Inventiones math. 57, 205–218 (1980)

Inventiones mathematicae

Group Completions and Limit Sets of Kleinian Groups

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1. Introduction

Finitely generated groups and actions of finitely generated groups often come up in studying topology and geometry. While the most important example may be as fundamental groups of compact manifolds, questions involving finitely generated groups also arise in transformation groups, dynamical systems, and Kleinian groups. A very striking example is Mostow's theorem [17], which says that, for dimensions greater than or equal to three, closed hyperbolic manifolds are determined up to isometry by their fundamental groups.

Jakob Nielsen, in a series of papers ([18–21]), used the Poincaré disk model of hyperbolic 2-space, H^2, as $\operatorname{int}(D^2)$ to study surfaces and their diffeomorphisms. Given a closed surface M^2 of genus $g \geqq 2$ and a diffeomorphism $f: M \to M$, he lifted f to a homeomorphism $f': H^2 \to H^2$ and showed that f extends to a homeomorphism of the circle $S^1 = \partial D^2$. Furthermore, the map on the circle does not depend on the particular diffeomorphism f, but only on its homotopy type. Nielsen made use of the extension of f' to D^2 and of the corresponding actions of $\Pi_1(M)$ and $\operatorname{Aut}(\Pi_1(M))$ on S^1 to systematically study topological properties of diffeomorphisms of surfaces.

The proof of Mostow's theorem also uses the action of $\Pi_1(M^n)$ on S^{n-1}. Given two closed hyperbolic n-manifolds M^n and N^n with $n \geqq 3$ and an isomorphism $\Phi: \Pi_1(M) \to \Pi_1(N)$, there is a homotopy equivalence $f: M \to N$ inducing Φ (since M and N are $K(\Pi, 1)$'s). f can be lifted to the universal covers to $\tilde{f}: H^n \to H^n$, and $\tilde{f}$ extends to a homeomorphism $f': S^{n-1} \to S^{n-1}$. The essence of Mostow's proof is to show that f' is conformal; this is done by first showing that f' is quasi-conformal and then using ergodicity of the action of $\Pi_1(M)$ on S^{n-1} to show conformality. It was realized by Margulis and at least implicity by Mostow that the homotopy equivalence $f: M \to N$ used to construct f' was not

* This work is a portion of the work in the author's doctoral dissertation, which was submitted to the faculty of Princeton University in candidacy for the degree of Doctor of Philosophy. This work was partially supported by a National Science Foundation Graduate Fellowship

0020-9910/80/0057/0205/$02.80

essential to the proof. Margulis indicated in [12] how to use the isomorphism $\Phi: \Pi_1(M) \to \Pi_1(N)$ and the actions of $\Pi_1(M)$ and $\Pi_1(N)$ on S^{n-1} to directly construct the homomorphism $f': S^{n-1} \to S^{n-1}$ and to show that it is quasi-conformal. Margulis's proof uses word metrics on finitely generated groups, which we will describe in Sect. 2.

Another result showing the interplay between finitely generated groups and geometry is the Fenchel-Nielsen isomorphism theorem (see [10] or [27] for a proof), which gives conditions under which an isomorphism of Fuchsian groups induces a homeomorphism $f: H^2 \to H^2$ and an extension (also a homeomorphism) $f': S^1 \to S^1$ which preserves stabilizers. A corresponding isomorphism theorem for Kleinian groups has been proved by Marden and Maskit [11].

In light of these results it is natural to ask whether one can recover much of the structure solely from the groups. For example, can you construct a closed hyperbolic manifold (either topologically or conformally) as a functor of its fundamental group? Similarly, for a Kleinian group G, when can you construct the limit set $\Lambda \Phi G)$ from the group itself?

If M is a closed hyperbolic n-manifold, then $\Pi_1(M)$ acts on $S^{n-1} = \partial D^n$, as can be seeen by using the Poincaré disk model of H^n as $\operatorname{int}(D^n)$ (when $n=3$, S^2 is the limit set of the Kleinian group $\Pi_1(M)$). This action of $\Pi_1(M)$ on S^{n-1} is ergodic (this is used in proving Mostow's theorem), and from the action one can construct topologically the Stiefel manifold of 2-frames on M. A reasonable initial goal for any construction on the group level would be whether one can construct S^{n-1}, together with the action of $\Pi_1(M)$ on S^{n-1}, as a functor of the group $\Pi_1(M)$.

The Hopf-Freudenthal theory ([4, 6]) of ends of groups does not suffice, since the fundamental groups mentioned above have only one end. What is needed is a refinement which detects different ways in the group of getting to infinity. In this paper we give and develop a group construction, based on an idea of Thurston's and inspired by a construction of Sullivan's (several mathematicians, including Gromov, Sullivan, and Thurston, have thought about these questions). Given a finitely generated group G with a chosen finite generating set Σ, we put a metric on the graph of G so that the completion $\bar{G} = \text{completion}(\text{graph}(G)) - \text{graph}(G)$ is a compact metric space with a natural G action. If M is a closed hyperbolic n-manifold and $G = \Pi_1(M)$, then there is a G-equivariant homeomorphism $\varphi: \bar{G} \to S^{n-1}$.

In Sect. 2 we give the construction and show some basic properties of it (e.g., independence of the generating set, behavior under finite index, and behavior under direct sum). We also show that it is natural in the sense that isomorphisms of groups induce homeomorphisms of their completions.

Section 3 gives a proof of the following

Theorem. *If G is a polycyclic group with one end, then the completion $\bar{G}$ is a point.*

We also briefly discuss how this relates to questions concerning polynomial growth and concerning flat manifolds.

