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Construction of Non-Linear Local Quantum Processes. II*

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Introduction

Part I of this series treated singular perturbations V of given self-adjoint operators H acting in $L_2(M)$, M being a given probability measure space. If A is a given self-adjoint operator in a Hilbert space $\mathbf{H}$ such that $A \geqq \varepsilon I$ for some $\varepsilon > 0$, the theory of Part I (referred to as "I" in the following) applied to operators H of the form $d\Gamma(A)$, acting in an associated quantum field space and canonically induced from A (see I for notation). As a corollary, certain non-linear local quantum processes could be constructed.

This article extends the theory of such singular perturbations, treating an abstract Hilbert space in which there is given a general kind of "calibration" by auxiliary norms which in I were the L_p-norms. This is natural from a purely mathematical viewpoint, and is useful for further applications in quantum field theory. In addition, the perturbation of a proper lowest vector is treated.

The theory applies in particular to any self-adjoint operator H in a Hilbert algebra $\mathbf{K}$ with unit having the properties that e^{-tH} is bounded from L_2 to L_p for some $t > 0$ and $p > 2$, and that e^{-tH} is bounded from L_p to L_p for all sufficiently small t by e^{at}, for some constant a. Concerning V, it suffices if it is of the form $L_v + R_v$, where v is an hermitian element of $\mathbf{K}$ such that $\|v\|_p + \|e^{-v}\|_p < \infty$ for all $p < \infty$ and L_v (respectively, R_v) denote the suitably formulated operations of left and right multiplication by v. Under these hypotheses, there exists a self-adjoint operator denoted $H \mp V$ having the properties, among others, that $H + f_n(V) \to H \mp V$ for every sequence $\{f_n\}$ of real bounded Baire functions on R^1 such that $f_n(\lambda) \to f(\lambda)$ and $|f_n(\lambda)| \leqq |\lambda|$ for all $\lambda \in R^1$; that $e^{-t(H \mp V)}$ is bounded from L_2 to $L_{2e^{\beta t}}$ by e^{bt} for some $\beta > 0$ and real constant b; for arbitrary $k \in [0, 1]$, $H \mp V \geqq (1-k) H - (k \inf H + 2 \log \|e^{-v}\|_{4/kc})$, where c is a constant dependent only on H; and $H \mp V \supset H + V$, which is essentially self-adjoint.

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In case **K** has the form $L_2(M)$, where M is a given probability measure space (or equivalently, if **K** is abelian), and H is "indecomposable" in a certain sense relevant to the theory of positivity-preserving operators, the same is true of $H \mp V$, implying the unicity and essential positivity of a proper lowest vector (PLV) for $H \mp V$, if any exists. If H is inverse compact (i.e. $(cI+H)^{-1}$ is compact for some constant c), a PLV for $H \mp V$ necessarily exists; it is here shown that this remains the case if H is in a certain sense "approximately inverse compact" (AIC). All this applies in particular to the case studied in I in which, briefly and informally, H is the hamiltonian of the covariant Weyl process determined by the differential equation $\Box \phi_0 = m^2 \phi_0$, with the condition $\phi_0 = \phi_0^*$, in two space-time dimensions, and V has the form $\int q \circ_{E_0} \phi_0(\vec{x}, 0) f(\vec{x}) d\vec{x}$, where q is a given non-negative polynomial on R^1, E_0 is the state determined by the PLV for H, $\phi_0(\vec{x}, t)$ is the generalized-operator-valued distribution on space, at each fixed t, determined by the process and satisfying the cited equation, f is a given non-negative, mildly regular, function of compact support on R^1, and the subscript "E_0" to the composition symbol $\circ$ signifies that renormalization is made, in the sense of [9], with respect to the state E_0, in formulating the *a priori* undefined non-linear function q of the operator-valued distribution of symbolic kernel $\phi_0(\vec{x}, 0)$. For any state E, renormalization of powers of a Weyl process in space with respect to E is a local operation in space and is independent of any dynamical structure; in the case of the particular state E_0, this renormalization coincides formally with the heuristically-established concept of "Wick ordering". It is here shown that the ("perturbed") process whose kernel is given symbolically by the equation

$$\phi(\vec{x}, t) = e^{it(H \mp V)} \phi_0(\vec{x}, 0)\, e^{-it(H \mp V)}$$

(which is rigorously valid when $\phi(\vec{x}, t)$, etc. is formulated in terms of generalized-operator-valued functions) has the property that for any fixed t, the renormalized powers with respect to the PLV for $H \mp V$ exist and enjoy certain regularity properties; and that $\phi(\vec{x}, t)$ satisfies the local partial differential equation

$$\Box \phi(\vec{x}, t) = m^2 \phi(\vec{x}, t) + q' \circ_E \phi(\vec{x}, t) + r(\vec{x}) \circ_E \phi(\vec{x}, t),$$

where $r(\vec{x})$ is for each $\vec{x}$ a polynomial on R^1 of degree less than that of q'. If space is taken as the circle T^1 in place of R^1, and if f is taken as constant, the same is true, and in addition $r(\vec{x})$ is independent of $\vec{x}$; it is left open whether the mapping $q \to q' + r$ carries the set **P** of all non-negative polynomials on R^1 *onto* **P**$'$. Thus, in this case, when space is T^1, there is a solution in a natural sense for the ("quantized") partial differential equation $\Box \phi = m^2 \phi + p(\phi)$, for an extensive class of certain polynomials

$p \in \mathbf{P}'$ — not however according to present knowledge, necessarily including the polynomials $p(\lambda) = g\lambda^{2n+1}$ ($g > 0$; $n = 1, 2, \ldots$).

Recent work of Gross [4] has initiated the extension of the classic theory of positivity-preserving operators due to Perron, Krein, *et al.*, to the (non-lattice) case of a probability Hilbert algebra, and at the same time eliminated the compactness assumption, replacing it by the assumption that the operator is bounded from L_2 to L_p for some $p > 2$. The existence as well as essential positivity of a highest proper vector is shown for any self-adjoint such operator, as well as, in the abelian case, the finite multiplicity of the corresponding proper value. These remarkably general results are in a direction similar to that of the section of the present work dealing with the PLV for $H \mp V$, and are indeed applicable to certain such cases, associated with Clifford as well as Weyl quantum processes. On the other hand, the present results, while limited to the abelian case, and in part making the assumption that H is AIC, yield a somewhat stronger conclusion about the spectrum near its infimum, may well be adaptable to the Clifford process case, and do not require boundedness from L_2 to L_p, or in the case of the existence result, a positivity-preserving assumption.

Recent work by Glimm and Jaffe [3] treats, in another form and spirit, the quantum process application described earlier, in the special case in which $q(\lambda) = g\lambda^4$, with $g > 0$. Unicity and existence of a PLV are shown with the use of Markov process theory and estimates dependent on the low degree of this q. It is stated that "the quantum field ϕ is a solution to [the equation] $\Box\phi = m^2\phi + 4g\phi^3$, provided that ϕ^3 is suitably interpreted". Actually ϕ^3 is defined in terms of its Cauchy data at time $t = 0$ and the operator $H \mp V$, rather than in terms of the Cauchy data for ϕ itself at time t; it is the "Wick-ordered" cube of the free field which is directly involved, rather than a generally defined operation on ϕ. In terms of the local renormalized operations treated here, the equation satisfied by ϕ is

$$\Box\phi = m^2\phi + 4g\phi^3 + r_0(x)\phi^2 + r_1(x)\phi + r_2(x),$$

where the $r_i(x)$ are certain f-dependent continuous functions on R^1, The "power" $\phi(\vec{x}, t)^r$ involved here is definable, say as the kernel of a distribution in space, as the limit of the operators

$$(\phi_h(\vec{x}, t)^r + s_0(\vec{x})\,\phi_h(\vec{x}, t)^{r-1} + \cdots + s_r(\vec{x})),$$

where $\phi_h(\vec{x}, t) = \int \phi(\vec{x} + \vec{y}, t)\, h(\vec{y})\, d\vec{y}$, and the $s_j(x)$ are certain well-defined continuous functions determined by the PLV for $H \mp V$, as the "test function" h converges to the delta-distribution. It is thus constructible in explicit terms from the Cauchy data for ϕ at time t in any neighborhood of $\vec{x}$.

1. Moderated Perturbations

A *calibrated* Banach space is here defined as a Banach space $\mathbf{B}$, a simply ordered set P with an infimum p_0, and a mapping $p \to \|.\|_p$ from P to non-negative functionals on $\mathbf{B}$, which functionals are (possibly-infinite-)valued norms, and which mapping has the properties:

1) $\|.\|_{p_0}$ is the given norm in $\mathbf{B}$;

2) $\|u\|_p$ is monotone increasing as a function of p, for every $u \in \mathbf{B}$;

3) the common part of the sets $\mathbf{B}_p = [u \in \mathbf{B}: \|u\|_p < \infty]$ as p ranges over P is dense in $\mathbf{B}$;

4) if $u_n \to u$ in $\mathbf{B}$, then $\|u\|_p \leq \sup_n \|u_n\|_p$ for all $p \in P$.

Example. In any probability measure space, or more generally in any probability gage space, say M, let $\mathbf{B} = L_{p_0}(M)$ for some $p_0 \in [1, \infty)$; let $P = [p_0, \infty)$, and let $\|u\|_p = (\int |u|^p)^{1/p}$. The foregoing conditions are then satisfied.

The subset $[u \in \mathbf{B}: \|u\|_p < \infty]$ will be denoted as $\mathbf{B}_p$. As a normed linear space with the norm $\|.\|_p$, $\mathbf{B}_p$ will be denoted as $[\mathbf{B}_p]$. The common part of the $\mathbf{K}_p$ as p varies over P will be denoted as $\mathbf{K}_\omega$. In the topology in which convergence means convergence in each $\|.\|_p$-norm, $\mathbf{K}_\omega$ will be denoted as $[\mathbf{K}_\omega]$. If T is any operator in $\mathbf{B}$, $\|T\|_{p,q}$ will denote its bound as an operator from $[\mathbf{K}_p]$ to $[\mathbf{K}_q]$, i.e. the supremum of $\|Tu\|_q/\|u\|_p$ as u varies over the non-zero elements of $\mathbf{K}_p$.

Let $V(t)$, $t \geq 0$, be a given continuous semigroup in $\mathbf{B}$, i.e. each $V(t)$ is a continuous linear operator in $\mathbf{B}$, $V(0) = I$, $V(t)\,V(t') = V(t+t')$ for all t and $t' \geq 0$, and $V(t)$ is a continuous function of t, in the strong operator topology. This topology for operators will be employed exclusively in the following unless otherwise specified. The semigroup $V(t)$, or the negative of its generator, will be said to be of type (Q, α, a, t_0), where Q is a given subinterval of P, which is here and henceforth assumed to be a real, finite or infinite, interval, and α, a, and t_0 are given real numbers, of which $t_0 > 0$, in case $\|V(t)\|_{p,\, p e^{\alpha t}} \leq e^{at}$ for all $t \in [0, t_0]$ and $p \in Q$ (in particular, it is assumed that $p e^{\alpha t} \in P$ for all such t and p). It will be said to be of type (Q, α, a) if there exists a $t_0 > 0$ such that it is of type (Q, α, a, t_0).

While the theory to be developed fits logically into a Banach space context, present applications are limited to the Hilbert space case, and only this case will here be treated. A sequence $\{A_n\}$ of self-adjoint operators in a complex Hilbert space $\mathbf{K}$ is said to converge to a self-adjoint operator A in $\mathbf{K}$ (symbolically, $A_n \to A$) in case any one of the following equivalent conditions holds: (a) $e^{itA_n} \to e^{itA}$ for all real t; (b) $f(A_n) \to f(A)$ for all bounded continuous functions f; (c) the same as (b) for all f which are continuous and vanish at infinity on R^1 (or any set of such f which separates R^1); (d) the spectral family $E_n(\lambda)$ for A_n converges to that, $E(\lambda)$, for A, for every λ which is not a point of discontinuity of $E(\lambda)$. This notion is treated in Kato [6], where it is called "generalized strong convergence" (in particular, a sufficient condition that $A_n \to A$ is that $A_n x \to Ax$ for all x in a domain $\mathbf{D}$ on which A is essentially self-adjoint). With regard to the indicated equivalences, see also Kallman [5]. In the present article, only the following result from Segal [11] will be used: if the A_n are uniformly bounded from below, then (a) holds if and only if $e^{-tA_n} \to e^{-tA}$ for all $t>0$ (or equivalently, for one $t>0$).

Let H and V be given self-adjoint operators in the complex Hilbert space $\mathbf{K}$, and let $\mathbf{S}$ be a given set of sequences $\{V_n\}$ of bounded self-adjoint operators, each sequence being convergent to V. The *moderated perturbation of H by V*, relative to the given set $\mathbf{S}$, is said to exist if there exists a self-adjoint operator H' in $\mathbf{K}$ such that $H+V_n \to H'$, for all sequences $\{V_n\} \in \mathbf{S}$. In this event, H' is called the *moderated sum* of H and V, and denoted as $H \dotplus V$.

