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Modular Forms and Root Systems

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§0. Introduction

The coefficients $p_d(m)$ of the formal expansion $\sum_{m=0}^{\infty} p_d(m)x^m$ of the d-th power of $\prod_{n=1}^{\infty}(1-x^n)$ have been the subjects of many investigations. Macdonald gave in his paper [7] a formula for $p_d(m)$ if d is the dimension of a simple Lie algebra. These identities were obtained by Macdonald, specializing a more general formula (see [7], (8.1)) which results from his theory of affine root systems. Since Dedekind's η-function, which is equal to $\prod_{n=1}^{\infty}(1-x^n)$ up to a factor $x^{\frac{1}{24}}$, is closely connected with the theory of modular forms, the question arose whether these identities could be proved using modular forms. We will give a affirmative answer in the first place for formula (8.9) of [7], which is the identity for η^d, and thereafter also for formula (8.13) of [7] which is another remarkable η-function identity obtained by Macdonald. Basic in our proof are the classical transformation formulas of theta functions. The theta functions we will consider are functions not only of the complex variable τ, but they depend on a parameter from some real vector space, too. As a new ingredient we study the situation in which there is a finite reflection group, in particular a Weyl group, acting on this parameter.

It became clear that the method used in the proof of the identities mentioned above might be pursued to get new identities. On the one hand we are interested in spherical functions, on the other we want to consider polynomials that are skew-invariant under the action of the Weyl group. Using a theorem of Chevalley we determine the dimension of the space of homogeneous skew-invariant spherical polynomials in any degree. With this knowledge some new identities are obtained. This is done in the last section, where we first give two more or less general theorems, involving Eisenstein series besides the η-function. Thereafter we consider a few special cases. In particular we get identities for η^{39} and η^{45}.

§ 1. Notations

V is a finite dimensional real vector space of dimension l, $R \subset V$ is a root system (such that R spans V). R will be irreducible and R will be reduced too, i.e. $R \cap 2R = \emptyset$. Let $2r$ denote the number of roots in R, and $d = 2r + l$ is the dimension of a compact Lie group having R as its system of roots relative to a maximal torus. $W(R)$ is the Weyl group of R.

$\langle\,,\,\rangle : V \times V \to \mathbb{R}$ is a scalar product on V which is invariant under the action of $W(R)$. For each $v \in V$ let $q(v) = \frac{1}{2}\langle v, v\rangle$. $L \subset V$ is a lattice, i.e. a free $\mathbb{Z}$-module of rank l. Unless stated otherwise we assume that $sL = L$ for all $s \in W(R)$, and $q(L) \subset \mathbb{Z}$. L^* is the dual lattice of L (with respect to the bilinear form $\langle\,,\,\rangle$), i.e. $L^* = \{v \in V \mid \langle v, \lambda\rangle \in \mathbb{Z}$ for all $\lambda \in L\}$. $(\ \mid\ ) : V \times V \to \mathbb{R}$ is a scalar product on V which is invariant under the action of $W(R)$ and such that $(\alpha \mid \alpha) = 2$ for any short root α (when all roots have the same length let us call them all short).

$R^\vee$ is the dual root system, identified with a subset of V by means of the scalar product $(\ \mid\ )$, i.e. $R^\vee = \left\{ \alpha^\vee = \frac{2\alpha}{(\alpha \mid \alpha)} \,\middle|\, \alpha \in R \right\}$.

$\mathscr{P}(R)$ is the lattice of weights of R.
$\mathfrak{S}_l$ will be the permutation group on l symbols.
$\mathscr{H}$ will be the complex upper half plane.
Γ is the full modular group.

For any positive integer p let $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \pmod{p} \right\}$, and let $\Gamma(p)$ denote the congruence subgroup of level p. Let $\mathscr{M}_{2n}$ denote the space of modular forms of weight $2n$ for Γ.