In Sect. 4 we use group completions to study the limit sets of Kleinian groups. For a finitely generated Kleinian group G, we give general conditions for there to be a G-equivariant map $\varphi: \bar{G} \twoheadrightarrow \Lambda(G)$. Our main result is the

Theorem. *If G is a geometrically finite Kleinian group, then there is a G-equivariant map $\varphi: \bar{G} \twoheadrightarrow \Lambda(G)$. φ is 2-to-1 onto parabolic fixed points of rank one and injective everywhere else.*

I would like to thank Steve Kerckhoff, Dennis Sullivan, and (especially) Bill Thurston for numerous conversations during the course of this work.

2. Definition of $\bar{G}$ and Basic Results

Let G be a finitely generated group with a chosen finite generating set $\Sigma = \{g_1, \ldots, g_n\}$. Corresponding to Σ there is a left-invariant metric on G, called the word metric of G, and a 1-complex, $K(G, \Sigma)$, called the graph of G. We will put a metric on $K(G, \Sigma)$ so that the completion $\bar{G} = \text{completion}(K(G, \Sigma)) - K(G, \Sigma)$ is a compact metric space with a natural G action.

We will use the following ideas about metric spaces. Let X, Y be metric spaces with metrics $\rho(\ ,\ )$ and $\rho'(\ ,\ )$, respectively. A map $f: X \to Y$ is a *quasi-isometry* if there are constants $c, d > 0$ so that $c\rho(x, y) \leqq \rho'(f(x), f(y)) \leqq d\rho(x, y)$ for all $x, y \in X$. Two metrics on X are *commensurable* if the identity map on X is a quasi-isometry. X and Y are *Lipschitz equivalent* if there is a homeomorphism $f: X \to Y$ which is a quasi-isometry.

For $g \in G$ define the norm $|g| = $ minimal word length of (word in $g_1, \ldots, g_n$) $= g$, where multiplicities count in measuring the word length. Make this into a left-invariant metric on G, called the word metric, by setting $(a, b) = |a^{-1}b|$ for $a, b \in G$. The word metric depends on the choice of the generating set Σ, but any two word metrics on G are commensurable.

Define $K(G, \Sigma)$ as follows. Vertices of $K(G, \Sigma)$ correspond bijectively to elements of G (the vertices are labelled by this correspondence), and two vertices $a, b \in G$ are joined by an (unordered) edge if $a = bg^{\pm 1}$ for some $g \in \Sigma$. The standard definition of the graph of G uses ordered edges, but we forget the order here to avoid confusion. $K(G, \Sigma)$ is also called the Cayley diagram or the group diagram.

To construct $\bar{G}$, let $f: \mathbb{N} \to \mathbb{R}^+$ be a monic, summable function such that, given $k \in \mathbb{N}$, there exist $M, N > 0$ so that $Mf(r) \leqq f(kr) \leqq Nf(r)$ for all $r \in \mathbb{N}$ (for example, $f(r) = r^{-2}$). Put a metric on $K(G, \Sigma)$ by giving the edge between vertices $a, b \in G$ the length $\min\{f(|a|), f(|b|)\}$ (define $f(0) = f(1)$ to make this well-defined) and enlarging this to a metric $\langle\ ,\ \rangle$ on $K(G, \Sigma)$ by taking shortest paths. Then complete $K(G, \Sigma)$ as a metric space, and define $\bar{G}(\Sigma, f) = \text{completion}(K(G, \Sigma)) - K(G, \Sigma)$, giving $\bar{G}(\Sigma, f)$ the metric topology. We will often abuse notation and write $\bar{G}$ instead of $\bar{G}(\Sigma, f)$.

Here is an example. Let $G = \mathbb{Z} * \mathbb{Z}$ with a generating set $\Sigma = \{a, b\}$, and let $f(r) = r^{-2}$. $K(G, \Sigma)$ is a tree, and $\bar{G}(\Sigma, f)$ is the Cantor set that forms the ends of $\mathbb{Z} * \mathbb{Z}$. In Fig. 1 is the subset of the graph of G of vertices with norm $\leqq 4$ and the edges between them.

The following results come easily from the construction. Since the proofs are straightforward arguments mainly involving choosing generating sets and then looking at Cauchy sequences, most of the details will be left to the reader.

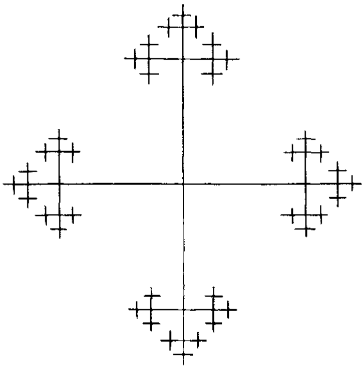

Fig. 1. Graph of $\mathbb{Z} * \mathbb{Z}$

Assume for the rest of this section that f is a monic, summable function, that G is a finitely generated group with a chosen finite generating set Σ, and that all groups mentioned are finitely generated.

Lemma 1. *$\bar{G}(\Sigma, f)$ is a compact metric space.*

Proof. Completion$(K(G, \Sigma))$ is compact since for every $\varepsilon > 0$ there is a finite set which is ε-dense, and $\bar{G}(\Sigma, f)$ is a closed subset of completion$(K(G, \Sigma))$. $\square$

Lemma 2. *If Σ' is a finite generating set for G, then $\bar{G}(\Sigma', f)$ is Lipschitz equivalent to $\bar{G}(\Sigma, f)$.*

Proof. By commensurability of different word metrics and the requirement that $Mf(r) \leqq f(kr) \leqq Nf(r)$. $\square$

Lemma 3. *If $H \xrightarrow{i} G \xrightarrow{p} J$ is an exact sequence of groups and H is a finite group, then $\bar{G}$ and $\bar{J}$ are Lipschitz equivalent.*

Lemma 4. *If Φ: $H \to G$ is a homomorphism of groups and a quasi-isometry of word metrics, then there is an induced, continuous map $\bar{\Phi}$: $\bar{H} \to \bar{G}$.*