In the case of a calibrated Hilbert space, it will be convenient to make the normalization $p_0 = 2$.

Theorem 1. *Let H and V be given self-adjoint operators in the given calibrated Hilbert space* $\mathbf{K}$. *Let* α, β, γ, a, b, *and* c *be given real numbers, such that* $\alpha \geqq 0$ *and* $\alpha+\gamma \geqq 0$. *Let* p_0 *and* q_0 *be given numbers in P such that* $2 < q_0 < p_0$.

Suppose that H is of type $(2, \alpha, a)$ *and also of type* $([2, p_0], \beta, b)$. *Let* $\mathbf{C}$ *denote the class of all self-adjoint operators W in* $\mathbf{K}$ *of type* $([2, p_0], \gamma, c)$, *such that* $\mathbf{K}_{q_0} \subset \mathbf{D}(W)$ *and* $\|W\|_{q_0, 2} < \infty$, *and let* $\mathbf{C}_0$ *denote the set of all bounded elements of* $\mathbf{C}$. *Suppose that* $V \in \mathbf{C}$, *and that* $\mathbf{S}$ *is a given non-empty set of sequences* $\{V_n\}$ *of elements of* $\mathbf{C}_0$, *such that for each sequence,* $V_n \to V$ *and* $\|V_n - V\|_{q_0, 2} \to 0$. *Then*

1) $H \dotplus V$ *exists, is* $\geqq -(a+c)I$, *and for any* $p_1 \in (2, p_0)$ *is of type* $(I, \beta+\gamma, b+c)$, *where* $I = [2, p_1]$ *if* $\beta+\gamma > 0$, $I = [p_1, p_0]$ *if* $\beta+\gamma < 0$, *and I may be taken as* $[2, p_0]$ *if* $\beta+\gamma = 0$.

2) *If* V' *and* $\mathbf{S}'$ *satisfy the same conditions as V and* $\mathbf{S}$, *with possibly different constants* γ' *and* c', *such that* $\alpha+\gamma+\gamma' > 0$, *then for all* $u \in \mathbf{K}_{p_0}$ *and all sufficiently small t,*

$$e^{-t(H \dotplus V)} u - e^{-t(H \dotplus V')} u = \int_0^t e^{-(t-s)(H \dotplus V')} (V - V') e^{-s(H \dotplus V)} u \, ds,$$

the integral being taken in the Riemann sense, the integrand being continuous.

Moreover, if $V_n + V_n' \to (V+V')^*$ *for all sequences and* $\{V_n\} \in \mathbf{S}$ *and* $\{V_n'\} \in \mathbf{S}'$, *then* $(H \dotplus V) \dotplus V' = H \dotplus (V+V')^*$.

3) *If* $\alpha+\gamma > 0$ *and* $\beta+\gamma > 0$, *and if H is of type* $(2, \alpha, a, t_0)$ *with* $2e^{\alpha t_0} > q_0$, *then every entire vector w for* $H \dotplus V$ *is in* $\mathbf{D}(H) \cap \mathbf{D}(V)$, *and* $(H \dotplus V) w = Hw + Vw$.

Lemma 1.1. *If $\{H_n\}$ is a sequence of self-adjoint operators converging to a self-adjoint operator H, and if V is a bounded self-adjoint operator, then $H_n + V \to H + V$.*

Proof. It suffices to show that $e^{it(H_n+V)} \to e^{it(H+V)}$ for all real t. By Duhamel's formula, for arbitrary $u \in \mathbf{K}$,

$$e^{it(H_n+V)}u = e^{itH_n}u + i\int_0^t e^{i(t-s)H_n} V e^{is(H_n+V)} u\, ds,$$

and similarly for $e^{it(H+V)}$. Subtracting, it follows that

$$e^{it(H_n+V)}u - e^{it(H+V)}u$$
$$= (e^{itH_n}u - e^{itH}u) + i\int_0^t (e^{i(t-s)H_n} V e^{is(H_n+V)} - e^{i(t-s)H} V e^{is(H+V)}) u\, ds.$$

Setting $g_n(t) = \|e^{it(H_n+V)}u - e^{it(H+V)}u\|$, and noting that the integrand may be written as

$$(e^{i(t-s)H_n} - e^{i(t-s)H}) V(e^{is(H+V)})u + e^{i(t-s)H_n} V(e^{is(H_n+V)} - e^{is(H+V)})u,$$

it follows that

$$g_n(t) \leqq A_n(t) + \|V\| \int_0^t g_n(s)\, ds,$$

where

$$A_n(t) = \|e^{itH_n}u - e^{itH}u\| + \int_0^t \|(e^{i(t-s)H_n} - e^{i(t-s)H}) V e^{is(H+V)} u\|\, ds.$$

The integrand in the last integral is bounded uniformly in n and s, and $\to 0$ as $n \to 0$ for each s. Hence $A_n(t) \to 0$ for each t, and boundedly in each finite t-interval. It follows from Gronwall's inequality that $g_n(t) \to 0$.

The following lemma is essentially a special case of Lemma 1.3, but it is convenient to state it separately.

Lemma 1.2. *Let A and B be self-adjoint operators in K, and suppose that B is bounded. If A is of type $(2, \varepsilon, a)$ and B is of type $([2, p_0], -\varepsilon, b)$ for some $p_0 > 2$, where $\varepsilon > 0$, then $A + B$ is of type $(2, 0, a+b, \infty)$.*

Proof. By the Lie-Trotter formula, $e^{-t(A+B)} = \lim_n (e^{-tB/n} e^{-tA/n})^n$, so it suffices to show that $\|e^{-tB/n} e^{-tA/n}\| \leqq e^{(a+b)t/n}$. Now

$$\|e^{-tB/n} e^{-tA/n}\| \leqq \|e^{-tA/n}\|_{2,p} \|e^{-tB/n}\|_{p,2}.$$

Taking $p = 2e^{\varepsilon t}$, the conclusion follows.

The hypothesis on B may evidently be weakened to the assumption that $\|e^{-tB}\|_{2e^{\varepsilon t},2} \leqq e^{bt}$ for all sufficiently small $t > 0$.

Lemma 1.3. *Let A and B be self-adjoint operators in $\mathbf{K}$, of which B is bounded. Suppose that A is of type $([2, r_0], \alpha, a)$ and that B is of type $([2, r_0], \beta, b)$, where $r_0 > 2$ and α positive. Then for arbitrary $r_1 \in (2, r_0)$, $A + B$*

is of type $(I, \alpha+\beta, a+b, t_1)$, *where* I *is of the form* $[2, r_1]$ *if* $\alpha+\beta>0$, *of the form* $[r_1, r_0]$ *if* $\alpha+\beta<0$, *and of the form* $[2, r_0]$ *if* $\alpha+\beta=0$; *and* $t_1 = |\alpha+\beta|^{-1}\log(r_0/r_1)$ *if* $\alpha+\beta>0$; $t_1=\infty$ *if* $\alpha+\beta=0$; *and* $t_1=|\alpha+\beta|^{-1}\log(r_1/2)$ *if* $\alpha+\beta<0$.

Proof. As in the proof of Lemma 1.2, if p is in the interior of $[2, r_0]$, then for sufficiently large n,

$$\|e^{-tB/n}e^{-tA/n}\|_{p,\,pe^{(\alpha+\beta)t/n}} \leqq \|e^{-tA}\|_{p,\,pe^{\alpha t/n}}\,\|e^{-tB}\|_{pe^{\alpha t/n},\,pe^{(\alpha+\beta)t/n}} \leqq e^{(a+b)t/n}.$$

It follows that if $[p, pe^{(\alpha+\beta)t}]\subset[2, r_0]$, then

$$\|(e^{-tB/n}e^{-tA/n})^n\|_{p,\,pe^{(\alpha+\beta)t}} \leqq e^{(a+b)t}.$$

As earlier, it results that $\|e^{-t(A+B)}\|_{p,\,pe^{(\alpha+\beta)t}} \leqq e^{(a+b)t}$.

If $\alpha+\beta>0$ and $p\in[2, r_1]$ with $2<r_1<r_0$, then $[p, pe^{(\alpha+\beta)t}]\subset[2, r_0]$ provided $r_1 e^{(\alpha+\beta)t}\leqq r_0$, giving the value of t_1 indicated. Analogous arguments apply in case $\alpha+\beta=0$ or is <0.

Proof of Theorem. Let W and W' be arbitrary in $\mathbf{C}_0$, and let u be arbitrary in $\mathbf{K}$. Then by Duhamel's formula,

$$e^{-t(H+W)}u - e^{-t(H+W')}u = \int_0^t e^{-(t-s)(H+W')}(W'-W)e^{-s(H+W)}u\,ds.$$

It follows that

$$\|e^{-t(H+W)}u - e^{-t(H+W')}u\| \leqq \int_0^t \|e^{-(t-s)(H+W')}\|_{2,2}\,\|W'-W\|_{q_0,2}\,\|e^{-s(H+W)}\|_{p_0,q_0}\,\|u\|_{p_0}\,ds.$$

By Lemma 1.2, $\|e^{-(t-s)(H+W')}\|_{2,2}\leqq e^{(a+c)t}$. By Lemma 1.3, if t is sufficiently small, say $t\in(0, t_0)$, $\|e^{-t(H+W)}\|_{p_0,q_0}\leqq e^{(a+b)t}$.

Hence if $u\in\mathbf{K}_{p_0}$, $\{V_n\}\in\mathbf{S}$, $t\in[0, t_1)$, $t_1<t_0$, then

$$\|e^{-t(H+V_n)}u - e^{-t(H+V_m)}u\|\to 0 \quad \text{as } m, n\to\infty.$$

It follows that $e^{-t(H+V_n)}u\to M_0(t)u$ for some operator $M_0(t)$, uniformly in t, for $t\in[0, t_1)$. Since $\|e^{-t(H+V_n)}\|_{2,2}$ is bounded as $n\to\infty$, $M_0(t)$ is a bounded linear operator, and can be uniquely extended to an everywhere-defined bounded linear operator $M(t)$, which is evidently self-adjoint and positive.

Straightforward approximation shows that if t, t', and $t+t'$ are in $[0, t_1)$, then $M(t)M(t')=M(t+t')$. Moreover, if $\varepsilon>0$, then $M(t)^\varepsilon=M(\varepsilon t)$, since the approximations $e^{-t(H+V_n)}$ have the same property. If now t is arbitrary in $(0,\infty)$, choose m so large that $t/m<t_1$, and define $M(t)=M(t/m)^m$; then $M(t)$ is independent of the choice of m, by the preceding sentence, and is a one-parameter semigroup. This semigroup is continuous, in view of the uniformity of the convergence in t, for $t\in[0, t_1)$,

and hence of the form (see Hille [8]) $M(t)=e^{-tH'}$ for a unique self-adjoint operator H'. Thus $H+V_n\to H'$.

If $\{V_n'\}$ is another sequence in $\mathbf{S}$, then taking $W=V_n$ and $W'=V_n'$ in the foregoing, and noting that $\|V_n-V_n'\|_{q_0,2}\to 0$, shows that the same operator $M_0(t)$, and hence the same $M(t)$, are obtained from $\{V_n'\}$. Thus $H\dot{+}V$ exists, and the type indicated for it follows from property 4) of a calibration. This completes the proof of conclusion 1).

Now substituting $W=V_n$ and $W'=V_n'$ in the foregoing, where $\{V_n'\}\in\mathbf{S}'$ in conclusion 2), and noting that $\|(V_n-V_n')-(V-V')\|_{q_0,2}\to 0$, the expression given for $e^{-t(H\dot{+}V)}u-e^{-t(H\dot{+}V')}u$ follows.

Now note the

Sublemma. *Let δ and d be given real numbers such that $\alpha+\delta>0$, $\alpha+\delta+\gamma>0$. Then for all sufficiently small t, and any given $u\in\mathbf{K}$,*

$$e^{-t((H\dot{+}W)\dot{+}V_n')}u\to e^{-t((H\dot{+}W)\dot{+}V')}u$$

uniformly for all self-adjoint operators W of type $([2,p_0],\delta,d)$.