§ 2. Modular Forms, in Particular Theta Functions

We will use the following kind of theta functions:

$$\Theta(\tau, L, P, \xi) = \sum_{\lambda \in L} P(\lambda + \xi) e^{2\pi i \tau q(\lambda + \xi)}, \tag{2.1}$$

where $\xi \in V$, and P must be a spherical function with respect to q. The quadratic form q is defined by means of the scalar product $\langle\,,\,\rangle$, and all scalar products on V which are invariant under the action of $W(R)$ are equal up to a positive factor (see [1], p. 66, Proposition 1 (ii)). Let $\varepsilon_1, \ldots, \varepsilon_l$ be an orthonormal basis of V [with respect to the scalar product $(\ \mid\ )$], and $x_1\varepsilon_1 + \ldots + x_l\varepsilon_l \in V$. Define the second order differential operator Δ by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_l^2}. \tag{2.2}$$

Then we have:
P is a spherical function with respect to $q \Leftrightarrow \Delta P = 0$.
From now on we take $\xi \in L^*$.

For the theta functions (2.1) we have the following transformation formulas:

$$\Theta(\tau+1, L, P, \xi)=e^{2\pi i q(\xi)}\Theta(\tau, L, P, \xi), \tag{2.3}$$

$$\Theta\left(-\frac{1}{\tau}, L, P, \xi\right) = \frac{(-i)^{\frac{l}{2}+2\deg(P)}}{v(L)}\tau^{\frac{l}{2}+\deg(P)} \sum_{\mu\in L^*/L} e^{2\pi i\langle\mu,\xi\rangle}\Theta(\tau, L, P, \mu), \tag{2.4}$$

where $v(L)$ is the measure of V/L with respect to $\langle\, ,\rangle$ and for any $\tau\in\mathbb{C}$ we take

$$\tau^{\frac{1}{2}}=\sqrt{|\tau|}\,e^{i\frac{\arg(\tau)}{2}}, \quad \text{where} \quad -\pi<\arg(\tau)\leqq\pi .$$

By $\sum_{\mu\in L^*/L}$ we indicate that a sum must be extended over a complete set of representatives of the (finite) quotient L^*/L. Formula (2.3) is an immediate consequence of $q(\lambda+\xi)\equiv q(\xi) \pmod{\mathbb{Z}}$ for all $\lambda\in L$. As for (2.4) we have

$$\Theta\left(-\frac{1}{\tau}, L, P, \xi\right) = \frac{(-i)^{\frac{l}{2}+2\deg(P)}}{v(L)}\tau^{\frac{l}{2}+\deg(P)} \sum_{\mu\in L^*} P(\mu)e^{2\pi i\tau q(\mu)+2\pi i\langle\mu,\xi\rangle} \tag{2.5}$$

(see for example [8], p. VI–14, Theorem 19)

$$= \frac{(-i)^{\frac{l}{2}+2\deg(P)}}{v(L)}\tau^{\frac{l}{2}+\deg(P)} \sum_{\mu\in L^*/L}\sum_{\lambda\in L} P(\mu+\lambda)e^{2\pi i\tau q(\mu+\lambda)+2\pi i\langle\mu+\lambda,\xi\rangle}$$

$$= \frac{(-i)^{\frac{l}{2}+2\deg(P)}}{v(L)}\tau^{\frac{l}{2}+\deg(P)} \sum_{\mu\in L^*/L} e^{2\pi i\langle\mu,\xi\rangle}\Theta(\tau, L, P, \mu).$$

For these theta functions we have the following theorem which was proved in the case that l is even by Schoeneberg and in the case that l is odd by Pfetzer:

(2.6) **Theorem.** *For all $\xi\in L^*$ the function $\Theta(\tau, L, P, \xi)$ is a modular form of weight $\frac{l}{2}+\deg(P)$ for the subgroup $\Gamma(N)$ of Γ, where N is the level of the quadratic form q, i.e. N is the least positive integer with $Nq(L^*)\subset\mathbb{Z}$.*

For a proof see [9] and [10].

It should be remarked that, when l is odd, modular form means modular form with a multiplier system.

Another function which will play an important role is Dedekind's η-function, defined by

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}). \tag{2.7}$$

For this function we have the following transformation formulas:

$$\eta(\tau+1)=e^{\frac{\pi i}{12}}\,\eta(\tau)\,, \tag{2.8}$$

$$\eta\left(-\frac{1}{\tau}\right)=(-i\tau)^{\frac{1}{2}}\eta(\tau) \qquad \text{(see for example [11])}. \tag{2.9}$$

Another property of $\eta(\tau)$ that will be used is:

the function $\eta(\tau)$ vanishes nowhere except at $i\infty$. (2.10)

Other functions that will be used are the (normalized) Eisenstein series, defined for any positive integer n by

$$\mathscr{E}_n(\tau)=1+\frac{(-1)^n 4n}{B_n}\sum_{r=1}^{\infty}\sigma_{2n-1}(r)e^{2\pi i r\tau},$$

where the B_n are the Bernoulli numbers, and $\sigma_{2n-1}(r)=\sum_{d|r} d^{2n-1}$.

