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Approximation Power of Smooth Bivariate PP Functions

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1. Introduction

Consider the space

$$S := \pi_{k,\Delta}^{\rho} := \{f \in C^{\rho} : \forall(\delta \in \Delta)\ f_{|\delta} \in (\pi_k)_{|\delta}\} \tag{1.1}$$

of piecewise polynomial (=:pp) functions of degree $\leqq k$ on the triangulation Δ and in C^{ρ}. We are interested in estimates of the standard form

$$\operatorname{dist}(f, S) \leqq \operatorname{const}_f |\Delta|^r$$

with

$$|\Delta| := \sup_{\delta \in \Delta} \operatorname{diam} \delta$$

the *meshsize*, and with the distance between functions measured in the max-norm on some domain G contained in $\bigcup_{\delta \in \Delta} \delta$. The exponent r depends on the smoothness of f, in general. We are interested in determining its largest possible value under the assumption that f is sufficiently smooth. We call this number the *approximation order* of S and denote it (again) by r.

It is well known that $r = k+1$ for $\rho \leqq 0$, but it is at present not clear what happens when $\rho \geqq 1$. Because of results of Ženíček [Z70], [Z73], it is believed that the polynomial degree k must be at least $4\rho + 1$ to obtain the *full* approximation order $r = k+1$. It is the purpose of this paper to show that actually a degree of $3\rho + 2$ suffices. Precisely, we prove the following.

Theorem. *If*

$$k > 3\rho + 1,$$

then there exists a constant const *which depends only on*

$$a := \textit{smallest angle in } \Delta$$

and k so that

$$\operatorname{dist}(f, S) \leqq \operatorname{const} \|D^{k+1} f\|\, |\Delta|^{k+1}$$

for all smooth f.

In addition, we show for small ρ that this result is sharp; i.e., we show that, on a very simple partition (the three-direction mesh), the approximation order is no better than k when $k=3\rho+1$ (and $\rho=1,2,3$).

Our argument suggests that, for pp functions of d variables, full approximation order is obtained as soon as $k \geqq (d+1)\rho+d$. This is to be compared with Le Méhauté's result [M83] who extends Ženíček's argument to the d-variate context and proves full approximation order only in case $k \geqq 2^d \rho+1$.

Our argument is unusual in that it uses duality to determine the exact approximation order (rather than just an upper bound for it). In consequence, we obtain the approximation order without exhibiting an approximation scheme that attains this order.

The duality argument requires a description of $S^{\perp}$, the collection of all linear functionals which vanish on the approximating space S. This is, in general, a hopeless task. In our case, though, it is sufficient to consider S as a linear subspace of

$$S_0 := \pi^0_{k,\Delta}. \tag{1.2}$$

This means that $S^{\perp}$ consists of the *smoothness conditions* which characterize S as $S_0 \cap C^{\rho}$. We are able to obtain a tractable description of these smoothness conditions because we employ the *B-net* representation familiar from Computer-Aided Geometric Design for the continuous *pp* functions, i.e., the elements of S_0. The corresponding description of the smoothness conditions reflects as much as possible and as cleanly as possible the geometry of the partition Δ.

2. Reduction to S_0

Denote by V_δ the vertex set of the triangle $\delta \in \Delta$ and by

$$V := \bigcup_{\delta\in\Delta} V_\delta \tag{2.1}$$

the collection of all meshpoints in the triangulation Δ. We define the *k-refinement*

$$V_k := \{v_\alpha : \alpha \in A\} \tag{2.2a}$$

of V as the collection of all points

$$v_\alpha := \sum_{v\in V} v\,\alpha(v)/|\alpha|, \qquad \alpha\in A, \tag{2.2b}$$

with

$$A := A_{k,\Delta} := \{\alpha \in \mathbb{Z}^V_+ : |\alpha| = k, \quad \operatorname{supp}\alpha \subset V_\delta \text{ for some } \delta\in\Delta\} \tag{2.2c}$$

and

$$|\alpha| := \sum_{v\in V} \alpha(v).$$

Then each $\delta\in\Delta$ contains exactly $\binom{k+2}{2} = \dim \pi_k$ points from V_k and, for given $f \in C$, there is exactly one $p_\delta \in \pi_k$ which agrees with f at these points. Further, if δ shares an edge with some $\delta' \in \Delta$, then, since $k+1$ points of V_k lie on this edge, p_δ and $p_{\delta'}$ agree on this edge. This means that the function Pf defined by

$$Pf := p_\delta \quad \text{on} \quad \delta, \qquad \text{for } \delta \in \Delta, \tag{2.3}$$

lies in S_0 and is the unique element of S_0 which agrees with f on V_k. Note that $\|P\| = \|P_\delta\|$ for any $\delta \in \Delta$, with the norm of the map $P_\delta\colon C(\delta) \to C(\delta): f_{|\delta} \mapsto p_\delta$ depending only on k.

Since, for smooth f,

$$\|f - Pf\|_\infty \leqq (1 + \|P\|) \operatorname{const}_k \|D^{k+1} f\| \, |\Delta|^{k+1}, \tag{2.4}$$

it follows that

$$\operatorname{dist}(f, S) = O(\|D^{k+1} f\| \, |\Delta|^{k+1}) + \operatorname{dist}(Pf, S). \tag{2.5}$$

Since the approximation order of S cannot exceed $k+1$, this shows that the approximation order is determined by how well one can approximate Pf from S, for smooth f.

Since $Pf \in S_0$, the Hahn-Banach Theorem provides the formula

$$\operatorname{dist}(Pf, S) = \max_{\lambda \in S^\perp} \frac{|\lambda Pf|}{\|\lambda\|} \tag{2.6}$$

with

$$S^\perp := \{\lambda \in S_0^* : \lambda(S) = 0\} \tag{2.7}$$

the smoothness conditions which single out the elements of S from those of S_0.

3. A "Good" Basis for $S^\perp$ Ensures Full Approximation Order

The collection $S^\perp$ of smoothness conditions satisfied by S (as a subspace of S_0) is spanned by the conditions which enforce some part of that smoothness across an edge common to two triangles in Δ. Thus, $S^\perp$ consists of linear functionals of the form

$$\lambda = \sum_{\tau \in T} c(\tau)\, \tau, \tag{3.1}$$

with each $\tau \in T$ having support in the union $\delta \cup \delta'$ of a pair of neighboring triangles in Δ. This implies that the sum makes sense even when T is infinite. Without loss, we assume that each τ has been normalized,

$$\|\tau\| = 1, \qquad \text{all } \tau \in T. \tag{3.2}$$

Lemma 3. *For smooth f,*

$$\frac{|\lambda Pf|}{\|\lambda\|} \leqq \frac{\|c\|_1}{\|\sum c(\tau)\,\tau\|} \operatorname{const} \|D^{k+1} f\| \, |\Delta|^{k+1}.$$

Proof. If $p \in \pi_k$, then $Pp = p \in S$, hence

$$\tau Pf = \tau P(f - p) \leqq \|\tau P\| \, \|(f-p)_{|\operatorname{supp} \tau P}\|_\infty.$$

Since $(Pg)_{|\delta}$ depends only on $g_{|\delta}$, the support of τP consists of the union of two neighboring triangles in Δ, hence

$$\tau Pf \leqq \|P\| \sup_{\delta,\delta' \in \Delta} \operatorname{dist}_{\delta \cup \delta'}(f, \pi_k) \leqq \|P\| \operatorname{const} \|D^{k+1} f\| \, |\Delta|^{k+1}.$$

Since λ is of the form (3.1), this finishes the proof. □

Corollary. *If*

$$\kappa_T := \sup_c \frac{\|c\|_1}{\|\sum_T c(\tau)\,\tau\|} \tag{3.3}$$

is finite, then S has full approximation order.

