j(X)$ which is the sum of all the $\mathfrak{r}^e$-submodules of $C_j(X)$ isomorphic to $M(\lambda)$, where $\lambda \in P_S$ and $\sigma(\lambda+\rho, \lambda+\rho) = \sigma(\mu+\rho, \mu+\rho)$. The $B_j(X)$ form an $\mathfrak{r}^e$-module subcomplex $B_*(X)$ of $C_*(X)$ all of whose maps are zero, and $C_*(X)$ is in a natural way the direct sum of $B_*(X)$ and another subcomplex whose homology is zero.

From Lemma 7.4 and the Remark after Lemma 7.8, we also have:

Theorem 8.7. *Let X, μ, S and j be as in Theorem 8.5. Let Ψ_μ^j be the (finite) family of elements of $(\mathfrak{h}^e)^*$, $\{w(\mu+\rho)-\rho\}$, as w ranges through the set of elements of W_S^1 of length j; the $w(\mu+\rho)-\rho$ are distinct and each lies in P_S. Then there is an exact sequence of $\mathfrak{g}^e$-modules*

$$\cdots \to E_1 \to E_0 \to X \to 0$$

where E_j is of type *Ψ_μ^j for each j (see the Definition preceding Lemma 7.3). As an $\mathfrak{r}^e$-module, each E_j is a direct sum of irreducible $\mathfrak{r}^e$-modules of the form $M(\lambda)$ with $\lambda \in P_S$.*

§ 9. The Macdonald-Kac Identities

In this section, we shall apply Theorem 8.6 (or Theorem 8.7) and an Euler-Poincaré principle, in order to derive the formal identities of Macdonald and Kac (see [11(b), 14]).

Let $(\mathfrak{h}^e)^*_{\mathbf{Z}}$ denote the set of integral linear forms (see § 2), i.e., the set of all $\lambda \in (\mathfrak{h}^e)^*$ such that $\lambda(h_i) \in \mathbf{Z}$ for all $i=1, \ldots, l$, and let $\mathscr{A}$ be the abelian group of all (possibly infinite) formal integral combinations of elements of $(\mathfrak{h}^e)^*_{\mathbf{Z}}$. Let $e(\lambda) \in \mathscr{A}$ denote the element corresponding to $\lambda \in (\mathfrak{h}^e)^*_{\mathbf{Z}}$. Denote by $\mathscr{A}^f$ the integral group algebra of $(\mathfrak{h}^e)^*_{\mathbf{Z}}$, so that $\mathscr{A}^f \subset \mathscr{A}$, and $\mathscr{A}^f$ consists of the formal combinations of *finitely many* of the $e(\lambda)$. In $\mathscr{A}^f$, we have of course $e(\lambda)e(\mu) = e(\lambda+\mu)$ for all λ, $\mu \in (\mathfrak{h}^e)^*_{\mathbf{Z}}$. We shall allow ourselves to multiply elements of $\mathscr{A}$ when the product is well defined.

Let $\mathscr{E}$ denote the category of all $\mathfrak{h}^e$-modules X such that X has a direct sum decomposition $\coprod_{\lambda \in (\mathfrak{h}^e)^*_{\mathbf{Z}}} X_\lambda$ and each X_λ is finite-dimensional (see § 3 for the definition of the weight space X_λ). The morphisms of this category are the $\mathfrak{h}^e$-module maps.

If X is an object of $\mathscr{E}$, then we can define its *formal character* $\mathscr{X}(X)$ as an element of $\mathscr{A}$, namely,

$$\mathscr{X}(X)=\sum_{\lambda\in(\mathfrak{h}^e)^*_{\mathbf{Z}}}(\dim X_\lambda)e(\lambda).$$

The following is clear:

Lemma 9.1. *Let*

$$0\to X_1\to X_2\to X_3\to 0$$

be an exact sequence in $\mathscr{E}$. *Then*

$$\mathscr{X}(X_2)=\mathscr{X}(X_1)+\mathscr{X}(X_3).$$

Let

$$\cdots\to C_{j+1}\xrightarrow{d_{j+1}}C_j\xrightarrow{d_j}\cdots\xrightarrow{d_1}C_0\to 0$$

be a chain complex in $\mathscr{E}$ (so that $d_j\circ d_{j+1}=0$ for all $j=1,2,\ldots$), and let H_j be the j-th homology group of this complex. Then $H_j\in\mathscr{E}$. We say that the chain complex C_* is *admissible* if $\coprod_j C_j$ is in $\mathscr{E}$. We have:

