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On the Difference between Consecutive Primes

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1. Introduction

Montgomery [5, 6, 7] has used ingenious techniques to estimate the number of zeros of Dirichlet series in certain rectangles, and he has shown the difference between consecutive primes to satisfy

$$p_{n+1} - p_n < p_n^{\delta} \tag{1.1}$$

for all sufficiently large n, whenever

$$\delta > 3/5. \tag{1.2}$$

A slight modification of Montgomery's argument allows us to conclude that (1.1) is true whenever

$$\delta > 7/12. \tag{1.3}$$

Let $N(\alpha, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle

$$\alpha \leqq \beta \leqq 1, \qquad -T \leqq \gamma \leqq T. \tag{1.4}$$

After Ingham [2] it is well known that any result

$$N(\alpha, T) \ll T^{\lambda(1-\alpha)} l^B \tag{1.5}$$

where l denotes $\log T$ and B is fixed, uniform in $\frac{1}{2} \leqq \alpha \leqq 1$, implies (1.1) for

$$\delta > 1 - \lambda^{-1}. \tag{1.6}$$

The use of (2.9) below in place of (2.7) in the proof of Theorem 1, Eq. (5) of [6] gives the following result:

$$N(\alpha, T) \ll T^{(5\alpha-3)(1-\alpha)/(\alpha^2+\alpha-1)} l^9 \tag{1.7}$$

uniformly in $\frac{3}{4} \leqq \alpha \leqq 1$. The range $\frac{1}{2} \leqq \alpha \leqq \frac{3}{4}$ is supplied by Ingham's theorem [3]

$$N(\alpha, T) \ll T^{3(1-\alpha)/(2-\alpha)} l^5, \tag{1.8}$$

and we have (1.5) with $\lambda = 12/5$.

On being shewn an earlier version of this paper, Montgomery proposed that

$$N(\alpha, T) \ll T^{3(1-\alpha)/(3\alpha-1)} l^{44} \tag{1.9}$$

uniformly in $3/4 \leqq \alpha \leqq 1$, which implies

$$N(\alpha, T) \ll T^{2(1-\alpha)} l^{44} \tag{1.10}$$

for $5/6 \leqq \alpha \leqq 1$. The inequality (1.10) is a form of the "density hypothesis", previously obtained for $\alpha \geqq 9/10$ by Montgomery [6] and for $\alpha \geqq 7/8$ by Jutila [4]. Our results take effect for $\alpha \geqq 3/4$, because this is the range in which the "large values" method of Halász, explained in [5, 7], gives a better bound for the class (i) zeros than do mean square estimates.

The proof of (1.7) adheres closely to that of Theorem 1 in [6], and is omitted. We prove (1.9) below; it is obtained by treating class (i) zeros by Jutila's method. Our results can be improved for $3/4 < \alpha < 1$ by using known bounds for $\zeta(\sigma + it)$, where the choice of σ depends on α, in the choice of suitable $b(1), \ldots, b(N)$ in (2.5) below, and appealing to Theorem 8.4 of [7]. In particular, using van der Corput's bound for $\zeta(1/6 + it)$ we can obtain the density hypothesis for $\alpha > 81/98$. Professor Bombieri has informed me in a letter that (1.10) can be obtained in a wider range without using deep bounds for $\zeta(s)$.

2. The Halász Inequality

Let $s = \sigma + it$ be a complex variable and

$$F(s) = \sum_{m=1}^{N} a(m)\, m^{-s}, \tag{2.1}$$

$$G = \sum_{m=1}^{N} |a(m)|^2. \tag{2.2}$$

Suppose that for $s = s_1, \ldots, s_R$ we have

$$|F(s)| \geqq V, \tag{2.3}$$

where $0 \leqq \sigma_r \leqq 1/4$ for $r = 1, \ldots, R$ and

$$1 \leqq |t_r - t_q| \leqq T \tag{2.4}$$

for $1 \leqq q < r \leqq R$. Values of s for which (2.3) holds are regarded as exceptional, and we obtain an upper bound for their number. The basic Halász inequality states