Note that the finiteness of κ_T not only requires that T be linearly independent, hence a basis for $S^\perp$, but that it be a "good" basis in the sense that the coordinate map $c \mapsto \sum c(\tau)\,\tau$ is bounded below (a nontrivial requirement when T is infinite). Further, we hope to bound κ_T just in terms of the smallest angle in any $\delta \in \Delta$, independently of any other details of Δ. This requires a careful study of $S^\perp$.

4. The B-net Representation for S_0 and $S^\perp$

The *B-form* for $p \in \pi_k$ with respect to some simplex readily provides information about the behavior of p near the boundary of that simplex. This is due to the fact that, in this form, the polynomial is described as a linear combination of all possible products of k linear polynomials, each of which vanishes on some facet of that simplex. This makes it easy to express the smoothness conditions across the boundary when putting together smooth pp functions on some triangulation. For this reason, this form is widely used in Computer-aided Geometric Design, where it carries the more detailed name of *Bernstein-Bézier-*, or *barycentric* form. For full supporting details of what is to follow, consult [F 79] or [B 87].

Explicitly, the B-form for $p \in \pi_k$ with respect to the simplex δ is

$$p = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \xi^\alpha c(\alpha) \tag{4.1 a}$$

with

$$\xi^\alpha := \prod_{v \in V_\delta} \xi_v^{\alpha(v)} \tag{4.1 b}$$

and ξ_v the linear polynomial satisfying

$$\xi_v(w) = \delta_{vw}, \qquad \text{all } w \in V_\delta. \tag{4.1 c}$$

Further, α is meant to be any vector indexed by V_δ with nonnegative integer entries, i.e., a multi-index, for short. The normalizing multinomial coefficients serve to have the sum add to 1 in case all the $c(\alpha)$ are 1.

The normalizing factors also make it possible to write the B-form more suggestively as

$$p = (\xi E)^k c(0), \tag{4.2 a}$$

with E the *shift* operator, i.e.,

$$E^{\beta} c(\alpha) = c(\alpha+\beta), \tag{4.2b}$$

hence

$$(\xi E)\, c(\alpha) = \sum_{v \in V_\delta} \xi_v c(\alpha + e_v). \tag{4.2c}$$

Here, e_v is a unit multi-index; specifically, $e_v(w) = \delta_{vw}$.

The formula (4.2a) is easily differentiated: For the directional derivative $D_y := \sum y(j)\, D_j$, one gets

$$D_y\, p = (\xi E)^{k-1} k(\mathrm{D}_y\, \xi E)\, c(0). \tag{4.3}$$

This is explicitly the B-form (wrto δ) of the polynomial $D_y p$. It shows that the B-form coefficients for $D_y p$ are obtained from those for p by a simple differencing, with the weight vector $\eta := \eta(y) := D_y \xi$ the unique solution to the linear system

$$y = \sum_v v \eta_v, \qquad 0 = \sum_v \eta_v.$$

Let F_W denote the face of δ spanned by the vertices in $W \subset V_\delta$. Then ξ_v vanishes on F_W iff $v \notin W$. Hence the behavior of p on F_W is entirely determined by the coefficients $c(\alpha)$ with $\operatorname{supp} \alpha \subset W$. In particular,

$$p(v) = c(k e_v). \tag{4.4}$$

This suggests the association of the coefficient $c(\alpha)$ with the refined meshpoint $v_\alpha = \sum_{v \in V_\delta} v\alpha(v)/|\alpha|$ introduced in (2.2), i.e., the introduction of the *mesh function*

$$b_p\colon\ v_\alpha \mapsto c(\alpha).$$

For, if we associate in the same way the coefficient $c'(\alpha)$ in the B-form for some $q \in \pi_k$ wrto some neighboring δ' with the refined meshpoint v_α, – keeping in mind that now $\operatorname{supp} \alpha \subset V_{\delta'}$, – then a continuous joining of the two polynomials at the interface $\delta \cap \delta'$ is equivalent to having b_p and b_q agree on the points common to their domains. In this way we arrive at the *B-net* for $f \in S_0$, viz. the mesh function b_f given by the rule

$$b_f = b_{f|\delta} \quad \text{on} \quad V_k \cap \delta. \tag{4.5}$$

The max-norm of $f \in S_0$ and the max-norm of its B-net b_f are equivalent, hence the map $f \mapsto b_f$ provides a linear homeomorphism between S_0 and $l_\infty(V_k)$. Consequently, we can think of $l_1(V_k)$ as a subspace of the continuous dual S_0^* of S_0.

In particular, any linear functional on S_0 with support in finitely many $\delta \leftarrow \Delta$ is (representable as) a finitely supported element of $l_1(V_k)$, hence we may think in this way of $S^\perp$ as a linear subspace of $l_1(V_k)$. For, a spanning set for $S^\perp$ is provided by the union over all pairs δ, δ' of neighboring triangles in Δ of the conditions which enforce C^ρ-continuity across the common edge $\delta \cap \delta'$. These conditions take (up to a normalizing factor) the convenient form:

$$b_f(v_{\beta + r e_{u'}}) = p_\beta(u'), \tag{4.6}$$

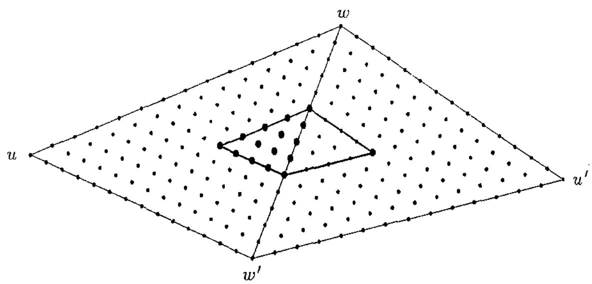

Fig. 4.1. Support diamond and actual support (heavy dots) of a smoothness condition of order 4 (for $k=14$)

for $\operatorname{supp}\beta\subset\delta\cap\delta'$, $r:=k-|\beta|\leqq\rho$. Here, u' is the vertex of δ' not in δ, and

$$p_\beta:=(\xi E)^{k-|\beta|}c(\beta) \tag{4.7}$$

is the polynomial whose B-form coefficients form the subtriangle

$$c(\beta+\gamma), \qquad \operatorname{supp}\gamma\subset\delta, |\gamma|=k-|\beta|,$$

of the B-form coefficients

$$c(\alpha), \quad |\alpha|=k,$$

for $f_{|\delta}(x)$. On comparing (4.7) with (4.2), one sees that $p_\beta(x)$ is generated at the $(k-|\beta|)$-th step of the evaluation of $f_{|\delta}(x)$.