Since an isomorphism of groups is a quasi-isometry of word metrics (look at what it does to a finite generating set), an immediate corollary to Lemma 4 is the

Theorem. Aut G *acts on $\bar{G}$, and so in particular G acts on $\bar{G}$ (using the action of G on itself by inner automorphisms).*

Lemma 5. *If $H \subset G$ is a subgroup of finite index, then $\bar{H}$ and $\bar{G}$ are Lipschitz equivalent.*

Proof. Since H contains a normal subgroup of finite index in G, we can assume that H is a normal subgroup of G. It is easily shown (for example, see [2] or [26]) that the inclusion i: $H \to G$ is a quasi-isometry of word metrics. The proof then follows by looking at Cauchy sequences. $\square$

Any point $\omega \in \bar{G}(\Sigma, f)$ can be represented by a Cauchy sequence $\{w_i\}$ in $K(G, \Sigma)$ with $w_i \in G$ (a vertex) for all $i \in \mathbb{N}$. We call such a sequence *shortest* if $|w_i|$

$=i$ and $(w_i, w_{i+1})=1$ for all $i\in\mathbb{N}$. If ω is defined by a shortest sequence $\{w_i\}$, then we can think of ω as an infinite word $g'_1 g'_2 \dots g'_n \dots$ in Σ which is written in shortest form at each finite stage. Alternately we can think of ω as a path in $K(G,\Sigma)$ which only goes one step at a time and never backtracks.

Lemma 6. *Every point in $\bar{G}$ can be represented by a shortest sequence.*

Proof. Let $\omega\in\bar{G}$ be represented by a Cauchy sequence $\{w_i\}$ of vertices, and write each w_i as a word in Σ in shortest form. Then use a variant of the Cantor diagonal argument. □

Lemma 7. *If G and H are infinite groups, then $\overline{G\oplus H}$ is a point.*

Proof. Choose finite generating sets Σ_1, Σ_2, and $\Sigma_1\cup\Sigma_2$ for G, H, and $G\oplus H$, respectively (using the natural inclusions of G and H in $G\oplus H$; the inclusions are isometries of these particular word metrics). Given $\varepsilon>0$, we will choose $N\in\mathbb{N}$ and a vertex $w_N\in G\oplus H$ so that any vertex $g\in G\oplus H$ with $|g|\geqq N$ is within ε of w_N in $K(G\oplus H, \Sigma_1\cup\Sigma_2)$.

Since G and H are infinite their completions are not empty. Pick $\omega\in\bar{G}$, $v\in\bar{H}$, and choose respective shortest sequences $\{w_i\}$ and $\{v_i\}$ defining them. Choose $N'\in\mathbb{N}$ so that $\sum_{j=N'}^{\infty} f(k)<\varepsilon/4$, and let $N=2N'$. Let $g=a\oplus b$ be a vertex with $|g|\geqq N$. If $|b|\geqq N/2$ then the path in $K(G\oplus H, \Sigma_1\cup\Sigma_2)$

$$g=a\oplus b \xrightarrow{a^{-1}w_N} w_N\oplus b \xrightarrow{b^{-1}} w_N$$

has length $\leqq\varepsilon$ (here $\xrightarrow{a^{-1}w_N}$ means a path given by writing $a^{-1}w_N$ as a word in shortest form and then taking the steps corresponding to this word in $K(G\oplus H, \Sigma_1\cup\Sigma_2)$). If $|a|\geqq N/2$ choose $M\in\mathbb{N}$ large enough that $f(M)<\varepsilon/4(w_N, a)$. Then the path

$$g=a\oplus b \xrightarrow{b^{-1}v_M} a\oplus v_M \xrightarrow{a^{-1}w_N} w_N\oplus v_M \xrightarrow{v_M^{-1}} w_N$$

has length $\leqq\varepsilon$. □

The infinite cyclic group $\mathbb{Z}$ has completion $\bar{\mathbb{Z}}=$disjoint union of two points (corresponding to the ends of $\mathbb{Z}$). An immediate consequence of Lemma's 5 and 7 is the

Corollary. *If G is a finitely generated abelian group of rank $\geqq 2$, then $\bar{G}$ is a point.*

Lemma 8. *Let f and g be monic, summable functions.*

1) *If $f(r)\leqq g(r)$ for all sufficiently large r, then there is a continuous map h: $\bar{G}(\Sigma, g)-\bar{G}(\Sigma, f)$.*

2) *If $f(r)/k\leqq g(r)\leqq kf(r)$ for some $k>0$ and all $r\in\mathbb{N}$, then $\bar{G}(\Sigma, f)$ and $\bar{G}(\Sigma, g)$ are Lipschitz equivalent.*

Proof. Uniform continuity. □

While the choice of the function f is not delicate, some care must be taken. If f is large then many sequences in $K(G,\Sigma)$ will not be Cauchy, and if f is too small then $\bar{G}(\Sigma, f)$ will be a point. The functions $f(r)=r^{-p}$, $p>1$, tend to work well. Since we are not concerned in this work with the effect of changing f, we will later restrict ourselves to the case $f(r)=r^{-2}$ for convenience.

A useful fact to note is that if M is a closed Riemannian manifold and the universal cover $\tilde{M}$ is given the induced metric from M, then each word metric on $\Pi_1(M)$ is commensurable to the induced metric on $\Pi_1(M)$ coming from viewing it as the orbit of a basepoint in $\tilde{M}$.

3. Polycyclic Groups

We have shown that a finitely generated abelian group with one end completes to a point and, more generally, that the direct sum of two infinite groups completes to a point. It seems that a group with enough "abelian behavior" completes to a point. In this section we prove a general result in this direction - a finitely generated polycyclic group with one end completes to a point. We prove the result first for nilpotent groups, and then go through the extension to polycyclic groups.