$$R^2 V^2 \leqq G \sum_{r=1}^{R} \sum_{q=1}^{R} \left| \sum_{m=1}^{N} b(m)\, m^{-\sigma_r - \sigma_q + it_r - it_q} \right|, \tag{2.5}$$

for any sequence $b(1), \ldots, b(N)$ of real numbers greater than one. Montgomery [5, 7] has shewn how to choose $b(1), \ldots, b(N)$ so that the sum over q in (2.5) is

$$\ll N + R T^{\frac{1}{2}} \log NT; \tag{2.6}$$

it follows that

$$R \ll GNV^{-2} \tag{2.7}$$

provided that

$$V^2 \geqq c_1 GT^{\frac{1}{2}} \log NT, \tag{2.8}$$

where c_1 is an absolute constant. If (2.8) is not satisfied we divide the interval $[t_1, t_1+T]$ into subintervals of length at most T_0, where T_0 is the value of T for which equality holds in (2.8). Hence

$$R \ll (T/T_0+1)\, GNV^{-2} \ll GNV^{-2} + G^3 NTV^{-6} \log^2 NT. \tag{2.9}$$

3. The Classification of Zeros

We follow Montgomery's method [6] of counting the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ in a rectangle

$$\alpha \leqq \beta \leqq 1, \qquad -T \leqq \gamma \leqq T, \tag{3.1}$$

where $\alpha > \frac{1}{2}$ and T will be assumed to be "sufficiently large". Here T is not necessarily the same T as in (2.4). Let X be a large integer to be chosen below, and let

$$M(s) = \sum_{m \leqq X} \mu(m)\, m^{-s} \tag{3.2}$$

be a partial sum for the Dirichlet series representing $(\zeta(s))^{-1}$. For $\sigma > 1$ we have

$$\zeta(s) M(s) = 1 + \sum_{m > X} b(m)\, m^{-s}, \tag{3.3}$$

where

$$b(m) = \sum_{\substack{d \mid m \\ d \leqq X}} \mu(d). \tag{3.4}$$

The integral transform

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(\rho+\omega) M(\rho+\omega)\, Y^{\omega}\, \Gamma(\omega)\, d\omega = e^{-1/Y} + \sum_{m > X} b(m)\, m^{-\rho} e^{-m/Y} \tag{3.5}$$

can be verified term by term. The zero of $\zeta(\rho+\omega)$ cancels the pole of $\Gamma(\omega)$ at $\omega=0$. When we move the line of integration to $\operatorname{Re}\omega = \frac{1}{2}-\beta$, the only pole of the integrand is at $\omega = 1-\rho$, with residue

$$M(1)\, Y^{1-\rho}\, \Gamma(1-\rho). \tag{3.6}$$

We shall suppose

$$\log X \leqq \log Y \leqq 2l, \tag{3.7}$$

where $l = \log T$ and T is sufficiently large. Then

$$|M(1)\, Y^{1-\rho}\, \Gamma(1-\rho)| < 1/10 \tag{3.8}$$

for

$$|\gamma| \geqq 100\, l, \tag{3.9}$$

and

$$\Big|\sum_{m>100\,l\,Y} b(m)\, m^{-\rho} e^{-m/Y}\Big| < 1/10. \tag{3.10}$$

The inequality (3.8) follows since for fixed σ

$$|\Gamma(\sigma+it)| \ll |t|^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \tag{3.11}$$

as $|t| \to \infty$. Since $\zeta(s)$ has $\ll l$ zeros in any unit square, there are $\ll l^2$ zeros for which (3.9) fails to be true.

We now subdivide classes (i) and (ii). We split the range $X < m \leqq 100\, l\, Y$ into $\leqq 2l$ intervals $I(n)$, the division between $I(n)$ and $I(n+1)$ being at $2^n Y$. We now see that all zeros of $\zeta(s)$ satisfying both (3.1) and (3.9) fall into at least one of the following classes.