We obtain one such condition for each β with $\operatorname{supp}\beta\subset\delta\cap\delta'$ and $|\beta|<k$, hence will denote it by

$$\tau_\beta, \quad \text{or more explicitly by } \tau_{\beta,\delta\cap\delta'},$$

when we think of it as an element of $l_1(V_k)$. Note that the support of τ_β is in a *diamond*,

$$\operatorname{supp}\tau_{\beta,\delta\cap\delta'}\subset\{v_{\beta+\gamma}: \operatorname{supp}\gamma\subset\delta \text{ or } \delta', |\beta+\gamma|=k\} \tag{4.8}$$

and that τ_β does not vanish at the two tips of this diamond, i.e.,

$$\tau_\beta(v_{\beta+re_v})\neq 0 \qquad \text{for } v\in\delta\setminus\delta' \text{ or } \delta'\setminus\delta, \; r:=k-|\beta|. \tag{4.9}$$

Further, all τ_β with β of the same length are just shifts of one another. Precisely,

$$\tau_\beta(v_\alpha)=\tau_{|\beta|\,e_w}(v_{\alpha+|\beta|\,e_w-\beta}) \qquad \text{for any } w\in V_\delta\cap V_{\delta'}. \tag{4.10}$$

To be sure, $\tau_\beta=0$ for $|\beta|=k$, since in this case (4.6) merely restates that the B-form coefficients of $f_{|\delta}$ and $f_{|\delta'}$ agree on $v_\alpha\in\delta\cap\delta'$. Hence we will not refer to it again. For $r=1$, (4.6) provides the geometrically quite striking condition that, for any β with $\operatorname{supp}\beta\subset\delta\cap\delta'$ and $|\beta|=k-1$, the four B-net points

$$(v_\gamma, b_f(v_\gamma)), \qquad \gamma=\beta+e_w, \; w\in\delta\cup\delta'$$

must be coplanar.

The whole collection $\{\tau_\beta : \operatorname{supp}\beta \subset \delta \cap \delta' \text{ with } |\beta| < k\}$ is linearly independent since, in any total ordering $\ll$ of this set in which

$$|\beta| > |\gamma| \Rightarrow \tau_\beta \ll \tau_\gamma,$$

τ_β is the first one to have $v_{\beta + re_u}$ in its support. (Here, once again, $r := k - |\beta|$ and $u \in \delta \setminus \delta'$.) Our difficulty in making up a basis for $S^\perp$ will come from the fact that τ_β associated with different edges will interfere, leading to linear dependence in quite complicated ways. Fortunately, we will not need to untangle these dependencies in full detail. But we will need a result concerning the local linear independence of certain τ_β which we now derive.

Lemma 4.1. *Let w, w' be the vertices common to δ and δ', and assume that $w' \notin [u, u']$. Then, for any s, the conditions*

$$\tau_i = \tau_{(k-i)e_w}, \qquad i = k, \ldots, k-2s+1, \tag{4.11}$$

are linearly independent over the point set

$$X := \{u_{k-s}, \ldots, u_{k-2s+1}, u'_{k-s}, \ldots, u'_{k-2s+1}\},$$

with

$$u_i := v_{(k-i)e_w + ie_u}, \qquad u'_i := v_{(k-i)e_w + ie_{u'}}.$$

Proof. For the proof, a look at Figure 4.2 might be helpful. The smoothness condition τ_i we consider has u_i and u'_i as the tips of its support diamond, and one of these tips is on the edge $[u, w]$.

For the proof, assume to the contrary that the sequence (4.11) is linearly dependent over X. Then, since there are as many points in X as there are conditions in the sequence, there exists $c: X \to \mathbb{R} \setminus 0$ so that

$$\sum_{x \in X} \tau_j(x)\, c(x) = 0, \qquad j = k, \ldots, k-2s+1.$$

Of all the conditions across the edge $\delta \cap \delta'$, those in the sequence (4.11) are the only ones with some support in X. Hence, on extending c to all of $V_k \cap (\delta \cup \delta')$ by setting it to zero off X, we obtain the B-net of some $g \in \pi^0_{k,\{\delta,\delta'\}}$ which satisfies all smoothness conditions across the edge $\delta \cap \delta'$, hence must be a polynomial

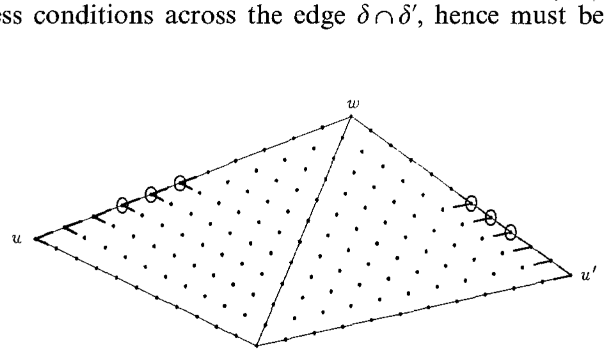

Fig. 4.2. The tips of the support diamonds of τ_i, $i = k, \ldots, k-2s+1$ and the points u_i, u'_i, $i = k-s, \ldots, k-2s+1$ over which they are linearly independent, for the case $s = 3$, $k = 11$

of degree $\leq k$, and must be nontrivial since $c \neq 0$. But, since most of its B-net values are zero, g must vanish to some order on $\delta \cap \delta'$ as well as on the edges $[u, w']$ and $[u', w']$, with w' the other vertex in $\delta \cap \delta'$. Precisely, g must be divisible by ξ_u^{k-2s+1} as well as by ξ_w^s and $(\xi'_w)^s$ (with ξ' the barycentric coordinates with respect to δ'). Hence, if ξ_w and ξ'_w are linearly independent, then g must be divisible by $\xi_u^{k-2s+1} \xi_w^s (\xi'_w)^s$ which is a polynomial of degree $> k$, an impossibility. □

For the exceptional case, i.e., when $w' \in [u, u']$, we need later the following observation.

