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Affine Root Systems and Dedekind's η-Function

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Introduction

Let R be a reduced root system in a real vector space V, as defined for example in [B], Chapter VI. Define positive and negative roots by choosing a Weyl chamber for R, and let ρ denote half the sum of the positive roots. Let $W(R)$ be the Weyl group of R, and let $\varepsilon(w)$ denote the determinant (equal to ± 1) of an element w of $W(R)$. Then there is a well-known polynomial identity, due to Hermann Weyl:

$$\sum_{w \in W(R)} \varepsilon(w)\, e^{w\rho} = \prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}), \tag{0.1}$$

where the e's are formal exponentials and the product on the right is over all the positive roots ([B], p. 185).

The main purpose of this paper is to establish an analogue of (0.1) for an "affine root system" S. For the precise definition of an affine root system we refer to § 2; all we shall say here is that the elements of S are affine-linear functions on a finite-dimensional real Euclidean space E and satisfy axioms analogous to those for a finite root system. The set S of "affine roots" is infinite, and the Weyl group $W(S)$ is an affine Weyl group, that is to say an infinite group of displacements of E generated by reflections.

One type of affine root system may be constructed as follows. Let R be a (not necessarily reduced) finite root system in V, and let $\langle x, y\rangle$ be a scalar product on V which is invariant under the action of the Weyl group $W(R)$. For each $\alpha \in R$ and integer k let $a_{\alpha,k}$ be the affine-linear function on V defined by

$$a_{\alpha,k}(x) = \langle \alpha, x\rangle + k.$$

Then the functions $a_{\alpha,k}$, where k is any integer if $\frac{1}{2}\alpha \notin R$, and k is an odd integer if $\frac{1}{2}\alpha \in R$, form an affine root system on V which we shall denote by $S(R)$.

For an affine root system S we can define positive and negative roots in the usual way, by choosing a Weyl chamber C for S. If S is irreducible, C is a rectilinear l-simplex, where l is the dimension of the Euclidean space E. A first objection to finding an analogue of Weyl's identity (0.1) in the affine case is that there are infinitely many positive roots and therefore no analogue of ρ. However, it is easy to banish ρ from (0.1). We have only to divide both sides by e^{ρ} and observe that $\rho - w\rho$ is equal to the sum, say $s(w)$, of the positive roots α such that $w^{-1}\alpha$ is negative. Hence (0.1) can be rewritten in the form

$$\sum_{w\in W(R)} \varepsilon(w)\, e^{-s(w)} = \prod_{\alpha>0} (1-e^{-\alpha}). \tag{0.2}$$

Both sides of (0.2) make sense for an affine root system S and its Weyl group $W(S)$: the left-hand side is a formal power series and the right-hand side a formal infinite product. So it is reasonable to ask whether (0.2) remains true as a formal identity in the affine case.

In fact, it doesn't quite; there is an extra factor which has to be inserted on the right-hand side. Assume for simplicity of description that S is irreducible, so that the chamber C is a simplex. Let $a_0, \dots, a_l$ be the positive affine roots which vanish on the walls of C. Then there is a unique relation of the form

$$\sum_{i=0}^{l} n_i a_i = c$$

in which the n_i are positive integers with no common factor $\neq 1$, and c is a constant function. Let X stand for the formal exponential e^{-c}, i.e. $X = \prod_{i=0}^{l} e^{-n_i a_i}$. Then the analogue of (0.2) for the affine root system S is

$$\sum_{w\in W(S)} \varepsilon(w)\, e^{-s(w)} = P(X) \prod_{a>0} (1-e^{-a}). \tag{0.3}$$

Here $\varepsilon(w)$ and $s(w)$ are defined exactly as before; the product on the right is over all the positive affine roots; and $P(X)$ is an infinite product of the form

$$P(X) = \prod_{n=1}^{\infty} p(X^n),$$

where $p(X)$ is a certain polynomial with integral coefficients, depending on S and of degree equal to $l = \dim E$. For example, if $S = S(R)$ the polynomial $p(X)$ is equal to $(1-X)^l$.

By computing the sum $s(w)$ explicitly we are led to write the identity (0.3) in a different form. Applied to the affine root system $S(R)$, where R is irreducible and reduced, this leads to the following result. Let $\|x\|^2 = \langle x, x\rangle$ for $x \in V$, let ϕ be the highest root of R and let $g = \frac{1}{2}(\|\phi+\rho\|^2 - \|\rho\|^2)$. Also let M be the lattice generated by the vectors $2g\,\alpha/\|\alpha\|^2$, where $\alpha \in R$. Then (0.3) takes the form

$$\prod_{n=1}^{\infty} \Big((1-X^n)^l \prod_{\alpha\in R} (1-X^n e^{\alpha})\Big) = \sum_{\mu\in M} \chi(\mu)\, X^{(\|\mu+\rho\|^2 - \|\rho\|^2)/2g}, \tag{0.4}$$

where

$$\chi(\mu)=\frac{\sum_{w\in W(R)} \varepsilon(w)\, e^{w(\mu+\rho)}}{\sum_{w\in W(R)} \varepsilon(w)\, e^{w\rho}}.$$

If we now replace each e^{α} by 1 in (0.4) we obtain a power-series for $\eta(X)^d$, where

$$\eta(X)=X^{1/24}\prod_{n=1}^{\infty}(1-X^n)$$

is Dedekind's η-function, and d is the dimension of the Lie algebra $\mathfrak{g}$ having R as its root system: namely

$$\eta(X)^d=\sum_{\mu\in M} d(\mu)\, X^{\|\mu+\rho\|^2/2g} \tag{0.5}$$

where

$$d(\mu)=\prod_{\alpha>0}\frac{\langle\mu+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

The simplest case of (0.4) is that in which R is of type A_1, with just two roots α, $-\alpha$. In this case (0.4) becomes

$$\prod_{n=1}^{\infty}((1-X^n)(1-X^n e^{\alpha})(1-X^{n-1}e^{-\alpha}))=\sum_{m\in\mathbb{Z}}(-1)^m X^{m(m-1)/2}\, e^{-m\alpha}, \tag{0.6}$$

which is one form of a famous theta-function identity due to Jacobi (see for example [3], p. 282). The identity (0.3) may therefore be regarded as a common generalization of Weyl's identity (0.1) and Jacobi's identity (0.6).

Likewise, when R is of type A_1, (0.5) becomes

$$\eta(X)^3=\sum_{n\equiv 1(4)} n\, X^{n^2/8}, \tag{0.7}$$

or equivalently

$$\prod_{n=1}^{\infty}(1-X^n)^3=\sum_{m=0}^{\infty}(-1)^m(2m+1)\, x^{m(m+1)/2}$$

which is also due to Jacobi. When R is of type BC_1 we obtain in the same way from the basic identity (0.3) the formulae

$$\eta(X)^5/\eta(X^2)^2=\sum_{n\equiv 1(6)} n\, X^{n^2/24},$$

$$\eta(X^2)^5/\eta(X)^2=\sum_{n\equiv 1(3)}(-1)^{n-1}\, n\, X^{n^2/3}.$$

Particular cases of (0.3) have been discovered by various people. The identity for R of type B_2 was found by Winquist [6], who used it (or rather the specialized form (0.5)) to give an elegant and elementary proof of Ramanujan's congruence $p(11n+6)\equiv 0 \pmod{11}$ for the partition function $p(n)$. (The congruence $p(7n+5)\equiv 0 \pmod 7$ comes from (0.5) for R of type $A_1\times A_1$ in the same way: see [3], p. 289.) Pursuing Winquist's methods, F. J. Dyson (unpublished) found many cases of (0.3), in particular those corresponding to affine root systems $S(R)$ with R of classical type (A_l, B_l, C_l and D_l). Others are due to A.O.L. Atkin (also unpublished).

One other special case of (0.5) may perhaps be mentioned here. When R is of type A_4, the dimension d of the Lie algebra is 24. Hence in this case (0.5) leads to the following formula (due originally to Dyson) for Ramanujan's τ-function:

$$\tau(n)=\frac{1}{1!\,2!\,3!\,4!}\sum\prod_{i<j}(u_i-u_j)$$

summed over integers $u_1,\ldots,u_5$ subject to the conditions

$$u_i\equiv i \pmod 5,\qquad \sum u_i=0,\qquad \sum u_i^2=10n.$$

The contents of the paper are as follows. § 1 establishes basic notation and terminology. In § 2 we define affine root systems. §§ 3–6 are devoted to their elementary properties and the classification of the irreducible reduced affine root systems: there are seven infinite families and seven "exceptional" systems. They are all either of the form $S(R)$ described earlier in this introduction, or are the duals of these (Theorem (5.2)).

Incidentally, the notion of an affine root system is equivalent to that of an "échelonnage" defined by Bruhat and Tits ([1], Chapter I, § 1.4). It follows therefore from their work that to each reductive group over a local field there is canonically associated an affine root system, and moreover that all affine root systems (including the non-reduced ones) arise in this way. It should also be remarked that the list of Dynkin diagrams of reduced irreducible affine root systems which we obtain in § 5 also occurs in the work of Moody ([7a, 7b]) on Euclidean Lie algebras.

The remainder of the paper is concerned with the identity (0.3). § 7 deals with the calculation of the exponent $s(w)$, and § 8 with the statement of the main theorem (Theorem (8.1)) and various specializations of it such as (0.5) mentioned above. The proof occupies §§ 9–12. Here the determination of the factor $P(X)$ offers most resistance, and is achieved by specializing the identity in two different ways and then comparing the results. The paper concludes with appendices which list the irreducible affine root systems.

As the reader will see for himself, our proof of the basic identity is a purely formal exercise and does not *explain* in any satisfactory way why there should be any relationship between powers of Dedekind's η-function and root systems of Lie algebras. For example, the presence of the factor $X^{d/24}$ on the left-hand side of (0.5) is accounted for on the right-hand side by the following "strange formula" ([2], p. 243):

$$\Phi_R(\rho, \rho) = d/24$$

where Φ_R is the scalar product on R induced by the Killing form on $\mathfrak{g}$.

Finally, it is a pleasure to acknowledge the benefit I have derived from correspondence with F. J. Dyson on this subject.

1. Notation and Terminology

Let E be an affine space over a field K: that is to say, E is a set on which a K-vector space V acts faithfully and transitively. The elements of V are called *translations* of E, and the effect of a translation $v \in V$ on a point $x \in E$ is written $x+v$. If $y=x+v$ we write $v=y-x$.

Let E' be another affine space over K, and V' its vector space of translations. A mapping $f\colon E \to E'$ is said to be *affine-linear* if there exists a K-linear mapping $Df\colon V \to V'$, called the *derivative* of f, such that

$$f(x+v) = f(x) + (Df)(v), \tag{1.1}$$

for all $x \in E$ and $v \in V$. In particular, a function $f\colon E \to K$ is affine-linear if and only if there exists a linear form $Df\colon V \to K$ such that (1.1) holds.

Let F denote the K-vector space of all affine-linear functions $f\colon E \to K$, and let V^* be the dual of the vector space V. Then D is a linear mapping of F onto V^*, and its kernel is the line F^0 in F consisting of the constant functions.

From now on K will be the field $\mathbb{R}$ of real numbers, and V will be a real vector space of finite dimension l, equipped with a positive definite bilinear form $\langle u, v\rangle$. Let $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. Then E is a Euclidean space of dimension l, and is a metric space with respect to the distance function $\|x-y\|$.

We shall identify V with its dual space V^* by means of the bilinear form $\langle u, v\rangle$. Then for any affine-linear function $f\colon E \to \mathbb{R}$, (1.1) now takes the form

$$f(x+v) = f(x) + \langle Df, v\rangle,$$

and Df is the *gradient* of f, in the usual sense of elementary calculus.

We define a bilinear form $\langle f, g\rangle$ on the space F as follows:

$$\langle f, g\rangle = \langle Df, Dg\rangle. \tag{1.2}$$

This bilinear form is positive semi-definite, and $f \in F$ is isotropic if and only if f is a constant function.

For each $v \neq 0$ in V we define

$$v^{\vee} = 2v/\langle v, v\rangle,$$

and for each non-constant $f \in F$ we define

$$f^{\vee} = 2f/\langle f, f\rangle$$

and

$$H_f = \{x \in E : f(x) = 0\}.$$

Then H_f is an affine hyperplane in E. The reflection in this hyperplane is the affine-linear isometry $w_f \colon E \to E$ given by

(1.3) $$w_f(x) = x - f^{\vee}(x)\, Df = x - f(x)\, Df^{\vee}.$$

By transposition, w_f acts on F: $w_f(g) = g \circ w_f^{-1} = g \circ w_f$. Explicitly,

(1.4) $$w_f(g) = g - \langle f^{\vee}, g\rangle f = g - \langle f, g\rangle f^{\vee}$$

for any $g \in F$.

For each $u \neq 0$ in V, let $w_u \colon V \to V$ be the reflection in the hyperplane orthogonal to u, so that

$$w_u(v) = v - \langle u, v\rangle\, u^{\vee}.$$

Then for any non-constant $f \in F$ we have

(1.5) $$D w_f = w_{Df}.$$

For if $v \in V$ and $x \in E$, then

$$\begin{aligned}(D w_f)(v) &= w_f(x+v) - w_f(x)\\ &= \big(x + v - f(x+v)\, Df^{\vee}\big) - \big(x - f(x)\, Df^{\vee}\big) \quad \text{by (1.3)}\\ &= v - \big(f(x+v) - f(x)\big)\, Df^{\vee}\\ &= v - \langle Df, v\rangle\, Df^{\vee} = w_{Df}(v).\end{aligned}$$

Finally, let $w \colon E \to E$ be an affine-linear isometry. Then its derivative Dw is a linear isometry of V, i.e. we have $\langle (Dw)u, (Dw)v\rangle = \langle u, v\rangle$ for all $u, v \in V$. The mapping w acts by transposition on F: $w(f) = f \circ w^{-1}$, and we have

(1.6) $$D\big(w(f)\big) = (Dw)(Df).$$

For if $v \in V$ and $x \in E$, then

$$\begin{aligned}\langle D(wf), v\rangle &= (wf)(x+v)-(wf)(x)\\ &= f\big(w^{-1}(x+v)\big)-f(w^{-1}x)\\ &= f\big(w^{-1}x+(Dw)^{-1}(v)\big)-f(w^{-1}x)\\ &= \langle Df, (Dw)^{-1}v\rangle\\ &= \langle (Dw)(Df), v\rangle.\end{aligned}$$

In § 2 we shall define affine root systems. To distinguish them from root systems of the usual sort, as defined for example in [B], p. 142, we shall call the latter *finite* root systems. If R is a (finite or affine) root system, we denote by $L(R)$ the lattice generated by R, and by $W(R)$ the Weyl group of R. If a basis of R has been fixed we denote by R^+ the set of positive roots relative to that basis, and we shall sometimes write $\alpha > 0$ to mean $\alpha \in R^+$.

2. Affine Root Systems

As in § 1 let E be a real Euclidean space of dimension l, and let V be its space of translations. We give E the usual topology, defined by the metric $\|x-y\|$, so that E is locally compact. As before, let F denote the vector space of affine-linear functions on E.

An *affine root system* on E is defined to be a subset S of F satisfying the following axioms (AR 1)–(AR 4):

(AR 1) *S spans F, and the elements of S are non-isotropic* (with respect to the scalar product (1.2)), i.e. they are non-constant functions.

(AR 2) *$w_a S = S$ for all $a \in S$.*

(AR 3) *$\langle a, b^\vee\rangle \in \mathbb{Z}$ for all $a, b \in S$.*

The elements of S are called *affine roots*, or just *roots*. Let $W(S)$ be the group of displacements of E generated by the reflections w_a ($a \in S$). The group $W(S)$ is called the *Weyl group* of S. The fourth axiom, which replaces the finiteness condition in the definition of an ordinary root system, is

(AR 4) *$W(S)$ (as a discrete group) acts properly on E.*

In other words ([B], p. 72), if K_1 and K_2 are compact subsets of E, then the set of elements $w \in W$ such that $w(K_1)$ meets K_2 is *finite.*

From (AR 3), just as in the finite case, we deduce that if a, λa are proportional affine roots, then λ is one of the numbers $\pm\frac{1}{2}$, ± 1, ± 2. If $a \in S$ and $\frac{1}{2}a \notin S$, the root a is said to be *indivisible.* We say that S is *reduced* if each $a \in S$ is indivisible, i.e. if the only roots proportional to a are $\pm a$.

If S is an affine root system on E, then

$$S^\vee = \{a^\vee : a \in S\}$$

is also an affine root system on E, called the *dual* of S. Clearly S and $S^\vee$ have the same Weyl group.

The *rank* of S is defined to be the dimension l of E. If S' is another affine root system on a Euclidean space E', then an *isomorphism* of S onto S' is a bijection of S onto S' which is induced by an affine-linear isometry of E onto E'.

The following proposition provides examples of affine root systems.

(2.1) **Proposition.** *Let R be a (finite) root system in a real finite-dimensional vector space V. For each $\alpha \in R$ and $n \in \mathbb{Z}$ let $a_{n,\alpha}$ be the affine-linear function on V defined by*

$$a_{n,\alpha}(x) = n + \langle \alpha, x \rangle$$

where $\langle u, v \rangle$ is a positive-definite bilinear form on V invariant under the Weyl group of R. Then the set $S(R)$ of functions $a_{n,\alpha}$, where $\alpha \in R$ and

$$n \in \mathbb{Z} \quad \text{if} \quad \tfrac{1}{2}\alpha \notin R; \qquad n \in 2\mathbb{Z}+1 \quad \text{if} \quad \tfrac{1}{2}\alpha \in R$$

is a reduced affine root system on V.

Proof. The fact that $S(R)$ satisfies (AR 1) and (AR 3) follows immediately from the corresponding axioms for R ([B], Chapter VI). As to (AR 2), let $a, b \in S(R)$, say $a = a_{m,\alpha}$ and $b = a_{n,\beta}$. Then a simple calculation shows that $w_a(b) = a_{k,\gamma}$ where

$$k = n - \langle \alpha^\vee, \beta \rangle m, \qquad \gamma = w_\alpha(\beta).$$

We have $k \in \mathbb{Z}$ in any case; and if $\frac{1}{2}\gamma \in R$ then also $\frac{1}{2}\beta \in R$ and therefore $\langle \frac{1}{2}\beta, \alpha^\vee \rangle$ is an integer, so that $\langle \alpha^\vee, \beta \rangle$ is an even integer; but n is odd, and therefore k is odd.

