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Ergodicity of Anosov Actions

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1. Introduction

In this paper we generalize some ergodicity results of Anosov and Sinai [1, 2] to group actions more general than Z and R. At the same time we provide what we consider to be a more natural proof of the central theorem in [1] concerning the absolute continuity of certain foliations – see (2.1).

Definition [5]. Let G be a Lie group acting differentiably on M, $A: G \to \mathrm{Diff}(M)$ where M is a compact smooth manifold. We assume that the orbits of G define a differentiable foliation $\mathscr{F}$, which is the case for instance if the G action is locally free (every isotropy group is discrete). The action is called Anosov if there exists an Anosov element – an element $g \in G$ such that $A(g) = f$ is hyperbolic at $\mathscr{F}$ [5] and

(1) the G action is locally free, or

(2) G is connected and g is central in G.

We recall that $A(g) = f$ is hyperbolic at $\mathscr{F}$ means that $Tf: TM \to TM$ leaves invariant a splitting

$$E^u \oplus T\mathscr{F} \oplus E^s = TM$$

contracting E^s more sharply than $T\mathscr{F}$, expanding E^u more sharply than $T\mathscr{F}$. ($T\mathscr{F}$ is the bundle of planes tangent to the leaves of $\mathscr{F}$.)

For example, if $\{\varphi_t\}$ is an Anosov flow on M then $t \mapsto \varphi_t$ defines an R-action on M and gives the foliation of M by the trajectories. Any φ_t, $t \neq 0$ is an Anosov element. Similarly, if f is an Anosov diffeomorphism of M then $n \mapsto f^n$ defines a Z-action on M which is Anosov. The leaves of the orbit foliation are the points of M. Further examples are given in [3, 5].

In [5] it was proven that Anosov actions are structurally stable, generalizing another part of the work of Anosov on flows and diffeomorphisms.

Definition. The action $A: G \to \mathrm{Diff}(M)$ is ergodic iff it is measure preserving and all invariant functions are constant. Precisely, we require

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(1) For each $g \in G$, $A(g)$ is measure preserving (respecting some fixed Lebesgue measure on M).

(2) If $f: M \to R$ is integrable and, for all $g \in G$, $f \circ A(g) = f$ almost everywhere on M then f equals a constant, almost everywhere.

Our main theorem is:

(1.1) Theorem. *Suppose* $A: G \to \mathrm{Diff}^2(M)$ *is a measure preserving Anosov action with an Anosov element in the centralizer of* G. *Then* A *is ergodic.*

In particular, if G *is abelian and* A *a measure preserving* C^2 *Anosov action then* A *is ergodic.*

Theorem (1.1) may be used in conjunction with [6] to give information about the ergodic elements of an Anosov action. We give one example:

(1.2) Theorem. *Suppose* $A: R^k \to \mathrm{Diff}^2(M)$ *is a measure preserving Anosov action. Then for every* $g \in R^k$ *off a countable family of hyperplanes in* R^k, $A(g)$ *is an ergodic diffeomorphism. We recall that a hyperplane is a translate of a hyperplane through zero.*

The idea of the proof is as follows. Let f be the Anosov element. Then f is hyperbolic at the orbit foliation and so, from [5], we deduce a stable manifold theory for f. By uniqueness and commutativity with f, the stable and unstable manifolds are A-invariant. We prove that any strong stable manifold foliation is absolutely continuous, and so is the center unstable foliation. Then we deduce ergodicity of A as Anosov and Sinai did, via Birkhoff's Theorem [2]. The center unstable case is harder than the strong stable, and it would be tempting to try avoiding it by using [8]. This would require measurability of the center unstable foliation in the sense of Sinai [8]. But measurability seems no easier to prove than absolute continuity, nor is it a consequence of being a foliation in the sense of Anosov [1, p. 18]. See §6 for an example of this.

2. Pre-Foliations

It is frequently useful and natural to deal with a localized version of a foliation – we call it a pre-foliation. It amounts to the continuous assignment of a disc through each point of a manifold.

Indeed, let M be a compact smooth Riemann manifold and let D^k be the k-disc. The set of all C^r, $r \geqq 0$, embeddings $D^k \to M$ carrying 0 onto some $p \in M$ forms a metric space

$$\mathrm{Emb}^r(D^k, 0; M, p).$$

The C^r distance between two embeddings is defined in the usual way – either via the Riemann metric or a fixed embedding of M into a Euclidean space. It is easy to see that $\mathrm{Emb}^r(D^k, M)$ is a C^r fiber bundle over M, $\pi(h)=h(0)$ being the projection.

Definition. A pre-foliation of M by C^r k-discs is a map $p \mapsto \mathscr{D}_p$ such that $\mathscr{D}_p$ is a C^r k-disc in M containing p and depending continuously on p in the following sense: M can be covered by charts, U, in which $p \mapsto \mathscr{D}_p$ is given by

$$\mathscr{D}_p = \sigma(p)(D^k) \qquad p \in U$$

and $\sigma \colon U \to \mathrm{Emb}^r(D^k, U)$ is a continuous section. If, in addition, these sections σ can all be chosen so that the maps $(p, x) \mapsto \sigma(p)(x)$ are of class C^s, $1 \leqq s \leqq r$, then the pre-foliation is said to be of class C^s.

Example 1. If $\mathscr{F}$ is a C^r k-foliation of M, $r \geqq 1$, let

$$\mathscr{F}_p(\delta) = \{x \in \mathscr{F}_p \colon d_{\mathscr{F}}(x, p) \leqq \delta\}$$

where $d_{\mathscr{F}}$ is the distance in the leaf measured respecting the Riemann structure in $T\mathscr{F}$ inherited from TM. Then, for small $\delta > 0$,

$$p \mapsto \mathscr{F}_p(\delta)$$

gives a C^r pre-foliation of M by C^r k-discs.

Example 2. Let N be a C^r sub-bundle of k-planes in TM. Then, for small $\delta > 0$,

$$p \mapsto \exp_p(N_p(\delta))$$

gives a C^r pre-foliation of M by C^∞ k-discs.

Example 3. Let $\mathscr{W}^u$ be the unstable manifold foliation of M for a C^r Anosov diffeomorphisms. For small $\delta > 0$

$$p \mapsto W_p^u(\delta) = \text{the } \delta\text{-local unstable manifold through } p$$

gives a pre-foliation of M by C^r k-discs. In general this pre-foliation is *not* of class C^1 [1, § 24].

On the same note, let us emphasize that for us, a "foliation of M by C^r k-leaves" need not be a C^r foliation. The leaves are C^r and they vary locally continuously in the C^r sense (this, for $r=1$, implies that the union of their tangent planes gives a continuous k-sub-bundle of TM) but their assembly is not necessarily C^r. Similarly for pre-foliations.

Now we shall explain the idea of Poincaré map along a pre-foliation. This is the usual "notion of translation in the transversal" for foliations. Let $\mathscr{G}$ be a pre-foliation of M by C^r k-discs, $r \geqq 1$, let $q \in \mathrm{Int}\, \mathscr{G}_p$, $\mathscr{G}_p$ = the $\mathscr{G}$-disc through p, and let D_p, D_q be two smooth $(m-k)$-discs embedded transverse to $\mathscr{G}_p$ at p, q. (See Fig. 1.)

$$T_p D_p \oplus T_p \mathscr{G}_p = T_p M, \qquad T_q D_q \oplus T_q \mathscr{G}_p = T_q M.$$

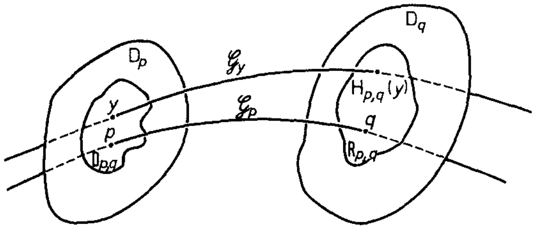

Fig. 1. The Poincaré map

Then there is defined a surjection $H_{p,q}$: $D_{p,q} \to R_{p,q}$ where $D_{p,q}$ is a neighborhood of p in D_p

$$\begin{array}{ccc} D_{p,q} & \longrightarrow & R_{p,q} \\ \cap & & \cap \\ D_p & & D_q \\ H_{p,q}(p) = q & & H_{p,q}(y) \in \mathcal{G}_y \cap D_q. \end{array}$$

Since $\mathcal{G}_y$ depends continuously on $y \in D_p$ in the C^r sense, $r \geqq 1$, and $\mathcal{G}_p$ transversally intersects D_q at q, there is uniquely defined a new point of transversal intersection, $H_{p,q}(y)$, depending continuously on y near p. The range of $H_{p,q}$, $R_{p,q}$, is not in general a neighborhood of q in D_q, nor is $H_{p,q}$ in general a local homeomorphism. On the other hand, $H_{p,q}$ is C^s when $\mathcal{G}$ is of class C^s and $H_{p,q}$ depends continuously on p, q, D_p, D_q in the C^s sense. Thus, if $\mathcal{G}$ is C^1 and q is near p then $H_{p,q}$ is a local diffeomorphism.

Next we explain the idea of absolutely continuous foliations. Recall that a bijection between measure spaces $h: U \to V$ is absolutely continuous if it is measurable and is a bijection between the zero sets of U and V.