Lemma 4.2. *If $w' \in [u, u']$, then $\tau_{\beta, \delta \cap \delta'}$ has support only on the edge of its support diamond parallel to $[u, u']$. Explicitly,*

$$\operatorname{supp} \tau_\beta = \{v_{\beta + re_u - j(e_u - e_{w'})} : j = 0, \ldots, r\} \cup \{v_{\beta + re_{u'}}\}. \tag{4.12}$$

5. A Bound for κ_T

We now consider the problem of choosing an appropriate basis T for $S^\perp$ as a subset of the smoothness conditions τ_β, $k - |\beta| \leq \rho$, across all the edges of the partition Δ in such a way that

$$\kappa_T := \sup_c \frac{\|c\|_1}{\|\sum_T c(\tau)\,\tau\|} \tag{3.3}$$

is bounded independently of Δ. Having identified S_0 with $l_1(V_k)$, the norm $\|\sum_T c(\tau)\,\tau\|$ appearing in the definition can be taken to be the l_1-norm. Hence

$$\kappa_T = \inf_U \kappa(T, U), \tag{5.1}$$

with

$$\kappa(R, U) := \sup_c \frac{\|c_{|R}\|_1}{\|\sum_{\tau \in R} c(\tau)\,\tau_{|U}\|_1} \tag{5.2}$$

and U any subset of V_k. Note that $\kappa(R, U) = \infty$ in case R fails to be linearly independent over U. On the other hand, if R is maximally linearly independent over U, i.e., if the matrix $A := (\tau(u))_{u \in U, \tau \in R}$ is invertible, then

$$\kappa(R, U) = \|A^{-1}\|_1.$$

Our results rely on the following observation, which allows us to use the support structure of the τ_β to subdivide the task of bounding $\kappa(T, U)$. For it, we use the abbreviation

$$\mu(R, U) := \sup_c \frac{\|\sum_R c(\tau)\,\tau_{|U}\|_1}{\|c\|_1} \tag{5.3}$$

Lemma 5. *If* $U = U' \dot{\cup} U''$ *and* $T = T' \dot{\cup} T'$ *in such a way that*

$$\alpha\beta < 1 \tag{5.4}$$

for

$$\alpha := 2\kappa(T', U')\,\mu(T'', U'), \qquad \beta := 2\kappa(T'', U'')\,\mu(T', U''), \tag{5.5}$$

then

$$\kappa(T, U) \leqq 2\max\{(1+\beta)\,\kappa(T', U'), (1+\alpha)\,\kappa(T'', U'')\}/(1-\alpha\beta). \tag{5.6}$$

Proof. We write

$$\Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U}\Big\|_1 = \Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U'}\Big\|_1 + \Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U''}\Big\|_1,$$

and bound the first term in the RHS from below:

$$\Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U'}\Big\|_1 = \Big\|\sum_{\tau\in T'} c(\tau)\,\tau_{|U'} + g\Big\|_1 \geqq \Big\|\sum_{\tau\in T'} (c+c')(\tau)\,\tau_{|U'}\Big\|_1/2$$
$$\geqq \|c_{|T'} + c'\|_1/(2\kappa(T', U')),$$

with $\sum_{\tau\in T'} c(\tau)\,\tau_{|U'}$ a best approximation to $g := \sum_{\tau\in T''} c(\tau)\,\tau_{|U'}$ from span $(T'_{|U'})$. The coefficient sequence c' of this best approximation satisfies

$$\|c'\|_1 \leqq \kappa(T', U')\Big\|\sum_{\tau\in T'} c'(\tau)\,\tau_{|U'}\Big\|_1 \leqq \kappa(T', U')\,2\,\|g\|_1 \leqq \kappa(T', U')\,2\mu(T', U')\,\|c_{|T''}\|_1,$$

i.e., with the abbreviation (5.5),

$$\|c'\|_1 \leqq \alpha\,\|c_{|T''}\|_1. \tag{5.7}$$

Analogously,

$$\Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U''}\Big\|_1 \geqq \|c'' + c_{|T''}\|_1/(2\kappa(T'', U'')),$$

with

$$\|c''\|_1 \leq \beta\,\|c_{|T'}\|_1.$$

Thus

$$\frac{\|c\|_1}{\Big\|\sum_{\tau\in T} c(\tau)\,\tau_{|U}\Big\|_1} \leqq \frac{2\,\|c\|_1}{\|c_{|T'} + c'\|_1/\kappa(T', U') + \|c'' + c_{|T''}\|_1/\kappa(T'', U'')}.$$

Next we write

$$\|c\|_1 = \|c_{|T'}\|_1 + \|c_{|T''}\|_1$$

and observe that, e.g., from (5.7),

$$\begin{aligned}\|c_{|T'}\|_1 &= \gamma\,\|c_{|T'}\|_1 + (1-\gamma)\,\|c_{|T'}\|_1\\ &\leqq \gamma\,\|c_{|T'} + c'\|_1 + \gamma\,\|c'\|_1 + (1-\gamma)\,\|c_{|T'}\|_1\\ &\leqq \gamma\,\|c_{|T'} + c'\|_1 + \gamma\alpha\,\|c_{|T''}\|_1 + (1-\gamma)\,\|c_{|T'}\|_1,\end{aligned}$$

hence,

$$\|c\|_1 \leqq \gamma\,\|c_{|T'} + c'\|_1 + \delta\,\|c'' + c_{|T''}\|_1 + (\gamma\alpha + 1 - \delta)\,\|c_{|T''}\|_1 + (1 - \gamma + \delta\beta)\,\|c_{|T'}\|_1,$$

with arbitrary positive γ, δ. The requirement that in this last inequality the terms involving $\|c_{|T'}\|_1$ and $\|c_{|T''}\|_1$ drop out leads to the choice

$$\gamma=(1+\beta)/(1-\alpha\beta), \qquad \delta=(1+\alpha)/(1-\alpha\beta),$$

and these are positive by assumption (5.4). Thus, altogether, with this choice,

$$\frac{\|c\|_1}{\|\sum_{\tau\in T} c(\tau)\,\tau_{|U}\|_1} \leqq 2\,\frac{\gamma\,\|c_{|T'}+c'\|_1+\delta\,\|c''+c_{|T''}\|_1}{\|c_{|T'}+c'\|_1/\kappa(T',U')+\|c''+c_{|T''}\|_1/\kappa(T'',U'')}$$
$$\leqq 2\max\{\gamma\kappa(T',U'),\delta\kappa(T'',U'')\}. \quad \square$$

Corollary. *If, in addition,*

$$T'=\{\tau\in T\colon \operatorm{supp}\tau\cap U'\neq\varnothing\},$$

hence

$$T''=\{\tau\in T\colon \operatorname{supp}\tau\subseteq U''\},$$

then

$$\kappa(T,U)\leqq 2\max\{(1+2\kappa_{T''})\,\kappa(T',U'),\kappa_{T''}\}.$$

Proof. In this case,

$$\kappa(T'',U'')=\kappa_{T''}, \qquad \mu(T'',U')=0,$$

while always (by the normalization $\|\tau\|_1=1$ for $\tau\in T$)

$$\mu(T',U'')\leqq 1.$$

Therefore, for this case, $\alpha=0$, and γ and δ can be taken to be

$$\gamma=1+2\kappa_{T''}, \qquad \delta=1. \quad \square$$

Remark. *If, in addition,* $T''=\{\tau\in T\colon \operatorname{supp}\tau\cap U''\neq\varnothing\}$, *then*

$$\kappa(T,U)=\max\{\kappa(T',U'),\kappa(T'',U'')\}.$$

The partition $T'\,\dot\cup\,T''$ of $T=V_k$ that we have in mind is based on the fact that the τ_β can be roughly classified by whether they are associated with a particular vertex or a particular edge of Δ. We are going to be precise about this eventually. For the time being, let

$$E_v$$

denote the collection of edges emanating from the vertex v, and define

$$R_v$$

to be the collection of all τ_β associated with some $e\in E_v$ and sharing part of its support diamond with that of some τ_γ associated with another edge in E_v. Because of the overlapping supports (which form a kind of *ring* around v), the collection R_v is far from linearly independent. For example, when $\rho=1=k$, we have $\#R_v=\#E_v$, yet $\dim\ \operatorname{span}(R_v)=\dim\pi^0_{k,\Delta_v}-\dim\pi_1=1+\#E_v-3=\#E_v-2$, with $\Delta_v:=\{\delta\in\Delta\colon v\in\delta\}$.