The Eisenstein series $\mathscr{E}_n$ are modular forms of weight $2n$ for the full modular group Γ (see [6], p. 53).

For the dimension of the space $\mathscr{M}_{2n}$ of modular forms for Γ of weight $2n$ we have the following formulas:

$$\begin{aligned}\dim \mathscr{M}_{2n} &= \left[\frac{n}{6}\right] \quad \text{if} \quad n\equiv 1 \pmod 6,\\ &= \left[\frac{n}{6}\right]+1 \quad \text{if} \quad n\not\equiv 1 \pmod 6 \qquad \text{(see [6], p. 26)}.\end{aligned} \tag{2.11}$$

Apart from Γ we shall need the subgroups $\Gamma(2)$, $\Gamma_0(2)$, and $\Gamma_0(3)$ in § 5. Now $\Gamma(2)$ is generated by the two transformations $\tau\mapsto\tau+2$ and $\tau\mapsto\dfrac{-\tau}{2\tau-1}$, and $\Gamma_0(p)$ is generated by $\tau\mapsto\tau+1$ and $\tau\mapsto\dfrac{-\tau}{p\tau-1}$ for $p=2, 3$. We can construct fundamental regions for these groups, and then it is easy to conclude

(2.12) **Lemma.** *$\{i\infty\}$ is a complete set of inequivalent parabolic vertices for Γ, $\{0, i\infty\}$ is one for both $\Gamma_0(2)$ and $\Gamma_0(3)$, and $\{0, 1, i\infty\}$ is one for $\Gamma(2)$.*

Let us now return to the theta functions under consideration. In (2.3) and (2.4) we got the behaviour of $\Theta(\tau, L, P, \xi)$ under the two transformations $\tau\mapsto\tau+1$ and $\tau\mapsto-\dfrac{1}{\tau}$ which generate Γ. Next we want to consider the effect on $\Theta(\tau, L, P, \xi)$ of the action of $W(R)$ on ξ. We suppose that P is a skew-invariant spherical function (the existence of such polynomials will be discussed in § 4).

(2.13) **Lemma.** (i) *$\Theta(\tau, L, P, \xi+\lambda)=\Theta(\tau, L, P, \xi)$ for all $\lambda\in L$,*
(ii) *$\Theta(\tau, L, P, s\xi)=\det(s)\Theta(\tau, L, P, \xi)$ for all $s\in W(R)$.*

Proof. (i) trivial,
(ii) is an immediate consequence of the fact that $sL=L$ and the skew-invariance of P.

Immediately we get the following

(2.14) **Corollary.** *If there exists* $s \in W(R)$ *such that* $\det(s) = -1$ *and* $s\xi - \xi \in L$, *then* $\Theta(\tau, L, P, \xi) = 0$.

Proof. $\Theta(\tau, L, P, \xi) = \Theta(\tau, L, P, s\xi) = -\Theta(\tau, L, P, \xi)$.

Remark. Of course (2.13) holds not only for $\xi \in L^*$, but for all $v \in V$.

In the next two sections we consider some properties of root systems. In particular, in view of (2.4) and (2.13) (ii) we are interested in the orbits of $W(R)$ in L^*/L.

§3. Root Systems

For any $\alpha \in R$ we denote by s_α the reflection in the hyperplane orthogonal to α, i.e. $s_\alpha(v) = v - \frac{2(v|\alpha)}{(\alpha|\alpha)}\alpha$. We now choose a Weyl chamber C for R. By doing so we get a total ordering on R, in particular we can define positive and negative roots. We introduce the following notations:

$\alpha_1, \ldots, \alpha_l$ is the basis of R corresponding to the chosen Weyl chamber C.

R_+ is the set of positive roots, so $R = R_+ \cup (-R_+)$.

$\varrho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha, \quad \sigma = \frac{1}{2} \sum_{\alpha \in R_+} \alpha^\vee$.

α_0 is the highest root of R.