Definition. A pre-foliation of M by C^r k-discs is absolutely continuous if each of its Poincaré maps $H_{p,q}$: $D_{p,q} \to R_{p,q}$ is absolutely continuous.

Definition. If, in addition, the Radon Nikodym derivative, J, is continuous and positive, $J: D_{p,q} \to R$,

$$\mu_{D_q}(S) = \int_{H_{p,q}^{-1}(S)} J \, d\mu_{D_p} \qquad S \subset R_{p,q}$$

then the pre-foliation is said to be measurewise C^1.

The measures μ_{D_q}, μ_{D_p} are the smooth ones induced by the Riemann structure on TM. Joint continuity in p, q, D_p, D_q, y is required. Variation of D_p, D_q is done in $\mathrm{Emb}^1(D^{m-k}, M)$. J is called the (generalized) Jacobian of H. Existence of such a J implies, of course, absolute continuity.

(2.1) Theorem. *Strong unstable and strong stable foliations are measurewise C^1. (In particular absolutely continuous.) Precisely: Suppose* *f is a C^s diffeomorphism of M, $s \geqq 2$, Tf leaves $E^u \oplus E^{ps} = TM$ invariant and*

$$\sup_{p\in M} \|T_p^{ps} f\|^j < \inf_{p\in M} m(T_p^u f) \qquad 0 \leqq j \leqq r \leqq s,\ r \geqq 1.$$

Then there is a unique f-invariant foliation of M by C^r leaves tangent to E^u, the strong unstable foliation, $\mathscr{W}^u$. It is measurewise C^1. Similarly for strong stable foliations.

Remarks. $m(T_p^u f)$ is the co-norm (or minimum norm) of $T_p f | E_p^u = T_p^u f$; that is, $m(T_p^u f) = \|T_{fp}^u f^{-1}\|^{-1}$. Our condition on Tf means that all vectors of E^u are expanded more sharply than any vectors in E^{ps}. The existence of a unique f-invariant foliation of M with C^r leaves tangent to E^u is proved in [5]. In general, there is no reason to believe E^{ps} can also be integrated. Notice that $\|T^{ps} f\|$ may be >1 which is why we write ps — to indicate pseudo-stable. A more or less explicit formula for the Jacobian J is developed in the proof of (2.1) given in §3. The inequality in the hypothesis of (2.1) can be weakened to

$$\inf_{p\in M} m(T_p^u f)\, \|T_p^{ps} f\|^{-j} > 1 \qquad 0 \leqq j \leqq r$$

but the proof of (2.1) becomes technically harder. If $\sup_p \|T_p^{ps} f\| \leqq 1$, notice that the hypothesis of (2.1) amounts to assuming $T^u f$ is an expansion.

Finally, we wish to point out that our proofs differ substantially from Anosov's [1] only in that they avoid using continuous differential forms, dealing directly with the Poincaré maps instead. In the same way, they differ from those in [8] in that no emphasis is laid on measure theoretic generality.

3. Proof that $\mathscr{W}^u$ Is Measurewise C^1

Although E^u, E^{ps} need not be smooth (this would imply measurewise C^1 at once) they are Hölder.

(3.1) Lemma [c.f. 1]. *E^u and E^{ps} are θ-Hölder continuous for some $\theta > 0$.*

Proof. Let $\tilde{E}^u$, $\tilde{E}^{ps}$ be smooth approximations to E^u, E^{ps} and let $\mathscr{D}_x = \{P \in L(\tilde{E}_x^{ps}, \tilde{E}_x^u) \colon \|P\| \leqq 1\}$. Then $\mathscr{D} = \bigcup \mathscr{D}_x$ is a smooth disc bundle over M and Tf^{-1} acts on $\mathscr{D}$ in the natural way

$$F\colon P \to (C_x + K_x P) \circ (A_x + B_x P)^{-1}$$

for

$$T_x f^{-1} = \begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix} \qquad \text{respecting } \tilde{E}^{ps} \oplus \tilde{E}^u.$$

F is a fiber contraction: it preserves fibers of $\mathscr{D}$, covers $f^{-1}: M \to M$, and the Lipschitz constant of $F|\mathscr{D}_x$ is $\leqq k < 1$. In fact k is approximately μ/λ when $\lambda = \inf m(T_x^u f)$, $\mu = \sup \|T_x^{ps} f\|$, and $\tilde{E}^u$, $\tilde{E}^{ps}$ are very near E^u, E^{ps}.

The bundle E^{ps}, represented as the graphs of linear maps $\tilde{E}^{ps} \to \tilde{E}^u$, is an F-invariant section of $\mathscr{D}$. But the Invariant Section Theorem [6.1 of 4] says that the unique F-invariant section of $\mathscr{D}$ is θ-Hölder continuous if F is C^1 and $k L(f)^\theta < 1$. Since f is at least C^2, this proves that, for some $\theta > 0$, E^{ps} is θ-Hölder. Similarly for E^u.

Following Anosov we write $\rightrightarrows$ to denote uniform convergence.

(3.2) Lemma [1, p. 136]. *Suppose $h: D^k \to R^k$ is a topological embedding and (g_n) is a sequence of C^1 embeddings $D^k \to R^k$ such that*

$$g_n \rightrightarrows h \qquad J(g_n) \rightrightarrows J$$

where $J(g_n)$ is the Jacobian of g_n. Then h is absolutely continuous and has Jacobian J.

Proof [1, p. 136]. We must show

$$\operatorname{mes}(hA) = \int_A J\, d\mu \qquad A \subset D^k, \qquad \text{measurable}$$

when $d\mu$ is Lebesgue measure on D^k. Since h is continuous, it suffices to prove this equality for $A =$ an arbitrary closed subdisc of D^k. Let $\varepsilon > 0$ be given and choose two other discs A', A'' such that A' is interior to A and A is interior to A''. They can be chosen so near A that

$$\int_{A''-A} J\, d\mu < \varepsilon/2$$

because J is continuous. Since g_n is a C^1 embedding, $\operatorname{mes}(g_n S) = \int_S J(g_n)\, d\mu$ for any measurable $S \subset D^k$, and since h is a topological embedding

$$g_n A' \subset h A \subset g_n A''$$

for large n. Thus

$$\begin{array}{ccccc} \int_{A'} J(g_n)\, d\mu & \leqq & \int_A J(g_n)\, d\mu & \leqq & \int_{A''} J(g_n)\, d\mu \\ \| & & & & \| \\ \operatorname{mes}(g_n A') & \leqq & \operatorname{mes}(hA) & \leqq & \operatorname{mes}(g_n A'') \end{array}$$

and so $\left|\operatorname{mes}(hA) - \int_A J(g_n)\, d\mu\right| < \varepsilon$ for large n. Since $\int_A J(g_n)\, d\mu \to \int_A J\, d\mu$, we have shown $\left|\operatorname{mes}(hA) - \int_A J\, d\mu\right| \leqq \varepsilon$ proving the lemma.

To state precisely the next lemma, we speak of angles between subspaces of TM. The Riemann structure on TM lets us define

$$\sphericalangle(A_p, B_p) = \max\{\sphericalangle(a, B_p) : a \in A_p - 0\} \cup \{\sphericalangle(b, A_p) : b \in B_p - 0\}$$

where A_p, B_p are linear subspaces of $T_p M$. This amounts to the Hausdorff metric on the Grassmanian. The angle between two subbundles A, B is the supremum of $\sphericalangle(A_p, B_p)$.

(3.3) Lemma. *Suppose $TM = N \oplus E^{ps} = E^u \oplus E^{ps}$ and N is smooth. Let $\mathscr{G}(\delta)$ be the smooth pre-foliation $p \mapsto \mathscr{G}_p(\delta) = \exp_p(N_p(\delta))$. Let β be given, $0 \leqq \beta < \pi/2$. For small $\delta > 0$, each Poincaré map $G_{p,q}: D_{p,q} \to R_{p,q}$ along $\mathscr{G}(\delta)$ is a smooth immersion if $\sphericalangle(TD_p, (E^u)^\perp) \leqq \beta$ and $\sphericalangle(TD_q, (E^u)^\perp) \leqq \beta$.*

Proof. The condition on D_p, D_q is that they be uniformly transverse to E^u. Since $G_{p,q}$ is smooth and its derivative is a continuous function of p, q, it suffices to prove that $T_y G_{p,q}$ is a bijection $T_y D_p \to T_{y'} D_q$ for $y' = G_{p,q}(y)$. Since $G_{p,q} = G_{y,y'}$ near y, it suffices to verify bijectivity at $y = p$. Clearly when $y = p = q$, this is true. But since the derivative of $G_{p,q}$ depends continuously on p, q, D_p, D_q and since M and $\{A_p \subset T_p M : \sphericalangle(A_p, (E^u)^\perp) \leqq \beta\}$ are compact, bijectivity on the diagonal $p = q$ propagates to some δ-neighborhood of the diagonal.