Lemma 9.2 (Euler-Poincaré). *Let* C_* *be an admissible chain complex in* $\mathscr{E}$. *Then* $\coprod_j H_j$ *is in* $\mathscr{E}$ *and*

$$\sum_{j\in\mathbf{Z}_+}(-1)^j\mathscr{X}(C_j)=\sum_{j\in\mathbf{Z}_+}(-1)^j\mathscr{X}(H_j).$$

Now we apply the Euler-Poincaré principle, in this form, to the chain complex displayed in Theorem 8.6, so that $C_j=C_j(X)=\Lambda^j(\mathfrak{u}^-)\otimes X$ and $H_j=H_j(\mathfrak{u}^-,X^{\mathfrak{r}})$, in the notation of Theorem 8.6. We use the special case $S=\emptyset$, so that $\mathfrak{r}^e=\mathfrak{h}^e$ and $\mathfrak{u}^-=\mathfrak{n}^-$ (see § 3). For the trivial one-dimensional $\mathfrak{g}^e$-module $X=k$, Theorem 8.6 yields:

Theorem 9.3 (Kac [11 (b)]). *We have*

$$\prod_{\varphi\in\Delta_+}(1-e(-\varphi))^{\dim\mathfrak{g}^\varphi}=\sum_{w\in W}(\det w)e(w\rho-\rho)$$
$$=\sum_{w\in W}(\det w)e(-\langle\Phi_w\rangle).$$

For X an arbitrary quasisimple $\mathfrak{g}^e$-module with highest weight $\mu\in P$ (see § 6), we get from Theorems 8.6 and 9.3 and the Euler-Poincaré principle:

Theorem 9.4 (Kac [11 (b)]). *We have*

$$\mathscr{X}(X)=\sum_{w\in W}(\det w)e(w(\mu+\rho))/\sum_{w\in W}(\det w)e(w\rho).$$

Remarks. (1) Macdonald's identities [14] comprise the special case of Theorem 9.3 in which $\mathfrak{g}$ is a Euclidean Lie algebra (see [15 (c), § 2]).

(2) When $\mathfrak{g}^e = \mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra (see § 2), Theorem 9.4 is just Weyl's character formula and Theorem 9.3 is Weyl's denominator formula (for the denominator in the character formula) (cf. [6, § 7.5] or [9, § 24.3]).

(3) If we apply the Euler-Poincaré principle to Theorem 8.6 for an arbitrary S of finite type and hence an arbitrary F-parabolic subalgebra (see § 3), we of course obtain analogues of Theorems 9.3 and 9.4 involving the formal characters of $\Lambda^j(\mathfrak{u}^-)$ and of the $\mathfrak{r}^e$-modules $M(w(\mu+\rho)-\rho)$ $(w \in W_S^1)$. (See [12, Proposition 7.3] for the corresponding result when $\mathfrak{g}^e = \mathfrak{g}$ is finite-dimensional complex semisimple.) These identities for any one S of finite type easily imply the corresponding identities for any other such S, without the use of homology theory.

(4) Macdonald's identities in their original form [14] are most transparent when S is taken to be an appropriate subset of $l-1$ elements in $\{1, \ldots, l\}$, so that the corresponding F-parabolic subalgebra is a "maximal parabolic subalgebra" of the Euclidean Lie algebra. (But cf. Remark (3).) For example, if the Euclidean Lie algebra is the Laurent series Lie algebra mentioned in the Introduction and treated in [7], the Euler-Poincaré identity for $X = k$ and an appropriate maximal parabolic subalgebra is formula [3.3] in [7], and it is this formula which after suitable specialization gives Macdonald's identities for powers of Dedekind's η-function. By contrast, formula [3.5] in [7] is the Euler-Poincaré identity corresponding to $S = \emptyset$, and when the same specialization is attempted in this case, we only get $0 = 0$.

(5) To write down one concrete example, we remark that Macdonald's specialized identity for the Laurent series Lie algebra corresponding to $\mathfrak{sl}(2)$ is Jacobi's identity

$$\prod_{n \geq 1} (1-x^n)^3 = \sum_{n \geq 0} (-1)^n (2n+1) x^{n(n+1)/2}$$

(x an indeterminate).

There is another important formulation of the Macdonald-Kac identities. To describe it, we define the natural generalization of Kostant's partition function, following [11(b)] and [15(c)]:

Let Λ be a set with a surjection $F\colon \Lambda \to \Delta_+$ onto the set of positive roots, such that for all $\varphi \in \Delta_+$, $F^{-1}(\varphi)$ has exactly $\dim \mathfrak{g}^\varphi$ elements.