Proposition. *If G is a finitely generated nilpotent group with one end, then $\bar{G}$ is a point.*

Proof. Since $\bar{G}$ does not change under finite extensions, we can assume ([8]) that G is torsion free. Recall that the lower central series $\{G_i\}$ for G is defined by $G_1=G$ and $G_{i+1}=[G_i, G]$. G is nilpotent if $G_{i+1}=\{1\}$ for some $c\in\mathbb{N}$; the least such c is the *class* of G.

We will prove the proposition by induction on the class of G, using for an induction hypothesis the stronger statement that given $\varepsilon>0$ there exists $N>0$ so that any two elements of G of norm $\geqq N$ are within ε of each other. We know the result for G abelian, so assume it for nilpotent groups of class $\leqq m-1$ and let G have class m. We can choose a generating set $\Sigma=\{x_1, \ldots, x_n\}$ for G so that each $g\in G$ can be written uniquely as $g=x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$ and $A_r=\{g\in G\colon g=x_1^{i_1}\ldots x_r^{i_r}\}$ is a normal subgroup for $1\leqq r\leqq n$ (see [25]). The center $Z_1=A_p$ for some p. Given $g=x_1^{i_1}\ldots x_n^{i_n}$, we will write $g=x\,y$, where $x=x_1^{i_1}\ldots x_p^{i_p}$ and $y=x_{p+1}^{i_{p+1}}\ldots x_n^{i_n}$.

Let $\varepsilon>0$. $Z_1 \xrightarrow{i} G \xrightarrow{\rho} G/Z_1$, and G/Z_1 is a nilpotent group of class $\leqq m-1$. Giving G/Z_1 the generating set $\rho(\Sigma)$, by induction there exists $N_1\in\mathbb{N}$ so that any two elements of G/Z_1 of norm $\geqq N_1$ are within $\varepsilon/4$ of each other in $K(G/Z_1, \rho(\Sigma))$. Choose N_2 so that $\sum_{k=N_2}^{\infty} f(k)<\varepsilon/12$ and let $N=3\max\{N_1, N_2\}$.

Let $g=x\,y\in G$ with $|g|\geqq N$. If $|\rho(g)|\geqq N_1$, then by induction g is within $\varepsilon/4$ of $x' x_n^{N_1}$ for some $x'\in Z_1$; if $|\rho(g)|\leqq N_1$, the path

$$g=x\,y \xrightarrow{y^{-1}x_n^{N_1}} x\,x_n^{N_1}$$

has length $\leqq\varepsilon/4$. Given $x x_n^{N_1}\in G$ with $x=x_1^{i_1}\ldots x_p^{i_p}\in Z_1$, choose $M\geqq N_1$ so that $(|i_1|+\ldots+|i_p|)f(M)<\varepsilon/12$. Then the path

$$x\,x_n^{N_1} \xrightarrow{x_n's} x\,x_n^{M} \xrightarrow{x^{-1}} x_m^{M} \xrightarrow{x_n^{-1}'s} x_n^{N_1}$$

has length $\leqq\varepsilon/4$, and hence $\langle g, x_n^{N_1}\rangle\leqq\varepsilon/2$. $\square$

We proceed to generalize the argument to polycyclic groups. Recall ([26]) that we can characterize polycyclic groups as the solvable groups for which every subgroup is finitely generated. Given a polycyclic group G', by passing to a subgroup G of finite index ([26]) we can assume that there is an exact sequence of groups $N \xrightarrow{i} G \xrightarrow{\rho} H$, with $N=[G,G]$ a torsion free nilpotent group and H a finitely generated free abelian group. If we try a proof by induction on the class of N, we are quickly led to the cases $\mathbb{Z}^n \to G \to \mathbb{Z}^r$.

Lemma. *If G is a polycyclic group with an exact sequence $\mathbb{Z}^n \xrightarrow{i} G \xrightarrow{\rho} \mathbb{Z}^r$, where $n, r \neq 0$ and $i(\mathbb{Z}^n)=[G,G]$, then $\bar{G}$ is a point.*

Proof. Choose a generating set $\Sigma=\Sigma_1 \cup \Sigma_2$ for G, where $\Sigma_1=\{x_1, \ldots, x_n\}$ comes from a basis for $\mathbb{Z}^n$ and $\Sigma_2=\{z_1, \ldots, z_r\}$ projects to a basis for $\mathbb{Z}^r$. Each $g \in G$ can be written $g=xz$, where x is a word in Σ_1 and z is a word in Σ_2, and $|g| \geqq |\rho(g)|$. Each z_i acts on $\mathbb{Z}^n$ be a conjugating matrix $\varphi_i \in GL(n, \mathbb{Z})$ (for $x \in i(\mathbb{Z}^n)$, $z_i x = \varphi_i(x) z_i$). Since $G \neq \mathbb{Z}^{n+r}$, after possible relabelling we can assume that $\varphi_r(x_1) \neq x_1$. We can also assume that for each power $p \in \mathbb{N}$ $\varphi_r^p(x_1) \neq x_1$.