Proof of (2.1). Let N be a smooth approximation to E^u. Choose β so that $0 < \beta < \pi/2$ and $\sphericalangle(E^{ps}, (E^u)^\perp) < \beta$, $\sphericalangle(E^{ps}, N^\perp) < \beta$. Then choose $\delta > 0$ according to (3.3). Let

$$\mathscr{G}: \mathscr{G}_y = \exp_y(N_y(\delta)) \qquad y \in M$$

be the resulting smooth pre-foliation. Let $\mathscr{G}^n$ be the pre-foliation gotten from iteration by f^n

$$\mathscr{G}^n: \mathscr{G}^n_y = f^n \mathscr{G}_{f^{-n}y}.$$

Let $\mathscr{G}^n(\varepsilon)$ be the restriction of $\mathscr{G}^n$ to radius ε

$$\mathscr{G}^n(\varepsilon): \mathscr{G}^n_y(\varepsilon) = \{x \in \mathscr{G}^n_y : d_{\mathscr{G}^n}(x, y) \leqq \varepsilon\}.$$

By [5], $\mathscr{G}^n(\varepsilon) \rightrightarrows \mathscr{W}^u(\varepsilon)$ and $T\mathscr{G}^n(\varepsilon) \rightrightarrows E^u$. Thus f acts on pre-foliations in a natural way and $\mathscr{W}^u$ is the attractive fixed point of this action.

Consider $q \in W^u_p$ and discs D_p, D_q transverse to E^u. We must study the Poincaré map $H_{p,q}: D_{p,q} \to R_{p,q}$ for the foliation $\mathscr{W}^u$. Because $\mathscr{W}^u$ is a foliation – not just a pre-foliation – $H_{p,q}$ is a homeomorphism and $R_{p,q}$ is a neighborhood of q in D_q.

The relation between $H_{p,q}$ and $H_{f^{-n}p, f^{-n}q}$ is expressed by commutativity of

$$\begin{array}{ccc} f^{-n} D_{p,q} & \xrightarrow{H_{f^{-n}p, f^{-n}q}} & f^{-n} R_{p,q} \\ \downarrow f^n & & \downarrow f^n \\ D_{p,q} & \xrightarrow{H_{p,q}} & R_{p,q} \end{array}$$

since $\mathscr{W}^u$ is f-invariant. Since f is a diffeomorphism existence of a continuous positive Jacobian for $H_{p,q}$ is equivalent to the question for $H_{f^{-n}p, f^{-n}q}$. Furthermore, as $n \to \infty$, $T(f^{-n} D_p)$ and $T(f^{-n} D_q) \rightrightarrows E^{ps}$ [5]. Thus it is no loss of generality to assume

$$q \in W_p^u(\varepsilon/2) \qquad \measuredangle(T(f^{-n} D_p), (E^u)^\perp) \leqq \beta \qquad \measuredangle(T(f^{-n} D_q), (E^u)^\perp) \leqq \beta \qquad (*)$$

for all $n \geqq 0$. Furthermore, we may shrink D_p so that $D_p = D_{p,q}$ and $R_{p,q} = \text{range } H_{p,q}$ is interior to D_q, for existence of $J(H_{p,q})$ is a local question.

Since $\mathscr{G}^n(\varepsilon) \rightrightarrows \mathscr{W}^u(\varepsilon)$, the Poincaré map $G_{p,q}^n$ of D_p to D_q along $\mathscr{G}^n(\varepsilon)$ is defined in a unique single valued continuous manner on the domain D_p, $n = 0, 1, 2, \ldots$. Thus it is clear that

$$g_n \rightrightarrows h$$

where $g_n = G_{p, Q_n}^n | D_p$, $Q_n = \mathscr{G}_p^n(\varepsilon) \cap D_q$, and $h = H_{p,q}$. We show that

$$g_n \text{ is an embedding}, \tag{a}$$

$$J(g_n) \rightrightarrows J = \operatorname{unif} \lim_{n \to \infty} \frac{\det(f^{-n} | T_y D_p)}{\det(f^{-n} | T_{hy} D_q)}. \tag{b}$$

Then, by (3.2), J is the Jacobian of $h = H_{p,q}$. Since the limit in (b) is uniform, J is continuous, and by symmetry positive. Thus, proof of (a), (b) demonstrates (2.1).

The proof of (a) is topological and thanks are due to R. Palais. By (3.3), (*), the choice of δ, and the naturality of Poincaré maps, g_n is at least immersion wherever defined. Moreover, both g_n and h are defined on a slightly larger disc $\hat{D}_p$, say

$$\hat{g}_n : \hat{D}_p \to D_q, \qquad \hat{h} : \hat{D}_p \to D_q$$

and $\hat{g}_n \rightrightarrows \hat{h}$. Since $\hat{g}_n$, $\hat{h}$ are locally injective, the theory of mapping degrees [7] is applicable. Let Y be a compact neighborhood of $R_{p,q} = h D_p$ interior to $\hat{h} \hat{D}_p$. For any $y \in Y$, degree $(\hat{h}, \hat{D}_p, y) = 1$ since $\hat{h}$ is a homeomorphism. For large n, $\hat{g}_n | \partial \hat{D}_p$ is very near $\hat{h} | \partial \hat{D}_p$ and so

$$\hat{g}_n | \partial \hat{D}_p \simeq \hat{h} | \partial \hat{D}_p \quad \text{in } D_q - Y.$$

Thus, for large n, degree $(\hat{g}_n, \hat{D}_p, y) = 1$ for all $y \in Y$, and thus $\hat{g}_n$ embeds $\hat{g}_n^{-1} Y$. The latter contains D_p, for large n, since $\hat{g}_n \rightrightarrows \hat{h}$ and $\hat{h}^{-1} Y$ contains D_p in its interior. This proves (a).

To prove (b) we express g_n in terms of the Poincaré map along $\mathcal{G}$, acted on by f^n – this is the straightforward thing to do. Consider $g_n: D_p \to D_q$ as

$$g_n = f^n \circ G^0_{p_n, q_n} \circ f^{-n}$$

where $p_n = f^{-n} p$, $q_n = f^{-n} Q_n$. (Recall that Q_n was the point $\mathcal{G}^n_p(\varepsilon) \cap D_q$.) Thus $q_n \in \mathcal{G}_{p_n}$ and so the Poincaré map along $\mathcal{G}$, $G^0_{p_n, q_n}$, is well defined on $f^{-n} D_p$. Moreover

$$q_n \in \mathcal{G}_{p_n}(\varepsilon_n), \qquad \varepsilon_n \to 0$$

as $n \to \infty$. For $\mathcal{G}_{p_n}$ is approximately tangent to E^u and is thus sharply expanded by f^n (see Fig. 2).

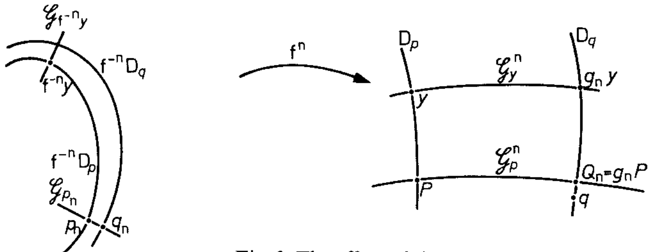

Fig. 2. The effect of f^n

Using the Chain Rule,

$$J_y(g_n) = \det\left(Tf^n \mid T_{f^{-n} g_n y}(f^{-n} D_q)\right) \det\left(TG^0_{p_n, q_n} \mid T_{f^{-n} y}(f^{-n} D_p)\right) \cdot \det(Tf^{-n} \mid T_y D_p)$$

for any $y \in D_p$. Since $T(f^{-n} D_p) \rightrightarrows E^{ps}$, $T(f^{-n} D_q) \rightrightarrows E^{ps}$, and $q_n \in \mathcal{G}_{p_n}(\varepsilon_n)$ with $\varepsilon_n \to 0$, the middle factor tends uniformly to 1. (b) is therefore equivalent to

$$\operatorname*{unif\,lim}_{n \to \infty} \frac{\det(Tf^{-n} \mid T_y D_p)}{\det(Tf^{-n} \mid T_{g_n y} D_q)} = \operatorname*{unif\,lim}_{n \to \infty} \frac{\det(Tf^{-n} \mid T_y D_p)}{\det(Tf^{-n} \mid T_{h y} D_q)}. \tag{b'}$$

Although (b′) could be proved directly, we first establish the special case (as does Anosov in [1]) $y = p$, $T_p D_p = E^{ps}_p$, $T_q D_q = E^{ps}_q$. We prove

$$\lim_{n \to \infty} \frac{\det(T^{ps}_p f^{-n})}{\det(T^{ps}_q f^{-n})} \text{ exists uniformly.} \tag{c}$$

$T^{ps} f^{-n}$ denotes $Tf^{-n} \mid E^{ps}$. By the Chain Rule (c) is equivalent to the uniform convergence of

$$\prod_{k=0}^{\infty} \frac{\det(T^{ps}_{f^{-k} p} f^{-1})}{\det(T^{ps}_{f^{-k} q} f^{-1})}$$

and this, in turn, is equivalent to the uniform convergence of

$$\sum_{k=0}^{\infty} |\det(T^{ps}_{f^{-k}p} f^{-1}) - \det(T^{ps}_{f^{-k}q} f^{-1})|.$$

Since E^{ps} is θ-Hölder with $\theta>0$ by (3.1), and f is C^2, $T^{ps}f^{-1}$ is θ-Hölder and so

$$|\det(T^{ps}_{f^{-k}p} f^{-1}) - \det(T^{ps}_{f^{-k}q} f^{-1})| \leqq C\, d(f^{-k}p, f^{-k}q)^\theta$$

for some constant C. Since $q\in W^u p$, $d(f^{-k}p, f^{-k}q) \leqq \lambda^{-k} d(p,q)$ where $\lambda = \inf m(T^u_x f) > 1$. Thus $\lambda^{-\theta} < 1$ and our series converges uniformly by comparison with $C\sum(\lambda^{-\theta})^k d(p,q)$. This proves (c).