Definition. Let the *partition function* $\mathscr{P}\colon (\mathfrak{h}^e)^* \to \mathbb{Z}_+$ be defined by the condition that for all $\varphi \in (\mathfrak{h}^e)^*$, $\mathscr{P}(\varphi)$ is the number of functions $f\colon \Lambda \to \mathbb{Z}_+$ such that

$$\varphi = \sum_{\gamma \in \Lambda} f(\gamma) F(\gamma).$$

Clearly $\mathscr{P}$ is independent of the choice of Λ and F, and reduces to Kostant's partition function (cf. [9, § 24.1]) when $\mathfrak{g}^e = \mathfrak{g}$ is finite-dimensional split semisimple. Note that $\mathscr{P}(\varphi) = 0$ unless φ is a nonnegative integral linear combination of positive roots.

Lemma 9.5. *We have*

$$\Big(\prod_{\varphi \in \Delta_+} (1-e(-\varphi))^{\dim \mathfrak{g}^\varphi}\Big)\Big(\sum_{\varphi \in (\mathfrak{h}^e)^*_{\mathbb{Z}}} \mathscr{P}(\varphi) e(-\varphi)\Big) = e(0).$$

Proof. The second factor on the left-hand side is just the product over $\varphi \in \Delta_+$ of the expressions $(\sum_{n \in \mathbb{Z}_+} e(-n\varphi))^{\dim \mathfrak{g}^\varphi}$. The rest is clear. Q.E.D.

In view of this lemma, Theorem 9.3 and hence Macdonald's identities may be reformulated as follows:

Theorem 9.6 (Kac [11(b)]; cf. *also* [15(c)]). *For all* $\lambda \in (\mathfrak{h}^e)^*$,

$$\sum_{w \in W} (\det w)\, \mathscr{P}(w\rho - (\lambda + \rho)) = 0$$

unless $\lambda = 0$, *in which case the sum is* 1.

Similarly, using Theorem 9.3 and Lemma 9.5, we see that Theorem 9.4 is equivalent to:

Theorem 9.7 (Kac [11(b), Theorem 1]). *Let* X *be a quasisimple* $\mathfrak{g}^e$*-module with highest weight* $\mu \in P$, *and let* $\lambda \in (\mathfrak{h}^e)^*$. *Then*

$$\dim X_\lambda = \sum_{w \in W} (\det w)\, \mathscr{P}(w(\mu + \rho) - (\lambda + \rho)).$$

Note that Theorem 9.6 is the special case $X = k$ of Theorem 9.7.

Since the right-hand side in Theorem 9.7 depends on X only through its highest weight μ, we have established what was promised in § 6 (see Proposition 6.2 and the subsequent Remark):

Corollary 9.8 (Kac [11(b), Corollary]). *Every quasisimple* $\mathfrak{g}^e$*-module is simple, and* P *bijectively indexes the set of equivalence classes of quasisimple* $\mathfrak{g}^e$*-modules.*

Remark. When $\mathfrak{g}^e = \mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra, Theorem 9.7 is Kostant's formula for the multiplicity of a weight in a finite-dimensional irreducible $\mathfrak{g}$-module (cf. [6, § 7.5] or [9, § 24.2]).

Finally, we indicate how our resolution, Theorem 8.7, can easily be used instead of the homology result, Theorem 8.6, to prove the Macdonald-Kac formulas. Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$.

Take $S = \emptyset$. Let $\lambda \in (\mathfrak{h}^e)^*_{\mathbb{Z}}$, and consider the $\mathfrak{g}^e$-module $V^{M(\lambda)}$ induced from the appropriate one-dimensional $\mathfrak{h}^e \oplus \mathfrak{n}$-module (see § 3). Let $\mathscr{N}^-$ be the universal enveloping algebra of $\mathfrak{n}^-$, and let k_λ be the field k regarded as an $\mathfrak{h}^e$-module with weight λ. Then $V^{M(\lambda)} \simeq \mathscr{N}^- \otimes k_\lambda$ as an $\mathfrak{h}^e$-module, and the following is straightforward:

Lemma 9.9. *We have*

$$\mathscr{X}(V^{M(\lambda)}) = \sum_{\varphi \in (\mathfrak{h}^e)^*_{\mathbb{Z}}} \mathscr{P}(\varphi)\, e(\lambda - \varphi).$$

Hence Lemma 9.5 implies:

Corollary 9.10. *We have*

$$\Big(\prod_{\varphi \in \Delta_+} (1 - e(-\varphi))^{\dim \mathfrak{g}^\varphi}\Big)\, \mathscr{X}(V^{M(\lambda)}) = e(\lambda).$$