For general ρ and k, a formula for dim span(R_v) can be given (which involves the details of E_v), but it is far from trivial to select a basis for span(R_v). We avoid this difficult task by constructing *reduced rings* r_v so that conditions from different reduced rings have disjoint support. This implies that, with T_v any basis for span(r_v), the set $T'' := \bigcup_{v \in V} T_v$ is linearly independent. In fact, from the above Remark

$$\kappa_{T''} = \sup_V \kappa_{T_v}.$$

Further, by a theorem of Auerbach, we can choose the (normalized) basis T_v for span(r_v) so that

$$\kappa_{T_v} \leqq \dim \operatorname{span}(r_v).$$

Since dim span$(r_v) \leqq \# r_v$, and this number can be bounded in terms of k, ρ, and $\# E_v$, while $\# E_v$ can be bounded in terms of a (the smallest angle in any triangle in Δ), this provides a Δ-independent bound on $\kappa_{T''}$.

The set T' is also put together of groups with disjoint supports, viz. a group T_e for each edge e. We identify these T_e in the next section.

6. Disentangling the Rings

As we now show, it is possible to disentangle neighboring rings as soon as there is a smoothness condition of maximal order, i.e., of order ρ, which belongs to no ring. This happens as soon as $k > 3\rho + 1$. The construction is based on selecting a subset T_e from the collection

$$S_e := \{\tau_{\beta,e} : \operatorname{supp} \beta \subset e, |\beta| \geqq k - \rho\}$$

of all smoothness conditions of order $\leqq \rho$ connected with the edge e.

Proposition 6. *If $k = 3\rho + 2$, then there exists, for each edge $e := \delta \cap \delta' =: [w, w']$, a set*

$$U_e \subset V_k \cap (\delta \cup \delta')$$

with the following properties:

(i) *U_e lies in the support of the unique $\tau \in S_e$ of order ρ which belongs to no ring.*

(ii) *The set*

$$T_e := \{\tau \in S_e : \operatorname{supp} \tau \cap U_e \neq \varnothing\}$$

is linearly independent over U_e.

(iii) *The remaining conditions in S_e fall into two classes,*

$$R_{e,x} := \{\tau_\beta \in S_e \setminus T_e : \beta(x) > \beta(x')\}, \quad \textit{with } (x, x') = (w, w') \textit{ or } (w', w),$$

and conditions from different classes have disjoint supports.

(iv) *T_e and U_e depend continuously on δ and δ' except when w or w' lies on the segment $[u, u']$ spanned by the vertices of δ and δ' not in e.*

Proof. While it would be possible to take for T_e the collection of all conditions belonging either to both rings R_w and $R_{w'}$ or to neither, it seems more efficient to be satisfied with a subset of these which is big enough to disentangle the rings.

We begin by looking more closely at the ring R_w. Note that $\tau_\beta=\tau_{\beta,e}$ belongs to R_w iff its tip $v_{\beta+re_u}$ is in the first ρ bands parallel to $[w,u]$, i.e., iff $(\beta+re_u)(w')\leqq\rho$, i.e., iff $\beta(w')\leqq\rho$. Since β only has support in $[w,w']$ and $|\beta|=k-r$, hence $\beta(w)+\beta(w')=k-r$, this says that τ_β belongs to neither ring iff $\rho<\beta(w)<k-r-\rho$. In particular, there are no such conditions of maximal order $r=\rho$ when $k<3\rho+2$. For $k=3\rho+2$ and $r=\rho$, this leaves just one choice, viz. $\beta(w)=\beta(w')=\rho+1$. Further, if $k=3\rho+2$, then no condition can belong to both rings. Thus, the only difficulty we have to overcome is to deal with conditions which belong to one ring, yet have some common support with some condition belonging to the other ring.

The general construction is based on Lemma 4.1, hence breaks down in case $[u,u']$ contains w or w'. In this latter case, though, the proposition is almost obvious since the smoothness conditions across the edge then have small support according to Lemma 4.2.

We consider the special case $w\in[u,u']$ first. In this case, according to Lemma 4.2, each τ_β has support only on the edge of its support diamond parallel to $[u,u']$. Thus the choice

$$U_e=\{v_{(\rho+1,\rho+i,\rho+1-i)}: i=1,\ldots,\rho\}$$

(i.e., the part of the support diamond strictly inside δ of the ρth order condition in neither ring) does the job in this case.

The idea in the contrary case is to construct both T_e and U_e step by step as a *sequence* in such a way that the resulting matrix $A_e:=(\tau(u))_{u\in U_e,\tau\in T_e}$ is block triangular with invertible diagonal blocks. More than that, for a sequence of integers m, the first m of the conditions in T_e are the only conditions (of order $<\rho$ and belonging to e) which have some support at the first m points in U_e. While the first condition put into T_e is that condition of maximal order belonging to neither ring, subsequent conditions may well belong to one ring or the other. Their removal into T_e reduces the number of conditions which are in one ring, yet share some support with a condition from the other ring. The process stops when there are no such conditions left. In the discussion to follow, Figure 6 may be of help.

We find it convenient to label the conditions in T_e and the points in U_e each by a number pair, and the ordering is the lexicographic one with respect to these labels.

The first condition is the sole condition of order ρ not belonging to either ring, and we give it the label (0, 1):

$$\tau_{0,1}:=\tau_{(\rho+1,\rho+1)}.$$

The corresponding first point in U_e is its support tip in δ:

$$u_{0,1}:=v_{(\rho+1,\rho+1,\rho,0)}.$$

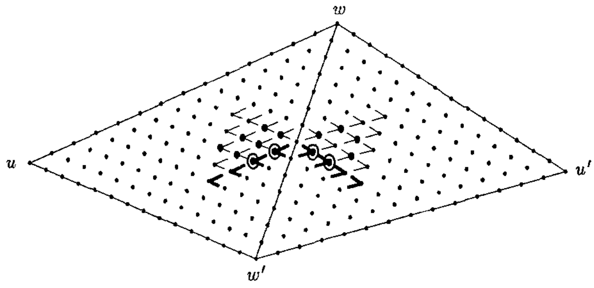

Fig. 6. The tips of the support diamonds of $\tau \in T_e$ and the set U_e (heavy dots), for $\rho = 4$, hence $k = 3\rho + 2 = 14$. The four heavily marked smoothness conditions are the only ones having support on the circled points and not already included in T_e

Note that $\tau_{0,1}(u_{0,1}) \neq 0$, by (4.9). Here and below, we use the ordering

$$w, w', u, u'$$

when describing the relevant part of meshfunctions on V_k.