As to (AR 4), it is clear from the definitions that the Weyl group W of $S(R)$ is the affine Weyl group ([B], p. 173) of the reduced root system consisting of the $\alpha \in R$ such that $2\alpha \notin R$. Hence W acts properly on V. Finally, it is clear that $S(R)$ is reduced.

3. Direct Sums. Reducibility

Let $E_1, \ldots, E_r$ be finite-dimensional real Euclidean spaces, and for each $i = 1, \ldots, r$ let V_i be the space of translations of E_i, and F_i the space of affine-linear functions on E_i. Let E be the product of the E_i and V the direct sum of the vector spaces V_i. Then E is naturally a Euclidean space with V as space of translations; the action of V on E is defined by

$$(x_1, \ldots, x_r)+(v_1, \ldots, v_r)=(x_1+v_1, \ldots, x_r+v_r)$$

$(x_i \in E_i,\ v_i \in V_i,\ 1 \leqq i \leqq r)$, and the bilinear form on V by

$$\langle (u_1, \ldots, u_r), (v_1, \ldots, v_r) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_r, v_r \rangle .$$

Let F be the space of affine-linear functions on E, and for each i let p_i be the projection of E onto E_i. Then the mappings π_i: $F_i \to F$ defined by $\pi_i(f_i)=f_i \circ p_i$ are injective linear isometries (for the scalar product (1.2)). The subspaces $\pi_i(F_i)$ generate F, are mutually orthogonal, and all contain the line F^0 of constant functions.

Now let S_i be an affine root system on E_i, for each $i=1, \ldots, r$, and let $S_i'=\pi_i(S_i) \subset F$. Let

(3.1) $$S=\bigcup_{i=1}^{r} S_i'.$$

Then it is a routine matter to check that S is an affine root system on E. This root system S is called the *direct sum* of the S_i, and we denote it by $\coprod_{i=1}^{r} S_i$. The subsets S_i' of S evidently satisfy

(3.2) *S_i', S_j' are orthogonal if $i \neq j$.*

Conversely, let E be a finite-dimensional real Euclidean space, V its space of translations, and S an affine root system on E. Let S_i' $(1 \leqq i \leqq r)$ be subsets of S satisfying (3.1) and (3.2). From (AR 1) it follows that the S_i' are pairwise disjoint, and therefore form a partition of S into mutually orthogonal sets of roots.

Let V_i be the subspace of V generated by the gradients of the roots belonging to S_i'. Let $V_i^\perp$ be the orthogonal complement of V_i in V, and let E_i be the space of orbits of E under the action of $V_i^\perp$. Then E_i has a natural structure of a Euclidean space with V_i as space of translations, such that the mapping p_i: $E \to E_i$, which assigns to each point of E its orbit under $V_i^\perp$, is affine-linear. The mapping $x \mapsto (p_1(x), \ldots, p_r(x))$ identifies E with the product Euclidean space $E_1 \times \cdots \times E_r$.

Let F_i be the space of affine-linear functions on E_i, and define as before the injective linear isometries π_i: $F_i \to F$. Then $S_i' \subset \pi_i(F_i)$, and $S_i=\pi_i^{-1}(S_i')$ is an affine root system on E_i, and finally the identification of E with $E_1 \times \cdots \times E_r$ identifies S with the direct sum $\coprod_{i=1}^{r} S_i$.

An affine root system S is said to be *irreducible* if S is not empty and is not the direct sum of two or more non-empty affine root systems. Equivalently, as we have just seen, S is irreducible if and only if S is not empty and there exists no partition of S into two or more non-empty subsets S_i' satisfying (3.2). Just as in the case of finite root systems we have

(3.3) **Proposition.** *Every affine root system is expressible as the direct sum of a finite family of irreducible affine root systems. This decomposition is unique to within isomorphism.*

The proof may be left to the reader.

The notion of isomorphism of root systems defined in § 2 is too restrictive for many purposes. For example, if S is an affine root system and λ is a non-zero real number, then S and $\lambda S = \{\lambda a: a \in S\}$ are not isomorphic, although they are effectively "the same". We therefore define a weaker equivalence relation, *similarity*, as follows. Let S be an affine root system, which by (3.3) we can write as a direct sum of irreducible systems S_i. Then a root system S' is said to be *similar* to S if S' is isomorphic to the direct sum $\coprod \lambda_i S_i$, where the λ_i are non-zero real numbers.

4. Chambers and Bases

Let S be an affine root system on a Euclidean space E of dimension l. The set

$$\mathfrak{H} = \{H_a: a \in S\}$$

of affine hyperplanes in E on which the affine roots vanish satisfies conditions (D 1) and (D 2) of [B], p. 72, because $w(H_a) = H_{wa}$ for all $w \in W(S)$ and $a \in S$. Hence (loc. cit., Lemma 1):

(4.1) **Proposition.** *$\mathfrak{H}$ is locally finite.*

It follows that the set $E - \bigcup_{a \in S} H_a$ is open in E, and therefore so are the connected components of this set, because E is locally connected. These components are called the *chambers* of the root system S, or of its Weyl group $W(S)$. We recall ([B], p. 74, Theorem 1):

(4.2) **Proposition.** *The Weyl group $W(S)$ acts faithfully and transitively on the set of chambers.*

Assume from now on that S is *irreducible*. This is purely a matter of convenience, to simplify statements of results. Then ([B], p. 86, Prop. 8) each chamber is an open rectilinear l-simplex. (If S is reducible, the chambers are orthogonal products of simplexes.) Choose a chamber C once and for all. Let $x_0, \ldots, x_l$ be the vertices of C, so that C is the set of all points $x \in E$ of the form $x = \sum_{i=0}^{l} \lambda_i x_i$ with $\sum \lambda_i = 1$ and each $\lambda_i > 0$.

Let $B = B(C)$ be the set of indivisible affine roots $a \in S$ which satisfy the following condition: *H_a is a wall of C, and $a(x) > 0$ for all $x \in C$.* Then

B consists of $l+1$ roots, one for each wall of C. Clearly B is a basis of the space F of affine-linear functions on E. Moreover ([B], p. 73, Lemma 2):

(4.3) **Proposition.** *The Weyl group $W(S)$ is generated by the reflections w_a for $a \in B$.*

(4.4) **Proposition.** *Let $b \in S$ be indivisible. Then $b = w\,a$ for some $w \in W(S)$ and some $a \in B$.*

Proof. The hyperplane H_b is a wall of some chamber C' on which b is positive. By (4.2), $C' = w\,C$ for some $w \in W(S)$. Hence $w^{-1} b$ is positive on C, and $H_{w^{-1}b} = w^{-1} H_b$ is a wall of C, so that $w^{-1} b \in B$.

Let L be the lattice in F generated by B. It is a free abelian group of rank $l+1$.

(4.5) **Proposition.** *L is equal to the lattice $L(S)$ in F generated by S.*

Proof. Clearly $L(S)$ is generated by the indivisible affine roots. Hence by (4.4) it is enough to show that L is stable under $W(S)$, and by (4.3) it is therefore enough to show that $w_a(L) \subset L$ for all $a \in B$. But if $b \in B$ we have $w_a(b) = b - \langle a^\vee, b\rangle\, a$ by (1.4), and $\langle a^\vee, b\rangle \in \mathbb{Z}$ by (AR 3). Hence $w_a(b) \in L$ and therefore $w_a(L) \subset L$.

An affine root a is said to be *positive* (resp. *negative*) (relative to the chamber C) if $a(x) > 0$ (resp. $a(x) < 0$) for all $x \in C$. Every affine root is either positive or negative.

The elements of B will be denoted by $a_0, \ldots, a_l$, the notation being chosen so that $a_i(x_j) = 0$ whenever $i \neq j$. Since $x_i \in \bar{C}$, we have $a_i(x_i) > 0$.

(4.6) **Proposition.** *Each affine root $a \in S$ is a linear combination of $a_0, \ldots, a_l$ with rational integer coefficients which are all $\geqq 0$ if a is positive, and all $\leqq 0$ if a is negative.*

Proof. By (4.5) we have $a \in L$, say

$$a = \sum_{j=0}^{l} \lambda_j a_j$$

with coefficients $\lambda_j \in \mathbb{Z}$. Evaluating both sides of this equation at x_i, we have $\lambda_i = a(x_i)/a_i(x_i)$. If a is positive then $a(x) \geqq 0$ for all x in the closure of C, and in particular $a(x_i) \geqq 0$. Since $a_i(x_i) > 0$, it follows that $\lambda_i \geqq 0$. Likewise, if a is negative, $\lambda_i \leqq 0$ for all i.

B is called a *basis* of S.

(4.7) *Example.* Let R be a finite irreducible root system, $\alpha_1, \ldots, \alpha_l$ a basis of R, and let ϕ be the highest root of R relative to this basis. Then the affine roots $a_0 = 1 - \phi$, $a_i = \alpha_i$ $(1 \leqq i \leqq l)$ form a basis of the affine root system $S(R)$ defined in (2.1).

5. Classification of Affine Root Systems

As before let S be an irreducible affine root system, C a chamber for S, $x_0, \ldots, x_l$ the vertices of C, and $B = \{a_0, \ldots, a_l\}$ the corresponding basis. For each i let F_i be the subspace of F consisting of the affine-linear functions on E which vanish at x_i, and put $S_i = S \cap F_i$. On F_i the bilinear form $\langle f, g \rangle$ is positive definite. Also let W_i be the subgroup of $W(S)$ which fixes x_i.

(5.1) **Proposition.** (1) *S_i is a (finite) root system in F_i, and is reduced if S is reduced.*

(2) *$B - \{a_i\}$ is a basis of S_i.*

(3) *W_i is the Weyl group of S_i.*

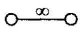
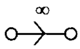
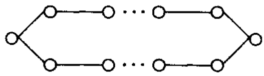
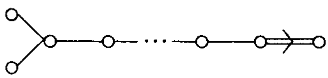
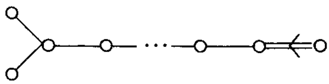
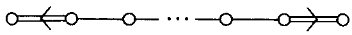
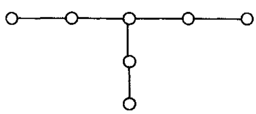
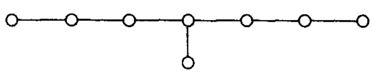
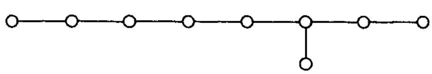
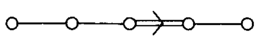
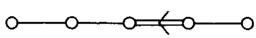
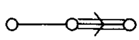
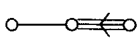
Proof. (1) Since $\mathfrak{H}$ is locally finite by (4.1) it is clear that S_i is finite; also S_i spans F_i and does not contain 0. If $a, b \in S_i$ then $w_a(b) = b - \langle a^\vee, b \rangle a$ belongs to S and vanishes at x_i, hence belongs to S_i. Finally it is clear that S_i is reduced if S is reduced.

(2) Let $a \in S_i$. Then $a = \sum \lambda_j a_j$ with the λ_j all integers of the same sign, by (4.6); evaluating both sides at x_i we see that $\lambda_i = 0$. Hence the a_j with $j \neq i$ form a basis of S_i, by [B], p. 162, Prop. 20, Cor. 3.

(3) By a basic theorem on reflection groups ([B], p. 75, Prop. 1), W_i is generated by the reflections belonging to $W(S)$ which fix x_i, that is to say by the reflections w_a where $a \in S_i$. Hence $W_i = W(S_i)$.

Now assume that S is *reduced.* We construct a Dynkin diagram for S according to the usual prescription: the nodes of the diagram correspond to the roots $a_0, \ldots, a_l$ belonging to the basis B, and bonds and arrows are inserted according to the same rules as for a finite root system. (When the rank of S is 1, we have to allow bonds of "infinite multiplicity".) By (5.1) this Dynkin diagram has the property that when any node is removed (together with the bonds issuing from that node) the resulting diagram is that of some finite reduced root system. Hence ([B], p. 196, Prop. 1) each of the S_i is determined up to similarity by the Dynkin diagram, and therefore so also is S. The similarity class of S is called the *type* of S.

In view of the known classifications of finite root systems and of affine Weyl groups, it is a straightforward matter to enumerate all possible Dynkin diagrams of irreducible reduced affine root systems. We have merely to take the Coxeter diagrams of the affine Weyl groups (which will be found for example on p. 199 of [B]) and replace each bond $\circ\overset{4}{\text{---}}\circ$ by either $\circ\!\Rightarrow\!\circ$ or $\circ\!\Leftarrow\!\circ$, and each bond $\circ\overset{6}{\text{---}}\circ$ by either $\circ\!\Rrightarrow\!\circ$ or $\circ\!\Lleftarrow\!\circ$. In this way we obtain the following list of possible connected Dynkin diagrams:

Type	
$A_1 = A_1^\vee$	
$BC_1 = BC_1^\vee$	
$A_l = A_l^\vee \quad (l \geqq 2)$	
$B_l \quad (l \geqq 3)$	
$B_l^\vee \quad (l \geqq 3)$	
$C_l \quad (l \geqq 2)$	
$C_l^\vee \quad (l \geqq 2)$	
$BC_l = BC_l^\vee \quad (l \geqq 2)$	
$D_l = D_l^\vee \quad (l \geqq 4)$	
$E_6 = E_6^\vee$	
$E_7 = E_7^\vee$	
$E_8 = E_8^\vee$	
F_4	
$F_4^\vee$	
G_2	
$G_2^\vee$	

Let X be any of the symbols $A_l, B_l, \ldots, G_2$. An affine root system S is said to be of *type* X (resp. *type* $X^\vee$) if S is similar to $S(R)$ (resp. $S(R)^\vee$) where R is a finite root system of type X. If S is of type X, it follows from

(4.7) that its Dynkin diagram is the "completed Dynkin diagram" ([B], p. 198) of type X. If S is of type $X^\vee$, its Dynkin diagram is obtained from the preceding one by reversing all the arrows.

The list above now shows that

(5.2) **Theorem.** *Every irreducible reduced affine root system is similar to either $S(R)$ or $S(R)^\vee$, where R is a finite irreducible root system.*

In § 6 we shall give another proof of (5.2) which does not depend on the classification of finite root systems and affine Weyl groups. We shall also classify the non-reduced irreducible root systems.

6. The Gradient Root System

Let S be an affine root system on E and let

$$\Sigma = DS = \{Da\colon\ a \in S\}$$

be the set of gradients (§ 1) of the affine roots.

(6.1) **Proposition.** (1) *Σ is a finite root system in V, the space of translations of E.*

(2) *If S is irreducible, so is Σ.*

(3) *The mapping $D\colon w \mapsto Dw$ is a homomorphism of $W(S)$ onto $W(\Sigma)$, the Weyl group of Σ, and the kernel of D is the subgroup T of translations in $W(S)$.*

Proof. (1) It follows from [B], p. 80, Theorem 3, that the number of families of parallel hyperplanes belonging to $\mathfrak{H}$ is finite. Also by (AR 3) we have $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma$, so that if β is proportional to α the number of possibilities for β is finite. Hence Σ is finite, and axioms (AR 1)–(AR 3) now imply directly that Σ is a root system in V.

(2) is obvious.

(3) Clearly D is a homomorphism. From (1.5), we have $D w_a = w_{Da}$ for all $a \in S$. Since $W(\Sigma)$ is generated by the reflections w_{Da}, it follows that D is surjective. Finally, $Dw = 1$ if and only if w is a translation, so that $\operatorname{Ker} D = T$.

We remark that Σ need not be reduced, even if S is reduced. If $S = S(R)$ (Prop. (2.1)) where R is a finite root system of type BC_l, then S is reduced, but $\Sigma = R$ is not reduced.

A point $x \in E$ is a *special point for S* if there exist affine roots $b_1, \ldots, b_l$ vanishing at x, whose gradients $Db_1, \ldots, Db_l$ form a basis of Σ.

(6.2) **Proposition.** (1) *There exist special points for S.*

(2) *If x is a special point for S, then x is a special point* ([B], p. 87) *for the Weyl group $W(S)$.*

(3) *If C is a chamber of S, there exists a vertex of C which is a special point for S.*

Proof. (1) Let $\{\beta_1, \dots, \beta_l\}$ be a basis of Σ, and for each i let b_i be an affine root with gradient β_i. Then since the β_i are linearly independent, the affine hyperplanes H_{b_i} intersect in a single point $x \in E$, which is therefore a special point for S.

(2) Let W_x be the subgroup of $W(S)$ which fixes x. Then $w_{b_i} \in W_x$ $(1 \leqq i \leqq l)$; also by (1.5) $D w_{b_i} = w_{D b_i}$, and by hypothesis the $D b_i$ are a basis of Σ, hence the reflections $w_{D b_i}$ generate $W(\Sigma)$. It follows that $D(W_x) = W(\Sigma)$, and since W_x contains no translations $\neq 1$, we conclude from (6.1.3) that D: $W_x \to W(\Sigma)$ is an isomorphism. Hence ([B], p. 87, Prop. 9) x is a special point for $W(S)$.

(3) From the previous paragraph, the reflections w_{b_i} generate W_x, and therefore the cone $\Gamma = \{y \in E\colon b_i(y) > 0 \ (1 \leqq i \leqq l)\}$ is a chamber for W_x. Hence ([B], p. 88, Prop. 11) there exists a unique chamber C' for $W(S)$ such that $C' \subset \Gamma$ and such that x is a vertex of C'. But $W(S)$ acts transitively on the set of chambers (4.2), hence there exists a vertex of the chamber C which is a special point for S.

Suppose from now on that S is irreducible. As in § 4, let $B = \{a_0, \dots, a_l\}$ be the basis of S corresponding to the chamber C. Then

(6.3) **Corollary.** *There exists $a_i \in B$ such that the gradients $D a_j$ for $j \neq i$ form a basis of Σ.*

Proof. Since S is irreducible, the chambers of S are simplexes. In the notation of the proof of (6.2.3), it is clear that $b_1, \dots, b_l$ belong to the basis B' of S determined by the chamber C'. Hence if $w C' = C$ we have $w B' = B$, and therefore B consists of $w b_1, \dots, w b_l$ together with say a_i. Then the set of gradients $\{D a_j\colon j \neq i\}$ is the image by $D w$ of the basis $\{\beta_1, \dots, \beta_l\}$ of Σ.