Write d for the first factor on the left-hand side. Now in the notation of Theorem 8.7, each E_j ($j \in \mathbb{Z}_+$) lies in the category $\mathscr{E}$, by the last assertion of that theorem and the definition of Ψ_μ^j. Moreover,

$$\mathscr{X}(E_j) = \sum_{\substack{w \in W \\ l(w)=j}} \mathscr{X}(V^{M(w(\mu+\rho)-\rho)}).$$

But Lemma 9.2 (Euler-Poincaré) applied to the exact sequence in Theorem 8.7 asserts that

$$\mathscr{X}(X) = \sum_{j \in \mathbb{Z}_+} (-1)^j \mathscr{X}(E_j).$$

Combining these last two formulas and multiplying through by d, we get, by Corollary 9.10,

$$d\mathscr{X}(X) = \sum_{w \in W} (\det w) e(w(\mu+\rho)-\rho).$$

Taking $X = k$ recovers Theorem 9.3, and then Theorem 9.4 is immediate.

Appendix

Here we give a proof of Theorem 8.6 (or more precisely, of Proposition 7.9) which is somewhat shorter than the one in the body of the paper. However, we do not get this way the resolution in terms of generalized Verma modules given in Theorem 8.7.

Let X be a quasisimple $\mathfrak{g}^e$-module with highest weight $\mu \in P$. Recall the $\mathfrak{g}^e$-module resolution

$$\cdots \xrightarrow{d_2^X} D_1^X \xrightarrow{d_1^X} D_0^X \xrightarrow{\varepsilon_0^X} X \to 0$$

in Proposition 7.1. For each $j \in \mathbb{Z}_+$,

$$D_j^X = \mathscr{G}^e \otimes_{\mathscr{P}^e} (\Lambda^j(\mathfrak{g}^e/\mathfrak{p}^e) \otimes X)$$

as a $\mathfrak{g}^e$-module. Moreover,

$$D_j^X \simeq \mathscr{U}^- \otimes_k \Lambda^j(\mathfrak{u}^-) \otimes_k X$$

as a $\mathscr{U}^-$-module and an $\mathfrak{r}^e$-module, with $\mathscr{U}^-$ acting by left multiplication on the first factor and $\mathfrak{r}^e$ acting naturally on the tensor product of the three $\mathfrak{r}^e$-modules. In particular, D_j^X is $\mathscr{U}^-$-free. The $\mathfrak{r}^e$-module complex $C_*(X)$ given by

$$\cdots \xrightarrow{1 \otimes d_2^X} k \otimes_{\mathscr{U}^-} D_1^X \xrightarrow{1 \otimes d_1^X} k \otimes_{\mathscr{U}^-} D_0^X \to 0$$

is naturally isomorphic to the standard $\mathfrak{r}^e$-module complex for computing $H_*(\mathfrak{u}^-, X^t)$ as an $\mathfrak{r}^e$-module, and for each $j \in \mathbb{Z}_+$, $C_j(X) \simeq \Lambda^j(\mathfrak{u}^-) \otimes X$ (see Proposition 7.2).

Since X and all the D_j^X lie in the category $\mathscr{C}$ of § 4, we can form the Casimir operators Γ_X and $\Gamma(j)=\Gamma_{D_j^X}$ as in § 4. For any vector space Y, let I_Y denote the identity transformation of Y. Then $\Gamma_X=\sigma(\mu+\rho,\mu+\rho)I_X$, by Corollary 4.3. Proposition 4.1 implies that

$$d_{j+1}^X\circ\Gamma(j+1)=\Gamma(j)\circ d_{j+1}^X \quad \text{for all } j\in\mathbb{Z}_+$$

and that

$$\varepsilon_0^X\circ\Gamma(0)=\sigma(\mu+\rho,\mu+\rho)\varepsilon_0^X.$$

But by Proposition 4.2, the $\Gamma(j)$ $(j\in\mathbb{Z}_+)$ are $\mathfrak{g}^e$-module maps, and hence $\mathscr{U}^-$-module maps. The fact that the D_j^X are $\mathscr{U}^-$-free thus allows us to construct $\mathscr{U}^-$-module homomorphisms $h_j\colon D_j^X\to D_{j+1}^X$ $(j\in\mathbb{Z}_+)$ such that

$$h_{j-1}\circ d_j^X+d_{j+1}^X\circ h_j=\Gamma(j)-\sigma(\mu+\rho,\mu+\rho)I_{D_j^X}$$

for all $j\in\mathbb{Z}_+$. (For $j=0$, the first term on the left is of course omitted.) It follows that the operator

$$1\otimes\Gamma(j)-\sigma(\mu+\rho,\mu+\rho)I_{C_j(X)}\in\operatorname{End}C_j(X)$$

induces the zero map on the j-th homology $H_j(\mathfrak{u}^-,X^t)$ of $C_*(X)$, for each $j\in\mathbb{Z}_+$.