Since we have identified $R^\vee$ with a subset of V, C is a Weyl chamber for $R^\vee$, too, and $\alpha_1^\vee, \ldots, \alpha_l^\vee$ is the corresponding basis. The lattice in V generated by R, respectively $R^\vee$ will be denoted by $L(R)$, respectively $L(R^\vee)$. It is clear that both $L(R)$ and $L(R^\vee)$ are invariant under the action of $W(R)$. We will use the following

(3.1) **Lemma.** *Let* L *be* $L(R)$ *or* $L(R^\vee)$. *For each* $v \in V$ *the subgroup*

$$\{s \in W(R) \mid sv - v \in L\} \quad \text{of} \quad W(R)$$

is generated by the reflections it contains.

Proof. See for example [1], p. 227, ex. 1.

From this lemma we see that if for some $\mu \in L^*$ the action of $W(R)$ on its orbit in L^*/L is not faithful, then there exists $s \in W(R)$ such that $\det(s) = -1$ and $s\mu - \mu \in L$, and by (2.14) we find that $\Theta(\tau, L, P, \mu) = 0$. So we are looking for the orbits in L^*/L on which $W(R)$ acts faithfully.

Let us first consider the lattice $L = L(R^\vee)$. Now for the affine Weyl group $W_a(R)$ of R we have $W_a(R) = W(R) \cdot L(R^\vee)$, the semi-direct product of $W(R)$ and the translation group generated by $L(R^\vee)$ (see [1], p. 173). So if we want to consider the orbits in L^*/L under the action of $W(R)$, we could as well consider the orbits in L^* under the action of $W_a(R)$. From [1], p. 66, Proposition 1 (ii), we know that every scalar product on V which is invariant under the action of $W(R)$ is equal to $(\ |\ )$ up to a positive factor. So let us define $\langle\ ,\ \rangle = k(\ |\ )$, where k is some positive integer. Then we get that $L^* = \{\xi \in V \mid \langle \xi, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in L(R^\vee)\} =$

$\frac{1}{k}\mathscr{P}(R)$. So the quotient L^*/L depends on the choice of k, and of course the number of orbits in L^*/L on which $W(R)$ might act faithfully depends on k, too.

We introduce another notation:

$$F(R)=\{v\in V\,|\,(\alpha_i|v)>0,\ 1\leqq i\leqq l,\ \text{and}\ (\alpha_0|v)<1\}.$$

Then

$$\bar{F}(R)=\{v\in V\,|\,(\alpha_i|v)\geqq 0,\ 1\leqq i\leqq l,\ \text{and}\ (\alpha_0|v)\leqq 1\}$$

is a fundamental domain for $W_a(R)$ (see [1], p. 175). We have

(3.2) **Lemma.** *By taking*

$k=1+(\alpha_0|\varrho)$ *when* R *is of type* A_l, D_l, E_6, E_7, E_8,

$\;\;=2+(\alpha_0|\varrho)$ *when* R *is of type* B_l, C_l, F_4,

$\;\;=3+(\alpha_0|\varrho)$ *when* R *is of type* G_2,

we find that there is exactly one orbit in $L(R^\vee)^*=\frac{1}{k}\mathscr{P}(R)$ *on which* $W_a(R)$ *acts faithfully, and that is the orbit of* $\frac{1}{k}\varrho$.

Proof. Let $\mu\in L^*$ such that $W_a(R)$ acts faithfully on the orbit of μ. We may assume that $\mu\in F(R)$. Then we have $(\alpha_i|\mu)>0$ and therefore $(\alpha_i^\vee|k\mu)>0$ for $1\leqq i\leqq l$. From this we get $(\alpha_i^\vee|\varrho-k\mu)<(\alpha_i^\vee|\varrho)=1$, hence $(\alpha_i^\vee|\varrho-k\mu)\leqq 0$ because it is an integer. On the other hand we have $(\alpha_0|\mu)<1$ and therefore $(\alpha_0|k\mu-\varrho)\leqq k-1-(\alpha_0|\varrho)$ [$(\alpha_0|k\mu-\varrho)$ must be an integer since $\alpha_0\in L(R^\vee)$].

We have a relation $\alpha_0=n_1\alpha_1^\vee+\ldots+n_l\alpha_l^\vee$, in which $n_i\in\mathbb{Z}$ and

$n_i\geqq 1$ if R is of type A_l, D_l, E_6, E_7, E_8,

$n_i\geqq 2$ if R is of type B_l, C_l, F_4,

$n_i\geqq 3$ if R is of type G_2.