Class (i, n): zeros ρ at which

$$\Big|\sum_{m\in I(n)} b(m)\, m^{-\rho} e^{-m/Y}\Big| > (6l)^{-1}, \tag{3.12}$$

Class (ii, n): zeros ρ at which

$$\max_{|\gamma-t|\leqq 2^n} |\zeta(\tfrac{1}{2}+it)\, M(\tfrac{1}{2}+it)| > c_2\, 2^n\, Y^{\alpha-\frac{1}{2}}, \tag{3.13}$$

where c_2 is chosen so that

$$8c_2 \max_{|t|\leqq 2} |\Gamma(\tfrac{1}{2}+it)| + \sum_{n=1}^{\infty} c_2\, 2^{2n+2} \max_{2^n\leqq|t|\leqq 2^{n+1}} |\Gamma(\tfrac{1}{2}+it)| \leqq 2\pi/3. \tag{3.14}$$

If (3.13) is falsified for $n=1, 2, \ldots$ then the left hand side of (3.5) is numerically less than 1/3. If (3.12) is also falsified for each integer n, then the right hand side of (3.5) has real part

$$\geqq 1 - 1/Y - 1/10 - 1/10 - (2l)(6l)^{-1} > 1/3 \tag{3.15}$$

if Y is sufficiently large, contradicting the assumption that ρ was a zero.

4. Class (i) Zeros

For each positive integer a we have

$$\Big(\sum_{U<m\leqq 2U} b(m)\, m^{-\rho} e^{-m/Y}\Big)^a = \sum_{U^a<m\leqq 2^a U^a} c(m)\, m^{-\rho} \tag{4.1}$$

where

$$c(m) = \sum_{m_1} \cdots \sum_{m_a} b(m_1)\ldots b(m_a)\, e^{-(m_1+\cdots+m_a)/Y}, \tag{4.2}$$

the sum being over all sequences $m_1, \ldots, m_a$ of a integers from the interval $(U, 2U]$ whose product is m. We have

$$|c(m)| \leqq d_{2a}(m)\, e^{-aU/Y}, \tag{4.3}$$

and hence

$$\sum |c(m)|^2 \leqq e^{-2aU/Y} \sum_1^{2^a U^a} (d_{2a}(m))^2 \leqq e^{-2aU/Y} \sum_1^{2^a U^a} d_{4a^2}(m)$$
$$\leqq e^{-2aU/Y} (a(2+\log U))^{4a^2-1}\, 2^a U^a. \tag{4.4}$$

We apply (2.9) to the sum on the right of (4.1) with $U=2^{n-1}Y$,

$$G \leqq (2^n Y)^{a(1-2\alpha)} e^{-a2^n} 2^{2a\alpha} (a(2+n\log 2+\log Y))^{4a^2-1}$$
$$\leqq (2^n Y)^{a(1-2\alpha)} e^{-a2^n} (3al)^{4a^2}, \tag{4.5}$$

the T of (2.4) being twice the T of (3.1), and with

$$V=(6l)^{-a}. \tag{4.6}$$

We choose a subsequence of the zeros which contains a proportion $\gg l^{-1}$ of all the class (i, n) zeros satisfying (3.1) in such a way that (2.4) holds for the subsequence. By (2.9) the number of zeros in class (i, n) is therefore

$$\ll (2^n Y)^{2a(1-\alpha)} e^{-a2^n} (3al)^{4a^2} (6l)^{2a} l$$
$$+ T(2^n Y)^{2a(2-3\alpha)} e^{-3a2^n} (3al)^{12a^2} (6l)^{6a} l^3. \tag{4.7}$$

5. The Jutila Breaks

Let

$$H(a)=(3al)^{4a^2}(6l)^{2a} l, \tag{5.1}$$

and let $J(a)$ be the interval $Y_a < m \leqq Y_{a+1}$, where Y_a is given by

$$Y_a^{(4a-2)\alpha-(2a-2)} = TH^3(a)(H(a+1)\,l)^{-1} \tag{5.2}$$

for $a=1, \ldots, A$, where A is determined by $X \in J(A)$. Eq. (5.2) is equivalent to

$$H(a+1)\, Y_a^{2(a+1)(1-\alpha)}\, l = H^3(a)\, Y_a^{2a(2-3\alpha)}\, T. \tag{5.3}$$