As in § 5, let S_i be the set of affine roots which vanish at x_i, and let $\Sigma_i = D S_i$ be the set of gradients of the roots of S_i. Then Σ_i is a subsystem of Σ, and the gradient map D: $S_i \to \Sigma_i$ is an isomorphism of finite root systems.

(6.4) **Proposition.** *Suppose that S is reduced and that the vertex x_i of C is special for S. Then Σ_i is the set of indivisible roots of Σ.*

Proof. Since S is reduced, so is S_i and therefore so also is Σ_i. Hence Σ_i is the reduced subsystem of Σ generated by a basis of Σ, whence the result.

We shall write $\alpha_j = D a_j$ for $0 \leqq j \leqq l$. Then the α_j for $j \neq i$ are a basis of Σ_i, because by (5.1) the a_j for $j \neq i$ are a basis of S_i. It follows that $\langle \alpha_i, \alpha_j \rangle \leqq 0$ for $0 \leqq i < j \leqq l$.

Now suppose that the vertex x_i of C is a special point for S. Since $\langle -\alpha_i, \alpha_j \rangle \geqq 0$ for all $j \neq i$, it follows that $-\alpha_i$ is a positive root of Σ relative to the basis $\{\alpha_j : j \neq i\}$. Hence we have

$$-\alpha_i = \sum_{j \neq i} n_j \alpha_j$$

where each n_j is a non-negative integer. In fact each n_j must be positive, because if $n_j = 0$ for some $j \neq i$, then the roots α_k with $k \neq j$ would be linearly dependent, which is impossible because they form a basis of Σ_j. The relation above may be written

(6.5) $$\sum_{i=0}^{l} n_i \alpha_i = 0$$

where each n_i is a positive integer, and $n_i = 1$ if the vertex x_i is special for S.

(6.6) **Proposition.** *Let S be an irreducible reduced affine root system, x_i a vertex of the chamber C. Then the following are equivalent:*

(1) *x_i is a special point for S;*

(2) *x_i is a special point for $W(S)$, and $n_i = 1$.*

Proof. (1) ⇒ (2) by (6.2.2) and the remark above.

(2) ⇒ (1). Since x_i is special for $W(S)$ it follows that $W(\Sigma_i) = W(\Sigma)$, by [B], p. 87, Prop. 9 (ii). Hence given $\alpha \in \Sigma$ there exists $\beta \in \Sigma_i$ such that $w_\alpha = w_\beta$, and therefore α is proportional to β. If $\alpha = \pm\beta$ or $\pm 2\beta$, then α is a linear combination of the basis elements α_j $(j \neq i)$ with integer coefficients all of the same sign. Suppose that $\alpha = \pm\frac{1}{2}\beta$. Since $n_i = 1$ it follows that α is a linear combination of the α_j $(j \neq i)$ with *integer* coefficients, i.e. $\alpha \in L(\Sigma_i)$ where $L(\Sigma_i)$ is the lattice in V generated by Σ_i. Hence $\beta \in 2L(\Sigma_i)$. But Σ_i is reduced (because S is reduced), hence there exists $w \in W(\Sigma_i)$ such that $w\beta = \alpha_j$ for some $j \neq i$. Consequently $\alpha_j \in 2L(\Sigma_i)$, which is absurd.

Hence every $\alpha \in \Sigma$ is a linear combination of the α_j $(j \neq i)$ with integer coefficients all of the same sign, and hence ([B], p. 162, Prop. 20, Cor. 3) the α_j $(j \neq i)$ form a basis of Σ, i.e. x_i is a special point for S.

The function

(6.7) $$c = \sum_{i=0}^{l} n_i a_i$$

is constant on E (because by (6.5) its gradient is zero) and positive (because it is positive on C). Moreover, every constant function c' in $L(S)$ is an integral multiple of c. For if $c' = \sum n'_i a_i$, then $\sum n'_i \alpha_i = 0$ and hence $n'_i = m n_i$ $(0 \leqq i \leqq l)$ for some $m \in \mathbb{Z}$. Hence c is the unique positive generator of $L(S) \cap F^0 \cong \mathbb{Z}$.

For each $a \in S$ let a_+ be the unique affine root such that $a_+ - a$ is constant, positive and as small as possible. Let $u_a = a_+ - a$, and let t_a be the translation $w_{a_+} w_a \in W(S)$. Then by (1.3) and (1.4) we have

$$(6.8)\qquad \begin{aligned} t_a(x) &= x - u_a D a^\vee, \\ t_a(f) &= f + u_a \langle a^\vee, f \rangle \end{aligned}$$

for any $x \in E$ and $f \in F$.

(6.9) **Proposition.** (1) *If $a \in S$ and $\lambda \in \mathbb{R}$, then $a + \lambda \in S$ if and only if $\lambda \in \mathbb{Z} u_a$.*

(2) *If $a \in S$ and $w \in W(S)$, then $u_{wa} = u_a$.*

(3) *If S is reduced, $a \in S$ and $2a + \lambda \in S$ for some $\lambda \in \mathbb{R}$, then $\lambda = m u_a$ where m is an odd integer.*

(4) *u_a is a positive integral multiple of the constant c.*

Proof. (1) If $m \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$ then by (6.8) we have

$$\begin{aligned} t_a^m(a+\lambda) &= a + \lambda + m u_a \langle a^\vee, a + \lambda \rangle \\ &= a + \lambda + 2 m u_a. \end{aligned}$$

Taking $\lambda = 0$ and $\lambda = u_a$, we see that $a + n u_a \in S$ for all $n \in \mathbb{Z}$. Conversely, if $a + \lambda \in S$ then $t_a^m(a+\lambda) \in S$ for all integers m. Choosing m suitably we obtain an affine root $a + \mu$ with $-u_a < \mu \leqq u_a$, and if μ is not equal to 0 or u_a this leads to a contradiction. Hence $\lambda \in \mathbb{Z} u_a$, and (1) is proved.

(2) We have $w(a + u_a) = wa + u_a$, hence $u_a \in \mathbb{Z} u_{wa}$ by (1) above. Similarly $u_{wa} \in \mathbb{Z} u_a$, hence $u_{wa} = u_a$.

(3) Let μ be the least real number $\geqq 0$ such that $2a + \mu \in S$. Since S is reduced and $a \in S$, we have $\mu > 0$. Now $w_a(2a + \mu) = -(2a - \mu)$, so that $2a - \mu \in S$. From the definition of μ it follows that there is no affine root $2a + \lambda$ with $|\lambda| < \mu$, and therefore (1) above (applied to $2a + \mu$) shows that $2a + \lambda \in S$ if and only if $\lambda = m\mu$ with m an odd integer. It remains to show that $\mu = u_a$.

Since $w_{2a+\mu}(a) = -(a + \mu)$, we have $a + \mu \in S$ and hence $\mu = r u_a$ for some integer $r \geqq 1$. If $r = 2$ then $2(a + u_a)$ and $a + u_a$ are affine roots, which is impossible because S is reduced. If $r \geqq 3$ there exists an integer s such that $\frac{1}{4} r \leqq s < \frac{1}{2} r$; now we have

$$w_{a + s u_a}(2a + r u_a) = -(2a + (4s - r) u_a)$$

and therefore $2a + (4s - r) u_a \in S$. But by our choice of s,

$$0 \leqq (4s - r) u_a < r u_a = \mu$$

which contradicts the definition of μ. Hence $r \geqq 3$ is impossible and so $r = 1$ and $\mu = u_a$.

(4) $u_a = a_+ - a \in L(S) \cap F^0$, hence u_a is an integer multiple of c.

From (6.9.1) it follows that u_a depends only on the gradient of a. Hence for each $\alpha \in \Sigma$ we may define u_α to be u_a for any $a \in S$ with $Da = \alpha$. For each $\alpha \in \Sigma$ let

$$\alpha_* = u_\alpha^{-1} \alpha, \qquad \alpha^* = u_\alpha \alpha^\vee = (\alpha_*)^\vee$$

and let

$$\Sigma^* = \{\alpha^* \colon \alpha \in \Sigma\}, \qquad \Sigma_* = \{\alpha_* \colon \alpha \in \Sigma\}.$$

(6.10) **Proposition.** *Σ^* and Σ_* are dual finite root systems in V. If S is reduced then Σ^* and Σ_* are reduced.*

Proof. It is clear that Σ^* is finite, does not contain 0, and spans V. By (6.9.2) and (1.6) we have $u_\alpha = u_{w\alpha}$ for all $\alpha \in \Sigma$ and $w \in W(\Sigma)$, so that Σ^* is stable under the action of $W(\Sigma)$. Next, if $a \in S$ has gradient α, then by (6.8) we have $t_a(b) = b + \langle \alpha^*, \beta \rangle$ for any $b \in S$ with gradient β. Since $t_a(b) \in S$, it follows from (6.9.1) that $\langle \alpha^*, \beta \rangle \in \mathbb{Z} u_\beta$, i.e. that $\langle \alpha^*, \beta_* \rangle \in \mathbb{Z}$. Hence Σ^* is a finite root system in V, and so also is $\Sigma_* = (\Sigma^*)^\vee$.

Finally, suppose that S is reduced. If Σ is reduced it is clear that Σ^* and Σ_* are reduced. If Σ is not reduced, let α and 2α both belong to Σ. Then it follows from (6.9.3) that $u_{2\alpha} = 2u_\alpha$, hence that $(2\alpha)^* = \alpha^*$ and $(2\alpha)_* = \alpha_*$. Hence Σ^* and Σ_* are reduced.

Let Λ be the set of vectors $\lambda \in V$ such that the translation $x \mapsto x + \lambda$ belongs to T, the translation subgroup of $W(S)$. Since T is a free abelian group on l generators, Λ is a lattice in V. For each $\lambda \in \Lambda$ let $t(\lambda) \colon x \mapsto x + \lambda$ be the corresponding element of T. From (6.8) we have $t(\alpha^*) = t_a^{-1}$ if a is an affine root with gradient α. Hence $\alpha^* \in \Lambda$ for all $\alpha \in \Sigma$, and therefore $L(\Sigma^*) \subset \Lambda$.

(6.11) **Proposition.** *If Σ is reduced then $\Lambda = L(\Sigma^*)$.*

Proof. Let T' be the subgroup of T consisting of the translations $t(\lambda)$ where $\lambda \in L(\Sigma^*)$. Let x_i be a vertex of C which is special for S. Then x_i is a special point for $W(S)$ by (6.2.2) and therefore $W(S)$ is the semi-direct product $T \cdot W_i$, where W_i is the subgroup of $W(S)$ which fixes x_i. Also W_i normalizes T', because $w\,t(\lambda)\,w^{-1} = t((Dw)\,\lambda)$ $(w \in W_i, \lambda \in L(\Sigma^*))$. Hence the subgroup of $W(S)$ generated by T' and W_i is the semi-direct product $T' \cdot W_i$.

Now since Σ is reduced, S is reduced and by (6.4) the gradient mapping is an isomorphism of S_i onto Σ. Hence for each $\alpha \in \Sigma$ there is an affine root a_α with gradient α which vanishes at x_i. By (6.9.1) the affine roots are $a_\alpha + k u_\alpha$ for all $\alpha \in \Sigma$ and $k \in \mathbb{Z}$. If $a = a_\alpha + k u_\alpha$, then from (6.8) we have $w_a = t(-k\alpha^*)\, w_{a_\alpha} \in T' \cdot W_i$. Hence $W(S) = T' \cdot W_i$, and therefore $T' = T$, so that $\Lambda = L(\Sigma^*)$.

We shall now give another proof of Theorem (5.2). Let S be a reduced irreducible root system and let Σ' be the set of indivisible roots of $\Sigma = DS$.

Let x_i be a special vertex of C. By (6.4), for each $\alpha \in \Sigma'$ there exists an affine root a_α with gradient α and vanishing at x_i, and by (6.9) the affine roots with gradient α are $a_\alpha + k u_\alpha$ ($k \in \mathbb{Z}$), and those with gradient 2α (if $2\alpha \in \Sigma$) are $2a_\alpha + (2k+1) u_\alpha$ ($k \in \mathbb{Z}$).

Consider the root systems Σ' and Σ_*. They are both reduced and irreducible, and from the proof of (6.10) we have $\Sigma_* = \{u_\alpha^{-1} \alpha : \alpha \in \Sigma'\}$. Hence there are just two possibilities: either (1) Σ_* is similar to Σ' or (2) Σ^* is similar to Σ'.

We shall take case (1) first. Here all the u_α ($\alpha \in \Sigma'$) are equal, and since we are concerned with S only up to similarity we may assume that $u_\alpha = 1$ for all $\alpha \in \Sigma'$. Then the identification of E with V obtained by taking x_i as origin in E clearly identifies $a_\alpha + k$ with $a_{k,\alpha}$ in the notation of (2.1), and hence identifies S with $S(R)^\vee$, where $R = \Sigma^\vee$.

In case (2) all the $u_\alpha / \|\alpha\|^2$ ($\alpha \in \Sigma'$) are equal. Suppose first that Σ is reduced, so that $\Sigma' = \Sigma$. Then the affine root system $S^\vee$ dual to S falls under case (1) above, and S is therefore similar to $S(R)$, where $R = \Sigma$.

Now suppose that Σ is not reduced. If $l = \dim E$ is equal to 1, then $\Sigma' = \{\alpha, -\alpha\}$ and we are still in case (1). If $l \geqq 2$, then Σ is of type BC_l and hence there exist two orthogonal roots $\alpha, \beta \in \Sigma$ of the same length, such that $\alpha \pm \beta$, 2α, 2β all belong to Σ. We may assume that $\|\alpha\| = \|\beta\| = u_\alpha = u_\beta = 1$, whence $u_{\alpha-\beta} = \|\alpha - \beta\|^2 = 2$. If a, b are the affine roots with gradients α, β respectively which vanish at x_i, then $a \pm b \in S$ by (6.4), and $2b - 1 \in S$ by (6.9.3). Now $w_{2b-1}(a+b) = a - b + 1$, hence $a - b + 1 \in S$ and therefore $u_{\alpha-\beta} \leqq 1$, contradicting $u_{\alpha-\beta} = 2$. Hence case (2) cannot arise when Σ is not reduced and of rank $\geqq 2$, and the proof is complete.

Finally, we shall briefly classify the non-reduced irreducible affine root systems. If S is irreducible and not reduced, let S' (resp. S'') be the set of $a \in S$ such that $\frac{1}{2} a \notin S$ (resp. $2a \notin S$). Then S', S'' are reduced systems, and $W(S') = W(S'') = W(S)$.

Since S is not reduced, neither is $\Sigma = DS$. Let Σ' (resp. Σ'') be the set of $\alpha \in \Sigma$ such that $\frac{1}{2}\alpha \notin \Sigma$ (resp. $2\alpha \notin \Sigma$). Then Σ is of type BC_l, Σ' of type B_l and Σ'' of type C_l. Also we have $\Sigma' \subseteq D(S') \subseteq \Sigma$ and $\Sigma'' \subseteq D(S'') \subseteq \Sigma$. Hence either $D(S') = \Sigma$, in which case S' is of type BC_l; or $D(S') = \Sigma'$, in which case S' is of type B_l or $C_l^\vee$. Hence S' is of one of the types BC_l, B_l, $C_l^\vee$, and likewise S'' is of one of the types BC_l, $B_l^\vee$, C_l. But since S' and S'' have the same Weyl group, it follows that if S' is of type B_l then S'' must be of type $B_l^\vee$; and if S' is of type BC_l or $C_l^\vee$ then S'' is of type BC_l or C_l. By examining the various possibilities we conclude that there are four series of irreducible non-reduced affine root systems, for which the types of S', S'' are respectively BC_l, C_l; $C_l^\vee, BC_l$; $B_l, B_l^\vee$; $C_l^\vee, C_l$. The first two are duals of each other; the last two are each self-dual (up to similarity). They are listed in Appendix 2.

7. The Function $s(w)$

In this section S is an irreducible reduced affine root system, C is a chamber of S, and $B=\{a_0, \ldots, a_l\}$ the basis of S determined by C. The vertices of the simplex C are $x_0, \ldots, x_l$, the vertex x_i being opposite the face on which a_i vanishes, so that $a_i(x_j)=0$ whenever $i \neq j$.

(7.1) **Proposition.** *There exists a unique point $r \in E$ at which the functions $a_i^\vee$ $(0 \leqq i \leqq l)$ all take the same value.*

Proof. Let $r \in E$, say $r=\Sigma \lambda_i x_i$ where $\Sigma \lambda_i=1$. Then $a_i^\vee(r)=\lambda_i a_i^\vee(x_i)$ so that $\lambda_i=a_i^\vee(r)/a_i^\vee(x_i)$ and therefore

$$\sum_{i=0}^{l} \frac{a_i^\vee(r)}{a_i^\vee(x_i)}=1.$$

If the $a_i^\vee(r)$ are all equal, let g^{-1} denote their common value. Then we have

$$(7.2)\qquad \begin{aligned} g &= \sum_{i=0}^{l} a_i^\vee(x_i)^{-1} \\ r &= g^{-1} \sum_{i=0}^{l} a_i^\vee(x_i)^{-1} x_i. \end{aligned}$$

Conversely, if r and g are given by (7.2) it is clear that $a_i(r)=g^{-1}$ for $0 \leqq i \leqq l$.

The point r lies in the simplex C, because $a_i^\vee(x_i)>0$ for all i. Its location in C may be described as follows. The gradients $D a_j$ $(j \neq i)$ are a basis of the finite root system Σ_i. Let ρ_i be half the sum of the positive roots of Σ_i relative to this basis. Then we have

(7.3) **Proposition.** $r=x_i+g^{-1}\rho_i$ $(0 \leqq i \leqq l)$.

Proof. Let $v=r-x_i \in V$. Then for any $j=0, \ldots, l$ we have

$$g^{-1}=a_j^\vee(r)=a_j^\vee(x_i)+\langle \alpha_j^\vee, v\rangle,$$

where $\alpha_j=D a_j$. Since $a_j^\vee(x_i)=0$ if $j \neq i$, it follows that $\langle \alpha_j^\vee, g v\rangle=1$ for all $j \neq i$. But also $\langle \alpha_j^\vee, \rho_i\rangle=1$ for all $j \neq i$ ([B], p. 168, Prop. 29(iii)). Hence $g v-\rho_i$ is orthogonal to $\alpha_j^\vee$ for all $j \neq i$, and hence $g v=\rho_i$.