Proof. In view of the uniformity of the bounds on the $e^{-t((H\dot{+}W)\dot{+}V_n')}$ for t in a sufficiently small interval J independent of W and n, it suffices to establish the conclusion for the case that $u\in\mathbf{K}_{p_0}$. In this case,

$$e^{-t((H\dot{+}W)\dot{+}V_n')}u-e^{-t((H\dot{+}W)\dot{+}V')}u = \int_0^t e^{-(t-s)((H\dot{+}W)+V_n')}(V-V_n)e^{-s((H\dot{+}W)\dot{+}V')}u\,ds,$$

for $t\in J$. Using again the uniformity of the bounds, and estimating as in the proof of 1), the required conclusion follows. *End.*

Resuming the proof of 2), since $V_n+V_n'\to(V+V')^*$ and

$$\|(V_n+V_n')-(V+V')^*\|_{q_0,2}\to 0,\qquad H\dot{+}(V_n+V_n')\to H\dot{+}(V+V')^*$$

by the proof of 1), and the observation that $(V+V')\subset(V+V')^*$. On the other hand, $e^{-t((H+V_m)+V_n')}u\to e^{-t((H+V_m)\dot{+}V')}u$ as $n\to\infty$, uniformly in m, while as $m\to\infty$ and n is held fixed there is convergence to $e^{-t((H\dot{+}V)\dot{+}V_n')}$ by Lemma 1.1. It follows that $e^{-t((H+V_m)+V_n')}u$ is convergent as a double sequence to $e^{-t((H\dot{+}V)+V')}u$; in particular, $e^{-t((H+V_n)+V_n')}u\to e^{-t((H\dot{+}V)\dot{+}V')}u$. Since it is evident that $((H+V_n)+V_n'=H+(V_n+V_n')$, it follows that $(H\dot{+}V)\dot{+}V'=H\dot{+}(V\dot{+}V')$.

To show 3), let w be an entire vector for $H'=H\dot{+}V$, i.e. $w\in\mathbf{D}(e^{tH'})$ for all t. Then $w=e^{-tH'}(e^{tH'}w)$. From Lemma 1.3 it follows that H' is of type $([2,p_1],\beta+\gamma,b+c,t_1)$, where $t_1=|\beta+\gamma|^{-1}\log(p_0/p_1)$, and $q_0<p_1<p_0$. Taking $t=t_1$, it follows that $w\in\mathbf{K}_p$ for all $p<p_0$. The Duhamel formula is therefore applicable to $e^{-t(H\dot{+}V)}w$ for all sufficiently small t, expressing it as

$$e^{-tH}w+\int_0^t e^{-(t-s)H}Ve^{-sH}w\,ds.$$

Hence

$$t^{-1}(e^{-tH}-I)w=t^{-1}(e^{-tH'}-I)w+t^{-1}\int_0^t e^{-(t-s)H}\,Ve^{-sH'}w\,ds.$$

Evidently, $t^{-1}(e^{-tH'}-I)w\to -H'w$ as $t\to 0$. Now $e^{-tH'}w$ is a continuous function of t near 0 with values in $[\mathbf{K}_{q_0}]$, since

$$e^{-tH'}w-w=e^{-sH'}(e^{-tH'}v-v)$$

for arbitrary $s>0$ and suitable v, and this shows that

$$\|e^{-tH'}w-w\|_{q_0}\leqq \text{const}\,\|e^{-tH'}v-v\|_2\to 0 \quad \text{as } t\to 0.$$

From this it follows by estimates used earlier that the integrand

$$e^{-(t-s)H}\,Ve^{-sH'}w$$

is a continuous function of s and t with values in $\mathbf{K}$, in the range $0\leqq s\leqq t$. At $s=t=0$, the integrand takes the value Vw, and it results that

$$t^{-1}\int_0^t e^{-(t-s)H}\,Ve^{-sH'}w\to Vw.$$

Thus, w is in the domain of H, and $Hw=H'w-Vw$.

Corollary 1.1. *If $H+V$ has a self-adjoint extension (in particular if for all $u\in\mathbf{D}(V)$, $V_n u\to Vu$ for all $\{V_n\}\in\mathbf{S}$), then $H+V\subset H\dotplus V$.*

Proof. Let $\mathbf{E}$ denote the domain of all entire vectors for H'. Then $H+V$ and $H\dotplus V$ agree on $\mathbf{E}$. But $H\dotplus V$ is the unique self-adjoint operator extending its restriction to $\mathbf{E}$, so any self-adjoint extension of $H\dotplus V$ must be identical with $H\dotplus V$.

If in particular, the indicated special condition is satisfied, then since $\langle(H+V_n)u,u\rangle\geqq\text{const}\langle u,u\rangle$ for all $u\in\mathbf{D}(H)$, it follows that if in addition $u\in\mathbf{D}(V)$, then $\langle(H+V)u,u\rangle\geqq\text{const}\langle u,u\rangle$. Thus $H+V$ is semi-bounded, and being hermitian, has therefore a self-adjoint extension.

Remark. Conceivably, $H+V$ always has a self-adjoint extension, but this is not known. By 3) of Theorem 1, a self-adjoint extension could fail to exist only when $\alpha+\gamma=0$.

Corollary 1.2. *If in 2), $(V-V')^*$ together with the $V-V'_n$, is self-adjoint and is for $\{V'_n\}\in\mathbf{S}'$, of type $([2,p_0],\delta,d)$, where $\alpha+\delta>0$, then*

$$(H\dotplus V)\dotplus(-V')=H\dotplus(V-V')^*.$$

Proof. By Lemma 2.1, the conclusion is valid if V' is bounded. In particular, $(H\dotplus V)-V'_n=H\dotplus(V-V'_n)$. But $H\dotplus(V-V'_n)\to H\dotplus(V-V')^*$, so that $(H\dotplus V)\dotplus(-V')$ exists and has the indicated value.

The notation R_w (resp. L_w) for any element w of a Hilbert algebra $\mathbf{K}$ will denote the operation of right (left) multiplication by w, in the sense indicated in Segal [12]. For $w=w^*$, R_w and L_w are self-adjoint and commute, so that defining M_w as the closure R_w+L_w, M_w also is self-adjoint.

Corollary 1.3. *Let* **K** *be a Hilbert algebra with unit e of unit norm, and let* $\|u\|_p$ *for arbitrary* $u\in\mathbf{K}$ *denote the* L_p*-norm:* $E(|u|^p)^{1/p}$*, where E denotes the trace.*

Let H be a self-adjoint operator in **K** *of types* $(2,\alpha,a)$ *and* $([2,p_0],\beta,b)$*, for some* $p_0>2$*, where* $\alpha>0$ *and* $\beta>0$*.*

Let v be a given hermitian element of **K** *such that* e^{-v} *and v are in* $\mathbf{K}_p$ *for all* $p<\infty$*. Let* **S** *denote the set of all sequences of the form* $\{M_{v_n}\}$*, where* $v_n=f_n(v)$*,* $\{f_n\}$ *being an arbitrary sequence of real bounded Baire functions on* R^1*, having the property that* $|f_n(\lambda)|\leqq|\lambda|$ *for all n and* $\lambda\in R^1$*, and that* $f_n(\lambda)\to f(\lambda)$ *as* $n\to\infty$*, for all* λ*. Then:*

1) $H\dot{+}M_v$ *exists, and for any* $p_1\in(2,p_0)$ *and all sufficiently small* $\varepsilon>0$*, is of type*

$$([2,p_1],\beta-\varepsilon,b+2\log\|e^{-v}\|_{2p_1/\varepsilon}).$$

2) *If* v' *satisfies the same conditions as v, then for all* $u\in\mathbf{K}$*,*

$$e^{-t(H\dot{+}M_v)}u-e^{-t(H\dot{+}M_{v'})}u=\int_0^t e^{-(t-s)(H\dot{+}M_{v'})}M_{v-v'}e^{-s(H\dot{+}M_v)}u\,ds$$

(Bochner integral), the integrand being bounded for $s\in[0,t]$*.*

Moreover, $(H\dot{+}M_v)\dot{+}M_{v'}=H\dot{+}M_{v+v'}$*.*

3) *Every analytic vector for* $H\dot{+}M_v$ *is in* $\mathbf{D}(H)\cap\mathbf{D}(M_v)$*, and* $H+M_v\subset H\dot{+}M_v$*.*

4) *For any* $\lambda\in[0,1]$*,*

$$H\dot{+}V\geqq(1-\lambda)H-(\lambda a+2\log\|e^{-v}\|_{4/\varepsilon\lambda}).$$

5) *Let* **B** *denote the class of all operators H satisfying the indicated conditions, and* **C** *the class of all v satisfying the indicated conditions, together with the conditions that* $\|v\|_p+\|e^{-v}\|_p<c_p$ *for* $p\in[2,\infty)$*, where for each p,* c_p *is a given constant. Let* ε *be an arbitrary positive number. Then there exist constants C and a, and a fixed index* $q\in[2,\infty)$ *such that*

$$\|e^{-t(H\dot{+}M_v)}-e^{-t(H'\dot{+}M_{v'})}\|\leqq C\big(\|(H-H')((1+a)I+H)^{-1}\|+\|v-v'\|_q\big).$$

Lemma 1.4. *For any* $\varepsilon>0$ *and* $t>0$*,*

$$\|e^{-tM_v}\|_{p,\,pe^{-\varepsilon t}}\leqq\|e^{-v}\|_{2p/\varepsilon}^{2t}.$$

Proof. Let an element u of **K** be called "special" if it lies in $\mathbf{K}_p$ for some $p>2$. Note first that if u is special, then $u\in\mathbf{D}(e^{-tM_v})$ and $e^{-tM_v}u=e^{-tv}u\,e^{-tv}$. To see this, observe that if v is bounded, then $e^{-tL_v}u=e^{-tv}u$ for all $u\in\mathbf{K}$. In particular, if $v_n=f_n(v)$, then $e^{-tL_{v_n}}u=e^{-tv_n}u$. By Hölder's inequality (see Kunze [7], for the form used here) $e^{-tv_n}u \to e^{-tv}u$. On the other hand, by commutative spectral theory, e^{-tM_v} is the closure of $e^{-tL_v}e^{-tR_v}$. Now, $e^{-tv_n}u \to e^{-tv}u$ by Hölder's inequality. On the other hand, $e^{-tL_{v_n}} = f_n(L_v)$. It follows from commutative spectral theory and Fatou's lemma that $u \in \mathbf{D}(e^{-tL_v})$, and it follows in turn that $e^{-tL_v}u = e^{-tv}u$. Since L_v and R_v commute, it results that $e^{-tM_v}u = e^{-tv}u\,e^{-tv}$.

By Hölder's inequality, if $1 \leqq q \leqq p$,

$$\|e^{-tv}u\,e^{-tv}\|_q \leqq \|e^{-tv}\|_r^2\,\|u\|_p$$

if $q^{-1} = 2r^{-1} + p^{-1}$. Substituting $q = p e^{-\varepsilon t}$ and using the monotone increasing character of $\|w\|_s$ as a function of s together with the inequality, $e^{t\varepsilon} - 1 > t\varepsilon$, the lemma follows.

Proof of Corollary. According to Lemma 1.4, M_v is for any $p_0 \in (2, \infty)$ and $\varepsilon > 0$ of type $([2, p_0], -\varepsilon, c)$ for finite c (as given in 1), which thereupon follows. Conclusion 2) follows from 2) of the theorem together with the observation that, by what has just been shown, $e^{-s(H \mp M_v)}u$ remains special for arbitrarily large s, if u is special. The boundedness of the integrand for $s \in [0, t/2]$ follows from the fact that $e^{-(t-s)(H+M_{v'})}$ is then uniformly bounded from L_2 to L_p for suitable $p > 2$, while for $s \in [t/2, t]$, this is true of $e^{-s(H \mp M_v)}$. The indicated associativity property follows directly from the theorem and Lemma 1.4.

The same argument as for Part 3 of the theorem shows that in the present case, inasmuch as q_0 can be chosen arbitrarily > 2, it suffices if w is an analytic vector; for then $w = e^{-tH'}(e^{tH'}w)$ for some $t > 0$, showing that w is special, which suffices.

The proof of 4) is virtually identical with that for Corollary 2.1 in Segal [9]. To prove 5), set $V = M_v$, $V' = M_{v'}$ and write

$$e^{-t(H \mp V)} - e^{-t(H' \mp V')} = (e^{-t(H \mp V)} - e^{-t(H' \mp V)}) + (e^{-t(H' \mp V)} - e^{-t(H' \mp V')}).$$

Applying the Duhamel formula to each of the two summands on the right, and estimating the integrand as earlier, in the intervals $[0, t/2]$ and $[t/2, t]$ separately, with the aid of 4) in dealing with the first summand, the conclusion follows.

Corollary 1.5. *If H and v are as in Corllary 1.3, then $F(t) = e^{-(H \mp t M_v)}$ has a holomorphic continuation from the half-line $t > 0$ to the half-plane* $\mathrm{Re}(t) > 0$, *in the uniform operator topology.*

Proof. If V is bounded, then $H \mp tV$ is the generator of a one-parameter semigroup, and is easily seen to be a holomorphic function of the complex variable t (e.g. by the Lie formula). Moreover, the application of the Lie formula as in the proofs of Lemmas 1.2 and 1.3 shows that the same results are valid with V replaced by tV in the hypothesis, t being complex, but with $(\mathrm{Re}\,t)V$ in the conclusion. Applying the Duhamel formula as earlier, it follows that $\lim_n e^{-(H+tM_{v_n})}$, where $v_n = f_n(v)$, the sequence f_n being as earlier, exists in the uniform operator topology, uniformly in any bounded t-region in which $\operatorname{Re} t$ is bounded away from 0. This limit is then a holomorphic function from the half-plane $\operatorname{Re} t > 0$ to the bounded operators in the uniform topology, extending $F(.)$.