The next conditions are the two of order $\geqq \rho - 1$ on the w-side of $\tau_{0,1}$, and we label them (1, 1) and (1, 2):

$$\tau_{1,1} := \tau_{(\rho+2,\rho)}, \qquad \tau_{1,2} := \tau_{(\rho+2,\rho+1)}.$$

The corresponding next two points are the two on which these two conditions are linearly independent by Lemma 4.1:

$$u_{1,1} := v_{(\rho+2,\rho+1,\rho-1,0)}, \qquad u_{1,2} := v_{(\rho+2,\rho+1,0,\rho-1)}.$$

Next come the two conditions of order $\geqq \rho - 1$ on the w'-side of $\tau_{0,1}$, and we label them (2, 1) and (2, 2):

$$\tau_{2,1} := \tau_{(\rho,\rho+2)}, \qquad \tau_{2,2} := \tau_{(\rho+1,\rho+2)}.$$

The corresponding next two points are the two on which these two conditions are linearly independent by Lemma 4.1:

$$u_{2,1} := v_{(\rho+1,\rho+2,\rho-1,0)}, \qquad u_{2,2} := v_{(\rho+1,\rho+2,0,\rho-1)}.$$

We conclude this step by taking

$$u_{2,3} := v_{(\rho+2,\rho+2,\rho-2,0)}$$

for our next point in U_e, and taking as our next condition the only one in S_e with some support at $u_{2,3}$ not yet included in T_e and labeling it (2, 3):

$$\tau_{2,3} := \tau_{(\rho+2,\rho+2)}.$$

In general, at step $2s-1$ of this procedure, we adjoin the next conditions of order $\geqq \rho-(2s-1)$ on the w-side and label them $(2s-1, j)$:

$$\tau_{2s-1,j} := \tau_{(\rho+1+s,\rho-s+j)}, \quad j=1, \ldots, 2s.$$

The corresponding next $2s$ points are those on which these conditions were shown to be linearly independent in Lemma 4.1:

$$u_{2s-1,j} := v_{(\rho+1+s,\rho+j,\rho+1-s-j,0)}, \quad u_{2s-1,s+j} := v_{(\rho+1+s,\rho+j,0,\rho+1-s-j)}, \quad j=1, \ldots, s.$$

In the general step $2s$ of this procedure, we first adjoin the next conditions of order $\geqq \rho-(2s-1)$ on the w'-side and label them $(2s, j)$:

$$\tau_{2s,j} := \tau_{(\rho-s+j,\rho+1+s)}, \quad j=1, \ldots, 2s.$$

The corresponding next $2s$ points are those on which these conditions were shown to be linearly independent in Lemma 4.1:

$$u_{2s,j} := v_{(\rho+j,\rho+1+s,\rho+1-s-j,0)}, \quad u_{2s,s+j} := v_{(\rho+j,\rho+1+s,0,\rho+1-s-j)}, \quad j=1, \ldots, s.$$

Figure 6 shows the situation at this point, with $s=2$. We conclude this step by taking

$$u_{2s,2s+1} := v_{(\rho+1+s,\rho+1+s,\rho-2s,0)}$$

for our next point in U_e, and taking as our next condition the only one in S_e with some support at $u_{2s,2s+1}$ not yet included in T_e and labeling it $(2s, 2s+1)$:

$$\tau_{2s,2s+1} := \tau_{(\rho+1+s,\rho+1+s)}.$$

Note that $\tau_{2s,2s+1}(u_{2s,2s+1}) \neq 0$, by (4.9).

With each step, the set U_e grows toward the edge e. We stop the process as soon as it would call for inclusion into U_e of a point on the edge, i.e., either in the middle of step ρ in case ρ is even, or else after step $\rho-1$. Since T_e contains all the conditions in S_e whose support diamond intersects U_e, this implies that the only conditions from S_e not included in T_e are those with support diamond either entirely to the w-side or else entirely to the w'-side of U_e, which proves (iii).

The remaining assertions follow from teh following

Claim. *The conditions $\tau_{m,i}$, $i=1, \ldots, m+1$ are linearly independent over the points $u_{m,i}$, $i=1, \ldots, m+1$. Further, except for conditions appearing earlier in T_e, they are the only conditions in S_e having support at these points.*

verified during the construction process. □

7. The Main Result

We are now prepared to prove the following

Theorem. *Let*

$$S := \pi^{\rho}_{k,\Delta} := \{f \in C^{\rho} : \forall(\delta \in \Delta)\, f_{|\delta} \in (\pi_k)_{|\delta}\}$$

be the space of piecewise polynomial C^ρ-functions of degree $\leqq k$ on the triangulation Δ of some domain G in $\mathbb{R}^2$. If

$$k > 3\rho + 1,$$

then there exists a constant const *which depends only on*

$$a := \text{smallest angle in } \Delta$$

and k so that

$$\operatorname{dist}(f, S) \leqq \text{const} \, \|D^{k+1} f\| \, |\Delta|^{k+1}$$

for all smooth f, with

$$|\Delta| := \sup_{\delta \in \Delta} \operatorname{diam} \delta.$$

Proof. By (2.5) and Lemma 3,

$$\operatorname{dist}(f, S) \leqq \text{const}_k \, \|D^{k+1} f\| \, \kappa_T |\Delta|^{k+1},$$

with T a (normalized) basis for $S^\perp$ and

$$\kappa_T := \sup_c \frac{\|c\|_1}{\|\sum_T c(\tau)\,\tau\|}.$$

By Proposition 6, we can find a suitable basis T for $S^\perp$ as follows. For each edge $e = \delta \cap \delta' = [w, w']$ in the partition, we can write the collection of all smoothness conditions of order $\leqq \rho$ across that edge as the disjoint union

$$R_{e,w} \,\dot\cup\, T_e \,\dot\cup\, R_{e,w'}, \tag{7.1}$$

with T_e maximally linearly independent over a certain subset U_e of V_k, and each condition in $R_{e,x}$ having its support diamond entirely to the x-side of U_e, $x = w, w'$.