Now we show how (c) implies (b′). Let π^{ps} be the projection of TM onto E^{ps} along E^u. Thus π^{ps} kills E^u and leaves E^{ps} fixed. Since Tf leaves $E^u \oplus E^{ps}$ invariant, Tf^{-n} commutes with π^{ps}. Thus

$$Tf^{-n}|T_y D_p = (\pi^{ps}|T_{f^{-n}y}(f^{-n}D_p))^{-1} \circ T^{ps}f^{-n} \circ (\pi^{ps}|T_y D_p)$$

for $y\in D_p$. Taking determinants gives

$$\det(Tf^{-n}|T_y D_p) = \frac{\det(T^{ps}_y f^{-n}) \det(\pi^{ps}|T_y D_p)}{\det(\pi^{ps}|T_{f^{-n}y} f^{-n} D_p)}.$$

As $n\to\infty$, $T(f^{-n}D_p) \rightrightarrows E^{ps}$ and so the denominator in the preceding fraction tends uniformly to 1. The same holds when y is replaced by a point of D_q. Thus, we are reduced to proving

$$\begin{aligned} &\operatorname*{unif\,lim}_{n\to\infty} \frac{\det(T^{ps}_y f^{-n}) \det(\pi^{ps}|T_y D_p)}{\det(T^{ps}_{g_n y} f^{-n}) \det(\pi^{ps}|T_{g_n y} D_q)} \\ &\quad = \operatorname*{unif\,lim}_{n\to\infty} \frac{\det(T^{ps}_y f^{-n}) \det(\pi^{ps}|T_y D_p)}{\det(T^{ps}_{hy} f^{-n}) \det(\pi^{ps}|T_{hy} D_q)}. \end{aligned} \tag{b''}$$

Since $g_n \rightrightarrows h$ and D_q is C^1, (b″) is equivalent to

$$\operatorname*{unif\,lim}_{n\to\infty} \frac{\det(T^{ps}_y f^{-n})}{\det(T^{ps}_{g_n y} f^{-n})} = \operatorname*{unif\,lim}_{n\to\infty} \frac{\det(T^{ps}_y f^{-n})}{\det(T^{ps}_{hy} f^{-n})}. \tag{b'''}$$

By (c)—applied to y, hy instead of p, q—the second limit exists and is uniform. To prove that the first exists and equals the second it suffices to show that

$$\operatorname*{unif\,lim}_{n\to\infty} \frac{\det(T^{ps}_{hy} f^{-n})}{\det(T^{ps}_{g_n y} f^{-n})} = 1. \tag{d}$$

(d) is equivalent to

$$\operatorname*{unif\,lim}_{n\to\infty} \sum_{k=0}^{n-1} |\det(T^{ps}_{f^{-k}hy} f^{-1}) - \det(T^{ps}_{f^{-k}g_n y} f^{-1})| = 0 \tag{d'}$$

by the Chain Rule, as before. Again, this sum is $\leqq$

$$C\sum_{k=0}^{n-1} d(f^{-k}h\,y, f^{-k}g_n\,y)^\theta$$

for some constant C, since E^{ps} is θ-Hölder. Let $\mu=\sup\|T_x^{ps}f\|$ and $\lambda=\inf m(T^u f)$. By hypothesis, $\mu<\lambda$ and $\lambda>1$. Choose

$$\max(\mu, 1)<\boldsymbol{\mu}<\boldsymbol{\lambda}<\lambda.$$

Since $f^{-n}h\,y\in W^u_{f^{-n}y}(\varepsilon_n)$, $f^{-n}(g_n\,y)\in\mathscr{G}_{f^{-n}y}(\varepsilon_n)$ and $\mathscr{G}$ is approximately tangent to E^u,

$$\varepsilon_n\leqq\boldsymbol{\lambda}^{-n}\quad\text{for large } n.$$

Thus, $d(f^{-n}(h\,y), f^{-n}(g_n\,y))\leqq\varepsilon_n<\boldsymbol{\lambda}^{-n}$ for large n. On the other hand, $d(f^{-k}(h\,y), f^{-k}(g_n\,y))=d(f^{n-k}(f^{-n}h\,y), f^{n-k}(f^{-n}g_n\,y))$, and for large k, $f^{-k}D_q, \ldots, f^{-n}D_q$ are nearly tangent to E^{ps}, so that

$$d(f^{-k}(h\,y), f^{-k}(g_n\,y))\leqq C'\boldsymbol{\mu}^{n-k}\boldsymbol{\lambda}^{-n}$$

for some constant C'. Thus

$$C\sum_0^{n-1} d(f^{-k}(h\,y), f^{-k}(g_n\,y))^\theta\leqq C(C')^\theta\left[\sum_0^{n-1}(\boldsymbol{\mu}^\theta)^{n-k}\right](\lambda^{-\theta})^n$$

$$=C''(\boldsymbol{\mu}^\theta+\cdots+\boldsymbol{\mu}^{n\theta})\,\lambda^{-n\theta}=C''\,\boldsymbol{\mu}^\theta\left(\frac{1-\boldsymbol{\mu}^{n\theta}}{1-\boldsymbol{\mu}^\theta}\right)\lambda^{-n\theta}$$

which tends to zero as $n\to\infty$. This proves (d′), hence (d), (b‴), (b″), (b′), and (b) – completing the proof of (2.1).

4. Measurewise Smoothness of Center Unstable Foliations

The main theorem of this section, (4.2), is an analogue of (2.1). Recall that a diffeomorphism f of M is normally hyperbolic at a foliation $\mathscr{F}$ of M iff Tf leaves invariant a splitting $TM=E^u\oplus E^c\oplus E^s$, expanding E^u more sharply than $E^c=T\mathscr{F}$, contracting E^s more sharply than E^c, and leaving $\mathscr{F}$-invariant. The following theorem was proved in [5].

(4.1) Theorem. *If $\mathscr{F}$ is C^1 and f is normally hyperbolic at $\mathscr{F}$ then there are unique f-invariant foliations of M, $\mathscr{W}^{cu}$ and $\mathscr{W}^{cs}$, tangent to $E^{cu}=E^u\oplus E^c$ and $E^{cs}=E^c\oplus E^s$. Each of their leaves is a union of $\mathscr{F}$-leaves and $W_p^{cu}=\bigcup_{q\in\mathscr{F}_p} W_q^u$, $W_p^{cs}=\bigcup_{q\in\mathscr{F}_p} W_q^s$.*

Here we shall prove

(4.2) Theorem. *If f is normally hyperbolic at $\mathscr{F}$, $\mathscr{F}$ is C^1, and f is C^2 then $\mathscr{W}^{cu}$, $\mathscr{W}^{cs}$ are measurewise C^1.*

Proof. We shall utilize a notion generalizing "pre-foliation by discs" to "pre-foliation by submanifolds". However, we shall not make the precise general definition of this, but confine ourselves to the case

$$\mathscr{H}: \mathscr{H}_p = \bigcup_{y \in \mathscr{F}_p} \exp_y(N_y(\delta))$$

where N is a smooth subbundle of TM approximating E^u. In § 3, we called

$$\mathscr{G}: \mathscr{G}_y = \exp_y(N_y(\delta))$$

the pre-foliation by u-discs. Now we are considering the union of all these u-discs as y ranges over the leaf $\mathscr{F}_p$. This gives the immersed manifold $\mathscr{H}_p$, nearly tangent to E^{cu}. Then let

$$\mathscr{H}^n: \mathscr{H}_p^n = \bigcup_{y \in \mathscr{F}} \mathscr{G}_y^n(\delta).$$

We know that $\mathscr{H}^n \rightrightarrows \mathscr{W}^{cu}$ and $T\mathscr{H}^n \rightrightarrows E^{cu}$ by [5].

Let D_p, D_q be s-discs transversal to E^{cu} through p, q with $q \in W_p^{cu}$. We must investigate the Poincaré map $H_{p,q}$ along $\mathscr{W}^{cu}$. As in § 3, we may assume

$$q \in W_{p'}^u(\varepsilon/2), \qquad p' \in \mathscr{F}_p(\varepsilon/2), \qquad D_p = \text{domain } H_{p,q}, \qquad \operatorname{diam}(D_p) < \varepsilon/2$$

without loss of generality. Consider the Poincaré maps $H_n = H_{p,q}^n$ along the $\mathscr{H}^n$ leaves through D_p. As in § 3, we must prove that

$$H_n \text{ is an embedding}, \qquad H_n \rightrightarrows H = H_{p,q}, \tag{A}$$

$$J(H_n) \rightrightarrows J > 0. \tag{B}$$

The proof of (A) is the same as (a) in § 3 because $\mathscr{H}^n \rightrightarrows \mathscr{W}^{cu}$ and $H_{p,q}$ is a homeomorphism.

Call $D = \bigcup_{y \in D_p} \mathscr{F}_y(\varepsilon)$. This D is a smooth disc transverse to E^u. It is smoothly fibered by the leaves of $\mathscr{F}$. For each $y \in D_p$, $\mathscr{F}_y \subset \mathscr{H}_y^n$ for all $n \geqq 0$. Thus, the Poincaré map along the leaves of $\mathscr{H}_y^n$, $y \in D_p$, would be *smooth* if the image disc, D_q lay in D.