This can be checked easily using the "Planches" of [1], or by giving a direct general proof. From this relation we get

$$(\alpha_0|k\mu-\varrho)+n_1(\alpha_1^\vee|\varrho-k\mu)+\ldots+n_l(\alpha_l^\vee|\varrho-k\mu)=0. \tag{3.3}$$

Now let k be as stated in (3.2), then it follows from (3.3) that $(\alpha_i^\vee|\varrho-k\mu)=0$ for $1\leqq i\leqq l$, hence $\varrho=k\mu$. So with this particular choice of k there is at most one orbit on which $W_a(R)$ acts faithfully. But it is clear that $\frac{1}{k}\varrho\in F(R)$, which proves the lemma.

These numbers k can also be written as $k=\frac{1}{2}(\alpha_0|\alpha_0)+(\alpha_0|\varrho)$, and from this we see that k depends on the chosen scalar product $(\ |\ )$. In Table 1 we list the value of k, the Coxeter number h and the degrees d_i of basic invariants.

Table 1

Type	k	h	Degrees of basic invariants
$A_l (l \geqq 1)$	$l+1$	$l+1$	$2, 3, \ldots, l+1$
$B_l (l \geqq 2)$	$4l-2$	$2l$	$2, 4, \ldots, 2l$
$C_l (l \geqq 2)$	$2l+2$	$2l$	$2, 4, \ldots, 2l$
$D_l (l \geqq 3)$	$2l-2$	$2l-2$	$2, 4, \ldots, 2l-2, l$
E_6	12	12	2, 5, 6, 8, 9, 12
E_7	18	18	2, 6, 8, 10, 12, 14, 18
E_8	30	30	2, 8, 12, 14, 18, 20, 24, 30
F_4	18	12	2, 6, 8, 12
G_2	12	6	2, 6

Remark. The quadratic form q is defined by $q(v)=\frac{1}{2}\langle v, v\rangle=\frac{1}{2}k(v|v)$. Now it is easy to check that $q(L(R^{\vee}))\subset\mathbb{Z}$.

In all cases but one the estimates of the n_i given in the proof of (3.2) are best possible, i.e. equality holds for at least one n_i. The only exception occurs when R is of type E_8. Then we even have $n_i \geqq 2$ for all i. So we get

(3.4) **Lemma.** *Let R be of type E_8. The only orbit in $\frac{1}{31}\mathscr{P}(R)$ on which $W_a(R)$ acts faithfully is the orbit of $\frac{1}{31}\varrho$.*

(3.5) **Lemma.** *The numbers k defined by (3.2) are such that $\frac{1}{2k}(\ |\ )$ is the canonical bilinear form Φ_R on V.*

(For the definition of Φ_R see [1], p. 172.)

Proof. We have $(\alpha|\alpha_0^{\vee})=0$ or 1 for all positive roots $\alpha \neq \alpha_0$ (see [1], p. 165, Proposition 25 (iv)). Let π be the sum of the positive roots not orthogonal to α_0. Then we have

$$\sum_{\alpha\in R_+}(\alpha|\alpha_0^{\vee})\alpha=\sum_{\substack{\alpha\in R_+\\ \alpha\neq\alpha_0}}(\alpha|\alpha_0^{\vee})\alpha+2\alpha_0=\pi+\alpha_0\,.$$

On the other hand we have $\pi=\varrho-s_{\alpha_0}\varrho=(\varrho|\alpha_0)\alpha_0^{\vee}$. So

$$\sum_{\alpha\in R_+}(\alpha|\alpha_0^{\vee})\alpha=(\varrho|\alpha_0)\alpha_0^{\vee}+\alpha_0=k\alpha_0^{\vee}\,.$$

Hence $\sum_{\alpha\in R_+}(\alpha|\alpha_0^{\vee})^2=k(\alpha_0^{\vee}|\alpha_0^{\vee})$.

By [1], p. 172, formula (17) we can write $\sum_{\alpha\in R_+}(\alpha|\alpha_0^{\vee})^2=\frac{2}{\Phi_R(\alpha_0,\alpha_0)}$ so that $\Phi_R(\alpha_0,\alpha_0)=\frac{(\alpha_0|\alpha_0)}{2k}$, which proves the lemma.