We sum the estimate (4.7) for the number of class (i, n) zeros over those n with $n \leqq 0$ for which $2^n Y \in J(a)$, and if $Y \in J(a)$, over all positive n also. The number of values of n is $\ll l$, and the sum is

$$\ll H(a)\, Y_{a-1}^{2a(1-\alpha)} e^{-aY_a/Y}\, l + H^3(a)\, Y_a^{2a(2-3\alpha)}\, T e^{-3aY_a/Y}$$
$$\ll H(a)\, Y_{a-1}^{2a(1-\alpha)}\, l + H(a+1)\, Y_a^{2(a+1)(1-\alpha)}\, l \tag{5.4}$$

for $\alpha \geqq 3/4$. If $Y \in J(a)$ we replace Y_{a-1} by Y, as the negative exponential makes the upper bound (4.7) decrease rapidly for $n>0$.

If $\alpha<1$ and X is chosen to make

$$A \leqq \tfrac{1}{6}((1-\alpha)\, l/\log l)^{\frac{1}{2}}, \tag{5.5}$$

then for $a=4, 5, \ldots, A$

$$\log H(a+1) \leqq \tfrac{2}{7}(1-\alpha)\, l \tag{5.6}$$

and
$$Y_a^{2(a+1)(1-\alpha)} H(a+1) \ll Y_3^{8(1-\alpha)} H(4). \tag{5.7}$$

If $Y \in J(2)$ the number of class (i) zeros totals
$$\begin{aligned} &\ll H(2)\, Y^{4(1-\alpha)}\, l + H(3)\, Y_2^{6(1-\alpha)}\, l + H(4)\, Y_3^{8(1-\alpha)}\, l^2 \\ &\ll Y^{4(1-\alpha)}\, l^{22} + T^{3(1-\alpha)/(3\alpha-1)}\, l^{44} + T^{4(1-\alpha)/(5\alpha-2)}\, l^{75}. \end{aligned} \tag{5.8}$$

6. The Zero-Density Result

We have
$$M(\tfrac{1}{2}+it) \ll X^{\frac{1}{2}}, \tag{6.1}$$

and by a lemma of Montgomery [6, 7] when $t_1, \ldots, t_R$ satisfy (2.4) then
$$\sum_1^R |\zeta(\tfrac{1}{2}+it_r)|^4 \ll T\, l^5. \tag{6.2}$$

If (3.13) holds for $t = t_1, \ldots, t_R$ then
$$R \ll 2^{-4n} X^2\, Y^{-2(2\alpha-1)}\, T\, l^5. \tag{6.3}$$

Each t_r corresponds to at most $\ll 2^n l$ class (ii, n) zeros, and so the total number of class (ii) zeros must be
$$\ll X^2\, Y^{-2(2\alpha-1)}\, T\, l^6. \tag{6.4}$$

A better estimate is possible, for example by raising the sum $M(\frac{1}{2}+it)$ to a high power and using (2.9). To obtain (1.9) for the range $3/4 \leqq \alpha \leqq 9/11$ we put
$$X = T^{(5-6\alpha)/(12\alpha-4)}, \tag{6.5}$$
$$Y = T^{3/(12\alpha-4)}, \tag{6.6}$$

the choice for X satisfying (5.5) if T is sufficiently large.

For $\alpha \geqq 9/11$ we use Haneke's bound [1] for $\zeta(s)$:
$$\zeta(\tfrac{1}{2}+it) \ll |t|^{6/37} \log(|t|+e) \ll |t|^{13/80}. \tag{6.7}$$

If T is sufficiently large and
$$X^{-\frac{1}{2}}\, Y^{\alpha-\frac{1}{2}} = T^{13/80}, \tag{6.8}$$

(3.13) cannot hold for any n, and all zeros are of class (i). The choice (6.6) makes
$$X = T^{-13/40}\, Y^{2\alpha-1} = T^{(21\alpha-17)/(120\alpha-40)} > Y_{300} \tag{6.9}$$

for $\alpha \geqq 9/11$. Our estimate for the number of zeros is obtained by summing (4.7) for $a = 2, 3, \ldots, 300$. The condition (5.5) is satisfied unless
$$\alpha \geqq 1 - 32400 \log l / l, \tag{6.10}$$

and it is well known that $N(\alpha, T)$ is zero when T is sufficiently large and (6.10) holds, and so (1.9) is still true.

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