For each $y \in D_p$, let y_n be the unique point of $\mathscr{F}_y(\varepsilon)$ such that

$$H_n y \in \mathscr{G}_{y_n}^n(\varepsilon)$$

and let y_* be the unique point of $\mathscr{F}_y(\varepsilon)$ such that

$$H y \in W_{y_*}^u(\varepsilon).$$

Clearly $y_n \rightrightarrows y_*$ and p_* is the point we called p'.

Choose smooth discs $\Sigma(y_n)$, $\Sigma(y_*)$ at y_n, y_* in D, transverse to E^c. We may assume them chosen so that

$$\Sigma(y_n) \rightrightarrows \Sigma(y_*), \qquad T\Sigma(y_n) \rightrightarrows T\Sigma(y_*).$$

Then we may factor H_n as $h_n \circ F_{y, y_n}$ where $F_{y, y_n}\colon D_p \to \Sigma(y_n)$ is the Poincaré map along $\mathscr{F}$ in D and $h_n\colon \Sigma(y_n) \to D_q$ is the Poincaré map along the leaves of $\mathscr{H}^n$ through D_p (see Fig. 3). Note that this factorization depends on y.

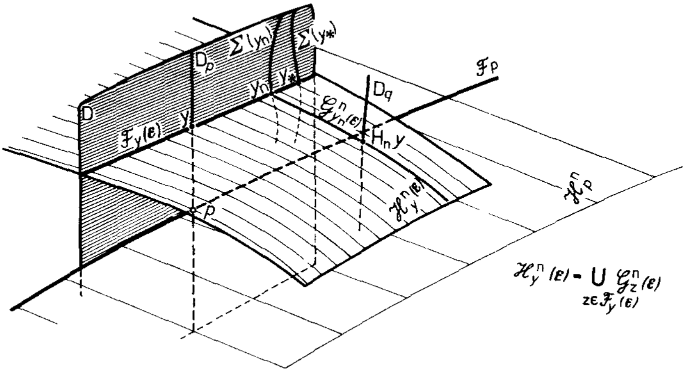

Fig. 3. Factorizing the Poincaré map H_n

Since $\Sigma(y_n) \rightrightarrows \Sigma(y_*)$ and $T\Sigma(y_n) \rightrightarrows T\Sigma(y_*)$ and $\mathscr{F}$ is C^1,

$$\det(T_y F_{y, y_n}) \rightrightarrows \det(T_y F_{y, y_*}) > 0 .$$

Thus (B) will follows from

$$\operatorname{unif}\lim_{n\to\infty} J_{y_n}(h_n) = \prod_{k=0}^{\infty} \frac{\det(Tf^{-1} \mid T_{f^{-k}y_*} f^{-k}\Sigma(y_*))}{\det(Tf^{-1} \mid T_{f^{-k}Hy} f^{-k} D_q)} \tag{B'}$$

when $H = H_{p,q}$.

As in § 3, let $\mu = \sup \|T^{cs}f\|$, $\lambda = \inf m(T^u f)$, and choose $\max(1, \mu) < \boldsymbol{\mu} < \boldsymbol{\lambda} < \lambda$. Then, as in § 3,

$$f^{-k} H_n y \in \mathscr{G}^{n-k}_{f^{-k}y_n}(\varepsilon\boldsymbol{\lambda}^{-k})$$
$$f^{-k} H y \in W^u_{f^{-k}Hy}(\varepsilon\boldsymbol{\lambda}^{-k})$$

for $0 \leqq k \leqq n$ and large n, because $\mathscr{G}^{n-k}$ is nearly tangent to E^u and is thus expanded by $\boldsymbol{\lambda}^k$ under f^k. We also claim that

$$\begin{aligned} d(f^{-k} y_n, f^{-k} y_*) &\leqq \boldsymbol{\lambda}^{-k}\varepsilon \\ d(f^{-k} H_n y, f^{-k} H y) &\leqq \boldsymbol{\lambda}^{-k}\varepsilon \end{aligned} \tag{$*$}$$

for $0 \leqq k \leqq n$ and n large. The proof is by induction on k. Since y_*, H_y, $H_n y$, y_n form a twisted trapezoid of small ($\leqq \varepsilon$) diameter whose nearly

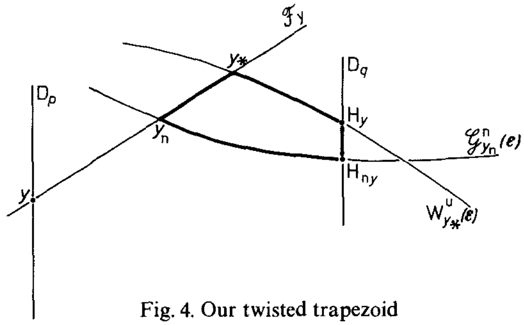

Fig. 4. Our twisted trapezoid

parallel opposite edges in $W^u_{y_*}$, $\mathcal{G}^n_{y_n}$ have length $\leqq \varepsilon$, the other edges – being in $\mathcal{F}$ and D_q must also have length $\leqq \varepsilon$ (see Fig. 4). This proves (*) for $k=0$.

Suppose (*) is valid for $k-1<n$. Let $\gamma = \sup \|T^c f^{-1}\|$. Then

$$d(f^{-k} y_*, f^{-k} y_n) \leqq \gamma\, d(f^{-k+1} y_*, f^{-k+1} y_n) \leqq \gamma\, \varepsilon\, \lambda^{-k+1}$$

by the induction assumption. Thus, $f^{-k} y_*$, $f^{-k} H y$, $f^{-k} H_n y$, $f^{-k} y_n$ forms a twisted trapezoid of small ($\leqq \gamma\varepsilon$) diameter whose nearly parallel opposite edges in $\mathcal{G}^{n-k}_{f^{-k} y_*}$, $\mathcal{G}^{n-k}_{f^{-k} y_n}$ have length $\leqq \varepsilon \lambda^{-k}$. Its other edges, being in $\mathcal{F}$ and $f^{-k} D_q$, must have length $\leqq \varepsilon \lambda^{-k}$; for $\mathcal{F}$, $f^{-k} D_q$ and $\mathcal{G}^{n-k}$ are essentially perpendicular to each other. This proves (*) for k. (See Fig. 5.) Note that we used $k \leqq n$ to assure $\mathcal{G}^{n-k}$ is defined and more or less tangent to E^u.

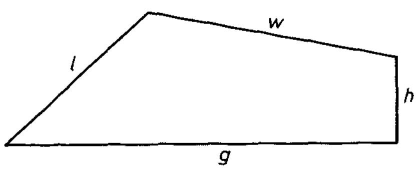

Fig. 5. General twisted trapezoid

$$l \perp g \qquad h \perp \operatorname{span}(l, g) \qquad \measuredangle(g, w) \to 0$$
$$\Rightarrow l/g \quad \text{and} \quad h/g \to 0.$$

Now we shall prove (B′). By the Chain Rule

$$J_{y_n}(h_n) = \frac{\det(T_{f^{-n} y_n} H^0_{f^{-n} y_n, f^{-n} H_n y}) \det(T f^{-n} \mid T_{y_n} \Sigma(y_n))}{\det(T f^{-n} \mid T_{H_n y} D_q)}$$

where $H^0_{f^{-n} y_n, f^{-n} H_n y}: f^{-n} \Sigma(y_n) \to f^{-n} D_q$ is the Poincaré map along the leaves of $\mathcal{H}^0$ through $f^{-n} D_p$. Since $d(f^{-n} y_n, f^{-n} H_n y) \rightrightarrows 0$ and $T\Sigma(y_n) \rightrightarrows T\Sigma(y_*)$, the first term of the numerator tends uniformly to 1.

Thus (B′) is equivalent to

$$\operatorname*{unif\,lim}_{n\to\infty} \frac{\det(Tf^{-n}|T_{y_n}\Sigma(y_n))}{\det(Tf^{-n}|T_{H_n y}D_q)} = \operatorname*{unif\,lim}_{n\to\infty} \frac{\det(Tf^{-n}|T_{y_*}\Sigma(y_*))}{\det(Tf^{-n}|T_{Hy}D_q)}. \qquad (\mathrm{B}'')$$