In Section 5 we will consider, besides the functions $\Theta(\tau, L(R^{\vee}), P, \xi)$, the theta functions $\Theta(\tau, L, P, \xi)$ where $L=L(R)$. There we will be interested in the orbits of $W(R)$ in $L^*/L(R^\vee)$, instead of L^*/L, and again the question arises whether we can choose the scalar product $\langle\,,\rangle$ such that there is exactly one orbit in $L^*/L(R^\vee)$ on which $W(R)$ acts faithfully. We have

(3.6) **Lemma.** *The only orbit in* $\frac{1}{h}\mathscr{P}(R^\vee)$ *on which* $W_a(R)$ *acts faithfully is the orbit of* $\frac{1}{h}\sigma$.

This lemma can be proved analogous to (3.2), using the relation $\alpha_0=m_1\alpha_1+\ldots+m_l\alpha_l$ such that $m_1+\ldots+m_l=h-1$ (see [1], p. 169, Proposition 31).

Remark. Again it is easy to check that for $q(v)=\frac{1}{2}h(v|v)$ we have $q(L(R))\subset\mathbb{Z}$.

(3.7) **Lemma.** *Let* $\xi\in F(R)$. *Then for all* $\lambda\in L(R^\vee)$, $\lambda\neq 0$, *we have*

$$(\xi+\lambda|\xi+\lambda)>(\xi|\xi).$$

Proof. Take $\lambda\in L(R^\vee)$ such that $(\xi+\lambda|\xi+\lambda)$ is minimal. There exists $s\in W(R)$ such that $s(\xi+\lambda)\in C$. Now $(s(\xi+\lambda)|\alpha_0)>1$ is impossible since then by applying the reflection w in the hyperplane $\{v\in V|(v|\alpha_0)=1\}$ we would get

$$(ws(\xi+\lambda)|ws(\xi+\lambda))=(s(\xi+\lambda)|s(\xi+\lambda))+\frac{4[1-(s(\xi+\lambda)|\alpha_0)]}{(\alpha_0|\alpha_0)}$$
$$<(\xi+\lambda|\xi+\lambda),$$

which contradicts the minimality of $(\xi+\lambda|\xi+\lambda)$ (we recall that $ws(\xi+\lambda)$ is expressible in the form $s'(\xi+\lambda')$ for some $s'\in W(R)$, $\lambda'\in L(R^\vee)$). So $(s(\xi+\lambda)|\alpha_0)<1$, and therefore $s(\xi+\lambda)\in F(R)$. Since $\bar{F}(R)$ is a fundamental domain for $W_a(R)$, and $\xi\in F(R)$, this implies $\lambda=0$, proving the lemma.

(3.8) **Lemma.** (i) $\frac{1}{k}\varrho\in F(R)$, (ii) $\frac{1}{h}\varrho\in F(R^\vee)$.

Proof. (i)Was already mentioned in the proof of (3.2), and (ii) follows immediately from (3.6) if we replace R by $R^\vee$.

In Section 5 we will consider in particular the cases that $L=L(R^\vee)$, $\xi=\frac{1}{k}\varrho$ and $L=L(R)$, $\xi=\frac{1}{h}\varrho$.

Then we see by (3.7) and (3.8) that the first term in the series $\Theta(\tau,L,P,\xi)$ is $P(\xi)e^{2\pi i\tau q(\xi)}$ provided that $P(\xi)\neq 0$; in particular $e^{2\pi i\tau q(\xi)}$ determines the behaviour of $\Theta(\tau,L,P,\xi)$ at $\tau=i\infty$. In these special cases $q(\xi)$ can be expressed in terms of k and h by some "strange formulas". The first one is

$$\Phi_R(\varrho,\varrho)=\frac{d}{24}\quad\text{(see [5], p. 243)}. \tag{3.9}$$

Another one, which was proved by Macdonald, is

$$\frac{(\varrho|\varrho)}{2h}=\frac{h+1}{24}\sum_{j=1}^{l}\frac{(\alpha_j|\alpha_j)}{2}\quad\text{(see [7], p. 120)}, \tag{3.10}$$

and as an immediate consequence of (3.10) we get

$$\frac{(\sigma|\sigma)}{2h}=\frac{h+1}{24}\sum_{j=1}^{l}\frac{2}{(\alpha_j|\alpha_j)}. \tag{3.11}$$

In (3.2) and (3.6) we found that by taking $\langle\,,\rangle=k(\;|\;)$, or $\langle\,,\rangle=h(\;|\;)$ respectively, there is exactly one orbit in $L(R^\vee)^*$ or $L(R)^*$ respectively on which $W_a(R)$ acts faithfully. We will give now one example (which will be used in Section 5 to prove formula 6(a) on p. 138 of [7]) in which there are more orbits. We consider the case that R is of type $B_l (l\geqq 2)$. Different from our normalisation of $(\;|\;)$ so far, we suppose in this particular case that the root lengths are 1 and 2. We can use the description given in [1], Planches.