Let S^+ (resp. S^-) be the set of positive (resp. negative) affine roots, and for each $w \in W(S)$ let $S(w)=S^+ \cap w S^-$. Then, for $a \in S$,

$$\begin{aligned} a \in S(w) &\Leftrightarrow a \text{ is positive on } C \text{ and negative on } w C \\ &\Leftrightarrow a>0 \text{ and the hyperplane } H_a \text{ separates } C \text{ and } wC. \end{aligned}$$

By (4.1) it follows that $S(w)$ is a finite set of affine roots (and the number of elements in $S(w)$ is equal to the length $l(w)$ of w as a reduced word in the generators w_a $(a \in B)$). Let

$$s(w) = \sum_{a \in S(w)} a$$

so that $s(w) \in L(S)$, the lattice spanned by S in F.

(7.4) **Lemma.** *Let* $w_1, w_2 \in W(S)$. *Then*

$$s(w_1 w_2) = w_1 s(w_2) + s(w_1).$$

Proof. Let

$$X = S^+ \cap w_1 S^- \cap w_1 w_2 S^+,$$
$$Y = S^+ \cap w_1 S^- \cap w_1 w_2 S^-,$$
$$Z = S^+ \cap w_1 S^+ \cap w_1 w_2 S^-.$$

Then

$$S(w_1) = X \cup Y, \qquad X \cap Y = \emptyset,$$
$$w_1 S(w_2) = (-X) \cup Z, \qquad (-X) \cap Z = \emptyset,$$
$$S(w_1 w_2) = Y \cup Z, \qquad Y \cap Z = \emptyset.$$

(7.4) follows directly from these equations.

Now let $\Phi: E \to R$ be the quadratic function

$$\Phi(x) = \tfrac{1}{2} g \, \|r - x\|^2$$

where r is the point defined in (7.1). For any $w \in W(S)$ let $w\Phi$ denote the function $x \mapsto \Phi(w^{-1}x) = \frac{1}{2} g \, \|wr - x\|^2$. Then we have the following formula for $s(w)$:

(7.5) **Proposition.** $s(w) = w\Phi - \Phi$.

Proof. We shall first verify (7.5) when w is the reflection w_a in a wall H_a of C. In this case $s(w_a) = a$, because H_a is the only hyperplane separating C and $w_a C$. Let $x \in E$ and put $v = x - r \in V$. Then

$$(w_a \Phi)(x) = \Phi(w_a x) = \tfrac{1}{2} g \, \|x - w_a r\|^2$$
$$= \tfrac{1}{2} g \, \|x - r + a^\vee(r) \, D a\|^2$$
$$= \tfrac{1}{2} g \, \|v + g^{-1} D a\|^2,$$

so that

$$(w_a \Phi - \Phi)(x) = \tfrac{1}{2} g (\|v + g^{-1} D a\|^2 - \|t\|^2)$$
$$= \langle v, D a \rangle + \tfrac{1}{2} g^{-1} \|D a\|^2$$
$$= \langle v, D a \rangle + a(r)$$

(because $a(r) = \frac{1}{2} \|a\|^2 a^\vee(r) = \frac{1}{2} g^{-1} \|D a\|^2$).

Hence $(w_a \Phi - \Phi)(x) = a(r + v) = a(x)$, and therefore (7.5) is true for $w = w_a$ $(a \in B)$.

For the general case we proceed by induction on the length $l(w)$ of w. We can write $w = w' w_a$ for some $a \in B$ and $l(w') < l(w)$. Then

$$s(w) = s(w' w_a) = w' s(w_a) + s(w') \qquad \text{by (7.4).}$$
$$= w'(w_a \Phi - \Phi) + (w' \Phi - \Phi) = w \Phi - \Phi$$

by the first part of the proof, and the inductive hypothesis.

(7.6) **Corollary.**

$$s(w)(r) = \tfrac{1}{2} g \, \|r - w r\|^2,$$
$$D s(w) = g(r - w r).$$

Proof. From (7.5) we have, for any $v \in V$

$$s(w)(r + v) = \tfrac{1}{2} g (\|r - w r + v\|^2 - \|v\|^2)$$
$$= \tfrac{1}{2} g \, \|r - w r\|^2 + \langle v, g(r - w r)\rangle$$

from which (7.6) follows.

The following formula will be useful later. As before, let W_i be the subgroup of $W(S)$ which fixes the vertex x_i of C. Let $w_i \in W_i$ and let $\lambda \in \Lambda$, so that the translation $t(\lambda)$: $x \mapsto x + \lambda$ belongs to $W(S)$. Then for any $w \in W(S)$ we have

(7.7) **Proposition.**

$$s(w_i \, t(\lambda) \, w) = \frac{1}{2g} (\|\mu\|^2 - \|\rho_i\|^2),$$
$$D s(w_i \, t(\lambda) \, w) = \rho_i - (D w_i) \, \mu,$$

where $\mu = g \lambda + \rho_i - D s(w)$.

Proof. From (7.5) we have

$$s(w_i \, t(\lambda) \, w)(x_i) = \tfrac{1}{2} g (\|r - (w_i \, t(\lambda) \, w)^{-1} x_i\|^2 - \|r - x_i\|^2)$$

and

$$r - (w_i \, t(\lambda) \, w)^{-1} x_i = r - w^{-1}(x_i - \lambda)$$
$$= w^{-1}(w r - x_i + \lambda)$$
$$= w^{-1}(r - g^{-1} D s(w) - x_i + \lambda) \qquad \text{by (7.6)}$$
$$= \frac{1}{g} w^{-1}(g \lambda + \rho_i - D s(w)) \qquad \text{by (7.3)}$$
$$= \frac{1}{g} w^{-1} \mu.$$

Hence $s(w_i \, t(\lambda) \, w)(x_i) = \dfrac{1}{2g} (\|\mu\|^2 - \|\rho_i\|^2)$, by (7.3) again.

Next, by (7.4) we have $s(w_i t(\lambda) w) = w_i t(\lambda) s(w) + s(w_i t(\lambda))$ so that

$$\begin{aligned} D\,s(w_i t(\lambda)\,w) &= D(w_i t(\lambda))\,D\,s(w) + D\,s(w_i t(\lambda)) \\ &= D\,w_i \cdot D\,s(w) + g(r - w_i t(\lambda)\,r) \quad \text{by (7.6)} \\ &= D\,w_i \cdot D\,s(w) + g(r - w_i(r+\lambda)). \end{aligned}$$

But

$$\begin{aligned} r - w_i r &= (x_i + g^{-1}\rho_i) - w_i(x_i + g^{-1}\rho_i) \quad \text{by (7.3)} \\ &= g^{-1}(\rho_i - (D\,w_i)\,\rho_i), \quad \text{since } w_i(x_i) = x_i. \end{aligned}$$

Hence

$$D\,s(w_i t(\lambda)\,w) = \rho_i - (D\,w_i)(g\,\lambda + \rho_i - D\,s(w)).$$

There is another expression for $s(w)$ which is also useful. For each $\alpha \in \Sigma$ let a_α be the smallest positive affine root with gradient α (so that a_α is positive and $a_\alpha - u_\alpha$ is negative). Define a quadratic function Ψ on E by

$$\Psi(x) = \tfrac{1}{4} \sum_{\alpha \in \Sigma} u_\alpha^{-1}\, a_\alpha(x)^2.$$

(7.8) **Proposition.** $s(w) = w\,\Psi - \Psi$.

Proof. As in the case of (7.5) we shall first verify this formula for $w = w_a$, where $a \in B$. If $\alpha = D\,a$ then $a_\alpha = a$ and $a_{-\alpha} = -a + u_\alpha$, so that $w_a(a_\alpha) = -a$, $w_a(a_{-\alpha}) = a + u_\alpha$, and the a_β for $\beta \neq \pm\alpha$ are permuted by w_a. Hence

$$\begin{aligned} w_a\,\Psi - \Psi &= \tfrac{1}{4} u_\alpha^{-1}\left((a+u_\alpha)^2 - (-a+u_\alpha)^2\right) \\ &= a = s(w_a) \end{aligned}$$

so that (7.8) is true for $w = w_a$. The rest of the proof is the same as in (7.5).

From (7.5) and (7.8) it follows that $\Phi - \Psi$ is invariant under all $w \in W(S)$, and is therefore a constant. On the other hand, $\Phi(r) = 0$, and therefore $\Phi(r+v) = \Psi(r+v) - \Psi(r)$ for all $v \in V$, so that

$$\begin{aligned} \tfrac{1}{2} g\,\|v\|^2 &= \tfrac{1}{4} \sum_{\alpha \in \Sigma} u_\alpha^{-1}\left(a_\alpha(r+v)^2 - a_\alpha(r)^2\right) \\ &= \tfrac{1}{4} \sum_{\alpha} u_\alpha^{-1}\left(2\,a_\alpha(r)\,\langle \alpha, v\rangle + \langle \alpha, v\rangle^2\right). \end{aligned}$$

Replacing v by $-v$ and adding, we obtain

$$\sum_{\alpha \in \Sigma} u_\alpha^{-1}\,\langle \alpha, v\rangle^2 = 2g\,\|v\|^2,$$

and by linearizing this identity

$$\sum_{\alpha \in \Sigma} \langle u_\alpha^{-1}\alpha, u\rangle\,\langle \alpha, v\rangle = 2g\,\langle u, v\rangle$$

for all $u, v \in V$. Letting u vary, we deduce that

(7.9) $$\sum_{\alpha \in \Sigma^+} \langle \alpha_*, v \rangle \alpha = g v$$

where Σ^+ denotes the set of positive roots in Σ relative to some basis.

Let Λ be the lattice in V determined by the translation subgroup of $W(S)$, as in § 6. Then if $\lambda \in \Lambda$ and $a \in S$ we have

$$t(\lambda)(a) = a - \langle \lambda, \alpha \rangle$$

where $\alpha = Da$; since $t(\lambda)(a) \in S$, it follows from (6.9) that $\langle \lambda, \alpha \rangle \in \mathbb{Z} u_\alpha$, so that $\langle \lambda, \alpha_* \rangle \in \mathbb{Z}$ for all $\alpha \in \Sigma$. Hence from (7.9) with $v = \lambda$ we see that

(7.10) $$g \Lambda \subset L(\Sigma)$$

where $L(\Sigma)$ is the lattice in V generated by Σ.

We shall use (7.9) to compute the constant g. For this purpose we need the following lemma:

(7.11) **Lemma.** *Let R be a finite root system and let $\alpha, \beta \in R$ be such that $\|\alpha\| \leqq \|\beta\|$. Then $|\langle \alpha, \beta^\vee \rangle| \leqq 1$ unless $\beta = \pm \alpha$.*

Proof. We have
$$\langle \alpha, \beta^\vee \rangle = \frac{2\langle \alpha, \beta \rangle}{\|\alpha\| \cdot \|\beta\|} \cdot \frac{\|\alpha\|}{\|\beta\|}.$$

Since $|\langle \alpha, \beta \rangle| \leqq \|\alpha\| \cdot \|\beta\|$, with equality if and only if α, β are proportional, it follows that $|\langle \alpha, \beta^\vee \rangle| \leqq 2$, with equality if and only if α and β are proportional and $\|\alpha\| = \|\beta\|$, i.e. if and only if $\beta = \pm \alpha$. Since $\langle \alpha, \beta^\vee \rangle$ is an integer, the result follows.

Choose a basis of Σ, and let $\phi \in \Sigma$ be such that $\phi_* = u_\phi^{-1} \phi$ is the highest root of Σ_*. Then neither $\frac{1}{2}\phi$ nor 2ϕ belong to Σ, so that ϕ is uniquely determined. (This is clear if Σ is reduced. If Σ is not reduced, let Σ' be the subsystem of Σ consisting of the indivisible roots. By (5.2) the u_α for $\alpha \in \Sigma'$ are all equal, hence Σ_* is similar to Σ', so that ϕ is the highest root of Σ'. But Σ is of type BC_l and Σ' is of type B_l, hence neither $\frac{1}{2}\phi$ nor 2ϕ belong to Σ.)

The vector ϕ_* lies in the positive chamber for Σ, hence $\langle \alpha_*, \phi^* \rangle \geqq 0$ for all $\alpha \in \Sigma^+$. By (7.11) applied to the root system Σ_* it follows that $\langle \alpha_*, \phi^* \rangle$ is equal to 0 or 1 for all $\alpha \neq \phi$ in Σ^+.

Let π be the sum of the positive roots of Σ not orthogonal to ϕ. From (7.9) we have
$$g \phi^* = \sum_{\alpha \in \Sigma^+} \langle \alpha_*, \phi^* \rangle \alpha = \phi + \pi.$$

On the other hand,
$$\pi = \rho - w_\phi \rho = \langle \rho, \phi \rangle \phi^\vee$$
where ρ is half the sum of the positive roots of Σ. Hence
$$g\phi^* = \phi + \langle \rho, \phi \rangle \phi^\vee,$$
so that

(7.12)
$$\begin{aligned} g &= u_\phi^{-1}(\tfrac{1}{2}\|\phi\|^2 + \langle \rho, \phi \rangle) \\ &= (2u_\phi)^{-1}(\|\phi+\rho\|^2 - \|\rho\|^2). \end{aligned}$$

Suppose that $S = S(R)$, where R is an irreducible finite root system. Then $\Sigma = R$ and $u_\alpha = 1$ for all $\alpha \in \Sigma'$, so that $\Sigma_* = \Sigma'$. Hence ϕ is the highest indivisible root of R, and $u_\phi = 1$, so that $g = \frac{1}{2}(\|\phi+\rho\|^2 - \|\rho\|^2)$ by (7.12). Also if R is *reduced* it is clear from (7.9) that $(2g)^{-1}\langle u, v \rangle$ is the canonical bilinear form on R ([B], p. 172).

If on the other hand $S = S(R)^\vee$, where R is irreducible and reduced, then $\Sigma = R^\vee$ and $u_\alpha = \frac{1}{2}\|\alpha\|^2$ for all $\alpha \in \Sigma$, so that $\alpha_* = \alpha^\vee$ and therefore $\Sigma_* = \Sigma^\vee = R$. Hence by (7.12) we have $g = 1 + \langle \phi^\vee, \rho \rangle$, where $\phi^\vee$ is the highest root of R. But $\langle \phi^\vee, \rho \rangle = h - 1$, where h is the Coxeter number of R ([B], p. 169, Prop. 31). Hence in this case $g = h$.

To summarize:

(7.13) **Proposition.** (1) *If $S = S(R)$ where R is an irreducible finite root system, then*
$$g = \tfrac{1}{2}(\|\phi+\rho\|^2 - \|\rho\|^2),$$
where ϕ is the highest indivisible root in R relative to some basis, and ρ is half the sum of the positive roots of R. If R is reduced, then $\Phi_R(u, v) = (2g)^{-1}\langle u, v \rangle$ is the canonical bilinear form on V.

(2) *If $S = S(R)^\vee$, where R is irreducible and reduced, then g is equal to the Coxeter number h of R.*

8. The Main Theorem

Let S be a reduced irreducible affine root system and $L(S)$ the lattice in F generated by S. For each $f \in L(S)$ let e^f denote the corresponding element of the integral group ring $\mathbb{Z}[L(S)]$, so that for all $f, g \in L(S)$ we have
$$e^f \cdot e^g = e^{f+g}, \qquad (e^f)^{-1} = e^{-f}, \qquad e^0 = 1.$$
We shall sometimes write $\exp(f)$ in place of e^f, when the exponent f is a complicated expression.

We write $f \geqq 0$ to mean that f takes values $\geqq 0$ on the chamber C. This will be the case if and only if $f = \sum_{i=0}^{l} m_i a_i$ with all coefficients $m_i \geqq 0$.

Let A be the ring of all formal infinite series $\sum_{f \geq 0} a_f e^{-f}$, with coefficients $a_f \in \mathbb{Z}$ and the obvious definitions of addition and multiplication. Alternatively, A is the formal power series ring $\mathbb{Z}[[e^{-a_0}, \ldots, e^{-a_l}]]$ in $l+1$ analytically independent variables e^{-a_i}.

We propose to compare the infinite product

$$\Pi = \prod_{a>0} (1-e^{-a})$$

taken over all the positive affine roots, with the infinite series

$$\Delta = \sum_{w \in W(S)} \varepsilon(w) e^{-s(w)}$$

where $\varepsilon(w) = (-1)^{l(w)}$ is the signature of w, and as in § 7 $s(w)$ is the sum of the positive affine roots $a \in S$ such that $w^{-1} a$ is negative. Both Π and Δ are elements of A.

Our main theorem is the following formal identity:

(8.1) **Theorem.** *Let S be a reduced affine root system. Then*

(8.1.1) $$\sum_{w \in W(S)} \varepsilon(w) e^{-s(w)} = P \cdot \prod_{a>0} (1-e^{-a})$$

where

(8.1.2) $$P = \prod_{n=1}^{\infty} \prod_{\alpha \in B(\Sigma)} (1-e^{-n u_\alpha}).$$

Here $B(\Sigma)$ is any basis of the gradient root system $DS = \Sigma$, and u_α is defined as follows (§ 6): if $a \in S$ has gradient $Da = \alpha$, then u_α is the least positive real number such that $a + u_\alpha \in S$.

In the statement of (8.1) we have not assumed that S is irreducible. If S is reducible, both sides of (8.1.1) factorize concordantly with the decomposition of S into a direct sum of irreducible subsystems (§ 3). Hence we shall continue to assume that S is irreducible (and reduced).

The proof of Theorem (8.1) will be given in §§ 9–12. In this section we shall calculate the two sides of (8.1) more explicitly, using the formulae established in § 7, and we shall then specialize the resulting identity in various ways.

Choose a vertex x_i of the simplex C, and take x_i as origin in E. Then the affine space E is identified with the vector space V: namely a point $x \in E$ is identified with the vector $x - x_i \in V$. The affine-linear functions on E which vanish at x_i are then identified with linear forms on V, and hence with elements of V via the identification (§ 1) of V with its dual space. If $f \in F$ vanishes at x_i, then f is identified with $Df \in V$. In particular, the root systems S_i and $\Sigma_i = DS_i$ are identified, and the subgroup W_i of

$W(S)$ which fixes x_i is identified with $W(\Sigma_i)$, by virtue of (5.1.3). The affine Weyl group $W(S)$ is identified with the semi-direct product of $W(\Sigma)$ and the translation group T.