Given $\varepsilon>0$, choose N_1 so that $\sum_{k=N_1}^{\infty} f(k)<\varepsilon/16$ and let $N=2N_1$. We will show that if $g \in G$ has $|g| \geqq N$, then for sufficiently large $p \in \mathbb{N}$, $\langle g, \varphi_r^p(x)\rangle < \varepsilon/2$ in $K(G, \Sigma)$. Write $g=xz$, let $\varphi \in GL(n, \mathbb{Z})$ be the conjugating matrix for z, and assume for convenience that the exponent of z_r in z is non-negative. Choose $p', q \in \mathbb{N}$ so that $|x \varphi \circ \varphi_r^q(x_1)| \geqq N+|\rho(z)|$ and $|\varphi_r^p(x_1)| \geqq N+|x \varphi \circ \varphi_r^q(x_1)|$ for all $p \in \mathbb{N}$ with $p \geqq p'$. Let $y=\varphi \circ \varphi_r^q(x_1)$. Then for $p \in \mathbb{N}$ with $p \geqq p'$, the path in $K(G, \Sigma)$ given by $g = xz \xrightarrow{z_r^q} xzz_r^q$

$$\xrightarrow{x_1} xzz_r^q x_1 \xrightarrow{z_r^{-q}} xzz_r^q x_1 z_r^{-q} = xyz \xrightarrow{z^{-1}} xy \xrightarrow{z_r^p} xyz_r^p$$
$$\xrightarrow{x_1} xyz_r^p x_1 \xrightarrow{z_r^{-p}} xyz_r^p x_1 z_r^{-p} \xrightarrow{(xy)^{-1}} \varphi_r^p(x_1)$$

has length $\leqq \varepsilon/2$. □

Theorem. *If G is a polycyclic group with one end, then the completion $\bar{G}$ is a point.*

Proof. As mentioned previously, we can assume that there is an exact sequence $N \to G \to H$, with $N=[G,G]$ a torsion free nilpotent group and H a finitely generated free abelian group. We prove the theorem by induction on the class of N. The above lemma gives a proof for N abelian. If N is not abelian, let $Z_1 = \text{center}(N)$. There is an exact sequence $Z_1 \to G \to G/Z_1$, where G/Z_1 is a polycyclic group whose commutator subgroup has class less than the class of N. Using this, the proof of the induction step is an easy consequence of the proofs of the above lemma and proposition. □

The above result leads to questions in two different areas, which we will now briefly discuss.

Given a finitely generated group G with a chosen, finite generating set and associated word metric and norm, define the growth function ([13, 15]) $\phi: \mathbb{N} \to \mathbb{N}$ by $\phi(t)=$ the cardinality of $\{g \in G: |g| \leqq t\}$. G has *polynomial growth* if there is a polynomial $f(t)$ with $\phi(t) \leqq f(t)$ for all $t \in \mathbb{N}$ (this is independent of the generating set). If G has a nilpotent subgroup of finite index then G has polynomial growth

(Bass [2], Wolf [26]); the converse is conjectured. The conjecture has been proved for solvable groups (Milnor [14] and Wolf [26]), linear groups (Tits [24]), and groups of differentiable germs (Plante and Thurston [22]). By the above result, a weaker conjecture would be the

Conjecture. *If G has polynomial growth then $\bar{G}$ is a point.*

J. Milnor [16] has shown the following result.

Theorem. *If the torsion free, finitely generated group G has a polycyclic subgroup of finite index, then G is the fundamental group of a complete, affinely flat manifold.*

In the same paper he raises the question of whether the converse is true, and notes that if the converse is false then $\mathbb{Z} * \mathbb{Z}$ gives a counterexample. Since $\overline{\mathbb{Z} * \mathbb{Z}}$ is a Cantor set, the converse is equivalent to the question of whether the fundamental group of a complete, affinely flat manifold completes to a point when the group has one end. A related question is whether the fundamental group of a compact, affinely flat manifold completes to a point.

4. Kleinian Groups

A *Kleinian group* G is a discrete subgroup of $PSL(2, \mathbb{C})$. If we think of the Poincaré disk model for hyperbolic 3-space, in which $H^3 = \mathrm{int}(D^3)$, with the metric given by $ds^2 = 4\,dx^2/(1-r^2)^2$★, then G acts on H^3 by orientation preserving isometries, and the action extends to an action on $S^2 = \partial D^3$ by conformal homeomorphisms. G acts discontinuously on H^3 by discreteness, but the action on S^2 need not be discontinuous. The *limit set* $\Lambda(G)$ is the set of accumulation points of the orbit of a point in $\mathrm{int}(D^3)$; it is independent of the point chosen. The *regular set* $\Omega(G) = S^2 - \Lambda(G)$ (we are not assuming that $\Omega(G) \neq \emptyset$ in our definition of a Kleinian group). $\Omega(G)$ is the largest open subset of S^2 on which the action is discontinuous. For details in the theory of Kleinian groups not mentioned here, see Harvey [5], Marden [9], and Thurston [23].

Let $x \in S^2$. A *horosphere* around x is a Euclidean sphere in $\mathrm{int}(D^3) = H^3$ which is tangent to $S^2 = \partial D^3$ at the point x. A *horoball* is the convex region in H^3 bounded by a horosphere.

Let $z \in \Lambda(G)$ and let L be a geodesic in H^3 which converges to z. z is a *point of approximation* if there is a sequence $\{h_i\}$, $h_i \in G$, so that $\{h_i(0)\}$ converges to z and stays within a finite distance of L (see Beardon and Maskit [3] for alternate, equivalent definitions). Fixed points of parabolic elements are not points of approximation. If $z \in \Lambda(G)$ is a parabolic fixed point with isotropy subgroup J ($J = \{g \in G \colon g(z) = z\}$), then z is a *cusped parabolic point* if either J has rank two (a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup) or if J has rank one (a finite extension of $\mathbb{Z}$) and there is a non-empty region U in H^3 which is the union of two disjoint half-spaces and is precisely invariant under J.

★ The standard metric for the Poincaré disk model is $dx^2/(1-r^2)^2$; we use this metric since it is compatible with the metric in the upper half space model for H^3

We can find a fundamental region for the action of G on H^3 as follows. Pick a basepoint $y \in H^3$ which is not fixed by any non-trivial element of G. Then the Poincaré fundamental polyhedron $P = P_y = \{x \in H^3: \rho(x, y) \leqq \rho(x, g(y))$ for all $g \in G\}$, where $\rho(\; , \;)$ is the hyperbolic distance, is a fundamental region for the action of G on H^3. The faces of P are identified in pairs under the action of G. G is *geometrically finite* if P has a finite number of faces. If G is geometrically finite, then (Beardon and Maskit [3]) every limit point is either a point of approximation or a cusped parabolic point.