Definition. An operator H in **K** is called *indecomposable* in case (a) if u is a non-negative hermitian element of **K** and if $t \in (0, \infty)$, then $e^{-tH}u$ is non-negative; (b) if $u, u' \in \mathbf{K}$ and $u, u' \geqq 0$, and if for all $t > 0$, $\langle e^{-tH}u, u' \rangle = 0$, then either $u = 0$ or $u' = 0$.

Example. Let **H** be a given complex Hilbert space, let A be a given self-adjoint operator in **H** such that $A \geqq \varepsilon I$ for some $\varepsilon > 0$; let $H = d\Gamma(A)$, in the notation of *I*; let M be the probability measure space associated with the "free" Weyl process over **H** as in *I*. Then (a) follows from Mehler's formula when u is a tame function, and thence for general non-negative f by a simple limiting argument; (b) follows from the fact that $e^{-tH}u \to \int u$ in **K**, as follows e.g. from the duality transform.

Corollary 1.6. *If H and v are as in Corollary* 1.3, *if H is indecomposable and* **K** *is abelian, then $H + M_v$ is indecomposable.*

Proof. The Lie formula shows that $e^{-t(H \mp sV)}$ is positivity-preserving for any $s > 0$ (i.e. satisfies condition (a)), setting $V = M_v$. Now if $v \leqq 0$, then by the Lie formula, $e^{-t(H \mp V)}u \geqq e^{-tH}u$ if $u \geqq 0$, and it follows that

$$\langle e^{-t(H \mp V)}u, u' \rangle \geqq \langle e^{-tH}u, u' \rangle,$$

from which (b) follows for $H \mp V$.

For the case of a general v, let $v_\pm$ be the positive and negative components of v, so that $V = V_+ - V_-$, where $V_\pm = M_{v_\pm}$. On replacing H by $H \mp (-V_-)$ through the use of the preceding paragraph, the question is reduced to the case in which $V \geqq 0$. Now assuming this, then by the Lie formula, $e^{-t(H \mp sV)}u$ is for fixed $t > 0$ and $u \geqq 0$, a monotone decreasing function of $s \geqq 0$. Since (a) holds for $H \mp V$, it can fail to be indecomposable only if there exist u and u' in **K**, each of which is non-negative and non-zero, such that $\langle e^{-t(H \mp V)}u, u' \rangle = 0$ for all $t > 0$. But

$$\langle e^{-t(H+V)}u, u' \rangle \geqq \langle e^{-t(H+sV)}u, u' \rangle \qquad \text{for } s \geqq 1,$$

and so vanishes for such s. From Corollary 1.4 it results that

$$\langle e^{-t(H \mp sV)}u, u' \rangle = 0 \qquad \text{for all } s > 0.$$

Letting $s \to 0$, it follows by continuity that $\langle e^{-tH}u, u' \rangle = 0$ for all $t > 0$, in contradiction with the assumed indecomposability of H. *End.*

Remark. The method used here for the proof of Corollary 1.5 appears to be potentially more powerful than that indicated in [11] for a special case, in that it applies to certain cases in which $H+V$ is not necessarily essentially self-adjoint, or in which H is not necessarily affiliated with the ring of operators determined by $H \dot{+} V$ and V. The earlier method is however quite sufficient for the present corollary.

More specifically, by [1, Theorem 3.3], if $H \dot{+} V$ is not indecomposable, there exists a measurable set N such that $e^{-t(H \dot{+} V)}$ leaves invariant the subspace **N** of all functions in $L_2(M)$ which vanish outside N. This means that **N** is invariant under the multiplication algebra **M** of M. On the other hand, H is affiliated with the ring of operators determined by $H \dot{+} V$ and V since H is the closure of $(H \dot{+} V) - V$; and hence with the ring determined by $H \dot{+} V$ and **M**. It follows that the e^{-tH} also leave **N** invariant, contradicting the indecomposability of N.

A quite brief proof may be given for the cited result of Ando in the particular case needed here, along similar lines, as follows.

Lemma 1.5. *Let H be a self-adjoint operator in $L_2(M)$ which is bounded from below and is such that e^{-tH} is positivity-preserving, for all $t>0$. Then H is indecomposable if and only if the e^{-tH}, $t>0$, together with the multiplication algebra of M, act (jointly) irreducibly on $L_2(M)$.*

Proof. Since an invariant subspace under the multiplication algebra consists of all vectors in $L_2(M)$ which vanish on some measurable set, the "only if" part is immediate. Suppose therefore that the e^{-tH} together with the multiplication algebra act irreducibly, but, as the basis of an indirect argument, that H is not indecomposable. Then there exist f, g in $L_2(M)$, neither zero and both non-negative, such that $\langle e^{-tH} f, g\rangle = 0$ for all $t>0$. Let N denote the essential union (i.e. union modulo null sets) of the supports of the $e^{-tH} f$ (equivalently, the union of the supports of the $e^{-tH} f$ for rational $t>0$); then N meets the support of g in a null set, so that N differs from all of M by more than a null set. To conclude the proof it suffices to show that the subspace **N** of all vectors h in $L_2(M)$ which vanish outside N is invariant under the e^{-tH} for $t>0$. Since the positive and negative parts of the real and imaginary parts of any such f are again in **N**, it is no essential loss of generality to consider the case in which $h \geqq 0$. It is easily seen that if $h \geqq 0$ is in **N**, then there exist a monotone increasing sequence $\{h_n\}$ of elements of $L_2(M)$ such that $0 \leqq h_n \leqq$ some finite linear combination of the e^{-tH} with positive coefficients, and such that $h_n \to h$. But for any $s>0$, $\{e^{-sH} h_n\}$ is a monotone increasing sequence of elements of $L_2(M)$ with the same property, which converges to $e^{-sH} f$, showing that the latter vector is an element of **N**.

In this connection, the following well-known result from the theory of positivity-preserving transformations should be noted (cf. Ando [1], Theorem 3.4).

If an indecomposable operator has a proper lowest vector, the corresponding invariant subspace is one-dimensional, and the vector may be chosen to be positive a.e.

Here the term "proper lowest vector" (abbr., "PLV") for a given self-adjoint operator T in a Hilbert space $\mathbf{K}$ is defined as a unit vector v in $\mathbf{K}$ such that $Tv=\lambda v$ for some $\lambda \in R^1$, while $T \geqq \lambda I$. If the invariant subspace corresponding to the proper value λ is one-dimensional, T is said to have a *unique* PLV. T is said to have the "spectral gap I" (resp. to be "compact in the spectral interval I") if $C_I(T)=0$, where C_I denotes the characteristic function of I (resp., $f(T)$ is compact for all continuous functions f on R^1 which vanish outside I). If $\lambda = \inf T$, and if $(\lambda, \lambda+\varepsilon)$ is a spectral gap for T, it is also said that T has a spectral gap "of width ε at λ".

A self-adjoint operator T in a Hilbert space $\mathbf{K}$ will be called inverse-compact if either one of the following two equivalent conditions holds: (a) $(cI-T)^{-1}$ exists and is compact for some $c>0$; (b) $(cI-T)^{-1}$ exists and is compact for all c not in the spectrum of T. The notation $\inf_{\mathbf{M}} T$ will denote the infimum of $\langle Tu, u\rangle$ as u varies over the unit vectors in $\mathbf{D}(T) \cap \mathbf{M}$.

Definition. A self-adjoint operator T in a Hilbert space $\mathbf{K}$ will be called approximately inverse-compact (abbr., AIC) if it is bounded from below, and if there exists a sequence $\{T_n\}$ of self-adjoint operators in $\mathbf{K}$ such that:

(a) $\|f(T_n)-f(T)\| \to 0$ for all continuous functions f of compact support;

(b) $\mathbf{K}$ is isomorphic to a direct product $\mathbf{K}_n' \times \mathbf{K}_n''$ in such a way that T_n is isomorphic to the closure of $T_n' \times I_n'' + I_n' \times T_n''$, where T_n' is self-adjoint and inverse compact in $\mathbf{K}_n'$ and T_n'' has a unique PLV and a spectral gap at $\inf T_n''$ of width $\varepsilon>0$ (uniformly in n).

Lemma 1.6. *If T is AIC, then T has a PLV, and is compact in the spectral interval* $(\inf T, \inf T+\varepsilon)$. *Moreover, if T_n has for each n a unique PLV w_n, then T has a unique PLV $w=\lim c_n w_n$, for a suitable sequence c_n of constants of absolute value one.*

Proof. Let λ_n (resp. λ_n', λ_n'') denote the infimum of the spectrum of T_n (resp. T_n', T_n''). Then $\lambda_n = \lambda_n' + \lambda_n''$, and any point in the spectrum of T_n in the range $(\lambda_n, \lambda_n+\varepsilon)$ must be of the form $\mu+\lambda_n''$, where μ is in the spectrum of T_n'. Since T_n' is inverse-compact, it follows that T_n is compact (or equivalently, has finite spectrum of finite multiplicity) in the spectral interval $(\lambda_n, \lambda_n+\varepsilon)$. By virtue of (a), $\lambda_n \to \inf T$, and T is compact in the spectral interval $[\inf T, \inf T+\varepsilon)$. In particular, T has a PLV. If T_n has a unique PLV, then it is easily seen to have the form $w_n' \times w_n''$, where w_n'

and w_n'' are unique PLVs for T_n' and T_n''. If f is a continuous function which is 1 in a sufficiently small interval around $\lambda = \inf T$, and which vanishes outside of a properly larger interval, then $\|f(T_n)-f(T)\| \to 0$ uniformly, implying that the projection P_n onto the one-dimensional subspace spanned by $w_n' \times w_n''$ converges to that onto the corresponding subspace for T. It follows that this subspace is one-dimensional and spanned by a vector of the indicated form.

Corollary 1.7. *Let M be a given probability measure space, and let v be a given real function in $L_p(M)$ for all $p<\infty$, such that $e^{-v} \in L_{4/\alpha}(M)$, where α is a given constant $\geqq 0$. Let H be a given self-adjoint operator in $L_2(M)$ of type $(2, \alpha, a)$. Suppose that H is AIC, with the properties that:* (a) *H_n is also of type $(2, \alpha, a)$, and $\|(H_n - H)(cI+H)^{-1}\| \to 0$ for some constant c;* (b) *there exist independent complemented σ-subrings $\mathbf{R}_n'$ and $\mathbf{R}_n''$ of the ring of measurable sets of M, which jointly generate the latter ring, such that $\mathbf{K}_n'$ and $\mathbf{K}_n''$ are naturally isomorphic to the subspaces of $\mathbf{K}$ consisting of the elements measurable with respect to these respective subrings.*

Remark. If $\mathbf{R}_n'$ and $\mathbf{R}_n''$ are as indicated, and M is the triple $(R, \mathbf{R}, r)$, then it is straightforward to verify that the mapping $f \times g \to fg$ extends uniquely to a unitary transformation from $L_2(M') \times L_2(M'')$ onto $L_2(M)$, where $M' = (R, \mathbf{R}', r)$ and $M'' = (R, \mathbf{R}'', r)$. This is the natural isomorphism referred to. Hypothesis (b) does not restrict materially further H, but effectively relates the action of M_v to the AIC formulation of H.

Proof. It will suffice to treat the case $\alpha > 0$, for the case $\alpha = 0$ is similar and simpler. Note that if v_n denotes the conditional expectation of v with respect to the subring $\mathbf{R}_n$, then $\|v_n\|_p \leqq \|v\|_p$, since conditional expectation is a contraction on any L_p-space. Similarly, $\|e^{-v_n}\|_p \leqq \|e^{-v}\|_p + 1$, for if $v = v_+ - v_-$, where $v_\pm$ are non-negative and have disjoint support, then $v_n = (v_+)_n - (v_-)_n$, so that

$$e^{-pv_n} = e^{p(v_-)_n} e^{-p(v_+)_n} \leqq e^{p(v_-)_n}.$$

It follows that

$$\int e^{-pv_n} \leqq \int e^{p(v_-)_n} = \int \left(\sum_{k=0}^{\infty} p^k (k!)^{-1} (v_-)_n^k \right),$$

which by monotone convergence equals

$$\sum_{k=0}^{\infty} p^k (k!)^{-1} \int (v_-)_n^k \leqq \sum_{k=0}^{\infty} p^k (k!)^{-1} = \int e^{pv_-} \leqq \int e^{pv} + 1.$$

It follows from Corollary 1.3, Part 5), that

$$\|e^{-t(H_n + M_{v_n})} - e^{-t(H+M_v)}\| \to 0,$$

which implies in turn, setting $M_{v_n}=V_n$ and $M_v=V$, that

$$\|f(H_n \mp V_n)-f(H \mp V)\| \to 0$$

for all continuous functions f of compact support. Now taking $T_n = H_n \mp V_n$, $T_n' = H_n' \mp V_n'$ where V_n' denotes the restriction of V_n to $\mathbf{K}_n \cap \mathbf{D}(V_n)$, it follows that the sequence $\{T_n\}$ satisfies the conditions given in the definition for AIC. Applying Scholium 1 the corollary follows.