(3.12) **Lemma.** *Let R be of type $B_l(l\geqq 2)$, and $\langle\,,\rangle=(2l+1)(\;|\;)$. Then the only orbits in $L(R)^*=\frac{1}{2l+1}\mathscr{P}(R^\vee)$ on which $W_a(R)$ acts faithfully are the orbits of $\frac{1}{2l+1}\sigma$ and $\frac{1}{2l+1}(\sigma+\omega_1)$ where ω_1 is the fundamental weight corresponding to α_1.*

Proof. From $\alpha_0=\alpha_1+2\alpha_2+\ldots+2\alpha_l$ (see [1], Planches) it is clear that for $\mu\in F(R)$ we must have $((2l+1)\mu|\alpha_i)=1$, $1\leqq i\leqq l$, which implies $(2l+1)\mu=\sigma$, or $((2l+1)\mu|\alpha_1)=2$ and $((2l+1)\mu|\alpha_i)=1$ for $2\leqq i\leqq l$, which implies $(2l+1)\mu=\sigma+\omega_1$ (since $\alpha_1^\vee=\alpha_1$).

Remark. The fact that there are two orbits in this case does not depend on the chosen normalisation of $(\;|\;)$. However, we did so in order to get

$$(\sigma|\sigma)\equiv(\sigma+\omega_1|\sigma+\omega_1)(\mathrm{mod}\,(2l+1)), \tag{3.13}$$

which will be used in Section 5.

§4. Skew-Invariant Spherical Polynomials

In this section all polynomials will be homogeneous. First we will consider some properties of skew-invariant polynomials and then we will determine for what degrees there are skew-invariant spherical polynomials. A crucial role is played by the polynomial $\prod_{\alpha\in R_+}(\xi|\alpha)$ which will be denoted shortly by $\boldsymbol{\Pi}$. We have

(4.1) **Theorem.** *Every skew-invariant polynomial is divisible by $\boldsymbol{\Pi}$.*

For a proof see for example [1], p. 185, Proposition 2 (ii).

Now we shall give a proof of formula (2.5) in the special case that $P=\boldsymbol{\Pi}$. The reason for doing so is that there is a kind of an analogue to the method used by Schoeneberg and Pfetzer. They proved formula (2.5) by applying one differential operator a number of times (see [9] and [10]), and we will prove it by applying several differential operators. For any indefinitely differentiable function $f: V\to\mathbb{C}$ and for any $\alpha\in R$ let

$$\mathscr{L}_\alpha f(\xi)=\lim_{t\to 0}\frac{f(\xi+t\alpha)-f(\xi)}{t}.$$

For all $\alpha,\beta\in R$ we have $\mathscr{L}_\alpha\mathscr{L}_\beta=\mathscr{L}_\beta\mathscr{L}_\alpha$,

$$\mathscr{L}_{-\alpha}=-\mathscr{L}_\alpha.$$

Now let $\mathscr{L}$ be the differential operator defined by

$$\mathscr{L}=\prod_{\alpha\in R_+}\mathscr{L}_\alpha .$$

(4.2) **Lemma.** *If $f: V\to\mathbb{C}$ is an indefinitely differentiable function, invariant under the action of $W(R)$, then $\mathscr{L}f(s\xi)=\det(s)\mathscr{L}f(\xi)$ for all $s\in W(R)$.*

Proof. Let $\alpha\in R$ be a simple root. Then

$$\mathscr{L}_\beta f(s_\alpha\xi)=\lim_{t\to 0}\frac{f(s_\alpha\xi+t\beta)-f(s_\alpha\xi)}{t}=\lim_{t\to 0}\frac{f(\xi+ts_\alpha\beta)-f(\xi)}{t}$$
$$=\mathscr{L}_{s_\alpha\beta}f(\xi).$$

Hence

$$\mathscr{L}f(s_\alpha\xi)=\Big(\prod_{\beta\in R_+}\mathscr{L}_{s_\alpha\beta}\Big)f(\xi)=-\Big(\prod_{\beta\in R_+}\mathscr{L}_\beta\Big)f(\xi)=-\mathscr{L}f(\xi),$$

because $s_\alpha\alpha=-\alpha$, and s_α permutes the other positive roots (see [1], p. 157, Corollary 1). This proves the lemma since $W(R)$ is generated by the s_α where α is a simple root.