As in § 3, we can easily demonstrate

$$\prod_{k=0}^{\infty} \frac{\det(T^s_{f^{-k}y_*}f^{-1})}{\det(T^s_{f^{-k}Hy}f^{-1})} \qquad (\mathrm{C})$$

converges uniformly. For $T^s f^{-1}$ is θ-Hölder, $\theta>0$, and

$$d(f^{-k}y_*, f^{-k}Hy) \leqq \lambda^{-k}.$$

From (C), it follows that the right hand side of (B″) exists. E^s is an exponential attractor, under Tf^{-1}, for any plane in TM complementary to E^{cu}. In fact

$$\begin{aligned} &\measuredangle(Tf^{-k}\Sigma(y_*), E^s) \leqq (\mu/\lambda)^k \\ &\measuredangle(Tf^{-k}\Sigma(y_n), E^s) \leqq (\mu/\lambda)^k \\ &\measuredangle(Tf^{-k}D_q, E^s) \leqq (\mu/\lambda)^k \end{aligned} \qquad (**)$$

for $k \leqq n$ and k large, since $T\Sigma(y_n) \rightrightarrows T\Sigma(y_*)$ and $T\Sigma(y_*)$ is complementary to E^{cu}. Since $\det(Tf^{-1}|P)$ is a smooth function of the plane P

$$\begin{aligned} &|\det(Tf^{-1}|T_{f^{-k}y_*}f^{-k}\Sigma(y_*)) - \det(T^s_{f^{-k}y_*}f^{-1})| \leqq C(\mu/\lambda)^k \\ &|\det(Tf^{-1}|T_{f^{-k}Hy}f^{-k}D_q) - \det(T^s_{f^{-k}Hy}f^{-1})| \leqq C(\mu/\lambda)^k \end{aligned} \qquad (***)$$

for some constant C. By the Chain Rule, the r.h.s. of (B″) converges uniformly iff

$$\prod_{k=0}^{\infty} \frac{\det(Tf^{-1}|T_{f^{-k}y_*}f^{-k}\Sigma(y_*))}{\det(Tf^{-1}|T_{f^{-k}Hy}f^{-k}D_q)}$$

does. Convergence of this infinite product follows from comparison with (C) via (***). Similarly, convergence of the l.h.s. of (B″) to the same limit is assured if

$$\begin{aligned} 0 = \operatorname*{unif\,lim}_{n\to\infty} \sum_{k=0}^{n-1} &|\det(Tf^{-1}|T_{f^{-k}y_n}f^{-k}\Sigma(y_n)) \\ &- \det(Tf^{-1}|T_{f^{-k}y_*}f^{-k}\Sigma(y_*))| \end{aligned} \qquad (\mathrm{D}\colon y_n)$$

$$\begin{aligned} 0 = \operatorname*{unif\,lim}_{n\to\infty} \sum_{k=0}^{n-1} &|\det(Tf^{-1}|T_{f^{-k}H_n y}f^{-k}D_q) \\ &- \det(Tf^{-1}|T_{f^{-k}Hy}f^{-k}D_q)|. \end{aligned} \qquad (\mathrm{D}\colon H_n y)$$

Express the k-th term in $(\mathrm{D}: y_n)$ as

$$\begin{aligned}
&|\det(Tf^{-1}|T_{f^{-k}y_n} f^{-k}\Sigma(y_n)) - \det(Tf^{-1}|T_{f^{-k}y_*} f^{-k}\Sigma(y_*))| \\
&\quad \leqq |\det(Tf^{-1}|T_{f^{-k}y_n} f^{-k}\Sigma(y_n)) - \det(T^s_{f^{-k}y_n} f^{-1})| \\
&\qquad + |\det(T^s_{f^{-k}y_n} f^{-1}) - \det(T^s_{f^{-k}y_*} f^{-1})| \\
&\qquad + |\det(T^s_{f^{-k}y_*} f^{-1}) - \det(Tf^{-1}|T_{f^{-k}y_*} f^{-k}\Sigma(y_*))| \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By Hölder continuity of $T^s f^{-1}$,

$$\begin{aligned}
\mathrm{II} &\leqq C' d(f^{-k} y_n, f^{-k} y_*)^\theta = C' d(f^{n-k} f^{-n} y_n, f^{n-k} f^{-n} y_*)^\theta \\
&\leqq C' \boldsymbol{\mu}^{(n-k)\theta} d(f^{-n} y_n, f^{-n} y_*)^\theta \leqq C' [\boldsymbol{\mu}^{n-k} \boldsymbol{\lambda}^{-n} \varepsilon]^\theta
\end{aligned}$$

for some constant C'. Thus, the sum in $(\mathrm{D}: y_n)$, is

$$\begin{aligned}
\sum_{k=0}^{n-1} &\leqq \sum_{k=0}^{K} + \sum_{k=K+1}^{n-1} (\mathrm{I} + \mathrm{II} + \mathrm{III}) \\
&\leqq \sum_{k=0}^{K} + 2C \sum_{k=K+1}^{\infty} (\boldsymbol{\lambda}^{-1} \boldsymbol{\mu})^k + C' \boldsymbol{\lambda}^{-n\theta} \sum_{k=0}^{n-1} \boldsymbol{\mu}^{(n-k)\theta}
\end{aligned}$$

for any K, $0 \leqq K \leqq n-1$. We used (***) to estimate I, III. This gives a bound for the $\limsup_{n\to\infty} \sum_{k=0}^{n-1}$ in $(\mathrm{D}: y_n)$, which can be made arbitrarily small by taking K large, fixing K, and then letting n tend to ∞. Thus $(\mathrm{D}: y_n)$ is proved. The proof of $(\mathrm{D}: H_n y)$ is the same. This completes the proof of (D), (B″), (B′), (B) and hence of (4.2).

5. Ergodicity

We now proceed to prove (1.1)—ergodicity of an Anosov action $A: G \to \mathrm{Diff}^2(M)$ with Anosov element f in the centralizer of the Lie group G.

The foliation $\mathscr{F}$ of M by the components of the A-orbits is C^2. (In fact, we only need $\mathscr{F} \in C^1$; it is f which must be C^2.) We shall adopt the usual, confusing notation that $g \in G$ is also considered as the diffeomorphism $A(g)$. This is all right if A is the only action considered.

Let

$$\gamma = \sup \|T^s f\| \qquad \eta = \inf m(T^c f) \qquad \mu = \sup \|T^c f\| \qquad \lambda = \inf m(T^u f)$$

and choose

$$\gamma < \boldsymbol{\gamma} < \boldsymbol{\eta} < \min(1, \eta) \qquad \max(1, \mu) < \boldsymbol{\mu} < \boldsymbol{\lambda} < \lambda.$$

Since f is normally hyperbolic at $\mathscr{F}$, we get the f-invariant foliations $\mathscr{W}^u$, $\mathscr{W}^s$. They are also G-invariant because of their exponential characterization [5]

$$W_p^u = \{x \in M : d(f^{-n}x, f^{-n}p)\,\lambda^n \to 0 \text{ as } n \to \infty\}$$

$$W_p^s = \{x \in M : d(f^n x, f^n p)\,\gamma^{-n} \to 0 \text{ as } n \to \infty\}.$$

For $g \in G$ commutes with f and so

$$d(f^{-n}g\,x, f^{-n}g\,p)\,\lambda^n = d(g\,f^{-n}x, g\,f^{-n}p)\,\lambda^n \leqq L(g)\,d(f^{-n}x, f^{-n}p)\,\lambda^n \to 0$$

iff $x \in W_p^u$. (As usual, $L(g)$ is the Lipschitz constant of g.) Thus, $g\,W_p^u = W_{gp}^u$. Similarly, $g\,W_p^s = W_{gp}^s$.

Since the f-invariant foliations $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ are defined by

$$W_p^{cu} = \bigcup_{q \in \mathcal{F}_p} W_q^u \qquad W_p^{cs} = \bigcup_{q \in \mathcal{F}_p} W_q^s$$

it is clear that $g\,W_p^{cu} = W_{gp}^{cu}$, $g\,W_p^{cs} = W_{gp}^{cs}$.

By (2.1), (4.2) the foliations $\mathcal{W}^u$, $\mathcal{W}^s$, $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ are absolutely continuous, in fact measurewise C^1. This will let us use the following Fubini-type lemmas.

(5.1) Lemma. *Let $\mathcal{F}$ be an absolutely continuous foliation of M. A set $Z \subset M$ has measure zero iff almost all leaves of $\mathcal{F}$ meet Z inessentially. If the essential maximum of a function Φ: $M \to R$ on almost every $\mathcal{F}$-leaf is $\leqq c$ then the essential maximum of Φ is $\leqq c$.*

(5.2) Lemma. *If $\mathcal{F}^1$, $\mathcal{F}^2$ are absolutely continuous, complementary foliations of M and Φ: $M \to R$ is a function that is essentially constant on almost every leaf of $\mathcal{F}^1$ and $\mathcal{F}^2$ then Φ is essentially constant.*

Remarks. By "almost all $\mathcal{F}$-leaves" we mean all $\mathcal{F}$ leaves not lying in a set composed of whole $\mathcal{F}$-leaves and having measure zero. An intersection is essential if it has positive or infinite measure, inessential if it has zero leaf-measure. The essential maximum of a function Φ: $M \to R$ is $\inf\{\sup \Phi|(M-Z)$: mes $Z=0\}$, and the essential minimum is $\sup\{\inf \Phi|(m-Z)$: mes $Z=0\}$. Since a countable number of zero sets forms a zero set, inf{ } and sup{ } can be replaced by min{ } and max{ }.

Proof of (5.1). For completeness, we reproduce part of [1, pp. 156–157]. It is obviously no loss of generality to restrict our attention to a neighborhood U of $p \in M$, where the components of the leaves of $\mathcal{F}$ are discs, $\mathcal{F}_q^U$, and where there is a smooth foliation $\mathcal{G}$ by discs complementary to $\mathcal{F}$. Thus, there is a local product structure

$$\pi: D^k \times D^{m-k} \to U$$

sending horizontal discs to $\mathcal{F}$-leaves, vertical discs to $\mathcal{G}$-leaves, and being smooth on $D^k \times 0$, $0 \times D^{m-k}$. The measure on the $\mathcal{F}$-leaves and $\mathcal{G}$-leaves is the Riemann measure induced by the Riemann structure on TM. The measures on D^k, D^{m-k} are the pull-backs via

$$D^k \leftrightarrow D^k \times 0 \xrightarrow{\pi} \mathcal{F}_p^U \qquad D^{m-k} \leftrightarrow 0 \times D^{m-k} \xrightarrow{\pi} \mathcal{G}_p$$

and the measure on $D^k \times D^{m-k}$ is the product measure. Thus,

$$\pi^{-1} | \mathscr{F}_p^U : \mathscr{F}_p^U \to D^k$$

is absolutely continuous, in fact measure preserving.