Next we obtain a useful decomposition $\Gamma_Y=\Gamma_{Y,1}+\Gamma_{Y,2}$ of Γ_Y, for $Y\in\mathscr{C}$: Recall that $\Delta_+(S)=\{\varphi\in\Delta_+\mid\mathfrak{g}^\varphi\subset\mathfrak{u}\}$ (§ 8), and set

$$\Gamma_{Y,1}=2\sum_{\varphi\in\Delta_+(S)}\omega_\varphi\in\operatorname{End}Y,$$

in the notation of § 4. The proof of Proposition 4.2 shows easily that $\Gamma_{Y,1}$ commutes with the action of $\mathfrak{r}^e$ on Y. Thus $\Gamma_{Y,2}\in\operatorname{End}Y$, defined to be $\Gamma_Y-\Gamma_{Y,1}$, commutes with the action of $\mathfrak{r}^e$, by Proposition 4.2.

Set $\Gamma_2(j)=\Gamma_{D_j^X,2}$ $(j\in\mathbb{Z}_+)$. Then as $\mathfrak{r}^e$-module endomorphisms of

$$C_j(X)=k\otimes_{\mathscr{U}^-}D_j^X,\qquad 1\otimes\Gamma(j)=1\otimes\Gamma_2(j),$$

so that

$$1\otimes\Gamma_2(j)-\sigma(\mu+\rho,\mu+\rho)I_{C_j(X)}$$

induces the zero map on the j-th homology of $C_*(X)$.

For an $\mathfrak{r}^e$-module Y and $\lambda\in P_S$, let $Y_{(\lambda)}$ denote the sum of all the $\mathfrak{r}^e$-submodules of Y isomorphic to $M(\lambda)$ (see Proposition 3.1). As in Proposition 7.9, set

$$B_j(X)=\prod_{\substack{\lambda\in P_S\\ \sigma(\lambda+\rho,\lambda+\rho)=\sigma(\mu+\rho,\mu+\rho)}}C_j(X)_{(\lambda)},$$

and also let

$$B_j'(X)=\coprod_{\substack{\lambda\in P_S\\ \sigma(\lambda+\rho,\lambda+\rho)\neq\sigma(\mu+\rho,\mu+\rho)}}C_j(X)_{(\lambda)},$$

for all $j\in\mathbb{Z}_+$. Then $C_j(X)=B_j(X)\oplus B_j'(X)$, by Proposition 6.3, and $C_*(X)$ is the direct sum of the $\mathfrak{r}^e$-module subcomplexes $B_*(X)$ and $B_*'(X)$, defined in the obvious way. When restricted to $C_j(X)_{(\lambda)}$ $(\lambda\in P_S)$, the operator $1\otimes\Gamma_2(j)$ acts as the scalar $\sigma(\lambda+\rho, \lambda+\rho)$, as we can see by applying the operator to a highest weight vector. Hence $1 \otimes \Gamma_2(j) - \sigma(\mu+\rho, \mu+\rho) I_{C_j(X)}$ is a nonsingular operator on $B_j'(X)$, and the conclusion is:

Lemma A.1. *The homology of $B_*'(X)$ is zero. In particular, the homology $H_*(\mathfrak{u}^-, X^{\mathfrak{r}})$ of $C_*(X)$ is naturally $\mathfrak{r}^e$-module isomorphic to the homology of its subcomplex $B_*(X)$.*

Thus we have recovered Proposition 7.9, and the arguments of § 8 prove Theorem 8.6.

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Notes Added in Proof. The present paper is summarized in "The Macdonald-Kac formulas as a consequence of the Euler-Poincaré principle," to appear in a collection of papers in honor of E. Kolchin's 60th birthday.

V.G. Kac has asked us to mention that the following three additions, supplied by Kac, should be made in his paper [11(b)]: In the second sentence after formula (1), after "G_α and $G_{-\alpha}$ are dual," insert: "modulo the radical R of $\langle .,. \rangle$". In the first sentence after formula (3), after "bases," insert "modulo R". In the third sentence after formula (3), after "that," insert "if M is simple". See also the first Remark in $s2$ of the present paper.