Corollary 1.8. *The operator $H'(H, V(f))$ indicated in Corollary* 3.3 *of* [9] *admits a unique PLV, and has compact spectrum in the interval of width m above its infimum.*

Proof. Suppose first that $\mathbf{H}$ is an arbitrary complex Hilbert space, $\mathbf{M}$ and $\mathbf{M}^{\perp}$ are arbitrary orthocomplementary closed linear subspaces, and A_0 is a non-negative self-adjoint operator in $\mathbf{H}$ which leaves $\mathbf{M}$ invariant and whose point spectrum (if any) omits 0. If $(\mathbf{K}, W, \Gamma, v)$ denotes the free Weyl process over $\mathbf{H}$, if J denotes an arbitrary conjugation on $\mathbf{H}$ leaving $\mathbf{M}$ invariant, and if $\mathbf{H}'=[x\in\mathbf{H}: Jx=x]$, $\mathbf{M}'=[x\in\mathbf{M}: Jx=x]$, and $\mathbf{M}^{\perp\prime}=[x\in\mathbf{M}^{\perp}: Jx=x]$, then the duality transform represents $\mathbf{K}$ as $L_2(\mathbf{H}', g)$, where $(\mathbf{H}', g)$ denotes the isonormal probability space over $\mathbf{H}'$, in a fashion which induces representations of $\mathbf{K}(\mathbf{M})$ and $\mathbf{K}(\mathbf{M}^{\perp})$ as $L_2(\mathbf{M}', g)$ and $L_2(\mathbf{M}^{\perp\prime}, g)$. Thus $\mathbf{K} \cong \mathbf{K}' \times \mathbf{K}''$, with $\mathbf{K}' = L_2(\mathbf{M}', g)$ and $K'' = L_2(\mathbf{M}^{\perp\prime}, g)$, in such a way that $d\Gamma(A_0)$ is decomposable with $H' = d\Gamma(A_0')$ and $H'' = d\Gamma(A_0'')$, where A_0' and A_0'' are the restrictions of A to $\mathbf{M}$ and $\mathbf{M}'$, respectively. The operators H' and H'' have unique PLVs, and if $A_0 \geqq \varepsilon I$ for some $\varepsilon > 0$, are of type $([2, \infty), \alpha, 0)$ for a fixed α (dependent on ε).

Turning now to the specific case cited, let z denote any fixed real cyclic vector for the (so-called single-particle hamiltonian) operator A, and set $\mathbf{M}_n$ for the closed linear manifold spanned by the vectors $C_{[m/2^n, (m+1)/2^n]}(A)$, where C_I denotes the characteristic function of the interval I and $m=0, \ldots, 2^n n$. Let $A_n = \sum_{0 \leqq m \leqq 2^n n} C_{[m/2^n, (m+1)/2^n]}(A)$; then A_n leaves invariant the finite-dimensional submanifold $\mathbf{M}_n$, and so has self-adjoint restrictions A_n' and A_n'' to $\mathbf{M}_n$ and $\mathbf{M}_n^{\perp}$. The reality conditions of the previous paragraph are fulfilled, and setting $H_n = d\Gamma(A_n)$, and defining H_n' and H_n'' similarly, all the hypotheses of Corollary 1.6 follows, except for the inverse compactness of H_n' and the suitable convergence of H_n to H. The first of the latter conditions follows from the finite-dimensionality of $\mathbf{M}_n$. The second is easily checked in the particle representation. Finally, since $\inf A = m$, the same is true of A_n'', whence H_n'' has the spectral gap m above $\inf H_n'' = 0$.

2. Nonlinear Perturbations of Weyl Processes

We now specialize to the case in which there is given a complex Hilbert space $\mathbf{H}$, a self-adjoint operator B in $\mathbf{H}$ such that $B \geqq \varepsilon I$ for some $\varepsilon > 0$, and a conjugation J on $\mathbf{H}$ commuting with B. Let $(\mathbf{K}, W, \Gamma, v)$ denote the free Weyl process over $\mathbf{H}$; and as earlier, let $\Psi(z)$ denote the self-adjoint generator of the one-parameter unitary group

$$[W(tz): t \in R^1], \qquad H = d\Gamma(B), \quad \text{and} \quad \mathbf{H}' = [z \in \mathbf{H}: Jz = z].$$

As usual, for any positive definite self-adjoint operator C in a Hilbert space $\mathbf{K}$, $[\mathbf{D}(C)]$ denotes the completion of $\mathbf{D}(C)$ as a pre-Hilbert space with the inner product: $\langle x, y \rangle_C = \langle Cx, Cy \rangle$. It is evident that C has a unique continuous linear extension from $\mathbf{D}(C)$ to all of $[\mathbf{D}(C)]$ into $\mathbf{K}$; in the present contexts it will cause no confusion to denote this extension also as C.

For arbitrary $z \in [\mathbf{D}(B^{-\frac{1}{2}})]$, define

$$\Phi_0(z, t) = \Gamma(t)\, \Psi(B^{-\frac{1}{2}} z)\, \Gamma(-t)$$

where $\Gamma(t)$ is an abbreviated notation for $\Gamma(e^{itB})$, and for $z \in \mathbf{D}(B^{\frac{1}{2}})$, define

$$\dot{\Phi}_0(z, t) = \Gamma(t)\, \Psi(i B^{\frac{1}{2}} z)\, \Gamma(-t).$$

The continuous (antilinear) extension of J to all of $[\mathbf{D}(B^a)]$ (for $a \neq 0$) will also be denoted as J, and a J-invariant element will be called "real"; the set of all real elements in a given domain will be denoted by the subscript "r".

Lemma 2.1. *If $x \in \mathbf{D}_r(B^{\frac{1}{2}})$, and if the u_i $(i = 1, 2)$ are arbitrary elements of $\mathbf{D}(H^{\frac{1}{2}})$, then $\langle \Phi_0(x, t)\, u_1, u_2 \rangle$ is a differentiable function of $t \in R^1$, with derivative $\langle \dot{\Phi}_0(x, t)\, u_1, u_2 \rangle$. If in addition $x \in \mathbf{D}(B^{\frac{3}{2}})$, then the latter expression is also differentiable, with derivative $-\langle \Phi_0(B^2 x, t)\, u_1, u_2 \rangle$.*

Proof. This is straightforward, hence omitted.

If $\mathbf{A}$ is an abelian ring of operators in a Hilbert space $\mathbf{K}$, and v is a unit vector in $\mathbf{K}$, the space of all normal operators T in K which are affiliated with $\mathbf{A}$ and have v in the domain of $|T|^{p/2}$ will be denoted as $L_p(\mathbf{A}, v)$, and considered as a Banach space with the norm: $\|T\|_p = \||T|^{p/2} v\|^{2/p}$. If M is any measure space, or couple $(\mathbf{A}, v)$, the notation $L_I(M)$, I being a subset of $(0, \infty]$, will denote the common part of the spaces $L_p(M)$, as p ranges over I, in the topology of convergence in each such space.

Theorem 2. *With the same notation as earlier as regards $(\mathbf{H}, B, J)$, etc., let $\mathbf{A}$ denote the ring of operators generated by the bounded functions of the $\Psi(x)$ for $x \in \mathbf{H}'$, and let V denote a self-adjoint element of $L_{[2, \infty)}(\mathbf{A}, v)$ having the property that e^{-V} is in $L_{[2, \infty)}(\mathbf{A}, v)$.*

Let x be an element of $\mathbf{D}(B^{\frac{3}{2}})$ *such that the map*

$$s \to e^{is\dot{\Phi}_0(x,0)} V e^{-is\dot{\Phi}_0(x,0)}$$

is differentiable at $s=0$, *from* R^1 *into* $L_{[2,\infty)}(\mathbf{A}, v)$, *with derivative* $V(x)$. *Define*

$$\Phi(x,t) = e^{itH'} \Phi_0(x,0) e^{-itH'}, \qquad \dot{\Phi}(x,t) = e^{itH'} \dot{\Phi}_0(x,0) e^{-itH'}.$$

If the u_i $(i=1,2)$ *are arbitrary analytic vectors for* $H' = H \mp V$, *then the following equations hold:*

$$(\partial/\partial t)\langle \Phi(x,t) u_1, u_2\rangle = \langle \dot{\Phi}(x,t) u_1, u_2\rangle \qquad (x \in \mathbf{D}(B^{\frac{1}{2}}))$$

$$(\partial^2/\partial t^2)\langle \Phi(x,t) u_1, u_2\rangle + \langle \Phi(B^2 x, t) u_1, u_2\rangle + \langle e^{itH'} V(x) e^{-itH'} u_1, u_2\rangle = 0 \qquad (x \in \mathbf{D}(B^{\frac{3}{2}})).$$

Lemma 2.2. *If H is self-adjoint, V is bounded self-adjoint, and T is a bounded linear operator on* $\mathbf{K}$, *then for arbitrary* $t \in R^1$,

$$e^{it(H+V)} T e^{-it(H+V)} = e^{itH} T e^{-itH} + \int_0^t e^{i(t-s)(H+V)} [V, e^{isH} T e^{-isH}] e^{-i(t-s)(H+V)} ds$$

(integral taken in the strong operator topology).

Proof. If H is bounded, this is a special case of Duhamel's formula, applied to the perturbation ad V of the operator ad H in the Banach space of all bounded linear operators in $\mathbf{K}$. If H is unbounded, let $\{H_n\}$ be a sequence of bounded self-adjoint operators such that $H_n \to H$; then $H_n + V \to H + V$ by Lemma 1.1, and a limiting argument employing dominated convergence and the formation of matrix elements with arbitrary elements of $\mathbf{K}$ completes the proof.

Lemma 2.3. *For arbitrary* $t \in R^1$, *bounded linear operator G on* $\mathbf{K}$, *and analytic vectors* u_i $(i=1,2)$ *for* H', *the following equation holds:*

$$\langle e^{itH'} G e^{-itH'} u_1, u_2\rangle = \langle e^{itH} G e^{-itH} u_1, u_2\rangle + \int_0^t \langle [V, e^{isH} G e^{-isH}] e^{-i(t-s)H'} u_1, e^{-i(t-s)H'} u_2\rangle ds. \qquad (*)$$

Proof. When V is bounded, this is implied by Lemma 2. For unbounded V, let $V_n = f_n(V)$, where $\{f_n\}$ is a sequence of functions on R^1 such that $|f_n(\lambda)| \leq |\lambda|$ and $f_n(\lambda) \to \lambda$ for all λ. Then equation (*) holds with V replaced by V_n. The only question in passing to the limit as $n \to \infty$ is with the integral on the right side.

Setting $w_i(s) = e^{-i(t-s)H'} u_i$ (t being held fixed for the moment), the $w_i(s)$ are again analytic vectors for H, for all i and s. By an argument given in an earlier proof, they are therefore in $\mathbf{L}_p$ for some $p > 2$, and moreover

$$\|w_i(s)\|_p \leq \text{const} \, \|e^{\varepsilon H'} u_i\|;$$

here, p and the constant depend only on the $\varepsilon>0$ for which u_1 and u_2 are in $\mathbf{D}(e^{\varepsilon H'})$ and so are uniform in s. The integrand is

$$\langle e^{isH}\,G\,e^{-isH}\,w_1(s), V_n\,w_2(s)\rangle - \langle e^{isH}\,G\,e^{-isH}\,V_n\,w_1(s), W_2(s)\rangle;$$

applying Hölder's inequality and the fact that $V_n \to V$ in L_q, $q<\infty$, including the q such that $p^{-1}+q^{-1}=\frac{1}{2}$, the required dominated convergence follows.

Lemma 2.4. *For arbitrary analytic vectors u and u' for H',*

$$\begin{aligned}&\langle e^{itH'}\,\dot{\Phi}(x,0)\,e^{-itH'}\,u_1, u_2\rangle\\ &\quad=\langle\dot{\Phi}_0(x,t)\,u_1, u_2\rangle+\int_0^t\langle e^{i(t-s)H'}\,V(x)\,e^{-i(t-s)H'}\,u_1, u_2\rangle\,ds.\end{aligned}$$

Proof. Note first that the expressions involved in the lemma do in fact exist. For $e^{-itH'}u$ is again analytic for H', hence is contained in the domain of $(H'+c'I)$ for a sufficiently large constant c', and hence contained in the domain of $(H+cI)^{\frac{1}{2}}$ for sufficiently large c. This implies that it is in the domain of $\dot{\Phi}(x,t)$ for all t and $x\in\mathbf{D}(B^{\frac{1}{2}})$. An argument similar to that used for the proof of Lemma 3 shows that the integrand and the integral on the right side are well-defined.