Let $\bar{Z}$ be the set of $\mathscr{F}^U$-leaves intersecting Z essentially. We must show mes $Z = 0$ iff mes $\bar{Z} = 0$.

$$\operatorname{mes} Z = 0$$

$$\Updownarrow \quad \text{[smoothness of } \mathscr{G}\text{]}$$

$$\operatorname{mes}(\mathscr{G}_x \cap Z) = 0 \quad \text{for a.e. } x \in \mathscr{F}_p^U$$

$$\Updownarrow \quad [x \times D^{m-k} \xrightarrow{\pi} \mathscr{G}_x \text{ is absolutely continuous because } \mathscr{F} \text{ is absolutely continuous}]$$

$$\operatorname{mes}(x \times D^{m-k} \cap \pi^{-1} Z) = 0 \quad \text{for a.e. } x \in D^k$$

$$\Updownarrow \quad \text{[Fubini Theorem for a product]}$$

$$\operatorname{mes}(\pi^{-1} Z) = 0$$

$$\Updownarrow \quad \text{[Same]}$$

$$\operatorname{mes}(D^k \times y \cap \pi^{-1} Z) = 0 \quad \text{for a.e. } y \in D^{m-k}$$

$$\Updownarrow \quad [D^k \times y \xrightarrow{\pi} \mathscr{F}_y^U \text{ is absolutely continuous, in fact smooth, because } \mathscr{G} \text{ is smooth}]$$

$$\operatorname{mes}(\mathscr{F}_y^U \cap Z) = 0 \quad \text{for a.e. } y \in \mathscr{G}_p$$

$$\Updownarrow \quad \text{[absolute continuity of } \mathscr{F}\text{]}$$

$$\operatorname{mes}(\mathscr{F}_y^U \cap Z) = 0 \quad \text{for a.e. } y \in \mathscr{G}_x \quad (\forall\, x \in \mathscr{F}_p^U)$$

$$\Updownarrow \quad [\operatorname{mes}(\mathscr{F}_y^U \cap Z) = 0 \Leftrightarrow \operatorname{mes}(\mathscr{F}_y^U \cap \bar{Z}) = 0]$$

$$\operatorname{mes}(\mathscr{F}_y \cap \bar{Z}) = 0 \quad \text{for a.e. } y \in \mathscr{G}_x \quad (\forall\, x \in \mathscr{F}_p^U)$$

$$\Updownarrow \quad \text{[obvious]}$$

$$\operatorname{mes}(\bar{Z} \cap \mathscr{G}_x) = 0 \quad \text{for all } x \in \mathscr{F}_p^U$$

$$\Downarrow \quad [\mathscr{G} \text{ is smooth}] \qquad \nwarrow \quad [\bar{Z} \text{ is composed of whole } \mathscr{F}^U\text{-leaves}]$$

$$\operatorname{mes}(\bar{Z}) = 0 \Longrightarrow \operatorname{mes}(\bar{Z} \cap \mathscr{G}_x) = 0 \quad \text{for a.e. } x \in \mathscr{F}_p^U$$

$$[\mathscr{G} \text{ is smooth}]$$

Thus, mes $Z = 0$ iff mes $\bar{Z} = 0$, proving the first half of (5.1).

Now suppose $\Phi: M \to R$ has essential maximum $\leqq c$ on almost all $\mathscr{F}$-leaves—that is, for each $\mathscr{F}$-leaf $\mathscr{F}_p$, there is a set $Z_p \subset \mathscr{F}_p$ such that $\sup \Phi | (\mathscr{F}_p - Z_p) \leqq c$, and for all $\mathscr{F}_p$ not lying in a zero set of $\mathscr{F}$-leaves, $\mathscr{Z}$, mes $Z_p = 0$. Then $Z = \mathscr{Z} \cup \bigcup_p Z_p$ is a zero set by the first half of (5.1), and $\sup \Phi | (M - Z) \leqq c$, completing the proof of (5.1).

Proof of (5.2). For any $c \in R$, let $M^c = \Phi^{-1}((-\infty, c])$ and let $\overline{M}^c$ be the set of $\mathscr{F}^1$-leaves essentially contained in M^c. Then $Z = M^c \Delta \overline{M}^c = (M^c - \overline{M}^c) \cup (\overline{M}^c - M^c)$ has measure zero. Almost every $\mathscr{F}^2$ leaf meets Z inessentially by (5.1). Therefore, almost every $\mathscr{F}^2$-leaf meets M^c essentially iff it meets $\overline{M}^c$ essentially.

Let $\varepsilon > 0$ be small enough so that 2ε-local product structure for $\mathscr{F}^1$, $\mathscr{F}^2$ holds for all $p \in M$:

$$x_1 \in \mathscr{F}_p^1(\varepsilon) \quad x_2 \in \mathscr{F}_p^2(\varepsilon) \Rightarrow \mathscr{F}_{x_2}^1(2\varepsilon) \cap \mathscr{F}_{x_1}^2(2\varepsilon) \quad \text{is a unique point.}$$

Let $M_p(\varepsilon)$ be this product neighborhood of p in M. For small $\varepsilon > 0$, we also have

$$\mathscr{F}_{x_2}^1(\varepsilon/2), \quad \mathscr{F}_{x_1}^2(\varepsilon/2) \subset M_p(\varepsilon)$$

for all $x_1 \in \mathscr{F}_p^1(\varepsilon)$, $x_2 \in \mathscr{F}_p^2(\varepsilon)$ (see Fig. 6).

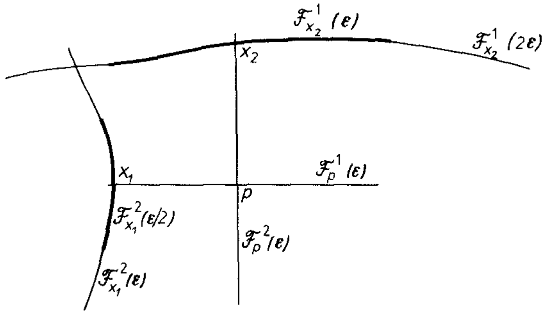

Fig. 6. Local product structure

Let p be a point of M and suppose $\mathscr{F}_p^2(\varepsilon)$ meets $\overline{M}^c$ essentially. By absolute continuity of $\mathscr{F}^1$, every other $\mathscr{F}_p^2(2\varepsilon)$ meets $\overline{M}^c$ essentially for $q \in \mathscr{F}_p^1(\varepsilon)$. Thus, most $\mathscr{F}_p^2$ meet M^c essentially for $q \in \mathscr{F}_p^1(\varepsilon)$, and on most $\mathscr{F}_p^2$, Φ is essentially constant. Therefore, the essential maximum of Φ on most $\mathscr{F}_p^2$, $q \in \mathscr{F}_p^1(\varepsilon)$, is $\leqq c$. By (5.1), the essential maximum of Φ on $M_p(\varepsilon)$ is also $\leqq c$.

On the other hand, suppose $\mathscr{F}_p^2(\varepsilon)$ meets $\overline{M}^c$ inessentially. By the absolute continuity of $\mathscr{F}^1$, every other $\mathscr{F}_q^2(\varepsilon/2)$ meets $\overline{M}^c$ inessentially for $q \in \mathscr{F}_p^1(\varepsilon)$. Thus, most $\mathscr{F}_q^2(\varepsilon/2)$ meet M^c inessentially for $q \in \mathscr{F}_p^1(\varepsilon)$ and on most $\mathscr{F}_q^2$, Φ is essentially constant. Therefore, the essential minimum of Φ on most $\mathscr{F}_q^2$, $q \in \mathscr{F}_p^1(\varepsilon)$ is $>c$. By (5.1), the essential minimum of Φ on $M_p(\varepsilon)$ is also $>c$.

Consequently, for every $c \in R$,

$$\text{ess}\max(\Phi|M_p(\varepsilon)) \leqq c \quad \text{or} \quad \text{ess}\min(\Phi|M_p(\varepsilon)) > c.$$

Hence Φ is essentially constant on $M_p(\varepsilon)$, and so is essentially constant on each component of M.

Suppose Φ: $M \to R$ is an A-invariant integrable function. Ergodicity of A means Φ must be constant almost everywhere. Let $\text{Inv}(g)$ = all integrable g-invariant functions $M \to R$ for $g \in G$. We are trying to show $\bigcap_{g \in G} \text{Inv}(g)$ is the set of constant functions.

According to [2, p. 144], we may define a projection I_g: $L^1(M) \to \text{Inv}(g)$ by

$$I_g \varphi(x) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \varphi(g^k x) \qquad g \in G.$$

That is, the limit exists almost everywhere, is integrable, and $\varphi \mapsto I_g \varphi$ is a continuous linear map onto the fixed points of I_g, $\text{Inv}(g)$. Moreover, the limits

$$I_g^{\pm} \varphi(x) = \lim_{n \to \pm\infty} \frac{1}{|n|+1} \sum_{k=0}^{n} \varphi(g^k x) \qquad g \in G$$

exist almost everywhere and $I_g^{\pm} \varphi(x) = I_g \varphi(x)$ for almost all x. That is, $I_g^+ = I_g^- = I_g$ as maps $L^1(M) \to \text{Inv}(g)$.