Choose

$$U' := \bigcup_{e \in E} U_e, \qquad T' := \bigcup_{e \in E} T_e.$$

Both of these unions are disjoint, hence

$$\kappa(T', U') = \sup_e \kappa(T_e, U_e). \tag{7.2}$$

For each vertex v of Δ, let T_v be a basis for the linear span of the *reduced ring*

$$r_v := \bigcup_{e \in E_v} R_{e,v}$$

of smoothness conditions for that vertex, with E_v the collection of edges emanating from v. By a theorem of Auerbach, we can choose T_v so that

$$\kappa_{T_v} \leqq \dim \operatorname{span}(r_v).$$

This bound, in turn, is bounded by $\# r_v$, hence can be bounded in terms of $\# E_v$ and k, hence ultimately in terms of a and k. Since conditions from different reduced rings have disjoint support, it follows that the condition $\kappa_{T''}$ of

$$T'' := \bigcup_{v \in V} T_v$$

is bounded in terms of a and k. Further,

$$T := T' \cup T''$$

spans $S^\perp$, and this union is disjoint, and

$$T' = \{\tau \in T : \operatorname{supp} \tau \cap U' \neq \emptyset\}.$$

We can therefore conclude, by the Corollary to Lemma 5, that

$$\kappa_T \leqq \text{const}_{a,k} \sup_{e \in E} \kappa(T_e, U_e). \tag{7.3}$$

The determination of $\sup_e \kappa(T_e, U_e)$ is a *local* problem. By construction,

$$\kappa(T_e, U_e) = \|A_e^{-1}\|_1,$$

with the matrix $A_e := (\tau(v))_{v \in U_e, \tau \in T_e}$ depending, continuously for the most part, on the four vertices w, w', u, u' of the quadrilateral $\delta \cup \delta'$. The exception occurs when one of the endpoints, w or w', of e is contained in the segment $[u, u']$, i.e., when the other two edges emanating from w (or w') are parallel.

Since A_e is invariant under rigid motions of the plane and under scaling, we may assume that w and w' are fixed, e.g., $w = 0$ and $w' = (0, 1)$. Then u, u' can be bounded in terms of the lower bound a on all angles in all triangles. Thus it is sufficient to bound the map

$$(u, u') \mapsto \|A_e^{-1}\|_1 \tag{7.4}$$

over a closed and bounded set.

Suppose to begin with that the exceptional case is excluded, e.g., suppose that all angles in the quadrilateral (w, u, w', u') are $\leqq \pi - b$ for some positive b. Then the function (7.4) is continuous on its domain, therefore bounded, and we are done.

We are similarly done if we restrict the angle

$$b_{w,e} := \angle u w u'$$

at w to be $\geqq \pi + b$ for some positive b. Thus the final problem we have to settle concerns a Δ-independent bound on $\kappa(T_e, U_e)$ when $b_{w,e} \sim \pi$.

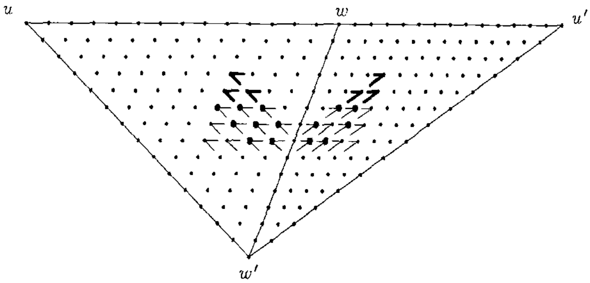

Fig. 7. The tips of the support diamonds of $\tau \in T_e$ and of its subset $T_{e,w}$ (heavily marked), and the set U_e (heavy dots), for $\rho = 4$, hence $k = 3\rho + 2 = 14$

As $b_{w,e}$ approaches π, $\kappa(T_e, U_e)$ approaches infinity for the simple reason that, for a certain nonempty subset $T_{e,w}$ of T_e (and in terms of the definition (5.3)),

$$\mu(T_{e,w}, U_e) \to 0.$$

To be precise, these are the conditions on the w-side whose support tips are not in U_e, i.e., the conditions $\tau_{2s-1,j}, j = 1, \ldots, s; s = 1, \ldots, \lfloor \rho \rfloor$ (see Figure), which, by Lemma 4.2, have no support in U_e when $b_{w,e} = \pi$. This is the reason why, in Proposition 6, we had to switch to a different construction when $b_{w,e}$ (or $b_{w',e}$) equals π.

On the other hand, this means that, by adjoining these conditions to the reduced ring r_w instead, $\mu(r_w, U_e)$, while not guaranteed to be zero anymore, would be small for $b_{w,e} \sim \pi$, hence a suitable bound is available from Lemma 5. The only difficulty still to be overcome stems from the fact that Lemma 5 does not require $\mu(r_w, U_e)$ to be small, but $\mu(T_w, U_e)$ with T_w the basis for $\operatorname{span}(r_w)$ chosen by Auerbach's theorem. Since this (normalized) basis is constructed only with regard to its condition κ_{T_w}, the best that we can offhand say about $\mu(T_w, U_e)$ is that it is $\leqq 1$, and that is not good enough for an application of Lemma 5.

We deal with this final difficulty as follows. With a positive

$$b$$

to be chosen shortly, we modify our definition of T_e in Proposition 6 in case $|b_{w,e} - \pi| \leqq b$ to exclude the conditions $T_{e,w}$. Note that this, correspondingly, modifies $R_{e,w}$, hence ultimately r_w, to include $T_{e,w}$. With this modification, T_e stays linearly independent over U_e as long as $b \leqq a$ since

$$2\pi - 2a \geqq b_{w,e} + b_{w',e},$$

hence $\pi - b_{w',e} \geqq 2a - b \geqq a$, i.e., $b_{w',e}$ must stay away from π when $b_{w,e}$ is close to π. Consequently, $\kappa(T_e, U_e)$ is bounded in terms of a (and k).

Since now the elements of r_w may have some support in U', we need to be more careful in the choice of the basis T_w for $\operatorname{span}(r_w)$. Among the many possible bases T_w for $\operatorname{span}(r_w)$ with $\kappa_{T_w} \leqq \# r_w$ (of which there is at least one by Auerbach's theorem), we choose one that minimizes $\mu(T_w, U')$. Since

$\mu(R, U_e)=0$ for an arbitrary basis R of $\operatorname{span}(r_w)$ in case $b_{w,e}=\pi$, there exists, for given $\varepsilon>0$, a positive $b=b(\varepsilon)$ so that

$$\mu(T_w, U')\leqq\varepsilon. \tag{7.5}$$

Choose now a positive b so that (7.5) holds with

$$\varepsilon:=1/(8\sup_e \kappa(T_e, U_e)\max_v \# r_v). \tag{7.6}$$

Then Lemma 5 provides a bound for $\kappa(T, U)$ in terms of $\sup_e \kappa(T_e, U_e)$ and $\max_v \# r_v$, hence, ultimately, in terms of a and k. □

8. Sharpness

In this final section,w e show that, already on the three-direction mesh $\Delta=\Delta_3$, the approximation order from π^{ρ}_{k,Δ_3} is no better than k when $k<3\rho+2$ (and $\rho=1,2,3$).

The argument is that of [BH83] where this was shown for $k=3$, $\rho=1$, and runs in general as follows. Recall from Sect. 2 that

$$\operatorname{dist}(f, S)=O(|\Delta|^{k+1})+\operatorname{dist}(Pf, S) \tag{2.5}$$

and that

$$\operatorname{dist}(Pf, S)=\max_{\lambda\in S^{\perp}}\frac{|\lambda Pf|}{\|\lambda\|}. \tag{2.6}$$

Thus it is sufficient to exhibit a smooth function f and a linear functional $\lambda=\lambda_{\Delta}\in S^{\perp}$ for which

$$\frac{|\lambda Pf|}{\|\lambda\|}\geqq \operatorname{const}_f|\Delta|^k \tag{8.1}$$

for some positive const_f.