Now set $G=e^{ir\dot{\Phi}(x,0)}$, $r\in R^1$; then it follows from Lemma 3 that

$$\begin{aligned}\langle e^{itH'}\,r^{-1}(G-I)\,e^{-itH'}\,u_1, u_2\rangle&=\langle e^{itH}\,r^{-1}(G-I)\,e^{-itH}\,u_1, u_2\rangle\\ &\quad+r^{-1}\int_0^t\langle[V, e^{isH}\,G\,e^{-isH}]\,e^{-i(t-s)H'}\,u_1, e^{-i(t-s)H'}\,u_2\rangle\,ds.\end{aligned}\tag{$*'$}$$

If w is analytic for H', it is in the domain of $\dot{\Phi}(x,s)$ by the argument above for all s, so that $e^{isH}r^{-1}(G-I)e^{-isH}w$, which is the same as $r^{-1}(e^{ir\dot{\Phi}(x,s)}-I)\,w$, converges as $r\to 0$ to $i\dot{\Phi}(x,s)\,w$. Thus, as $r\to 0$ the left side of Eq. ($*'$) converges to the left side of Eq. ($*$).

Setting $G(s)=e^{isH}\,G\,e^{-isH}$ and defining the $w_i(s)$ as earlier, the integrand on the right may be written as

$$-r^{-1}\,\langle\!\big(G(s)\,VG(s)^{-1}-V\big)\,G\,w_1(s), w_2(s)\rangle.$$

For any fixed s, $r^{-1}\big(G(s)\,VG(s)^{-1}-V\big)$ converges as $r\to 0$ to $V(x)$ in L_p for all $p<\infty$. It follows as earlier that if $z\in\mathbf{K}_p$, $p>2$, then

$$\big\|r^{-1}\big(G(s)\,VG(s)^{-1}-I\big)\,z-V(x)\,z\big\|\leqq c\,\|z\|_p$$

as $r\to 0$, where c depends only on x and V and not on z.

Now writing

$$\begin{aligned}&r^{-1}\big(G(s)\,VG(s)^{-1}-I\big)\,G(s)\,w\\ &\quad=r^{-1}\big(G(s)\,VG(s)^{-1}-I\big)\,w+r^{-1}\big(G(s)\,VG(s)^{-1}-I\big)\big(G(s)-I\big)\,w,\end{aligned}$$

the first term on the right converges to $V(x)w$, uniformly in s, by the observation just made. To examine the second term on the right, note that $\|r^{-1}(G(s)VG(s)^{-1}-I)\|_{L_p(\mathbf{A},E)}$ remains bounded as $r\to 0$, for all $p\in[2,\infty)$; while $(G(s)-I)w_2(s)\to 0$. Since

$$r^{-1}\langle (G(s)VG(s)^{-1}-I)(G(s)-I)w_1(s), w_2(s)\rangle$$
$$=\langle (G(s)-I)w_1(s), r^{-1}(G(s)VG(s)^{-1}-I)w_2(s)\rangle\to 0$$

boundedly in s, it follows that

$$r^{-1}\int_0^t \langle [V, G(s)]\,w_1(s), w_2(s)\rangle\,ds\to\int_0^t \langle V(x)\,w_1(s), w_2(s)\rangle\,ds.$$

Proof of Theorem. Evidently, for arbitrary $\varepsilon>0$,

$$\langle \varepsilon^{-1}(e^{i(t+\varepsilon)H'}\dot{\Phi}(x,0)\,e^{-i(t+\varepsilon)H'}-e^{itH'}\dot{\Phi}(x,0)\,e^{-itH'})\,u_1, u_2\rangle$$
$$=\langle \varepsilon^{-1}(e^{i\varepsilon H'}\dot{\Phi}(x,0)\,e^{-i\varepsilon H'}-\dot{\Phi}(x,0))\,w_1, w_2\rangle,$$

where $w_i=e^{-itH'}u_i$. By Lemma 2.4, the last expression is

$$\langle \varepsilon^{-1}(\dot{\Phi}_0(x,\varepsilon)-\dot{\Phi}_0(x,0))\,w, w'\rangle+\varepsilon^{-1}\int_0^\varepsilon \langle e^{i(\varepsilon-s)H'}V(x)\,e^{-i(\varepsilon-s)H'}w_1, w_2\rangle\,ds.$$

By observations concerning Φ_0 already made,

$$\langle \varepsilon^{-1}(\dot{\Phi}_0(x,\varepsilon)-\dot{\Phi}_0(x,0))\,w_1, w_2\rangle\to-\langle \Phi_0(B^2x)\,w_1, w_2\rangle.$$

Since $\langle e^{i(\varepsilon-s)H'}Ve^{-i(\varepsilon-s)H'}w_1, w_2\rangle$ is a jointly continuous function of ε and s,

$$\varepsilon^{-1}\int_0^\varepsilon \langle e^{i(\varepsilon-s)H'}V(x)\,e^{-i(\varepsilon-s)H'}w_1, w_2\rangle\,ds\to\langle V(x)\,w_1, w_2\rangle.$$

It follows that $(\partial/\partial t)\langle \dot{\Phi}(x,t)\,u_1, u_2\rangle$ exists and equals

$$-\langle \Phi_0(B^2x)\,w_1, w_2\rangle-\langle V(x)\,w_1, w_2\rangle=-\langle \Phi(B^2x,t)\,u_1, u_2\rangle$$
$$-\langle e^{itH'}V(x)\,e^{-itH'}u_1, u_2\rangle.$$

Finally, $(\partial/\partial t)\langle \Phi(x,t)\,u_1, u_2\rangle=\langle \dot{\Phi}(x,t)\,u_1, u_2\rangle$, for

$$(\partial/\partial t)\langle \Phi(x,t)\,u_1, u_2\rangle=(\partial/\partial t)\langle e^{itH'}\Phi(x,0)\,e^{-itH'}u_1, u_2\rangle$$
$$=-i\langle e^{itH'}\Phi(x,0)\,H'e^{-itH'}u_1, u_2\rangle+i\langle \Phi(x,0)\,e^{-itH'}u_1, e^{-itH'}H'u_2\rangle$$

(note that $H'u_i$ is again an analytic vector for H', so the indicated derivative exists and the expression given for it is well-defined). Now if u is analytic for H', then it is in the domain of H and of V, and $H'u=Hu+Vu$.

It follows that the last expression may be written as

$$(*)=-i\langle \Phi(x,0)(H+V)\,w_1, w_2\rangle+i\langle \Phi(x,0)\,w_1, (H+V)\,w_2\rangle.$$

Note next that $H w_i$ and $V w_i$ are in $\mathbf{D}(\Phi(x,0))$ $(i=1,2)$. For if w is any analytic vector for H', $w \in L_p$ for some $p>2$, so by Hölder's inequality and the assumption that $V \in \mathbf{C}$, $Vw \in L_q$ for all $q<p$, showing that $Vw \in \mathbf{D}(\Phi(x,0))$, inasmuch as $\Phi(x,0)$ is the operation of multiplication by an element of $\mathbf{D}$. On the other hand, $Hw = H'w - Vw$; $H'w$ is again analytic for H', hence is in L_{p_1} for some $p_1>2$, and consequently in $\mathbf{D}(\Phi(x,0))$.

It follows that

$$(*) = -i\langle \Phi(x,0)\, V w_1, w_2\rangle + i\langle \Phi(x,0)\, w_1, V w_2\rangle - i\langle \Phi(x,0)\, H w_1, w_2\rangle + i\langle \Phi(x,0)\, w_1, H w_2\rangle .$$

Now V and $\Phi(x,0)$ commute, in the sense that their spectral projections do so; it follows that

$$-i\langle \Phi(x,0)\, V w_1, w_2\rangle + i\langle \Phi(x,0)\, w_1, V w_2\rangle = 0 .$$

Thus,

$$\begin{aligned}(*) &= \frac{\partial}{\partial t}\langle e^{itH}\,\Phi(x,0)\, e^{-itH} w_1, w_2\rangle|_{t=0}\\ &= \langle \dot{\Phi}_0(x,0)\, w_1, w_2\rangle = \langle e^{itH'}\,\dot{\Phi}_0(x,0)\, e^{-itH'} u_1, u_2\rangle\\ &= \langle \dot{\Phi}(x,t)\, u_1, u_2\rangle . \qquad \textit{End}.\end{aligned}$$

In the Lemmas and Theorem of this section, the processes $\Phi_0(x,t)$ and process $\Phi(x,t)$ are treated for suitably regular x. As a function of x for fixed t, these processes are operator-valued distributions. In an analogous classical context, these distributions may be established as functions. Although it is out of the question to define the present processes as strict operator-valued functions, they may be identified with generalized operator-valued functions, in the case of a two-dimensional space time. This is a corollary to the following results, together with the fact that the delta distribution on space is in the domain of B^{-1} in the two-dimensional space time case.

It is convenient to use the following notational conventions. If B is a given sesquilinear form with domain $\mathbf{G}$ in the Hilbert space $\mathbf{K}$, the value of B on the given ordered pair of vectors u_1, u_2 in $\mathbf{G}$ will be denoted as $\langle B u_1, u_2\rangle$; and $\langle u_1, B u_2\rangle$ will denote $\overline{\langle B u_2, u_1\rangle}$. If P and Q are operators in $\mathbf{K}$ of which P is bounded while Q and Q^* are defined on $\mathbf{G}$, then $\langle [P,Q]\, u_1, u_2\rangle$ will denote $\langle Q u_1, P^* u_2\rangle - \langle P u_1, Q^* u_2\rangle$.

If B is a self-adjoint operator in a Hilbert space $\mathbf{H}$ such that $B \geqq \varepsilon I$ for some $\varepsilon>0$, the notation $D_{-\infty}(B)$ will refer to the union of the $[\mathbf{D}(B^{-k})]$, $k=1,2,\ldots$, with the usual identifications between these respective domains; in the topology of convergence in some one $[\mathbf{D}(B^{-k})]$, this space will be denoted as $[\mathbf{D}_{-\infty}(B)]$; C will denote $B^{\frac{1}{2}}$.

Scholium 2.1. *The sesquilinear forms $\langle \Phi_0(x,t)\, u_1, u_2\rangle$ and $\langle \dot{\Phi}_0(x,t)\, u_1, u_2\rangle$ on $[\mathbf{D}_\infty(H)]$ $(u_i \in \mathbf{D}_\infty(H))$ extend continuously (and uniquely so) from the given domains for x to the domain $[\mathbf{D}_{-\infty}(B)]$; and on this domain*

$$(\partial/\partial t)\langle \Phi_0(x,t)\, u_1, u_2\rangle = \langle \dot{\Phi}_0(x,t)\, u_1, u_2\rangle,$$
$$(\partial^2/\partial t^2)\langle \Phi_0(x,t)\, u_1, u_2\rangle = -\langle \Phi_0(B^2 x,t)\, u_1, u_2\rangle.$$

Proof. This is by recursion from the relations

$$\langle \dot{\Phi}_0(x,t)\, u_1, u_2\rangle = \langle [i H, \Phi_0(x,t)]\, u_1, u_2\rangle,$$
$$\langle \Phi_0(B^2 x,t)\, u_1, u_2\rangle = \langle [i H, \dot{\Phi}_0(x,t)]\, u_1, u_2\rangle.$$

Thus $\langle \dot{\Phi}_0(x,t)\, u_1, u_2\rangle$, originally defined only for $x \in \mathbf{D}(B^{\frac{1}{2}})$, is bounded by const $\|C^{-1} x\|\, \|(I+H)\, u_1\|\, \|(I+H)\, u_2\|$, and so extends in a unique continuous fashion to $\mathbf{D}(B^{-\frac{1}{2}})$. Similarly, $\langle \Phi_0(x,t)\, u_1, u_2\rangle$ extends in a unique continuous fashion to $\mathbf{D}(B^{-\frac{1}{2}})$. Similarly, $\langle \Phi_0(x,t)\, u_1, u_2\rangle$ extends in a unique continuous fashion from $x \in \mathbf{D}(B^{-\frac{1}{2}})$ to $x \in \mathbf{D}(B^{-\frac{3}{2}})$. The original relations remain valid for these extensions, and the procedure indicated may be iterated, yielding ultimately the indicated extensions.