More generally, any $f\in F$ is identified with the function $v\mapsto f(x_i)+\langle Df, v\rangle$ on V. We shall denote this function by $f(x_i)+Df$.

As in § 7, let ρ_i denote half the sum of the positive roots of Σ_i, relative to the basis $\{Da_j: j\neq i\}$. Let $w_i\in W(\Sigma_i)$, $\lambda\in\Lambda$ and $w\in W(S)$. Then by (7.7) and the identifications we have just described, we have

$$s(w_i\, t(\lambda)\, w)=\frac{1}{2g}(\|\mu\|^2-\|\rho_i\|^2)+\rho_i-w_i\mu, \tag{8.2}$$

where $\mu=g\lambda-Ds(w)+\rho_i$.

Now let W^i be a set of right coset representatives of $W(\Sigma_i)$ in $W(\Sigma)$. Then every element w of $W(S)$ is uniquely expressible as $w=w_i\, t(\lambda)\, w^i$ with $w_i\in W_i$, $\lambda\in\Lambda$ and $w^i\in W^i$.

Hence

$$\Delta=\sum_{w_i,\,\lambda,\,w^i}\varepsilon(w_i)\,\varepsilon(w^i)\,e^{-s(w_i t(\lambda) w^i)}$$

and therefore by (8.2)

$$\Delta=e^{-\rho_i}\sum_{w^i}\varepsilon(w^i)\sum_{\lambda}J_i(\mu)\exp-\frac{1}{2g}(\|\mu\|^2-\|\rho_i\|^2) \tag{8.3}$$

where $\mu=g\lambda-Ds(w^i)+\rho_i$ and

$$J_i(v)=\sum_{w\in W(\Sigma_i)}\varepsilon(w)\,e^{wv}$$

for any v in the weight lattice of Σ_i.

Next consider the product Π. Take first the factors $(1-e^{-a})$ in Π for which $a(x_i)=0$. These form the product $\Pi(1-e^{-\alpha})$ taken over the positive roots of Σ_i, which by Weyl's identity (0.1) is equal to $e^{-\rho_i}J_i(\rho_i)$. The remaining factors in Π are $(1-e^{-a})$ for all $a\in S$ such that $a(x_i)>0$ (for if $a(x_i)>0$, the root a must be positive). Hence Theorem (8.1) takes the following form:

$$\begin{aligned} &P\cdot\prod_{a(x_i)>0}(1-e^{-a})\\ &=e^{\|\rho_i\|^2/2g}\sum_{w^i}\varepsilon(w^i)\sum_{\lambda\in\Lambda}\chi_i(g\lambda-Ds(w^i))\exp-\frac{1}{2g}\|g\lambda-Ds(w^i)+\rho_i\|^2\end{aligned} \tag{8.4}$$

where

$$\chi_i(v)=J_i(v+\rho_i)/J_i(\rho_i) \tag{8.5}$$

is Weyl's character formula for the root system Σ_i.

If x_i is a special point for $W(S)$, then $W(\Sigma_i)=W(\Sigma)$ and so (8.3) and (8.4) take the simpler forms

$$\text{(8.3')}\qquad \Delta=e^{-\rho_i}\sum_{\lambda\in\Lambda} J_i(g\lambda+\rho_i)\exp-\frac{1}{2g}(\|g\lambda+\rho_i\|^2-\|\rho_i\|^2),$$

$$\text{(8.4')}\qquad P\cdot\prod_{a(x_i)>0}(1-e^{-a})=e^{\|\rho_i\|^2/2g}\sum_{\lambda\in\Lambda}\chi_i(g\lambda)\exp-\frac{1}{2g}\|g\lambda+\rho_i\|^2.$$

By (5.2), S is similar to either $S(R)$ or $S(R)^\vee$, where R is an irreducible finite root system. Suppose first that $S=S(R)$, where R is *reduced.* Choose a basis of R, which determines a basis of S as in (4.7), and take x_i to be the origin in V (in other words $i=0$). This is a special point for S, and therefore also for $W(S)$. We have $\rho_i=\rho$, half the sum of the positive roots of R. Let

$$d=l+\operatorname{card}(R)$$

which is the dimension of a compact Lie group having R as its system of roots relative to a maximal torus. By (7.13), $\Phi_R(u,v)=(2g)^{-1}\langle u,v\rangle$ is the canonical bilinear form on V defined by R. Now there is the following "strange formula":

$$\text{(8.5)}\qquad \Phi_R(\rho,\rho)=d/24$$

(see Freudenthal and de Vries [2], p. 243, where it is deduced from Weyl's character formula for the adjoint representation). Moreover, each u_α is equal to 1, and the affine roots which are positive at the origin are $n+\alpha$ for $n\geqq 1$ and all $\alpha\in R$. Hence, writing $X=\exp(-1)$, the left-hand side of (8.4') is seen to be equal to $\prod_{n=1}^{\infty} p(X^n)$, where

$$\text{(8.6)}\qquad p(X)=(1-X)^l\prod_{\alpha\in R}(1-Xe^\alpha)$$

is in a formal sense the characteristic polynomial of the adjoint representation of G.

Moreover, from (7.9) we have

$$g\lambda=\sum_{\alpha>0}\langle\alpha,\lambda\rangle\,\alpha.$$

Also by (6.11) we have $\Lambda=L(R^\vee)$, the lattice spanned by the dual root system $R^\vee$. Hence $M=g\Lambda$ is the image of $L(R^\vee)$ in $L(R)$ under the mapping $\lambda\mapsto\sum_{\alpha>0}\langle\alpha,\lambda\rangle\,\alpha$.

The identity (8.4′) now takes the following form:

(8.7) **Theorem.** *Let R be the system of roots of a compact Lie group G of dimension d relative to a maximal torus. Let Φ be the canonical bilinear form associated with R, and let M be the sublattice of $L(R)$ defined above. Then*

$$\sum_{\mu\in M}\chi(\mu)\,X^{\Phi(\mu+\rho,\mu+\rho)}=X^{d/24}\prod_{n=1}^{\infty}p(X^n),$$

where $p(X)$ is the characteristic polynomial (8.6) *of the adjoint representation of G, and $\chi(\mu)=J(\mu+\rho)/J(\rho)$, where ρ is half the sum of the positive roots of R relative to some basis, and $J(v)=\sum_{w\in W(R)}\varepsilon(w)\,e^{wv}$ for any v in the weight lattice of R.*

Both sides of (8.7) may be regarded as functions on G with values in the power series ring $\mathbb{C}[[X]]$. Let us evaluate both sides at the identity element of G. Then $\chi(\mu)$ is replaced by

$$d(\mu)=\prod_{\alpha\in R^+}\frac{\langle\mu+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} \tag{8.8}$$

and $p(X)$ becomes simply $(1-X)^d$. Hence, introducing Dedekind's η-function

$$\eta(X)=X^{1/24}\prod_{n=1}^{\infty}(1-X^n)$$

we obtain from (8.7) the formula

$$\sum_{\mu\in M}d(\mu)\,X^{\Phi(\mu+\rho,\mu+\rho)}=\eta(X)^d. \tag{8.9}$$

Next, take $S=S(R^\vee)^\vee$ with R reduced and irreducible. This time $g=h$, the Coxeter number of R (7.13), and $\Sigma=R$. We have $u_\alpha=\frac{1}{2}\|\alpha\|^2$ for all $\alpha\in R$, so that $\alpha^*=u_\alpha\alpha^\vee=\alpha$ and therefore (6.11) $\Lambda=L(R)$. Assume that $\|\alpha\|^2\in\mathbb{Z}$ for all $\alpha\in R$. Then, writing $X=\exp(-\frac{1}{2})$, the left hand side of (8.4′) is equal to $\prod_{n=1}^{\infty}q(X^n)$, where

$$q(X)=\prod_{\beta\in B(R)}(1-X^{\|\beta\|^2})\prod_{\alpha\in R}(1-X^{\|\alpha\|^2}e^\alpha), \tag{8.10}$$

$B(R)$ being any basis of R. Hence in this case the identity (8.1) takes the following form:

(8.11) **Theorem.** *Let R be a reduced irreducible root system, $q(X)$ the polynomial defined by* (8.10). *Then*

$$\sum_{\lambda\in L(R)}\chi(h\lambda)\,X^{h\|\lambda\|^2+2\langle\lambda,\rho\rangle}=\prod_{n=1}^{\infty}q(X^n).$$

As before, we shall specialize this identity by mapping each e^{α} to 1. Since the action on R of a Coxeter element of the Weyl group is to partition R into l orbits, each containing h roots and each containing exactly one root from the basis $B(R)$ ([B], p. 170, Prop. 33), it follows that $q(X)$ specializes to the polynomial

$$\prod_{\beta\in B(R)} (1-X^{\|\beta\|^2})^{h+1}.$$

On the other hand the exponent of X on the left-hand side in (8.10) is equal to $h^{-1}(\|h\lambda+\rho\|^2-\|\rho\|^2)$. Now we have another strange formula:

(8.12) $$\|\rho\|^2=\frac{h(h+1)}{24}\sum_{\beta\in B(R)}\|\beta\|^2.$$

Assuming this for the moment, we obtain from (8.10)

(8.13) $$\sum_{\lambda\in L(R)} d(h\lambda)\, X^{h^{-1}\|h\lambda+\rho\|^2}=\prod_{\beta\in B(R)}\eta(X^{\|\beta\|^2})$$

where as before $\eta(X)$ is Dedekind's η-function.

To prove (8.12), we may assume that the bilinear form $\langle u, v\rangle$ is the canonical bilinear form $\Phi_R(u, v)$, i.e. that

$$\sum_{\alpha\in R}\langle\alpha, u\rangle\langle\alpha, v\rangle=\langle u, v\rangle.$$

This implies that the matrix $(\langle\alpha,\beta\rangle)_{\alpha,\beta\in R}$ is idempotent: since its rank is l, it follows that its trace is equal to l, i.e. that

$$\sum_{\alpha\in R}\|\alpha\|^2=l.$$

Hence, considering the orbits of the action of a Coxeter element as before, we see that

$$\sum_{\beta\in B(R)}\|\beta\|^2=l/h.$$

On the other hand, by (9.5),

$$\|\rho\|^2=d/24=l(h+1)/24$$

and (8.12) is proved.

Finally, we shall make one other specialization of the identity (8.11). Let $\omega=\exp(2i\pi/h)$ be a primitive h-th root of unity (here of course exp is the usual complex exponential). We shall specialize each e^{α} to $\omega^{\langle\alpha,\sigma\rangle}$, where σ is half the sum of the positive roots of the dual root system $R^{\vee}$. We recall that $\langle\alpha,\sigma\rangle$ is the *height* of $\alpha\in R$; in particular it is an integer. Consider first the effect of the specialization on $\chi(h\lambda)=J(h\lambda+\rho)/J(\rho)$.

Now $J(h\lambda+\rho)$ specializes to

$$\sum_{w\in W(R)} \varepsilon(w)\,\omega^{\langle w(h\lambda+\rho),\sigma\rangle}$$

and since $\langle\lambda,\sigma\rangle\in\mathbb{Z}$ for $\lambda\in L(R)$, we have $\omega^{\langle w(h\lambda+\rho),\sigma\rangle}=\omega^{\langle w\rho,\sigma\rangle}$. Hence $\chi(h\lambda)$ specializes to 1.

Next consider the polynomial $q(X)$ defined by (8.10). We need the following result:

(8.14) **Proposition.** *Let R be an irreducible reduced finite root system of rank l, with Coxeter number h. For each $p=1,\ldots,h$ let η_p be the number of roots of height p in R, relative to some basis of R. Then $\eta_p+\eta_q=l$ if $p+q=h+1$.*

Proof. This is an easy consequence of the following two facts:

(a) If $m_1\geqq m_2\geqq\cdots\geqq m_l$ are the exponents of R, then $m_i+m_j=h$ if $i+j=l+1$;

(b) the partitions $(m_1,\ldots,m_l)$ and $(\eta_1,\ldots,\eta_{h-1})$ are conjugate (in other words, if Γ is the set of points $(i,j)\in\mathbb{Z}^2$ such that $1\leqq j\leqq m_i$, then Γ is also the set of $(i,j)\in\mathbb{Z}^2$ such that $1\leqq i\leqq\eta_j$).

For a proof of (a) see [B], p. 118; for (b) see [4] or [5].

Assume now that $\|\alpha\|=1$ for all $\alpha\in R$. Then the polynomial $q(X)$ specializes to

(8.15) $$(1-X)^l\prod_{\alpha\in R}(1-\omega^{\langle\alpha,\sigma\rangle}X).$$

It follows from (8.14) that, for $1\leqq p\leqq h-1$,

$$\eta_p+\eta_{h-p}=\begin{cases} l & \text{if } p \text{ is not an exponent } m_i\\ l+1 & \text{if } p \text{ is an exponent.}\end{cases}$$

For $\eta_p=\eta_{p-1}$ if p is not an exponent, and $\eta_p=\eta_{p-1}+1$ if p is an exponent. Hence (8.15) is equal to

$$(1-X^h)\prod_{i=1}^{l}(1-\omega^{m_i}X)=(1-X^h)^l c(X),$$

where $c(X)$ is the characteristic polynomial of a Coxeter element of $W(R)$. Hence

(8.16) **Theorem.** *Let R be a reduced irreducible finite root system such that $\|\alpha\|=1$ for all $\alpha\in R$. Then*

$$\sum_{\lambda\in L(R)} X^{h^{-1}\|h\lambda+\rho\|^2}=\eta(X^h)^l\cdot X^{l/24}\prod_{n=1}^{\infty}c(X^n)$$

$$=\eta(X^h)^l\prod_{i=1}^{l}\eta(\omega_i X)$$

where $c(X)$ is the characteristic polynomial and $\omega_1, \ldots, \omega_l$ the eigenvalues of a Coxeter element of $W(R)$.

When R contains roots of different lengths, the formula corresponding to (8.16) is more complicated, and we shall not reproduce it here.

9. Proof of the Main Theorem

The proof of Theorem (8.1) will occupy §§ 9–12. As we remarked in § 8, we may assume that S is irreducible. Let

$$X = e^{-c} = \prod_{i=0}^{l} e^{-n_i a_i} \tag{9.1}$$

where c is the constant defined in (6.7). By (6.9.4) it follows that $e^{-n u_\alpha}$ is a positive integral power of X, for all $\alpha \in \Sigma$, and therefore the product P defined in (8.1.2) belongs to $\mathbb{Z}[[X]]$. The first stage in the proof of (8.1) is to show that (8.1.1) holds for *some* $P \in \mathbb{Z}[[X]]$. The proof that P is given by the product (8.1.2) is then achieved by specializing (8.1.1) in two different ways (in §§ 10 and 11) and comparing the results.

When multiplied out, the product $\Pi = \prod_{a>0} (1 - e^{-a})$ is of the form

$$\Pi = \sum_{f \geqq 0} a_f e^{-f} \tag{9.2}$$

with coefficients $a_f \in \mathbb{Z}$. Consider the effect of transforming Π by an element w of $W(S)$:

$$w\Pi = \prod_{a>0} (1 - e^{-wa}).$$

For each affine root $a > 0$ such that $wa < 0$ we write

$$1 - e^{-wa} = -e^{-wa}(1 - e^{wa})$$

and then it is clear that

$$w\Pi = \varepsilon(w)\, e^{-\Sigma wa}\, \Pi,$$

the sum in the exponent being over all $a > 0$ such that $wa < 0$. Writing $wa = -b$, we have $b > 0$ and $w^{-1} b < 0$, so that $-\Sigma\, wa = s(w)$, and therefore

$$w\Pi = \varepsilon(w)\, e^{s(w)}\, \Pi.$$

Hence from (9.2) we obtain

$$a_f = \varepsilon(w)\, a_{wf + s(w)}. \tag{9.3}$$

Define a (non-linear) action of the Weyl group $W(S)$ on the space F of affine-linear functions on E, as follows:

$$w \circ f = wf + s(w) = w(f + \Phi) - \Phi \tag{9.4}$$

by (7.6). In this notation (9.3) takes the form

(9.5) $$a_{w \circ f} = \varepsilon(w)\, a_f.$$

We shall now describe this action of $W(S)$ on F more concretely. Define $\psi: F \to E$ by the rule

$$\psi(f) = r - g^{-1} Df$$

where r is the point in the simplex C defined in § 7. Clearly ψ is surjective and its fibres are the cosets of F^0 (the line of constant functions) in F. The action of $W(S)$ on F defined by (9.4) passes to the quotient F/F^0, for if $f_1 - f_2$ is constant then so is $w \circ f_1 - w \circ f_2$. Hence it induces an action of $W(S)$ on E, and this action is just the usual one: namely

(9.6) $$\psi(w \circ f) = w(\psi(f)).$$

For

$$\begin{aligned} \psi(w \circ f) &= \psi(wf + s(w)) \\ &= r - g^{-1}(Dw \cdot Df + Ds(w)) \\ &= r - g^{-1} Dw \cdot Df - (r - wr) \qquad \text{by (7.6)} \\ &= w(r - g^{-1} Df) = w(\psi(f)). \end{aligned}$$

From (9.6) it follows that the action (9.4) of $W(S)$ on F is that of a reflection group, the reflecting hyperplanes being all parallel to the line F^0. Hence, by a basic property of reflection groups:

(9.7) **Lemma.** *For each $f \in F$ the group*

$$W_f = \{w \in W(S):\ w \circ f = f\}$$

is generated by the reflections it contains.

For each $f \in F$ let

$$\Delta_f = \sum_{w \in W(S)} \varepsilon(w)\, e^{-w \circ f}.$$

(9.8) **Lemma.** $\Delta_f \neq 0$ *if and only if* $W_f = \{1\}$.

Proof. Suppose $\Delta_f = 0$. Then the term e^f in Δ_f must cancel with $\varepsilon(w)\, e^{w \circ f}$ for some $w \neq 1$. Hence $f = w \circ f$ (and $\varepsilon(w) = -1$) for some $w \neq 1$ in $W(S)$, and therefore $W_f \neq \{1\}$.

Conversely, if $W_f \neq \{1\}$ then by (9.7) there exists a reflection $w \in W(S)$ such that $w \circ f = f$. Hence $\Delta_f = \Delta_{w \circ f} = \varepsilon(w)\, \Delta_f = -\Delta_f$ and therefore $\Delta_f = 0$.