We pick the domain G on which we want to approximate to be a square,

$$G=[0, M]^2,$$

say. We choose $\Delta=\Delta_3$, with Δ_3 the *three-direction mesh*, i.e., the partition of $\mathbb{R}^2$ provided by the three meshline families

$$x(1)=n, \quad x(2)=n, \quad x(1)-x(2)=n, \qquad \text{for all } n\in\mathbb{Z}.$$

Further, we pick some $v\in S^{\perp}\setminus 0$ with the following three properties:

(i) $\operatorname{supp} v\subset[-1, 1]^2$.

(ii) For some homogeneous polynomial g of degree $k+1$, $vg\neq 0$.

(iii) $\sum_{n\in\mathbb{Z}^2} vE^n=0$.

Here, E denotes the *shift*, i.e.,

$$E^z f(x) := f(x+z).$$

Set

$$\lambda := \sum_{n \in N} \nu E^n$$

with

$$N := \{n \in \mathbb{Z}^2 : \operatorname{supp}(\nu E^n) \subset G\}.$$

Then, since $(1-E^2)\pi_{k+1} \subseteq \pi_k$, and P is the identity on π_k, and ν vanishes on S and therefore on π_k,

$$\lambda P g = \sum_{n \in N} \nu E^n P g = \# N \nu g \sim M^2 \nu f.$$

On the other hand, from Property (iii), λ has support only near the boundary of G, hence

$$\|\lambda\|_1 \sim \text{perimeter}(G) \sim 4M \|\nu\|_1.$$

Consequently,

$$|\lambda P g| / \|\lambda\|_1 \sim \text{const}\, M$$

for some positive const. Finally, a re-scaling of the plane by $1/M$ carries G to the unit square $[0, 1]^2$ and carries g to g/M^{k+1}, hence provides the sought-for inequality

$$|\lambda P g| / \|\lambda\|_1 \geqq \text{const}\, |\Delta|^k.$$

For $\rho = 1$, i.e., $k = 4$, a suitable choice for ν is the sum

$$\nu = \tau_{(2,1),1} - \tau_{(2,1),2} + \tau_{(2,1),3} - \tau_{(2,1),4} + \tau_{(2,1),5} - \tau_{(2,1),6} + \tau_{(3,0),4} - \tau_{(3,0),1},$$

in which the various smoothness conditions $\tau_{\beta,e}$ are described in shorthand, as follows. The vertices of Δ involved are enumerated as

$$c := 0,\ v_1 := (1, 0),\ v_2 := (1, 1),\ v_3 := (0, 1),$$
$$v_{3+j} := -v_j, \qquad \text{all } j.$$

These are the center and the six corners of a regular hexagon. The edge $e = [c, v_j]$ is specified by the number j, and only the two possibly non-zero entries of β are given, with the first one always specifying $\beta(c)$. One checks that, with the normalization

$$\tau_{(3,0),j}(v) = \begin{cases} -1, & \text{for } v = 0, v_j/4, \\ 1, & \text{for } v = v_{j\pm 1}/4, \\ 0 & \text{otherwise,} \end{cases}$$

the linear functional ν takes the simple form

$$\nu(v) = \begin{cases} (-1)^j, & \text{if } v = v_j/2; \quad j = 1, \ldots, 6, \\ 0 & \text{otherwise,} \end{cases}$$

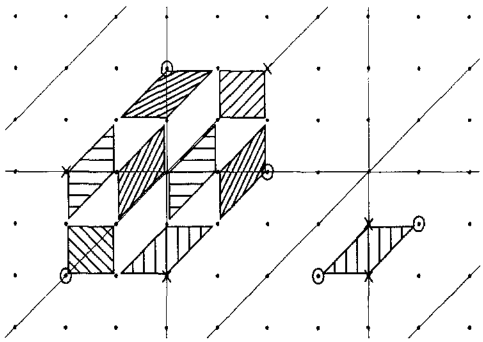

Fig. 8.1. A suitable linear functional ν as the sum of C^1-smoothness conditions $\tau_{\beta,e}$ for $\rho=1$, $k=4$. The actual support of ν is indicated by circles (weight -1) and crosses (weight 1). The typical $\tau_{\beta,e}$ is shown to the right in the same way

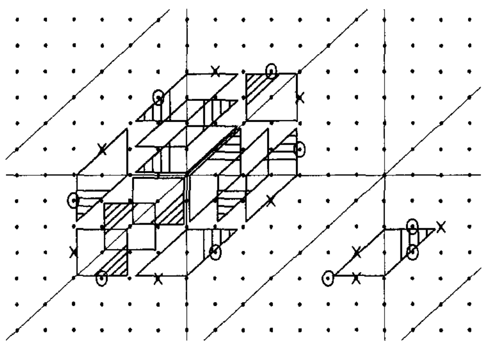

Fig. 8.2. A suitable linear functional ν as the sum of certain $\sigma_{\beta,e}$ when $\rho=2$, $k=7$. The actual support of ν is indicated by circles (weight -1) and crosses (weight 1). The typical $\sigma_{\beta,e}$ is shown to the right in the same way

and that therefore $\nu g \neq 0$ for the monomial $g := ()^{(3,2)}$, and also that Property (iii) holds. This case was earlier treated in [J 84].

For $\rho=2$, i.e., $k=7$, the construction is almost as simple. Using the same notation for the details of the Δ_3 partition, we consider here the element $\sigma_{\beta,j} := \tau_{\beta,j} - \tau_{\beta+e_c,j} - \tau_{\beta+e_{v_j},j}$ in $S^\perp$, which, for $\beta=(5,0)$, is of the following form:

$$\sigma_{(5,0),j}(v) = \begin{cases} 1, & \text{if } v=(2/7)\,v_{j-1}, v_{j+1}/7, (v_{j+1}+v_j)/7, \\ -1, & \text{if } v=(2/7)\,v_{j+1}, v_{j-1}/7, (v_{j-1}+v_j)/7, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, a suitable ν takes the following simple form:

$$\nu = \sum_{j=1}^{6} \sigma_{(3,2),j} + \sum_{j=1,3,5} (\sigma_{(5,0),j} - \sigma_{(4,1),j}).$$

One verifies that

$$\nu(v)=\begin{cases}\pm 1, & \text{if } v=(3v_j \pm v_{j\pm 1})/7,\\ 0 & \text{otherwise.}\end{cases}$$

This makes it easy to verify that Property (iii) holds and that $\nu g \neq 0$ for all the 8-th degree monomials $g=()^{(j,8-j)}$ with $j \neq 0, 4, 8$.

For $\rho=3$, i.e., $k=10$, the construction becomes complicated enough to make us desist writing it down here. We have not yet obtained a construction for general ρ.

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