Scholium 2.2. *Let $\mathbf{E}$ denote the domain of all entire vectors for H', in the topology in which a generic neighborhood of $u \in \mathbf{E}$ is*

$$[u' \in \mathbf{G}\colon \|e^{tH'}(u-u')\| < \delta]$$

for some t, $\delta > 0$. Then the sesquilinear forms $\langle \Phi(x,t)\, u_1, u_2\rangle$ and $\langle \dot{\Phi}(x,t)\, u_1, u_2\rangle$ on $\mathbf{E}$ $(u_i \in \mathbf{E})$ extend continuously, and uniquely so, from the given domains for x to the domains $D(B^{-\frac{3}{2}})$ and $\mathbf{D}(B^{-\frac{1}{2}})$, respectively. Moreover,

$$\frac{\partial}{\partial t}\langle \Phi(x,t)\, u_1, u_2\rangle = \langle \dot{\Phi}(x,t)\, u_1, u_2\rangle \qquad \textit{if}\ \ x \in \mathbf{D}(B^{-\frac{1}{2}}).$$

Proof. Note first that for arbitrary $z \in \mathbf{D}(B)$,

$$\|(1+H)^{-1}\, \Psi(Bz)(1+H)^{-1}\| \leqq \text{const}\, \|z\|.$$

For by Lemma 3.1 of [10, II], with $n=0$

$$\|\Psi(z)\, u\| \leqq \text{const}\, \|z\|\, \|(1+H)^{\frac{1}{2}}\, u\| \leqq c\, \|z\|\, \|(1+H)\, u\|.$$

It follows that $\|\Psi(z)(1+H)^{-1} u\| \leqq c\, \|z\|$, and by taking adjoints, $\|(1+H)^{-1}\, \Psi(z)\, u\| \leqq c\, \|z\|$. On the other hand,

$$\Psi(i B z)\, w = i[H, \Psi(z)]\, w = i[1+H, \Psi(z)]\, w$$

for $w \in \mathbf{D}_\infty(H)$, whence

$$(1+H)^{-1}\Psi(iBz)(1+H)^{-1}w = i\Psi(z)(1+H)^{-1} - i(1+H)^{-1}\Psi(z).$$

Taking bounds, the result follows.

Note next that if u_1 and u_2 are in $\mathbf{D}(H)$ and if $z \in \mathbf{H}$, then

$$|\langle \Psi(z)u_1, u_2\rangle| \leqq \text{const}\,\|B^{-1}z\|\,\|(1+H)u_1\|\,\|(1+H)u_2\|.$$

This conclusion follows by replacing Bz in the preceding paragraph by z.

Finally, note that if u is entire for H', then by part of Theorem 1, $\|Hu\| \leqq \text{const}\,\|e^{t_0 H'}u\|$ for sufficiently large t_0. Combining these observations, the existence of the extensions described in Scholium 3 follows. To conclude the proof, note that in order to show that $f'(t) = g(t)$, f and g being given numerical functions on R^1, it suffices to show that there are sequences $f_n(t)$ and $g_n(t)$ of continuous such functions such that $f_n(t) \to f(t)$ and $g_n(t) \to g(t)$ pointwise and boundedly on finite intervals and $f_n'(t) = g(t)$. For

$$f(t) = \lim \int_0^t g_n(s)\,ds = \int_0^t g(s)\,ds.$$

Now we have pointwise convergence if $x_n \in \mathbf{H}$, $x_n \to x$ in $[\mathbf{D}(B^{-\frac{1}{2}})]$:

$$\langle \dot{\Phi}(x_n, t)u_1, u_2\rangle \to \langle \dot{\Phi}(x, t)u_1, u_2\rangle$$
$$\langle \Phi(x_n, t)u_1, u_2\rangle \to \langle \Phi(x, t)u_1, u_2\rangle;$$

and

$$|\langle \dot{\Phi}(x_n, t)u_1, u_2\rangle| = |\langle \Psi(iB^{\frac{1}{2}}x_n, t)u_1, u_2\rangle| \leqq c\,\|B^{-\frac{1}{2}}x_n\|\,\|e^{\varepsilon H'}u_1\|\,\|e^{\varepsilon H'}u_2\|. \quad \textit{End.}$$

The differential equation obtained in Theorem 2 involves a "source" term $\langle e^{itH'}V(t)e^{-itH'}u_1, u_2\rangle$ which is in no effective sense a function of $\Phi(x, t)$ and $\dot{\Phi}(x, t)$. It is remarkable that in the case of the perturbations V treated in I, the equation can be given the form of a local differential equation in a natural sense. In order to exhibit this form, the concept of renormalized (or generalized Wick) power of a Weyl process in space, with respect to a relatively general vacuum, must be further developed.

3. Renormalized Powers with Respect to General Vacuums of a Scalar Relativistic Quantum Process in a Two-Dimensional Space-Time

This section is concerned with an extension of the results of treating the existence and properties of renormalized powers with respect to the free vacuum, to the case of the renormalized powers with respect to a vacuum vector in a general class. These powers were defined in [10], but *a priori* it is not clear how extensive their domains of definition (consisting of "test functions" in space) may be, or indeed if any vectors other than 0 are in these domains. For finite-dimensional systems, parallel algebraic existence, etc. is established in [10]; in question here are the existence and properties of certain self-adjoint operators in Hilbert space, constituting an infinite-dimensional analogue to the algebraic results. In the case of the renormalized powers relative to the free vacuum, the possibility of defining the powers as a limit of standardized rearrangements of non-commuting monomials, in line with Wick's formal ideas, makes possible a variety of approaches to the question. In the general case, lack of any simple expression for the expectation value of a product of renormalized powers appears to necessitate an approach which depends more on functional analysis and less on combinatorics or explicit expressions.

The definition of the strong renormalized power for an arbitrary Weyl process may be briefly recalled as follows. If Φ and $\dot{\Phi}$ is an arbitrary Heisenberg pair over a linear space $\mathbf{L}$ of functions on a manifold S (paired with itself via the usual inner product relative to a given measure on S), and v a given vector in the representation space $\mathbf{K}$ in the domains of all the monomials in the $\Phi(x)$ and $\dot{\Phi}(y)$ for arbitrary x and y in $\mathbf{L}$, then $\Phi^{(n)}$ is defined recursively by the conditions: (f here is a suitable real element of $\mathbf{L}$)

$\langle \Phi^{(n)}(f)\, v, v\rangle = 0;$

$\Phi^{(n)}(f)$ is a self-adjoint operator affiliated with the ring of operators determined by the $\Phi(x)$, $x \in \mathbf{L}$;

$$e^{i\dot{\Phi}(g)}\,\Phi^{(n)}(f)\,e^{-i\dot{\Phi}(g)} = \Phi^{(n)}(f) + n\,\Phi^{(n-1)}(fg) + \cdots + \binom{n}{r}\,\Phi^{(n-r)}(fg^r) + \cdots$$

where $\Phi^{(0)}(f) = \int f$. In other words, having defined the $\Phi^{(m)}(.)$ for $m<n$, including the domains of functions f to which they are applicable (and possibly consisting only of 0), a given f is said to be in the domain of $\Phi^{(n)}$ if there exists a self-adjoint operator T affiliated with $\mathbf{A}$ such that

$$e^{i\dot{\Phi}(g)}\,T e^{-i\dot{\Phi}(g)} = \text{closure of } T + \text{the indicated operators}$$

(note that all the operators in question are affiliated with the abelian ring $\mathbf{A}$, so there is no problem with their addition and closure). This means in particular that fg^r must be in the domain of $\Phi^{(n-r)}$.

There is also a more general concept of weak renormalized power, where sesquilinear forms relative to a given domain are involved; this is not required here.

On the other hand, the treatment of the general case given here depends on the prior existence of a treatment for the case of the free vacuum, and indeed it is necessary to make certain aspects of this treatment more precise. In presently relevant terms, the basic existential result of [10, I] may be stated as follows.

Recall first that for any measure space M, the notation $L_{[p,q]}(M)$, where $p \leqq q$, denotes the common part of the spaces $L_r(M)$, $r \in [p,q]$, in the topology of convergence in each such space $L_r(M)$. If G is a locally compact abelian group, and I is any interval in R^1, the notation $\hat{L}_I(G^*)$ refers to the space of all functions on G^* which are Fourier transforms of elements of $L_I(G)$, in the topology in which Fourier transformation is a homeomorphism.

Let G be a locally compact abelian group, and let B denote a real self-adjoint operator in $L_2(G)$ which commutes with all translation operators[1] in $L_2(G)$. Suppose that the spectral function $B(.)$ for B on the dual group G^* has the property that $B(.)^{-1} \in L_{(1,\infty]}(G^*)$. Let $(\mathbf{K}, \Phi_0, \dot{\Phi}_0, v_0)$ denote the normal static Heisenberg process[2] associated with the given pair (G, B). Let $\mathbf{R}$ denote the ring of operators on $\mathbf{K}$ generated by the bounded Baire functions of the $\Phi(x)$.

Then there exists for each $n = 1, 2, \ldots$ a unique mapping $\Phi_0^{(n)}$ from the space of all real elements of $L_1(G) \wedge L_2(G)$ to the self-adjoint operators in $\mathbf{K}$ affiliated with $\mathbf{R}$, having the property that for arbitrary[3]

$$g \in \mathbf{D}(B^{\frac{1}{2}}) \wedge \hat{L}_{[1,\infty]}(G^*), \qquad e^{-i\dot{\Phi}_0(g)}\, \Phi_0^{(n)}(f)\, e^{i\dot{\Phi}_0(g)}$$

is the closure of

$$\sum_{j=0}^{n} \binom{n}{j} \Phi_0^{(n-j)}(f g^j).$$

Moreover,

$$\Phi_0^{(n)}(f) \in L_{[1,\infty)}(\mathbf{R}, v_0).$$

There does not appear to be any uniquely convenient spaces in which to take the functions f and g which enter into the theorem, due to the circumstance that the g's are appropriately chosen to be such that the operation M_g of multiplication by g is continuous on the space chosen

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[1] (i.e. those of the form $f(x) \to f(a^{-1}x)$, for some $a \in G$, and arbitrary $f \in L_2(G)$).

[2] i.e., these represent the "free scalar quantum field" for a scalar particle in the space G, whose energy-momentum dependence function is $B(.)$ at an arbitrary fixed time, e.g. $t=0$; cf. below.

[3] The condition that g lie in $\mathbf{D}(B^{\frac{1}{2}})$, in addition to $\hat{L}_{[1,\infty]}(G^*)$ was inadvertently omitted in the formulation of Theorem 2.1 in [10, I]; as $\Phi(g)$ is defined only for $g \in \mathbf{D}(B^{\frac{1}{2}})$, the operator on the left in the statement of the conclusion is undefined, although the right side is defined. An extended definition as a limit can readily be supplied ($\dot{\Phi}_0(G)$ can be formulated e.g. as the generator of an automorphism of the Weyl algebra treated elsewhere; alternatively, sesquilinear forms may be employed); but it is quite sufficient for our purposes here to have this conclusion for an arbitrary set of vectors g which is dense in $[\mathbf{D}(C)]$.

for the f's, in addition to other natural conditions. In particular, if a single space **A** is desired as a domain for all the $\Phi^{(n)}$, $n>0$, and for $\dot{\Phi}$ as well, the following desiderata appear conservatively applicable: (a) **A** should be a real algebra of functions on G; (b) **A** should be contained and dense in $[\mathbf{D}(C)]$, which is the natural space for the g's; (c) **A** should be translation invariant; (d) $\mathbf{A}^2$ should be contained in $L_1(G)$, so that the constant terms entering into the conclusion will be defined. When (b) holds, $\mathbf{A}\subset L_2(G)\subset[\mathbf{D}(C_n^{-1})]$, where C_n is the operator defined below, such that $\Phi^{(n)}$ is in a certain sense naturally definable on $[\mathbf{D}(C_n^{-1})]$. Making the conservative assumption that $B(.)$ is continuous on G^*, a simple algebra **A** satisfying these conditions is that of all functions on G which are Fourier transforms of hermitian-symmetric continuous functions of compact support on G^*. In the important special case $G=R^1$ or T^1 and $B(k)=(m^2+k^2)^{\frac{1}{2}}$, m being a nonzero constant, a more convenient space, which has the additional properties of being a Banach algebra and of being contained in $L_1(G)$ (so that $\Phi_0^{(0)}$ is also defined on it) is $[\mathbf{D}(B)]$. The preceding observations follow from

Lemma 3.1. *For arbitrary* $n>0$, *the mapping* $f\to\Phi_0^{(n)}(f)$ *extends uniquely from* $L_1(G)\wedge L_2(G)$ *to a continuous linear mapping from* $[\mathbf{D}(C_n^{-1})]$ *to* $L_{[1,\infty)}(\mathbf{R},v)$, *where* C_n *is the translation invariant self-adjoint operator in* $L_2(G)$ *whose spectral function is*

$$(B^{-1}(.)*\cdots*B^{-1}(.))^{-\frac{1}{2}} \qquad (\textit{n-fold convolution}).$$

Proof. For $f\in L_1(G)\wedge L_2(G)$,

$$\|\Phi_0^{(n)}(f)\|_2^2=\mathrm{const}\int(b^{-1}*\cdots*b^{-1})\,|\hat{f}(k)|^2\,dk,$$

by [10, I] (cf. I); $=\mathrm{const}(n)\,\|C_n f\|_2^2$. On the other hand, $L_1(G)\wedge L_2(G)$ is a dense subset of $[\mathbf{D}(C_n^{-1})]$, for if $y\in[\mathbf{D}(C_n^{-1})]$ is orthogonal to all $f\in L_1(G)\wedge L_2(G)$, then $\langle C_n^{-1}f, C_n^{-1}y\rangle=0$ (where $\langle .,.\rangle$ denotes the L_2-inner product), whence $\langle f, C_n^{-2}y\rangle=0$, and $C_n^{-2}y=0$, so that $y=0$.