From (9.2) and (9.5) we have

(9.9) $$\Pi = \sum_f a_f \Delta_f$$

where the sum is over a set of representatives of the orbits of $W(S)$ for the action (9.4) which intersect $L(S)^+ = \{f \in L(S): f \geqq 0\}$. By (9.8) we need consider only those orbits on which $W(S)$ acts faithfully, and these are identified by

(9.10) **Lemma.** *The only orbits in $L(S)$ on which $W(S)$ acts faithfully (for the action* (9.4)*) are those which intersect the line F^0 of constant functions.*

Proof. By (9.6) we may equivalently consider the orbit of $r - g^{-1} Df$ in E under the usual action of $W(S)$. If W acts faithfully on this orbit then we may assume that the representative f of the orbit is such that $r - g^{-1} Df \in C$, i.e. such that $a_i^\vee(r - g^{-1} Df) \geqq 0$ for $0 \leqq i \leqq l$. But

$$a_i^\vee(r - g^{-1} Df) = a_i^\vee(r) - g^{-1}\langle \alpha_i^\vee, Df\rangle$$
$$= g^{-1}(1 - \langle \alpha_i^\vee, Df\rangle)$$

by (7.1), where α_i is the gradient of a_i. Hence $\langle \alpha_i^\vee, Df\rangle < 1$, hence is $\leqq 0$, because it is an integer. But from (6.5) there is a relation of the form $\sum_{i=0}^{l} m_i \alpha_i^\vee = 0$, in which each coefficient m_i is strictly positive. Hence

$$\sum_{i=0}^{l} m_i \langle \alpha_i^\vee, Df\rangle = 0$$

and therefore $\langle \alpha_i^\vee, Df\rangle = 0$ for $0 \leqq i \leqq l$, so that $Df = 0$.

From (9.8), (9.9) and (9.10) we have

$$\Pi = \sum_{n=0}^{\infty} a_{nc} \Delta_{nc},$$

and $w \circ (nc) = nc + s(w)$, so that

$$\Delta_{nc} = e^{-nc} \Delta = X^n \Delta.$$

Hence $\Pi = \sum_{n=0}^{\infty} a_{nc} X^n \Delta$, or say

$$\Pi = Q \Delta \tag{9.11}$$

where $Q = \sum_{n=0}^{\infty} a_{nc} X^n \in \mathbb{Z}[[X]]$.

Since $a_0 = 1$, Q is a unit in $\mathbb{Z}[[X]]$, and it remains therefore to show that $Q^{-1} = P$, where P is the product (8.1.2). We shall do this by specializing (9.11) in two different ways, in §§ 10 and 11. For the purposes of the calculations it seems to be necessary to assume that the gradient system Σ is *reduced*. The case where Σ is not reduced (i.e. where S is of type BC_l) will be dealt with separately in § 12.

10. First Specialization

Until further notice we shall assume that S is irreducible and that $\Sigma = DS$ is *reduced.* We can assume that the vertex x_0 of the simplex is a special point for S (§6). As in §8 we shall identify the affine space E with the vector space V by taking x_0 as origin in E. Then the root systems S_0 and Σ_0 are identified, and $\Sigma_0 = \Sigma$ (6.4). The gradients $\alpha_i = D a_i$ $(1 \leqq i \leqq l)$ form a basis of Σ. This choice of basis defines positive and negative roots for Σ, and hence also for Σ^* and Σ_*. The positive affine roots are

$$\alpha + n u_\alpha, \qquad -\alpha + (n+1) u_\alpha$$

where $\alpha \in \Sigma^+$ and n is any integer $\geqq 0$. In particular, the root $a_j \in B$ is now identified with α_j $(1 \leqq j \leqq l)$; also a_0 is identified with $\alpha_0 + u_{\alpha_0}$. Since $n_0 = 1$ in (6.7) it follows that $u_{\alpha_0} = c$. Hence $e^{-a_0} = X e^{-\alpha_0}$.

Let ρ^* be half the sum of the positive roots of Σ^*. If $\alpha \in \Sigma^+$, then $\langle \alpha, \rho^* \rangle = u_\alpha \langle \alpha_*, \rho^* \rangle$ is a positive integral multiple of u_α, and hence by (6.9.4), $e^{-\langle \alpha, \varrho^* \rangle}$ is a positive integral power of $X = e^{-c}$.

Define a homomorphism θ of $A = \mathbb{Z}[[e^{-a_0}, \ldots, e^{-a_l}]]$ into $\mathbb{Z}[[X]]$ as follows:

$$\theta(e^{-\alpha}) = e^{-\langle \alpha, \varrho^* \rangle} \quad (\alpha \in \Sigma^+),$$

$$\theta(X) = X^{h+1}$$

where as before h is the Coxeter number of Σ. (To show that θ does map A into $\mathbb{Z}[[X]]$ we observe that $\theta(e^{-a_0}) = \theta(X e^{-\alpha_0}) = X^{h+1} e^{-\langle \alpha_0, \varrho^* \rangle}$, and

$$\langle \alpha_0, \rho^* \rangle = u_{\alpha_0} \langle \alpha_{0*}, \rho^* \rangle \leqq c(h-1)$$

because the height of a root of Σ_* is at most $h-1$. Consequently $\theta(e^{-a_0}) = X^m$ where m is an integer $\geqq 2$.)

We have to calculate $\theta(\Delta)$ and $\theta(\Pi)$. Consider $\theta(\Pi)$ first. It is the product of factors

$$(1 - X_\alpha^{\langle \alpha_*, \varrho^* \rangle + n(h+1)}), \qquad (1 - X_\alpha^{-\langle \alpha_*, \varrho^* \rangle + (n+1)(h+1)})$$

for all $\alpha \in \Sigma^+$ and all integers $n \geqq 0$, where $X_\alpha = e^{-u_\alpha}$. To transform this product we need the following property of finite root systems:

Let R be an irreducible reduced finite root system. We shall say that a root $\alpha \in R$ is *short* if there exists $\beta \in R$ such that $\|\beta\| > \|\alpha\|$; otherwise α is *long*. (So if all the roots of R have the same length, they are all long roots.) Choose a basis of R, so that the heights (§8) of the roots $\alpha \in R$ are defined. Let h be the Coxeter number of R, and for $1 \leqq p \leqq h$ let η_{1p} (resp. η_{2p}) be the number of short (resp. long) roots of height p in R, and let l_1 $(= \eta_{11})$ (resp. l_2 $(= \eta_{21})$) be the number of short (resp. long) roots in a basis of R. Then

(10.1) **Lemma.** *If* $p + q = h + 1$ *then* $\eta_{ip} + \eta_{iq} = l_i$ $(i = 1, 2)$.

In view of (8.14) it is enough to establish (10.1) for the short roots, and this is a consequence of the following observation:

(10.2) **Observation.** *The number $h_1 = h/(l_1+1)$ is always an integer, and the sequence $(\eta_{1p})_{1 \leq p \leq h}$ consists of h_1 terms equal to l_1, followed by h_1 terms equal to $l_1 - 1$, and so on, ending with h_1 terms equal to* 0.

I do not know of any uniform explanation of (10.2). It is easily checked case by case (there are effectively only four cases to check).

Let $\alpha, \beta \in \Sigma$. If $\|\alpha_*\| = \|\beta_*\|$ then α_* and β_* are congruent under $W(\Sigma_*) = W(\Sigma)$, hence α and β are congruent under $W(\Sigma)$, and therefore $u_\alpha = u_\beta$, so that $X_\alpha = X_\beta$. Let X_1 (resp. X_2) denote X_α for α_* short (resp. α_* long). In the notation used above, with $R = \Sigma_*$, it is clear that $\theta(\Pi)$ is the product of η_{ip} factors equal to $(1 - X_i^{p+n(h+1)})$ and η_{ip} factors equal to $(1 - X_i^{h+1-p+n(h+1)})$, for $i = 1, 2$; $p = 1, \ldots, h$, and all integers $n \geqq 0$. Hence by (10.1) it is the product of l_i factors equal to $(1 - X_i^{p+n(h+1)})$ for the same range of values i, p and n.

On the other hand, from the definition (8.1.2) of $P = P(X)$, we have

$$P(X) = \prod_{n=1}^{\infty} (1 - X_1^n)^{l_1} (1 - X_2^n)^{l_2}.$$

Hence

(10.3) $$\theta(\Pi) = P(X)/P(X^{h+1}).$$

Now consider $\theta(\Delta)$. Each element of $W(S)$ is uniquely of the form $w\, t(\lambda)$, where $w \in W(\Sigma)$ and $\lambda \in \Lambda$. From (7.5) we have $\theta(e^{-s(w t(\lambda))}) = e^{-U(w, \lambda)}$, where

(10.4) $$U(w, \lambda) = (h+1)\left(\tfrac{1}{2} g \|\lambda\|^2 + \langle \rho, \lambda \rangle\right) + \langle \rho - w(g\lambda + \rho), \rho^* \rangle$$

in which ρ is half the sum of the positive roots of Σ. Hence $\theta(\Delta) = R(X)$, where

(10.5) $$R(X) = \sum_{w \in W(\Sigma)} \varepsilon(w) \sum_{\lambda \in \Lambda} e^{-U(w, \lambda)}.$$

By applying θ to both sides of (9.11) we obtain, from (10.3) and (10.5),

(10.6) $$\frac{P(X)}{P(X^{h+1})} = Q(X^{h+1})\, R(X).$$

For purposes of comparison later we need to express $U(w, \lambda)$ in a form which does not involve ρ^*. For this we require the following

(10.7) **Observation.** *There exists an element $w' \in W(\Sigma)$ such that*

$$g\rho^* = (h+1)\rho - w'\rho.$$

I do not know a uniform proof of (10.7). Suppose first that $S=S(R)^\vee$, so that $\Sigma=R^\vee$. Then $g=h$ (7.13), and $u_\alpha=\frac{1}{2}\|\alpha\|^2$ for all $\alpha\in\Sigma$, so that $\alpha^*=\alpha$ and hence $\rho^*=\rho$. So (10.7) is satisfied with $w'=1$.

Now let $S=S(R)$. Then each $u_\alpha=1$, so that $\alpha^*=\alpha^\vee$ and therefore ρ^* is half the sum of the positive roots of $R^\vee$. We need consider only root systems R which have roots of different lengths, since the others are already covered by the previous paragraph. We consider each case (types B_l, C_l, F_4, G_2) in turn, using the notation of the tables at the end of [B].

(i) If R is of type B_l, then $\rho=\sum_{i=1}^{l}(l+\frac{1}{2}-i)\varepsilon_i$ and $\rho^*=\sum_{i=1}^{l}(l+1-i)\varepsilon_i$. Also $g=2l-1$ and $h=2l$, so that

$$(h+1)\rho-g\rho^*=\sum_{i=1}^{l}(l+\tfrac{3}{2}-2i)\varepsilon_i=w'\rho$$

where w' is the element of $W(R)$ which maps $\varepsilon_1,\varepsilon_2,\varepsilon_3,\dots$ respectively to $\varepsilon_1,-\varepsilon_l,\varepsilon_2,-\varepsilon_{l-1},\dots$.

(ii) If R is of type C_l, then ρ and ρ^* in (i) are interchanged. Also $g=2l+2$ and $h=2l$, so that

$$(h+1)\rho-g\rho^*=\sum_{i=1}^{l}i\varepsilon_i=w'\rho$$

where $w'(\varepsilon_i)=\varepsilon_{l+1-i}$ $(1\leqq i\leqq l)$.

(iii) If R is of type F_4 then $\rho=\frac{1}{2}(11\varepsilon_1+5\varepsilon_2+3\varepsilon_3+\varepsilon_4)$, $\rho^*=8\varepsilon_1+3\varepsilon_2+2\varepsilon_3+\varepsilon_4$. Also $g=9$ and $h=12$, so that

$$(h+1)\rho-g\rho^*=\tfrac{1}{2}(-\varepsilon_1+11\varepsilon_2+3\varepsilon_3-5\varepsilon_4)=w'\rho$$

where w' is the element of $W(R)$ which maps $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ to $\varepsilon_2,-\varepsilon_4,\varepsilon_3,-\varepsilon_1$ respectively.

(iv) If R is of type G_2, then $\rho=5\alpha_1+3\alpha_2$, $\rho^*=9\alpha_1+5\alpha_2$. Also $g=4$ and $h=6$, so that

$$(h+1)\rho-g\rho^*=-\alpha_1+\alpha_2=w'\rho$$

where w' is the reflection associated with the root $3\alpha_1+\alpha_2$. This completes the verification of (10.7).

Using (10.7), we obtain by a straightforward calculation from (10.4) the following expression for $U(w,\lambda)$:

(10.8) $$U(w,\lambda)=\frac{1}{2g(h+1)}\left(\|(h+1)(w(g\lambda+\rho)-\rho)+w'\rho\|^2-\|\rho\|^2\right).$$

11. Second Specialization

We retain the notation and assumptions of § 10. Let $\omega = \exp(2\pi i/(h+1))$, where exp is the ordinary complex exponential. Define a homomorphism $\psi\colon A \to \mathbb{C}[[X]]$ as follows:

$$\psi(e^{-\alpha}) = \omega^{\langle\alpha,\sigma\rangle} \qquad (\alpha \in \Sigma)$$
$$\psi(X) = X,$$

where σ is half the sum of the positive roots of $\Sigma^\vee$, so that $\langle\alpha,\sigma\rangle$ is the height of $\alpha \in \Sigma$.

As in § 8 we split up the product Π into two parts, say Π_0 and Π_1, where Π_0 is the product of the factors $(1-e^{-a})$ for which $a(x_0)=0$, so that

(11.1)
$$\Pi_0 = \prod_{\alpha\in\Sigma^+} (1-e^{-\alpha}),$$

and Π_1 is the product of the $(1-e^{-a})$ for which $a(x_0)>0$, so that

$$\Pi_1 = \prod_{n=1}^{\infty} \prod_{\alpha\in\Sigma} (1-e^{-\alpha-nu_\alpha}).$$

Hence

(11.2)
$$\psi(\Pi_1) = \prod_{n=1}^{\infty} \prod_{\alpha\in\Sigma} (1-\omega^{\langle\alpha,\sigma\rangle} e^{-nu_\alpha}).$$

This product can be simplified means of

(11.3) **Lemma.** *Let R be a finite irreducible reduced root system, B a basis of R, and h the Coxeter number of R. For each $\alpha \in R$ let X_α be an indeterminate, such that $X_{w\alpha} = X_\alpha$ for all $\alpha \in R$ and $w \in W(R)$. Then*

$$\prod_{\alpha\in R} (1-\omega^{\langle\alpha,\sigma\rangle} X_\alpha) = \prod_{\beta\in B} (1-X_\beta^{h+1})/(1-X_\beta)$$

where $\omega = \exp(2\pi i/(h+1))$ and σ is half the sum of the positive roots of $R^\vee$.

Proof. This is a consequence of (10.1). Let X_1 (resp. X_2) denote X_α for α short (resp. α long). Then the product $\Pi(1-\omega^{\langle\alpha,\sigma\rangle} X_\alpha)$ consists of η_{ip} factors $(1-\omega^p X_i)$ and η_{ip} factors $(1-\omega^{-p} X_i)$, for $i=1,2$ and $1 \leqq p \leqq h$. Since $\omega^{-p} = \omega^{h+1-p}$ we have by (10.1) altogether l_i factors $(1-\omega^p X_i)$ for $i=1,2$ and $1 \leqq p \leqq h$, whence the result.

From (11.2) and (11.3) it is clear that

(11.4)
$$\psi(\Pi_1) = P(X^{h+1})/P(X).$$

Next, consider Δ. From (8.3′), we have

$$\Delta = e^{-\rho} \sum_{\mu\in M} J(\mu+\rho) \exp -\frac{1}{2g} (\|\mu+\rho\|^2 - \|\rho\|^2)$$

where $M = g\Lambda$ and

$$J(\mu+\rho) = \sum_{w\in W(\Sigma)} \varepsilon(w)\, e^{w(\mu+\varrho)}.$$

Now

$$\psi(J(\mu+\rho)) = \sum_{w\in W(\Sigma)} \varepsilon(w)\, \omega^{\langle w(\mu+\varrho),\, \sigma\rangle}$$

which by Weyl's formula (0.1) is equal to

$$\prod_{\alpha\in\Sigma^+} (\omega^{\langle\mu+\varrho,\, \alpha^\vee/2\rangle} - \omega^{-\langle\mu+\varrho,\, \alpha^\vee/2\rangle}).$$

Hence $\psi(J(\mu+\rho)) \neq 0$ if and only if μ satisfies the following condition:

(11.5) $\qquad \langle\mu+\rho, \alpha^\vee\rangle \not\equiv 0 \bmod (h+1) \quad$ *for all* $\alpha\in\Sigma$.

[The scalar product $\langle\mu+\rho, \alpha^\vee\rangle$ is an integer because $M = g\Lambda \subset L(\Sigma)$ by (6.11).]

We have to investigate which elements μ of M satisfy (11.5).

(11.6) **Lemma.** *Let* $\mu\in L(\Sigma)$. *Then* μ *satisfies* (11.5) *if and only if there exist* $w\in W(\Sigma)$ *and* $\lambda\in L(\Sigma)$ *such that*

(11.6.1) $$\mu+\rho-w\rho = (h+1)\lambda.$$

Proof. If $\mu+\rho-w\rho\in(h+1)L(\Sigma)$, then clearly

$$\langle\mu+\rho, \alpha^\vee\rangle \equiv \langle w\rho, \alpha^\vee\rangle \bmod (h+1) \quad \text{for all } \alpha\in\Sigma.$$

But $\langle w\rho, \alpha^\vee\rangle$ is equal to the height of $w^{-1}\alpha^\vee$, so that $1 \leqq |\langle w\rho, \alpha^\vee\rangle| \leqq h-1$. Hence $\langle\mu+\rho, \alpha^\vee\rangle$ is not divisible by $h+1$.