Now let G be a finitely generated Kleinian group with a chosen, finite generating set Σ, $|g|$ = word norm of $g \in G$, $h(g) = \rho(0, g(0))$, and $r(g)$ = the Euclidean distance between 0 and $g(0)$. If G is geometrically finite assume that Σ is the set of face pairing transformations. In what follows, by $\bar{G}$ we will mean the completion with respect to the summable function $f(r) = r^{-2}$. $h(g)$ and $r(g)$ are related by the equations

$$h(g) = \log|(1+r(g))(1-r(g))^{-1}|, \qquad r(g) = (e^{h(g)} - 1)(e^{h(g)} + 1)^{-1}.$$

Proposition. *If there are constants N, k so that $2\log|g| - k \leqq h(g)$ for all $g \in G$ with $|g| \geqq N$, then there is a continuous, G-equivariant surjection $\varphi: \bar{G} \twoheadrightarrow \Lambda(G)$.*

Proof. Define $\phi: K(G, \Sigma) \to \operatorname{int}(D^3)$ as follows. Map the vertex labelled by $a \in G$ to $a(0) \in D^3$ and map the edge joining the vertices $a, b \in G$ to the hyperbolic geodesic are between $a(0)$ and $b(0)$. We are interested in ϕ as a map of metric spaces, where we are thinking of D^3 with the Euclidean metric.

Suppose $2\log|g| - k \leqq h(g)$ for all $g \in G$ with $|g| \geqq N$. Let $a \in G$ have $|a| = n \geqq N$, and let L be an edge in $K(G, \Sigma)$ joining the vertices $a, b \in G$. L has length either n^{-2} or $(n+1)^{-2}$, depending on $|b|$. Since the hyperbolic length of $\phi(L)$ is bounded by $\max\{h(g): g \in \Sigma\}$, there exists $K > 0$ so that $\phi(L)$ has Euclidean length $\leqq K(1 - r(a)^2) = 4K(e^{h(a)} + 2 + e^{-h(a)})^{-1} \leqq 4K e^k n^{-2}$. Thus ϕ is Lipschitz and induces a map ϕ': completion $(K(G, \Sigma)) \to$ completion $(\phi(K(G, \Sigma)))$. ϕ' restricts to a map $\varphi: \bar{G} \to \Lambda(G)$. φ is onto since $\Lambda(G)$ is the set of accumulation points of the orbit of 0, and φ is equivariant since $\phi | G \subset K(G, \Sigma)$ commutes with the action of G. □

Lemma. *If the geometrically finite Kleinian group G has no parabolic elements, then there are constants $k, k' > 0$ so that $k|g| \leqq h(g) \leqq k'|g|$ for all $g \in G$.*

Proof. The inequality $h(g) \leqq k'|g|$ holds with $k' = \max\{h(a): a \in \Sigma\}$ by the triangle inequality. To prove the other inequality, let $\mathcal{N} \subset H^3$ be the Nielsen convex region ($\mathcal{N}$ is the convex hull of $\Lambda(G)$), and assume for convenience that the basepoint $0 \in \mathcal{N}$. $\mathcal{N}$ is invariant under the action of G, $P' = P \cap \mathcal{N}$ is compact, and the geodesic arc L between 0 and $g(0)$ is contained in $\mathcal{N}$ for each $g \in G$ ([23]).

Let $d = \operatorname{diam}(P')$ and let $C = \max\{|g|: g \in G,\ h(g) \leqq 7d\}$. Given $g \in G$ and a geodesic arc L between 0 and $g(0)$, divide L into intervals of length $5d$ (with one shorter interval), and connect each endpoint of each interval to the closest point in the orbit of 0. This gives the estimate $|g| \leqq C(1 + h(g)/5d)$, which establishes the lemma. □