Thus the map $f\to\Phi_0^{(n)}(f)$ is continuous from $[\mathbf{D}(C_n^{-1})]$ to $L_2(\mathbf{R},v_0)$ and defined on the indicated dense subset, and the conclusion follows, as regards extension to a continuous mapping from all of $[\mathbf{D}(C_n^{-1})]$ into $L_2(\mathbf{R},v_0)$. On the other hand, by Corollary 1.1 of I, $\|\Phi_0^{(n)}(f)\|_p=\mathrm{const}(n,p)\,\|\Phi_0^{(n)}(f)\|_2$ for arbitrary $p\in[1,\infty)$, and the entire conclusion follows. *End.*

The following result, intuitively to the effect that $\langle:\phi_0(x)^n:\,v,v\rangle$ exists and is a function, is needed in the proof of the next Theorem.

Lemma 3.2. *Let* v *be an arbitrary unit vector in* $\mathbf{K}_p$ *for some* $p>2$. *Then there exists a unique function* $h_n\in[\mathbf{D}(C_n)]$ *such that* $\langle\Phi_0^{(n)}(f)\,v,v\rangle=\langle f,h_n\rangle$ *for all* $f\in L_2(G)$.

Proof. By Hölder's inequality, $|\langle \Phi^{(n)}(f)\, v, v\rangle| \leqq \text{const}\, \|\Phi^{(n)}(f)\|_q$ for some $q<\infty$, which in turn by I is $\leqq \text{const}\, \|\Phi^{(n)}(f)\|_2 = \text{const}\, \|C_n^{-1} f\|_2^2$. Thus $\langle \Phi^{(n)}(f)\, v, v\rangle$ as a function of $f\in[\mathbf{D}(C_n^{-1})]$ is a continuous linear functional, hence of the form $\langle C_n^{-1} f, C_n^{-1} g\rangle$ for some $g\in[\mathbf{D}(C_n^{-1})]$. Setting $h_n = C_n^{-1}(C_n^{-1} g)$, the stated conclusion follows. *End.*

One direction of refinement of the transformation rule for $\Phi_0^{(n)}(f)$ under $e^{i\dot{\Phi}_0(g)}$ may be indicated as follows.

Lemma 3.3. *If $g\in\mathbf{D}(C)$, and if the operation M_g of multiplication by g is bounded on $\bigcap_n [\mathbf{D}(C_n^{-1})]$, in the topology of convergence in each $[\mathbf{D}(C_n^{-1})]$, and maps this space into $L_1(G)$, then*

$$e^{i\dot{\Phi}_0(g)}\, \Phi_0^{(n)}(f)\, e^{-i\dot{\Phi}_0(g)} = \text{closure of} \sum_{j=0}^{n} \Phi^{(n-j)}(f g^j) \binom{n}{j}.$$

Proof. The validity of the foregoing equation for suitably regular f and g as in the cited theorem implies it for all f in $\bigcap_n [\mathbf{D}(C_n^{-1})]$, and the indicated g, inasmuch as transformation by $e^{i\dot{\Phi}(g)}$ acts continuously on $L_2(\mathbf{R}, v)$.

Remark. The further development of the precise spaces on which renormalized products are conveniently defined leads to as yet apparently untreated questions concerning the Soboleff-Calderon spaces $L_{p,n}$ [2]. The structure of the space of all multiplication operators M_g which carry $L_{p,n}$ into $L_{q,m}$, for given p, n, q, and m is relevant, particularly in the cases $p=q=2$.

In the case of general vacuum, the basic existential result applicable to a two-dimensional space time is the

Theorem 3. *Let G and B be as earlier. Let $(K, \Phi_0, \dot{\Phi}_0, v_0)$ denote the normal static Heisenberg process associated with this pair, and the ring of operators generated by the bounded functions of the $\Phi_0(x)$.*

Suppose $(K, \Phi, \dot{\Phi}, v)$ is another Heisenberg process with $v\in\mathbf{K}_p$ for some $p>2$, and with Φ and $\dot{\Phi}$ of the form:

$$\Phi(x) = Z\Phi(x)\, Z^{-1}, \quad x\in L_2(G); \qquad \dot{\Phi}(y) = Z\dot{\Phi}_0(y)\, Z^{-1}, \quad y\in\mathbf{D}(B),$$

Z being a unitary operator on K such that $Zv=v$. Then the renormalized powers $\Phi^{(n)}$ relative to this latter Heisenberg process have all of $L_2(G)$ in their domains.

For complete specificity and the reader's convenience, the following terminological notes are made.

1) $(\mathbf{K}, \Phi, \dot{\Phi}, v)$ is a Heisenberg process with the indicated domains means that Φ and $\dot{\Phi}$ are mappings from the real vectors in $L_2(G)$ and

$\mathbf{D}(B)$ respectively to the self-adjoint operators in $\mathbf{K}$, satisfying the relations

$$e^{i\Phi(x+x')} = e^{i\Phi(x)} e^{i\Phi(x')}, \qquad e^{i\dot{\Phi}(y+y')} = e^{i\dot{\Phi}(y)} e^{i\dot{\Phi}(y')},$$
$$e^{i\Phi(x)} e^{i\dot{\Phi}(y)} = e^{i\langle x, y\rangle} e^{i\dot{\Phi}(y)} e^{i\Phi(x)}$$

for all $x \in L_2(G)$ and $y \in \mathbf{D}(B)$; that v is a cyclic vector signifies that the only closed linear manifold in $\mathbf{K}$ which contains v and is invariant under all bounded Baire functions of the $\Phi(x)$ and of the $\dot{\Phi}(y)$ is all of $\mathbf{K}$. In addition, the mappings $x \to e^{i\Phi(x)}$ and $y \to e^{i\dot{\Phi}(y)}$ are required to be continuous.

2) $\mathbf{K}_p$ is the subset of $\mathbf{K}$ consisting of all vectors of the form Tv_0, with $T \in L_p(\mathbf{R}, v_0)$.

3) The normal static Heisenberg process associated with the pair (G, B) is the process derived from the normal (free Weyl) process $(\mathbf{K}, W, \Gamma, v_0)$ over $L_2(G)$ as follows. If $\Psi(z)$ denotes the self-adjoint generator of the unitary group $[W(tz): t \in R^1]$, then for real z:

$$\Phi_0(z) = \Psi(B^{-\frac{1}{2}}), \qquad \dot{\Phi}_0(z) = \Psi(i B^{\frac{1}{2}} z).$$

4) $\Phi^{(n)}(.)$ is defined recursively as follows. $\Phi^{(0)}(f) = \int f$. Now assuming that $\Phi^{(r)}(f)$ is defined for $r < n$ and real $f \in L_2(M)$ as a self-adjoint operator in $\mathbf{K}$ affiliated with the abelian ring of operators $\mathbf{R}$ generated by the bounded Baire functions of the $\Phi(f)$, $f \in L_2(M)$, and is such that for all $g \in \mathbf{N}$,

$$e^{i\dot{\Phi}(g)} \Phi^{(r)}(f) e^{-i\dot{\Phi}(g)} = \text{closure of } \sum_{s=0}^{r} \binom{r}{s} \Phi^{(r-s)}(f g^s),$$

for all n, a given element f of $L_2(M)$ is said to be in the domain of definition of $\Phi^{(n)}(.)$ in case there exists a self-adjoint operator T affiliated with $\mathbf{A}$ such that for all $g \in \mathbf{D}(B)$,

$$e^{i\dot{\Phi}(g)} T e^{-i\dot{\Phi}(g)} = \text{closure of } T + \binom{n}{s} \sum_{s=1}^{n} \Phi^{(n-s)}(f g^s);$$

such an operator T is necessarily unique, and is defined as $\Phi^{(n)}(f)$.

Proof of Theorem. This is by induction. Note first that it suffices to consider the case $Z = I$, by virtue of the invariance of the notion of renormalized power, relative to unitary transformations which leave invariant the basic unit vector v. This invariance results directly from the definition of the renormalized powers, together with their unicity.

Now set $\Phi^{(0)}(f) = \Phi_0^{(0)}(f) = \int f$. By Lemma 3.2 there exists a unique $h_n \in \mathbf{D}(C_n)$ such that

$$\langle \Phi^{(n)}(f) v, v \rangle = \langle f, h_n \rangle$$

for all $f \in L_2(G)$. Now as an induction hypothesis, suppose it has been shown that for $r<n$, $\Phi^{(r)}(f)$ exists satisfying the conclusions of the Theorem, and such that moreover:

$$\Phi^{(r)}(f) = \text{closure of } \Phi_0^{(r)}(f) + \binom{r}{1} \Phi^{(r-1)}(f h_1) + \cdots + \binom{r}{r-1} \Phi^1(f h_{r-1}) + \Phi^0(f h_r).$$

Now let

$$T = \text{closure of } \Phi_0^{(n)}(f) + \binom{n}{1} \Phi^{(n-1)}(f h_1) + \cdots + \Phi^{(0)}(f h_{n+1}).$$

Then

$$\begin{aligned} e^{i\dot{\Phi}(g)} T e^{-i\dot{\Phi}(g)} &= \text{closure of } \Phi_0^{(n)}(f) + \binom{n}{1} \Phi_0^{(n-1)}(fg) + \cdots \\ &\quad + \binom{n}{1} \left[\Phi^{(n-1)}(f h_1) + \binom{n-1}{1} \Phi^{(n-2)}(f h_1 g) + \cdots\right] \\ &\quad + \cdots \\ &= \text{closure of } \left[\Phi_0^{(n)}(f) + \binom{n}{1} \Phi^{(n-1)}(f h_1) + \cdots\right] \\ &\quad + \binom{n}{1} \left[\Phi_0^{(n-1)}(fg) + \binom{n-1}{1} \Phi^{(n-2)}(f h_1 g) + \cdots\right] \\ &\quad + \cdots \\ &= \text{closure of } T + \binom{n}{1} \Phi^{(n-1)}(fg) + \cdots . \end{aligned}$$

This means that $\Phi^{(n)}(f)$ exists, and has the form required to complete the induction.

Corollary 3.1. *In the preceding theorem,*

$$\Phi^{(n)}(f) = \text{closure of } \Phi_0^{(n)}(f) + \sum \Phi^{(n-j)}(f h_j) \binom{n}{j},$$

where $\langle \Phi_0^{(n)}(f)\, v, v\rangle = \int_G f h_j$.

Remark. It follows that

$$\Phi^{(n)}(f) = \sum_{r=0}^{n} \Phi_0^{(n-r)}(f k_r),$$

where $k_0 = 1$ and the other k_r are in $L_{[1,\infty)}$. Moreover, the $\Phi_0^{(m)}$ may be similarly expressed in terms of the $\Phi^{(n)}$.

Notation. If $q(\lambda) = \sum_j a_j \lambda^j$, then the symbolic kernel of the distribution, $f \to \sum_j a_j \Phi^{(j)}(f)$, will be denoted as $q \circ_E \phi(x)$, where E is the state determined by the vector v.

Remark. The symbolic expression $q \circ_E \phi(x)$ representing a distribution may be strictly defined as a function of x, whose values are sesquilinear forms or generalized operators, rather than strict operators, in the same way as in the case $E = E_0 =$ the PLV for H which is treated in [10, II].

Corollary 3.2. *Suppose that the V of the theorem has the form $V = \int q \circ_{E_0} \phi(x, 0) f(x)\, dx$, treated in* I. *Then $\phi(x, t)$ satisfies the differential equation*

$$\square \phi(x, t) = m^2 \phi(x, t) + q' \circ_E \phi(x, t) + r(x) \circ_E \phi(x, t),$$

where for each x, $r(x)$ is a polynomial of degree lower than q', whose coefficients, as functions of x, are in $L_{[1, \infty)}(G)$, and E is the state determined by the PLV for H'.

In case $G = T^1$ and $f(x)$ is constant, then it follows from spatial invariance that the functions r_j are constants, where $r(x)(\lambda) = \sum r_j(x) \lambda^j$. In this case there then results a well defined mapping $p \to \tilde{p}$ from the real polynomials on R^1 of the form $p = q'$, $q \geqq 0$, to similar polynomials $\tilde{p}$, such that starting as in I with a hamiltonian involving p (more precisely, $\int p$), there is obtained a solution ϕ of the equation

$$\square \phi = m^2 \phi + \tilde{p} \circ_E \phi.$$

The question of whether the equation $\square \phi = m^2 \phi + r \circ_E \phi$, where r is a given polynomial, is soluble in the sense treated here, is equivalent to the question of whether r is of the form $\tilde{p}$ for some p. The mapping $p \to \tilde{p}$ has not yet been explored, particularly in the more relevant aspect of its behavior for large coefficients a_j of q, and the cited question is presently open.

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