Conversely, suppose that μ satisfies (11.5). Then $\mu+\rho$ does not lie on any of the hyperplanes

$$H_{\alpha,n} = \{x\in V\colon \langle\alpha^\vee, x\rangle + n(h+1) = 0\}$$

where $\alpha\in\Sigma$ and $n\in\mathbb{Z}$. Let G be the group of isometries of V generated by the reflections in these hyperplanes. Then G is an affine Weyl group and is the semi-direct product of $W(\Sigma)$ and the group of translations of the form $t((h+1)\lambda)$, where $\lambda\in L(\Sigma)$. If $\phi^\vee$ is the highest root of $\Sigma^\vee$, then the open simplex Γ in V consisting of all $x\in V$ such that

$$\langle\alpha_i^\vee, x\rangle > 0 \quad (1\leqq i\leqq l), \qquad \langle\phi^\vee, x\rangle < h+1$$

is a chamber for the group G. Since $\mu+\rho$ does not lie on any of the reflecting hyperplanes for G, it lies in $s\Gamma$ for some $s\in G$; hence there exist $w\in W(\Sigma)$ and $\lambda\in L(\Sigma)$ such that $\mu+\rho\in(h+1)\lambda+w\Gamma$, or equivalently $w^{-1}(\mu+\rho-(h+1)\lambda)\in\Gamma$.

Let $\nu = w^{-1}(\mu + \rho - (h+1)\lambda) - \rho$. Since $w^{-1}\rho - \rho \in L(\Sigma)$ it follows that $\nu \in L(\Sigma)$, and since $\nu + \rho \in \Gamma$ we have

$$\langle \alpha_i^\vee, \nu + \rho \rangle > 0 \quad (1 \leqq i \leqq l), \qquad \langle \phi^\vee, \nu + \rho \rangle < h+1.$$

But $\langle \alpha_i^\vee, \rho \rangle = 1$ and $\langle \phi^\vee, \rho \rangle = h-1$, so that

$$\langle \alpha_i^\vee, \nu \rangle \geqq 0 \quad (1 \leqq i \leqq l), \qquad \langle \phi^\vee, \nu \rangle \leqq 1.$$

Hence $\langle \phi^\vee, \nu \rangle$, being an integer, must be either 0 or 1.

Suppose that $\langle \phi^\vee, \nu \rangle = 1$. Writing $\phi^\vee$ as a linear combination of the $\alpha_i^\vee$, say $\phi^\vee = \sum m_i \alpha_i^\vee$, we have

$$\sum_{i=1}^{l} m_i \langle \alpha_i^\vee, \nu \rangle = 1.$$

Since each m_i is a positive integer and each $\langle \alpha_i^\vee, \nu \rangle$ is a non-negative integer, it follows that $\langle \alpha_i^\vee, \nu \rangle = 0$ for $i = 1, \ldots, l$ with just one exception, and that for this one value of i we have $m_i = \langle \alpha_i^\vee, \nu \rangle = 1$. Hence ν is a fundamental weight ϖ_i ([B], p. 167). But the fundamental weights ϖ_i for which $m_i = 1$ are never in the root lattice $L(\Sigma)$ ([B], p. 177, Cor. to Prop. 6). This is a contradiction, since $\nu \in L(\Sigma)$.

Hence we must have $\langle \phi^\vee, \nu \rangle = 0$ and therefore $\langle \alpha_i^\vee, \nu \rangle = 0$ for $1 \leqq i \leqq l$, whence $\nu = 0$ and so $\mu + \rho = w\rho + (h+1)\lambda$, as required.

From (11.5) and (11.6) it follows that if $\psi(J(\mu+\rho)) \neq 0$, then there exists $w \in W(\Sigma)$ such that $\langle w_1(\mu+\rho), \sigma \rangle \equiv \langle w_1 w \rho, \sigma \rangle \bmod (h+1)$ for all $w_1 \in W(\Sigma)$, and consequently $\psi(J(\mu+\rho)) = \varepsilon(w)\, \psi(J(\rho))$.

On the other hand, from (11.1) and (0.1) we have $\Pi_0 = e^{-\varrho} J(\rho)$, and therefore

$$\psi(\Pi_0^{-1} \Delta) = \sum_{\mu, w} \varepsilon(w) \exp -\frac{1}{2g}(\|\mu+\rho\|^2 - \|\rho\|^2), \tag{11.7}$$

the summation being over all $\mu \in M = gA$ and $w \in W(\Sigma)$ which satisfy (11.6.1) for some $\lambda \in L(\Sigma)$.

The next step is therefore to find all solutions $(\lambda, \mu, w) \in L(\Sigma) \times M \times W(\Sigma)$ of (11.6.1). Given $w \in W(\Sigma)$, one solution may be obtained as follows. From (10.7) we have

$$g w'^{-1} \rho^* + \rho = (h+1) w'^{-1} \rho.$$

Operating on either side with w and subtracting, we get

$$g w'^{-1}(\rho^* - w_1 \rho^*) + \rho - w\rho = (h+1) w'^{-1}(\rho - w_1 \rho)$$

where $w_1 = w' w w'^{-1}$. Hence if we put

$$
\begin{aligned}
(11.8) \qquad & \lambda_0 = w'^{-1}(\rho - w_1 \rho) \in L(\Sigma), \\
& \mu_0 = g\, w'^{-1}(\rho^* - w_1 \rho^*) \in g\, L(\Sigma^*) = g\Lambda = M
\end{aligned}
$$

we have

$$\mu_0 + \rho - w\rho = (h+1)\,\lambda_0 .$$

Hence λ, μ, w satisfy (11.6.1) if and only if

$$(11.9) \qquad \mu - \mu_0 = (h+1)(\lambda - \lambda_0).$$

Now we have

(11.10) **Observation.** $M \cap (h+1)\, L(\Sigma) = (h+1)\, M$.

As in the case of (10.7), I do not know a uniform proof. First, if $S = S(R)^\vee$ then $g = h$ and $\Lambda = L(\Sigma)$, so that $M = h\, L(\Sigma)$. Hence

$$M \cap (h+1)\, L(\Sigma) = h\, L(\Sigma) \cap (h+1)\, L(\Sigma) = h(h+1)\, L(\Sigma) = (h+1)\, M .$$

The other possibilities for S (i.e. $S = S(R)$ where R is of one of the types B_l, C_l, F_4, G_2) have to be checked case by case. This presents no difficulty, and we shall omit the details.

From (11.9) and (11.10) it follows that λ, μ, w satisfy (11.6.1) if and only if $\lambda - \lambda_0 \in M$, say $\lambda - \lambda_0 = g\, \lambda_1$ for some $\lambda_1 \in \Lambda$. Hence

$$
\begin{aligned}
\|\mu + \rho\| &= \|(h+1)\,\lambda + w\,\rho\| \\
&= \|(h+1)\big(g\,\lambda_1 + w'^{-1}(\rho - w_1\rho)\big) + w\,\rho\| \\
&= \|(h+1)\big(w_1^{-1}(g\, w'\,\lambda_1 + \rho) - \rho\big) + w'\,\rho\|
\end{aligned}
$$

and therefore, from (10.8)

$$\frac{1}{2g}\left(\|\mu + \rho\|^2 - \|\rho\|^2\right) = (h+1)\, U(w_1^{-1}, w'\lambda_1).$$

Hence, from (11.7), we have

$$
\begin{aligned}
\psi(\Pi_0^{-1} \Delta) &= \sum_{w \in W(\Sigma)} \varepsilon(w) \sum_{\lambda \in \Lambda} \exp -(h+1)\, U(w, \lambda) \\
&= R(X^{h+1})
\end{aligned}
$$

by (10.5). From this and (11.4) and (9.11) we obtain

$$(11.11) \qquad \frac{P(X^{h+1})}{P(X)} = Q(X)\, R(X^{h+1}).$$

12. End of the Proof

We can now complete the proof of (8.1) in the case where Σ is reduced. Let $u(X)=P(X)\,Q(X)$ and $v(X)=P(X)\,R(X)^{-1}$. Then from (10.6) we have

$$u(X^{h+1})=v(X),$$

and from (11.11) we have

$$u(X)=v(X^{h+1}).$$

Hence

$$u(X^{(h+1)^2})=u(X)$$

from which it follows immediately that $u(X)$ is a constant. Since P, Q each have constant term 1, it follows that $u(X)=1$, whence $Q=P^{-1}$ and therefore $\Delta=P\Pi$ by (9.11).

Finally, we have to deal with the case where Σ is not reduced. We may take $S=S(R)$ where R is of type BC_l. Let $\varepsilon_1, \ldots, \varepsilon_l$ be an orthonormal basis of V. We may take the elements of R to be the vectors $\pm\varepsilon_i$, $\pm 2\varepsilon_i$ $(1\leqq i\leqq l)$, $\pm\varepsilon_i\pm\varepsilon_j (1\leqq i<j\leqq l)$. Then the affine roots are $n\pm\varepsilon_i$, $(2n+1)\pm 2\varepsilon_i$, $n\pm\varepsilon_i\pm\varepsilon_j$ for all integers n. We take as basis of S the affine roots $a_0=1-2\varepsilon_1$, $a_i=\varepsilon_i-\varepsilon_{i+1}$ $(1\leqq i\leqq l-1)$, $a_l=\varepsilon_l$. Then $g=2l+1$, $h=2l$ and $c=1$.

Let R' be the subsystem (of type B_l) of R obtained by deleting the roots $\pm 2\varepsilon_i$ $(1\leqq i\leqq l)$. As in § 11 let Π_1 denote the product of the $1-e^{-a}$ where $a\in S$ and $a(x_0)>0$, and let Π_1' denote the corresponding product for the subsystem $S(R')$. Then $\Pi_1=\Pi_1'\cdot\Pi_1''$, where

$$\Pi_1''=\prod_{n=0}^{\infty}\prod_{i=1}^{l}(1-e^{-2\varepsilon_i}X^{2n+1})(1-e^{2\varepsilon_i}X^{2n+1}).$$

We shall apply the homomorphism ψ of § 11 to the identity (9.11):

$$\psi(e^{-\alpha})=\omega^{\langle\alpha,\sigma\rangle},\qquad \psi(X)=X$$

where σ is half the sum of the positive roots of $(R')^{\vee}$, so that

$$\sigma=\sum_{i=1}^{l}(l+1-i)\,\varepsilon_i,$$

and $\omega=\exp(2\pi i/(2l+1))$. Then we find easily that

$$\psi(\Pi_1'')=\prod_{n=0}^{\infty}(1-X^{(2n+1)(2l+1)})/(1-X^{2n+1}); \tag{12.1}$$

also from (11.4) we have

$$\psi(\Pi_1')=P(X^{2l+1})/P(X),$$

where $P(X)=\prod_{n=1}^{\infty}(1-X^n)^l$. Hence if we put

$$\pi(X)=\prod_{n=0}^{\infty}(1-X^{2n+1})=\prod_{n=1}^{\infty}(1+X^n)^{-1}$$

we have

(12.2) $$\psi(\Pi_1)=\frac{P(X^{2l+1})\,\pi(X^{2l+1})}{P(X)\,\pi(X)}.$$

Next, consider $\psi(\Pi_0^{-1}\Delta)$. From § 8,

$$\Pi_0^{-1}\Delta=\sum_{\mu\in M}\chi(\mu)\exp-\frac{1}{4l+2}(\|\mu+\rho\|^2-\|\rho\|^2),$$

where $\rho=\sum_{i=1}^{l}(l+\frac{1}{2}-i)\,\varepsilon_i$ is half the sum of the positive roots of R', and $M=(2l+1)\Lambda$, where $\Lambda=\sum_{i=1}^{l}\mathbb{Z}\,\varepsilon_i$. From the definition of ψ it follows that $\psi(e^\mu)=1$ for all $\mu\in M$, whence $\psi(J(\mu+\rho))=\psi(J(\rho))$ and therefore $\psi(\chi(\mu))=1$. Hence

(12.3) $$\psi(\Pi_0^{-1}\Delta)=\sum_{\lambda\in\Lambda}X^{f(\lambda)}$$

where

$$f(\lambda)=\frac{1}{4l+2}(\|(2l+1)\,\lambda+\rho\|^2-\|\rho\|^2)$$
$$=\tfrac{1}{2}(2l+1)\,\|\lambda\|^2+\langle\lambda,\rho\rangle.$$

Hence if $\lambda=-\sum_{i=1}^{l}n_i\,\varepsilon_i$ then

$$f(\lambda)=\sum_{i=1}^{l}\left((2l+1)\,\tfrac{1}{2}n_i(n_i-1)+i\,n_i\right)$$

and therefore from (12.3) we have

$$\psi(\Pi_0^{-1}\Delta)=\prod_{i=1}^{l}\sum_{n\in\mathbb{Z}}X^{((2l+1)n(n-1)/2)+in}$$

which by Jacobi's identity (0.6) is equal to

$$\prod_{i=1}^{l}\prod_{n=1}^{\infty}(1-X^{(2l+1)n})(1+X^{(2l+1)(n-1)+i})(1+X^{(2l+1)n-i}).$$

Hence we find

(12.4) $$\psi(\Pi_0^{-1}\Delta)=P(X^{2l+1})\,\pi(X^{2l+1})/\pi(X).$$

From (12.2), (12.4) and (9.11) it follows that $Q(X)=P(X)^{-1}$. This completes the proof of (8.1).

Appendix 1

Reduced Irreducible Affine Root Systems

For each type of reduced irreducible affine root system we shall exhibit:

(1) an affine root system S on a Euclidean space E.

(2) a basis $\{a_0, \dots, a_l\}$ of S.

(3) the values of g, h and c for S.

(4) the lattice $M = g\Lambda$.

(5) the Dynkin diagram. The black nodes correspond to vertices x_i of the chamber C which are special points for $W(S)$. The numbers attached to the nodes are the coefficients n_i of (6.5).

(6) η-function identities obtained by specializing the identity (8.1) for S. In each of these, c_0 denotes a numerical constant whose value can be written down by considering the term of lowest degree in the power series.

Notation. $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are a sequence of orthonormal vectors in a real Hilbert space. If $v = (v_1, \dots, v_l)$ then

$$\|v\|^2 = \Sigma\, v_i^2, \qquad \chi_B(v) = \prod_i v_i \prod_{i<j} (v_i^2 - v_j^2), \qquad \chi_D(v) = \prod_{i<j} (v_i^2 - v_j^2).$$

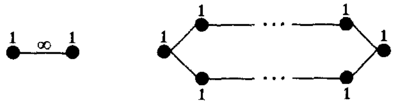
Type A_l $(l \geqq 1)$

(1) Basis of E: $\varepsilon_{i-1} - \varepsilon_i$ $(1 \leqq i \leqq l)$.
Affine roots: $n \pm (\varepsilon_i - \varepsilon_j)$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 - \varepsilon_0 + \varepsilon_l$, $a_i = \varepsilon_{i-1} - \varepsilon_i$ $(1 \leqq i \leqq l)$.

(3) $g = h = l+1$; $c = 1$.

(4) $M = \left\{ (l+1) \sum_{i=0}^{l} n_i \varepsilon_i : \sum n_i = 0 \right\}$.

(5) 

$(l=1)$ $(l \geqq 2)$

(6) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

(a) l even:

$$\eta(X)^{l^2+2l} = c_0 \sum_v \prod_{i<j} (v_i - v_j)\, X^{\|v\|^2/2(l+1)}$$

summed over $v = (v_0, \dots, v_l) \in \mathbb{Z}^{l+1}$ satisfying

$$v_i \equiv i \pmod{l+1} \quad (0 \leqq i \leqq l) \quad \text{and} \quad \sum v_i = 0.$$

(b) l odd:

$$\eta(X)^{l^2+2l} = c_0 \sum_v \prod_{i<j} (v_i - v_j)\, X^{\|v\|^2/8(l+1)}$$

summed over $v=(v_0, \ldots, v_l) \in \mathbb{Z}^{l+1}$ satisfying

$v_i \equiv 2i+1 \pmod{2l+2}$ $(0 \leqq i \leqq l)$ and $\sum v_i = 0$.

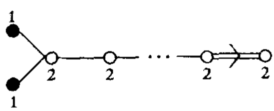
Type B_l $(l \geqq 3)$

(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.
Affine roots: $n \pm \varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \varepsilon_l$.

(3) $g = 2l-1$, $h = 2l$, $c = 1$.

(4) $M = \left\{ (2l-1) \sum_{i=1}^{l} n_i \varepsilon_i : \sum n_i \equiv 0 \pmod 2 \right\}$.

(5) 

(6) (a) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\eta(X)^{2l^2+l} = c_0 \sum_v \chi_B(v)\, X^{\|v\|^2/8(2l-1)}$$

summed over $v=(v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv 2i-1 \pmod{4l-2}$ $(1 \leqq i \leqq l)$ and $\sum v_i \equiv l^2 \pmod{8l-4}$.

(b) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l-1)$, $e^{a_l} \mapsto -1$:

$$\left(\eta(X)^{2l-3}\, \eta(X^2)^2\right)^l = c_0 \sum_v \chi_D(v)\, X^{\|v\|^2/8(2l-1)}$$

summed over the same set of $v \in \mathbb{Z}^l$ as in (a).

(c) Specialization $e^{a_i} \mapsto 1$ $(0 \leqq i \leqq l-1)$:

$$\left(\eta(X^{1/2})^2\, \eta(X)^{2l-3}\right)^l = c_0 \sum_v (-1)^{\Sigma v_i} \chi_D(v)^{\|v\|^2/(4l-2)}$$

summed over $v=(v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv i-1 \pmod{2l-1}$ $(1 \leqq i \leqq l)$.

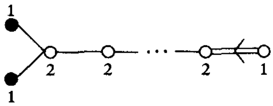
Type $B_l^\vee$ $(l \geqq 3)$

(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.
Affine roots: $2n \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = 2\varepsilon_l$.

(3) $g = h = 2l$; $c = 1$.

(4) $M = \left\{ 2l \sum_{i=1}^{l} n_i \varepsilon_i : \sum n_i \equiv 0 \pmod 2 \right\}$.

(5) 

(6) (a) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\left(\eta(X)^{l-1} \eta(X^2)\right)^{2l+1} = c_0 \sum_v \chi_B(v) X^{\|v\|^2/4l}$$

summed over $v = (v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv i \pmod{2l}$ $(1 \leqq i \leqq l)$ and $\sum v_i \equiv \frac{1}{2} l(l+1) \pmod{4l}$.

(b) Specialization $e^{a_i} \mapsto 1$ $(0 \leqq i \leqq l-1)$:

$$\left(\eta(X)^{l+1} \eta(X^2)^{-1}\right)^{2l-1} = c_0 \sum_v (-1)^{\Sigma(v_i - i + 1)/2l} \chi_D(v) X^{\|v\|^2/4l}$$

summed over $v = (v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv i - 1 \pmod{2l}$ $(1 \leqq i \leqq l)$.