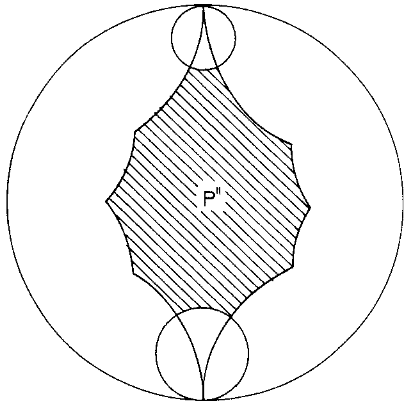

Fig. 2. $P'' = P \cap \mathcal{N}'$

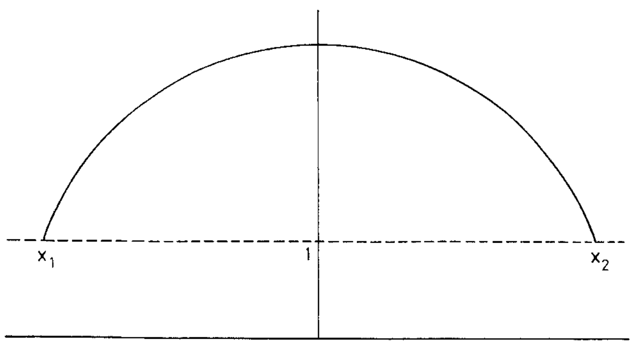

Fig. 3. Hyperbolic distance and distance on the horosphere

If G is geometrically finite and has parabolic elements, then P' will not be compact because it will have cusps. Remove a disjoint equivariant family of open horoballs around the parabolic fixed points ([9, 23]), and let $\mathcal{N}' = \mathcal{N}$-horoballs. $P'' = P \cap \mathcal{N}'$ is compact (see Fig. 2). Define a metric $\rho'(\ ,\ )$ on $\mathcal{N}'$ by letting $\rho'(a, b)$ be the minimum length of a path in $\mathcal{N}'$ between a and b, and let $h'(g) = \rho'(0, g(0))$ (having chosen the horoballs small enough so that the geodesic arc between 0 and $g(0)$ is in $\mathcal{N}'$ for each $g \in \Sigma$). As in the above lemma there are constants $k, k' > 0$ with $k|g| \leqq h'(g) \leqq k'|g|$ for all $g \in G$. $h'(g)$ differs from $h(g)$ whenever the geodesic arc L between 0 and $g(0)$ leaves $\mathcal{N}'$ and cuts across a horosphere. To see just how they differ we need to compute, for two points x_1 and x_2 on a horosphere, their hyperbolic distance and the distance of a path between them which stays on the horosphere.

We can conjugate to the following situation; x_1 and x_2 lie in the $y-z$ plane in the upper half space model for H^3, at height 1 (see Fig. 3). In particular let $x_1 = (0, -\sqrt{r^2-1}, 1)$ and $x_2 = (0, \sqrt{r^2-1}, 1)$, where r is the Euclidean radius of the geodesic arc between them. The path between x_1 and x_2 on the horosphere (defined by $z=1$, the parabolic fixed point is at ∞) has length $2\sqrt{r^2-1}$, and

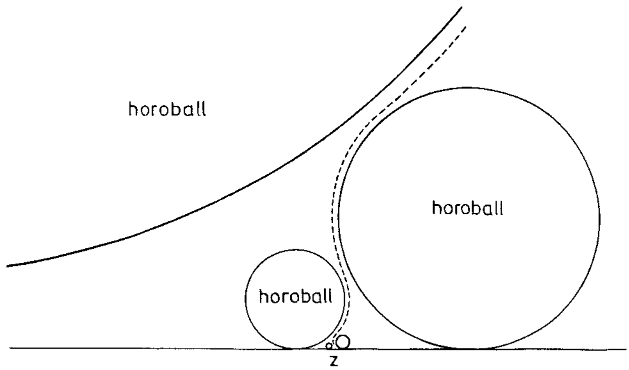

Fig. 4. A point of approximation z with a path in $\mathcal{N}'$ converging to z

their hyperbolic distance $\rho(x_1, x_2) = 2\log(\sqrt{r^2-1}+r)$ (the hyperbolic metric is given by $ds^2 = z^{-2}dx^2$). Since $h(g)$ only differs from $h'(g)$ when the arc L cuts across a horosphere, this computation shows that $h'(g) \leqq e^{h(g)/2}$, and so $k|g| \leqq e^{h(g)/2}$ and $2\log|g| + 2\log k \leqq h(g)$. This establishes the

Proposition. *If G is a geometrically finite Kleinian group, then there are constants N, $k>0$ so that $2\log|g| - k \leqq h(g)$ for all $g \in G$ with $|g| \geqq N$.*

Corollary. *If G is a geometrically finite Kleinian group, then there is a continuous, G-equivariant map $\varphi: \bar{G} \twoheadrightarrow \Lambda(G)$.*

φ will not be a homeomorphism in general, for the following reason. The group $\mathbb{Z}$ has completion $\bar{\mathbb{Z}}$ = two points. If we take a hyperbolic element $g \in PSL(2, \mathbb{C})$, then the Kleinian group $\mathbb{Z}$ generated by g has a limit set with two points, and $\varphi: \bar{\mathbb{Z}} \to \Lambda(\mathbb{Z})$ is a homeomorphism. But if instead we take a parabolic element $g \in PSL(2, \mathbb{C})$, then the Kleinian group $\mathbb{Z}$ generated by g has limit set a single point, and $\varphi: \bar{\mathbb{Z}} \to \Lambda(\mathbb{Z})$ is 2-to-1. This is precisely the way in which φ fails to be a homeomorphism when G is geometrically finite.

Theorem. *If G is a geometrically finite Kleinian group, then the induced map $\varphi: \bar{G} \twoheadrightarrow \Lambda(G)$ is 2-to-1 onto parabolic points of rank one, and injective everywhere else.*

Proof. Suppose G is geometrically finite. By passing to a subgroup of finite index ($\bar{G}$ and $\Lambda(G)$ do not change under finite index), we can assume that G is torsion free.

Let $\omega \in \bar{G}$ such that $z = \varphi(\omega)$ is a point of approximation (see Fig. 4), and let $\{w_i\}$ be a shortest sequence which defines ω. We claim there is a subsequence of $\{w_i(0)\}$ which converges to z in a cone (this is equivalent to converging within a finite distance of a geodesic asymptotic to z). To see this connect each $w_i(0)$ to $w_{i+1}(0)$ by a geodesic arc to form a path v starting at 0 and converging to z. Since the word metric on G is commensurable to the induced metric coming from paths in $\mathcal{N}'$, for $a, b \in v$ their distance along v is commensurable to $\rho'(a, b)$. Let v_i be the geodesic arc between 0 and $w_i(0)$. As $i \to \infty$ the number of vertices

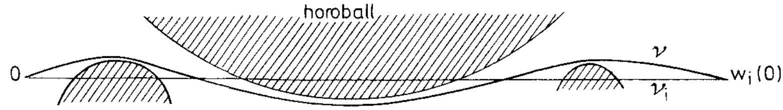

Fig. 5. v comes near v_i between horoballs

of v, between 0 and $w_i(0)$, within a bounded distance of v_i goes to infinity (see Fig. 5), since the length of a path in H^3 goes up exponentially with its distance from the geodesic (the point is that v must come near v_i between horoballs, and either v_i crosses a lot of horoballs or the length of $v_i|\mathcal{N}'$ goes to infinity). This implies that a subsequence of $\{w_i(0)\}$ converges to z in a cone.