Since the continuous functions are dense in $L^1(M)$, their I_g-images are dense in $\text{Inv}(g)$. Therefore, it is useful to prove

If φ is continuous then $I_f \varphi$ is essentially constant along $\mathscr{W}^u$ and $\mathscr{W}^s$. (*)

For any $x, y \in W_p^u$ and any continuous φ: $M \to R$ it is clear that either both $I_f^- \varphi(x)$, $I_f^- \varphi(y)$ are defined, or neither, and if defined they are equal. Since $I_f^- \varphi$ is defined almost everywhere $I_f^- \varphi$ is defined and constant on almost all $\mathscr{W}^u$-leaves. Since $\mathscr{W}^u$ is absolutely continuous and $I_f^- \varphi = I_f \varphi$ almost everywhere, $I_f \varphi$ is essentially constant on almost every $\mathscr{W}^u$ leaf by (5.1). Similarly for $\mathscr{W}^s$, proving (*).

By density Φ is the limit, almost everywhere of $I_f \varphi$ with φ continuous. Therefore, on almost every $\mathscr{W}^u$ leaf and $\mathscr{W}^s$ leaf, Φ is the pointwise limit, almost everywhere on the leaf, of essentially constant functions. Hence Φ is essentially constant along $\mathscr{W}^u$, $\mathscr{W}^s$ and $\mathscr{F}$: say Φ is essentially constant on all $\mathscr{W}^u$ leaves, $\mathscr{W}^s$ leaves, and $\mathscr{F}$-leaves, not essentially intersecting Z, $\text{mes}\, Z = 0$.

The foliations $\mathscr{F}|W_p^{cu}$, $\mathscr{W}^u|W_p^{cu}$ are both (!) smooth. $\mathscr{F}$ is smooth on M so it is certainly smooth on W_p^{cu}; $\mathscr{W}^u|W_p^{cu}$ is smooth because W_p^u

is smooth and all the other W_q^u, $q \in \mathscr{F}_p$, are gotten from W_p^u as $g\,W_p^u = W_q^u$ for g in the identity component of G.

By absolute continuity of $\mathscr{W}^{cu}$ and (5.1), almost every W_p^{cu} meets Z inessentially; by (5.1) on such a W_p^{cu}, almost every $\mathscr{F}_y$, W_q^u in W_p^{cu} meet $Z \cap W_p^{cu}$ inessentially. Therefore, by (5.2) on W_p^{cu}, Φ is essentially constant on W_p^{cu}. Thus Φ is essentially constant along $\mathscr{W}^{cu}$.

By (5.2) on M and the absolute continuity of $\mathscr{W}^{cu}$, $\mathscr{W}^s$, Φ is essentially constant on M.

6. A Pathological Foliation

Here we give an example to show that there are foliations by smooth discs which are not measurable in the sense of Sinai [8]. It seems to us that verification of a foliation's measurability is generally no easier than verification of its measurewise smoothness. A conversation with N. Kopell was helpful in cooking up our example.

Let $I=[0,1]$ and $h: I \times I \to I$ be continuous with

(i) $h_t = h(t,\cdot): I \to I$ is a homeomorphism, $0 \leqq t \leqq 1$.

(ii) $h_t =$ identity for $t \leqq \frac{1}{3}$, $h_t = h_1$ for $t \geqq \frac{2}{3}$.

(iii) h_1 is not absolutely continuous.

(iv) $h_t | U$ is a C^∞ embedding for some open dense $U \subset I$, $0 \leqq t \leqq 1$.

(v) dh_t/dt is continuous.

It is easy to construct such an h – we do it at the end of this section.

Consider the foliation $\mathscr{F}$ of $I \times I$ whose leaves are the graphs

$$\beta(y) = \{t, h_t y): t \in I\} \qquad y \in I.$$

By (v), the foliation has a continuous tangent bundle. Since dh_t/dt is smooth on the dense strips $\{(t, h_t y): t \in I, y \in U\}$ there is no curve everywhere tangent to leaves but not contained in a leaf. Thus, we have a foliation in the sense of Anosov [1, p. 18].

Let μ be the usual measure on R^2. Let $d\sigma_\beta$ be the smooth induced Riemann measure on the leaf β. Let $d\mu_\beta$ be the quotient measure on the space of leaves, $\mathscr{B}$. If B is a collection of (whole) leaves, then $\mu_{\mathscr{B}}(B) = \mu(\bigcup_{\beta \subset B} \beta)$. Suppose that $\mathscr{F}$ were measurable in the sense of Sinai. Then there would be a measurable function $K: I \times I \to R$ such that

(1) K is positive almost everywhere on $I \times I$.

(2) K is integrable on every leaf β not belonging to a set $\mathscr{Z}$ of leaves having $\mu(\mathscr{Z})=0$ and, for $\beta \not\subset \mathscr{Z}$, $\int_\beta K\, d\sigma_\beta = 1$.

(3) $\mu(A, \beta) \overset{\text{def}}{=} \int_{A \cap \beta} K\, d\sigma_\beta$ is an integrable function of $\beta \in \mathscr{B}$ if $\beta \not\subset \mathscr{Z}$ and if A is measurable in $I \times I$.

(4) $\mu(A \cap B) = \int_B \mu(A, \beta)\, d\mu_{\mathcal{B}}$ for any measurable set $A \subset I \times I$ and any measurable $B \subset \mathcal{B}$.

Let N be the set where K is not defined or is not positive, $\mu(N) = 0$. By (4) with $B = I \times I$

$$\mu(N) = \int_B \mu(N, \beta)\, d\mu_{\mathcal{B}}$$

and so, for a set of leaves $\mathcal{Z}_1$ such that $\mu(\mathcal{Z}_1) = 0$,

$$\beta \not\subset \mathcal{Z}_1 \Rightarrow \mu(N, \beta) = 0.$$

Let Z be a zero set of I such that $h_1 Z$ has positive linear measure and let $B_Z = \bigcup_{y \in Z} \beta(y)$. Then $\mu(B_Z) > 0$ because $[\frac{2}{3}, 1] \times h_1(Z) \subset B_Z$. Also $\mu(B_{Z'}) > 0$ for

$$Z' = \{y \in Z \colon \beta(y) \not\subset \mathcal{Z} \cup \mathcal{Z}_1\}$$

$$B_{Z'} = \bigcup_{y \in Z'} \beta(y) = B_Z - (\mathcal{Z} \cup \mathcal{Z}_1).$$

Now let $A = [0, \frac{1}{3}] \times I$, $B = B_{Z'}$. Then $A \cap B = [0, \frac{1}{3}] \times Z'$ so $\mu(A \cap B) = 0$. Since each $\beta \subset B_{Z'}$ lies outside $\mathcal{Z}_1$, $K|\beta$ is almost everywhere positive on β. In particular, $K|A \cap \beta$ is almost everywhere positive, $\beta \subset B_{Z'}$. That is

$$\mu(A, \beta) > 0 \qquad \text{for all } \beta \subset B_{Z'}.$$

Since $\mu_{\mathcal{B}}(B_{Z'}) > 0$, this proves that

$$\int_{B_{Z'}} \mu(A, \beta)\, d\mu_{\mathcal{B}} > 0$$

contradicting (4) for this A and B.

Now we construct the homotopy h used to find the foliation. Let U be an open dense subset of I with measure $\frac{1}{2}$ and let

$$u(x) = \int_0^x [1 - \chi_U(s)]\, ds$$

where χ_U is the characteristic function of U. This map $u \colon I \to [0, \frac{1}{2}]$ collapses U onto a countable set $C \subset [0, \frac{1}{2}]$, $u|(I - U)$ preserves measure, and $u^{-1}(u U) = U$. Let $g \colon [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ be a homeomorphism with $g(0) = 0$, $g(\frac{1}{2}) = \frac{1}{2}$, that is not absolutely continuous. Find an open set $V \subset I$, $\mu V = \frac{1}{2}$, and a collapsing map $v \colon I \to [0, \frac{1}{2}]$ with $v V = g C$, $v|(I - V)$ measure preserving, and $v^{-1}(v V) = V$. Then define $h_1 \colon I \to I$ so that

$$\begin{array}{ccc} I & \xrightarrow{h_1} & I \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ [0, \frac{1}{2}] & \xrightarrow{g} & [0, \frac{1}{2}] \end{array}$$

commutes and h_1 carries $u^{-1}(c)$ onto $v^{-1}(g\,c)$ diffeomorphically for all $c \in C$. Finally, put

$$h_t(y) = [1 - \varphi(t)]\, y + \varphi(t)\, h_1(y)$$

for $t, y \in I$ and φ a C^∞ function $R \to [0, 1]$ with $\varphi = 0$ for $t \leqq \frac{1}{3}$, $\varphi = 1$ for $t \geqq \frac{2}{3}$. Clearly h_t is a homeomorphism for all $t \in I$ and h_t is smooth in t. That is, (i)–(v) are verified.

Post Script. It seems likely that these method apply to metric transitivity questions for Anosov Actions if such questions make any sense.

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