Type C_l $(l \geqq 2)$

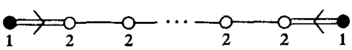
(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.

Affine roots: $n \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 - 2\varepsilon_1$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = 2\varepsilon_l$.

(3) $g = 2l + 2$, $h = 2l$, $c = 1$.

(4) $M = (2l+2) \sum_{i=1} \mathbb{Z}\, \varepsilon_i$.

(5) 

(6) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\eta(X)^{2l^2 + l} = c_0 \sum_v \chi_B(v) X^{\|v\|^2/4(l+1)}$$

summed over $v = (v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv i \pmod{2l+2}$ $(1 \leqq i \leqq l)$.

Type $C_l^\vee$ ($l \geqq 2$)

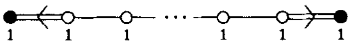
(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.

Affine roots: $\frac{1}{2}n \pm \varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = \frac{1}{2} - \varepsilon_1$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \varepsilon_l$.

(3) $g = h = 2l$; $c = \frac{1}{2}$.

(4) $M = 2l \sum_{i=1}^{l} \mathbb{Z}\, \varepsilon_i$.

(5) 

(6) (a) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\left(\eta(X)\, \eta(X^2)^{l-1}\right)^{2l+1} = c_0 \sum_v \chi_B(v)\, X^{\|v\|^2/8l}$$

summed over $v = (v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying

$v_i \equiv 2i - 1 \pmod{4l}$ $(1 \leqq i \leqq l)$.

(b) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l-1)$, $e^{a_l} \mapsto -1$:

$$\left(\eta(X)^{-1}\, \eta(X^2)^{l+1}\right)^{2l-1} = c_0 \sum_v \chi_D(v)\, X^{\|v\|^2/8l}$$

summed over the same $v \in \mathbb{Z}^l$ as in (a).

Type BC_l ($l \geqq 1$)

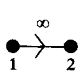
(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.

Affine roots: $n \pm \varepsilon_i$ $(1 \leqq i \leqq l)$, $2n + 1 \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$,
$n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

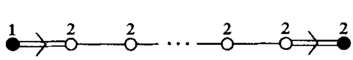
(2) $a_0 = 1 - 2\varepsilon_1$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \varepsilon_l$.

(3) $g = 2l + 1$, $h = 2l$, $c = 1$.

(4) $M = (2l+1) \sum_{i=1}^{l} \mathbb{Z}\, \varepsilon_i$.

(5) 

($l = 1$)



($l \geqq 2$)

(6) (a) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\left(\eta(X)^{2l+3}\,\eta(X^2)^{-2}\right)^l = c_0 \sum_v \chi_B(v)\, X^{\|v\|^2/8(2l+1)}$$

summed over $v=(v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying $v_i \equiv 2i-1 \pmod{4l+2}$ $(1 \leqq i \leqq l)$.

(b) Specialization $e^{a_i} \mapsto 1$ $(0 \leqq i \leqq l-1)$:

$$\left(\eta(X^{1/2})^2\,\eta(X)^{2l-3}\,\eta(X^2)^2\right)^l = c_0 \sum_v \chi_B(v)\, X^{\|v\|^2/(4l+2)}$$

summed over $v=(v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying $v_i \equiv i \pmod{2l+1}$ $(1 \leqq i \leqq l)$.

(c) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l-1)$, $e^{a_l} \mapsto -1$:

$$\eta(X)^{2l^2-l} = c_0 \sum_v (-1)^{\Sigma(v_i-1)/2}\, \chi_D(v)\, X^{\|v\|^2/8(2l+1)}$$

summed over the same $v \in \mathbb{Z}^l$ as in (a).

(d) Specialization $e^{a_0/2} \mapsto -1$, $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l-1)$:

$$\left(\eta(X^{1/2})^{-2}\,\eta(X)^{2l+3}\right)^l = c_0 \sum_v (-1)^{\Sigma v_i}\, \chi_B(v)\, X^{\|v\|^2/(4l+2)}$$

summed over the same $v \in \mathbb{Z}^l$ as in (b).

Type D_l $(l \geqq 4)$

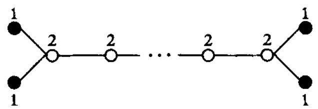
(1) Basis of E: $\varepsilon_1, \ldots, \varepsilon_l$.

Affine roots: $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 - \varepsilon_1 - \varepsilon_2$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \varepsilon_{l-1} + \varepsilon_l$.

(3) $g = h = 2l-2$; $c = 1$.

(4) $M = \left\{(2l-2)\sum_{i=1}^{i} n_i \varepsilon_i \colon \sum n_i \equiv 0 \pmod 2\right\}$.

(5) 

(6) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\eta(X)^{2l^2-l} = c_0 \sum_v \chi_D(v)\, X^{\|v\|^2/4(l-1)}$$

summed over $v=(v_1, \ldots, v_l) \in \mathbb{Z}^l$ satisfying $v_i \equiv i-1 \pmod{2l-2}$ $(1 \leqq i \leqq l)$.

In the next three types (E_6, E_7, E_8) let $\omega_i=\varepsilon_i-\frac{1}{9}\sum_{j=0}^{8}\varepsilon_j$, so that $\sum_{i=0}^{8}\omega_i=0$ and $\langle\omega_i,\omega_j\rangle=-\frac{1}{9}+\delta_{ij}$.

Type E_6

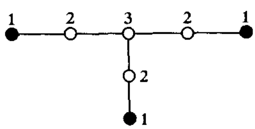
(1) Basis of E: $\omega_1,\dots,\omega_6$.

Affine roots: $n\pm(\omega_i-\omega_j)$ $(1\leqq i<j\leqq 6)$, $n\pm(\omega_i+\omega_j+\omega_k)$ $(1\leqq i<j<k\leqq 6)$, $n\pm(\omega_1+\cdots+\omega_6)$ $(n\in\mathbb{Z})$.

(2) $a_0=1-(\omega_1+\cdots+\omega_6)$, $a_i=\omega_i-\omega_{i+1}$ $(1\leqq i\leqq 5)$, $a_6=\omega_4+\omega_5+\omega_6$.

(3) $g=h=12$, $c=1$.

(4) $M=\left\{12\sum_{i=1}^{6}n_i\,\omega_i\colon \sum n_i\equiv 0 \pmod 3\right\}$.

(5) 

(6) Specialization $e^{a_i}\mapsto 1$ $(1\leqq i\leqq 6)$:

$$\eta(X)^{78}=c_0\sum_{u,v}u\prod_{i<j}(v_i-v_j)\prod_{i<j<k}(u+v_i+v_j+v_k)\,X^{(-u^2+\|v\|^2)/24}$$

summed over $u\in\mathbb{Z}$ and $v=(v_1,\dots,v_6)\in\mathbb{Z}^6$ satisfying

$u\equiv 1 \pmod{12}$, $v_i\equiv 9-i \pmod{12}$ $(1\leqq i\leqq 6)$, $3u+\sum v_i=0$.

Type E_7

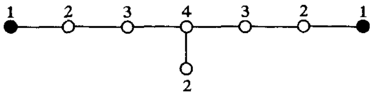
(1) Basis of V: $\omega_1,\dots,\omega_7$.

Affine roots: $n\pm(\omega_i-\omega_j)$ $(1\leqq i<j\leqq 7)$, $n\pm(\omega_1+\omega_j+\omega_k)$ $(1\leqq i<j<k\leqq 7)$, $n\pm(\omega_1+\cdots+\hat{\omega}_i+\cdots+\omega_7)$ $(1\leqq i\leqq 7)$ $(n\in\mathbb{Z})$.

(2) $a_0=1-(\omega_1+\cdots+\omega_6)$, $a_i=\omega_i-\omega_{i+1}$ $(1\leqq i\leqq 6)$, $a_7=\omega_5+\omega_6+\omega_7$.

(3) $g=h=18$, $c=1$.

(4) $M=\left\{18\sum_{i=1}^{7}n_i\,\omega_i\colon \sum n_i\equiv 0 \pmod 3\right\}$.

(5) 

(6) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq 7)$:

$$\eta(X)^{133} = c_0 \sum_{u,v} \prod_i (u+v_i) \prod_{i<j} (v_i - v_j) \prod_{i<j<k} (u+v_i+v_j+v_k) \cdot X^{(-u^2+\|v\|^2)/144}$$

summed over $u \in \mathbb{Z}$ and $v = (v_i, \ldots, v_7) \in \mathbb{Z}^7$ satisfying

$$u \equiv 23 \pmod{36}, \quad v_i \equiv 29 - 2i \pmod{36} \ (1 \leqq i \leqq 7), \quad 3u + \sum v_i = 0.$$

Type E_8

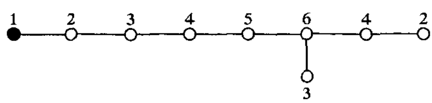
(1) Basis of V: $\omega_1, \ldots, \omega_8$.

Affine roots: $n \pm (\omega_i - \omega_j)$ $(0 \leqq i < j \leqq 8)$, $n \pm (\omega_i + \omega_j + \omega_k)$ $(0 \leqq i < j < k \leqq 8)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 + \omega_0 - \omega_1$, $a_i = \omega_i - \omega_{i+1}$ $(1 \leqq i \leqq 7)$, $a_8 = \omega_6 + \omega_7 + \omega_8$.

(3) $g = h = 30$, $c = 1$.

(4) $M = \left\{ 30 \sum_{i=1}^{8} n_i \omega_i : \sum n_i \equiv 0 \pmod 3 \right\}$.

(5) 

(6) Specialization $e^{a_i} \mapsto 1$ $(1 \leqq i \leqq l)$:

$$\eta(X)^{248} = c_0 \sum_{u,v} \prod_{i<j} (v_i - v_j) \prod_{i<j<k} (u+v_i+v_j+v_k) X^{(-u^2+\|v\|^2)/60}$$

summed over $u \in \mathbb{Z}$ and $v = (v_i, \ldots, v_9) \in \mathbb{Z}^9$ satisfying

$$u \equiv 8 \pmod{30}, \quad v_i \equiv i \pmod{30} \ (1 \leqq i \leqq 8), \quad v_9 = 0, \quad 3v_0 + \sum v_i = 0.$$

Type F_4

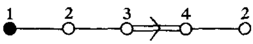
(1) Basis of E: $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.

Affine roots: $n \pm \varepsilon_i$ $(1 \leqq i \leqq 4)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq 4)$,
$n + \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ $(n \in \mathbb{Z})$.

(2) $a_0 = 1 + \varepsilon_1 - \varepsilon_2$, $a_1 = \varepsilon_2 - \varepsilon_3$, $a_2 = \varepsilon_3 - \varepsilon_4$, $a_3 = \varepsilon_4 - \sigma$, $a_4 = \sigma$ where $\sigma = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$.

(3) $g=9, \quad h=12, \quad c=1.$

(4) $M=\left\{9\sum_{i=1}^{4} n_i\,\varepsilon_i\colon \sum n_i\equiv 0 \pmod 2\right\}.$

(5) 

(6) Specialization $e^{a_i}\mapsto 1$ $(1\leqq i\leqq 4)$:

$$\eta(X)^{52}=c_0\sum_v \prod_i v_i \prod_{i<j}(v_i^2-v_j^2)\prod(v_1\pm v_2\pm v_3\pm v_4)\,X^{\|v\|^2/18}$$

summed over $v=(v_1, v_2, v_3, v_4)\in\mathbb{Z}^4$ satisfying
$v_i\equiv i \pmod 9$ $(1\leqq i\leqq 4)$, $\sum v_i\equiv 0 \pmod 2$.

Type $F_4^\vee$

(1) Basis of E: $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.

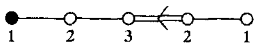
Affine roots: $2n\pm 2\varepsilon_i$ $(1\leqq i\leqq 4)$, $\quad n\pm\varepsilon_i\pm\varepsilon_j$ $(1\leqq i<j\leqq 4)$,

$2n\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4$ $(n\in\mathbb{Z})$.

(2) $a_0=1+\varepsilon_1-\varepsilon_2, \quad a_1=\varepsilon_2-\varepsilon_3, \quad a_2=\varepsilon_3-\varepsilon_4, \quad a_3=2\varepsilon_4,$
$a_4=-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4).$

(3) $g=h=12, \quad c=1.$

(4) $M=\left\{12\sum_{i=1}^{4} n_i\,\varepsilon_i\colon \sum n_i\equiv 0 \pmod 2\right\}.$

(5) 

(6) Specialization $e^{a_i}\mapsto 1$ $(1\leqq i\leqq 4)$:

$$(\eta(X)\,\eta(X^2))^{26}=c_0\sum_v \prod_i v_i \prod_{i<j}(v_i^2-v_j^2)\prod(v_1\pm v_2\pm v_3\pm v_4)\,X^{\|v\|^2/24}$$

summed over $v=(v_1, v_2, v_3, v_4)\in\mathbb{Z}^4$ satisfying

$v_i\equiv i \pmod{12}$ $(1\leqq i\leqq 4)$ and $\sum v_i\equiv 6 \pmod 8$.

In the last two types (G_2 and $G_2^\vee$) let $\phi_i=\varepsilon_i-\frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ $(i=1, 2, 3)$, so that $\sum\phi_i=0$ and $\langle\phi_i, \phi_j\rangle=-\frac{1}{3}+\delta_{ij}$.

Type G_2

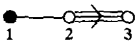
(1) Basis of E: ϕ_1, ϕ_2.

Affine roots: $n\pm\phi_i$ $(1\leqq i\leqq 3)$, $\quad n\pm(\phi_i-\phi_j)$ $(1\leqq i<j\leqq 3)$ $(n\in\mathbb{Z})$.

(2) $a_0=1-\phi_1+\phi_3, \quad a_1=\phi_1-\phi_2, \quad a_2=\phi_2.$

(3) $g=4,\quad h=6,\quad c=1.$

(4) $M=4\sum_{i=1}^{3}\mathbb{Z}\phi_i.$

(5) 

(6) Specialization $e^{a_i}\mapsto 1$ $(i=1,2)$:

$$\eta(X)^{14}=c_0\sum_v\prod_i v_i\prod_{i<j}(v_i-v_j)\,X^{\|u\|^2/72}$$

summed over $v=(v_1,v_2,v_3)\in\mathbb{Z}^3$ satisfying

$v_i\equiv 3i-2\pmod{12}$ and $\sum v_i=0.$

Type $G_2^\vee$

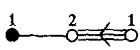
(1) Basis of E: ϕ_1,ϕ_2.

Affine roots: $3n\pm 3\phi_i$ $(1\leqq i\leqq 3)$, $n\pm(\phi_i-\phi_j)$ $(1\leqq i<j\leqq 3)$ $(n\in\mathbb{Z})$.

(2) $a_0=1-\phi_1+\phi_3,\quad a_1=\phi_1-\phi_2,\quad a_2=3\phi_2.$

(3) $g=h=6,\quad c=1.$

(4) $M=\left\{6\sum_{i=1}^{3}n_i\phi_i\colon \sum n_i\equiv 0\pmod 3\right\}.$

(5) 

(6) Specialization $e^{a_i}\mapsto 1$ $(i=1,2)$:

$$(\eta(X)\,\eta(X^3))^7=c_0\sum_v\prod_i v_i\prod_{i<j}(v_i-v_j)\,X^{\|u\|^2/12}$$

summed over $v=(v_1,v_2,v_3)\in\mathbb{Z}^3$ satisfying

$v_i\equiv i\pmod 6$ $(i=1,2,3)$ and $\sum v_i=0.$

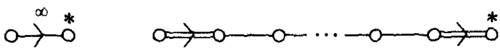
Appendix 2

Non-Reduced Irreducible Affine Root Systems

In the Dynkin diagrams below, an asterisk placed over a node indicates that if a is the affine root corresponding to that node in a basis of the affine root system, then $2a$ is also an affine root.

Type BCC_l $(l\geqq 1)$

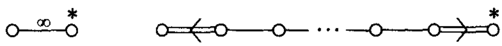
Affine roots $n\pm\varepsilon_i$, $n\pm 2\varepsilon_i$ $(1\leqq i\leqq l)$, $n\pm\varepsilon_i\pm\varepsilon_j$ $(1\leqq i<j\leqq l)$ $(n\in\mathbb{Z})$.



$(l=1)$ $(l\geqq 2)$

Type $C^\vee BC_l$ $(l \geqq 1)$

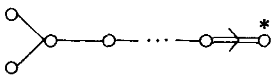
Affine roots $\frac{1}{2}n \pm \varepsilon_i$, $2n \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.



$(l=1)$ $(l \geqq 2)$

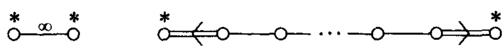
Type $BB_l^\vee$ $(l \geqq 3)$

Affine roots $n \pm \varepsilon_i$, $2n \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.



Type $C^\vee C_l$ $(l \geqq 1)$

Affine roots $\frac{n}{2} \pm \varepsilon_i$, $n \pm 2\varepsilon_i$ $(1 \leqq i \leqq l)$, $n \pm \varepsilon_i \pm \varepsilon_j$ $(1 \leqq i < j \leqq l)$ $(n \in \mathbb{Z})$.



$(l=1)$ $(l \geqq 2)$

References

B. Bourbaki, N.: Groupes et algèbres de Lie, Chapitres 4, 5, et 6. Paris: Hermann 1969.
1. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Publ. Math. I.H.E.S., 41 (to appear).
2. Freudenthal, H., Vries, H. de: Linear Lie groups. New York: Academic Press 1969.
3. Hardy, G. H., Wright, E. M.: Introduction to the theory of numbers (4th edition). Oxford: Oxford University Press 1959.
4. Kostant, B.: The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math. **81**, 973–1032 (1959).
5. Macdonald, I. G.: The Poincaré series of a Coxeter group (to appear).
6. Winquist, L.: Elementary proof of $p(11m+6) \equiv 0 \pmod{11}$. J. Comb. Theory **6**, 56–59 (1969).

7a. Moody, R. V.: A new class of Lie algebras. J. Alg. **10**, 211–230 (1968).

7b. – Euclidean Lie algebras. Can. J. Math. **21**, 1432–1454 (1969).

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