Suppose $\omega' \in \bar{G}$ with $\varphi(\omega) = z = \varphi(\omega')$, and let $\{w_i\}$, $\{w'_i\}$ be shortest sequences defining ω and ω'. Choose subsequences $\{w_j\}$ and $\{w'_j\}$ so that $\{w_j(0)\}$ and $\{w'_j(0)\}$ converge to z in a cone. Since they stay in a cone, one can find a sequence $\{m_j\}$, $m_j \in G$, such that each $m_j = w_{j_k}$ or w'_{j_k}, $\{m_j\}$ goes back and forth infinitely often between the w_j's and the w'_j's, and $\{m_j\}$ is a Cauchy sequence in $K(G, \Sigma)$. $\{m_j\}$ determines a point $\alpha \in \bar{G}$; clearly $\omega = \alpha = \omega'$. Thus φ is injective on the pre-image of points of approximation.

If $z \in \Lambda(G)$ is not a point of approximation, then it is a parabolic fixed point ([3]). Let $\omega \in \bar{G}$ with $\varphi(\omega) = z$, $\{w_j\}$ a shortest sequence in G defining ω, and v the path in $\mathcal{N}'$, starting at 0 and converging to z, as defined above. Since z is a parabolic fixed point, v stays outside of some horoball about z. By an argument similar to the one above, one can show that $\omega \in \bar{i}(\bar{J})$, where J is the isotropy subgroup of z (we are using the fact that the inclusion $i: J \to G$ is a quasi-isometry). Since G is torsion free J is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. If J is $\mathbb{Z} \oplus \mathbb{Z}$ then $\bar{J}$ is a point and φ is injective on the pre-image of z. If J is $\mathbb{Z}$, then $\bar{J}$ is two points and φ is at most 2-to-1 on the pre-image of z.

To prove that φ is exactly 2-to-1 onto z when z is a parabolic point of rank one, we will show that $\bar{i}: \bar{J} \to \bar{G}$ is injective. Let g generate J; the two points in $\bar{J}$ are defined by the Cauchy sequences $\{g^n\}$ and $\{g^{-n}\}$. Recall that φ was defined using a map $\phi: K(G, \Sigma) \to H^3 = \operatorname{int}(D^3)$. If we give $\phi(K(G, \Sigma))$ a metric by the Euclidean length of paths then ϕ is still uniformly continuous, so it suffices to show that $\operatorname{dist}(g^n(0), g^{-n}(0))$ does not go to zero. Since z is a cusped parabolic point, there are two disjoint half spaces U_1 and U_2 in H^3 which do not intersect $\phi(K(G, \Sigma))$ (see Fig. 6). $\phi(K(G, \Sigma))$ stays outside of a horoball around z, and the part of the bounding horosphere not in $U_1 \cup U_2$ is a strip of finite hyperbolic width. Since $\{g^n(0)\}$ and $\{g^{-n}(0)\}$ go out in different directions of the strip, $\operatorname{dist}(g^n(0), g^{-n}(0))$ does not go to zero (U_1 and U_2 prevent short cuts between the two ends of the strip). □

Here is an example to illustrate the theorem. Let S be a closed surface of genus $g \geqq 2$, and let $G = \Pi_1(S)$. $\bar{G} \approx S^1$, since we can represent G as a geometrically finite Fuchsian group with limit set S^1 and no parabolic points. Let Γ be a regular b-group which is isomorphic to G. We know that $\Lambda(\Gamma)$ is the image of a circle because of Abikoff's theorem [1] that $\Lambda(\Gamma)$ is locally connected. Since Γ is geometrically finite a corollary to the above theorem is the

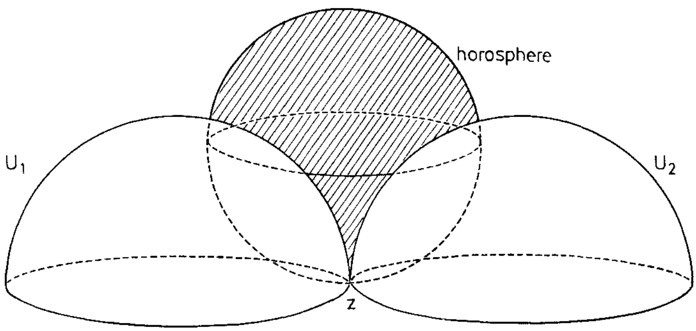

Fig. 6. A cusped parabolic point z of rank one

Proposition. *Let Γ be a regular b-group such that $\Gamma \approx \Pi_1(S)$, where S is a closed surface. Then there is a continuous map $\varphi: S^1 \twoheadrightarrow \Lambda(\Gamma)$. φ is 2-to-1 onto parabolic points of rank one, and injective elsewhere.*

One of our initial goals in studying completions was to construct S^{n-1} from the fundamental group $G = \Pi_1(M)$ when M is a closed hyperbolic n-manifold. In this case G acts on hyperbolic space H^n and $M \approx H^n/G$. By thinking of the Poincaré disk model of H^n as $\operatorname{int}(D^n)$, it is easy to see that the action of G on H^n extends to an action of G on $S^{n-1} = \partial D^n$. Since the Poincaré fundamental polyhedron for the action of G on H^n is compact, word metrics on G are commensurable to the induced metric coming from hyperbolic distance. Every point in S^{n-1} is a point of approximation, and as in the above theorem (the proof is easier here since there are no parabolic fixed points) we have the

Theorem. *Let $G = \Pi_1(M)$, where M is a closed hyperbolic n-manifold. Then there is a G-equivariant homeomorphism $\varphi: \bar{G} \to S^{n-1}$.*

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Received May 12, 1979