(4.3) **Lemma.** *For any $\tau\in\mathbb{C}$ we have $\mathscr{L}(e^{2\pi i\tau q(\xi)})=(2\pi i\tau)^r\prod_{\alpha\in R_+}\langle\xi,\alpha\rangle e^{2\pi i\tau q(\xi)}$.*

Proof. $\mathscr{L}(e^{2\pi i\tau q(\xi)})=\Big[(2\pi i\tau)^r\prod_{\alpha\in R_+}\langle\xi,\alpha\rangle+g(\xi)\Big]e^{2\pi i\tau q(\xi)}$, where $g(\xi)$ is a polynomial function of degree $<r$. Both $\mathscr{L}(e^{2\pi i\tau q(\xi)})$ and $\prod_{\alpha\in R_+}\langle\xi,\alpha\rangle$ are skew-invariant under the action of $W(R)$, and therefore $g(\xi)$ must be skew-invariant too. By (4.1) $g(\xi)$ must be divisible by $\boldsymbol{\Pi}$, which is of degree r, and therefore $g(\xi)=0$, which proves the lemma.

Remark. $\prod_{\alpha\in R_+}\langle\xi,\alpha\rangle$ is equal to $\boldsymbol{\Pi}$ up to a positive constant.

Now consider the function $\Theta(\tau,L,\xi)=\sum_{\lambda\in L}e^{2\pi i\tau q(\lambda+\xi)}$, where $\tau\in\mathscr{H}$ and $\xi\in V$.

This function Θ is analytic in $\mathscr{H}$ and periodic on V, i.e. $\Theta(\tau,L,\xi+\lambda)=\Theta(\tau,L,\xi)$ for all $\lambda\in L$. Therefore we obtain its Fourier series and we have (see [8], p. VI–10, Proposition 23)

$$\Theta(\tau,L,\xi)=\frac{1}{v(L)}\left(\frac{\tau}{i}\right)^{-\frac{l}{2}}\sum_{\mu\in L^*}e^{-\frac{2\pi i}{\tau}q(\mu)+2\pi i\langle\xi,\mu\rangle}. \tag{4.4}$$

By applying the differential operator $\mathscr{L}$ to both sides of (4.4) we obtain

$$(2\pi i\tau)^r\sum_{\lambda\in L}\prod_{\alpha\in R_+}\langle\lambda+\xi,\alpha\rangle e^{2\pi i\tau q(\lambda+\xi)}$$
$$=\frac{i^{\frac{l}{2}}}{v(L)}\tau^{-\frac{l}{2}}(2\pi i)^r\sum_{\mu\in L^*}\prod_{\alpha\in R_+}\langle\mu,\alpha\rangle e^{-\frac{2\pi i}{\tau}q(\mu)+2\pi i\langle\mu,\xi\rangle},$$

from which (2.5) (with $P=\boldsymbol{\Pi}$) follows immediately.

We now return to the problem of finding skew-invariant spherical polynomials. Let S be the $\mathbb{C}$-algebra of homogeneous invariant polynomials. We have the following theorem, due originally to Chevalley (see [2]):

(4.5) **Theorem.** *S is a polynomial ring, i.e. $S=\mathbb{C}[I_1,\ldots,I_l]$, where $I_1,\ldots,I_l$ are algebraic independent invariant polynomials.*

For a proof see also [1], Chapter V, § 5.

We remark that the number of invariants in a basic set is equal to the rank of the root system R. Now the set $I_1,\ldots,I_l$ of basic polynomial invariants is not uniquely determined. However, their degrees $d_1,\ldots,d_l$ are uniquely determined (see [1], p. 103, corollaire). The degrees d_i are listed in Table 1.

Let S_n be the space of polynomials in S of degree n. From Table 1 it is very easy to determine the dimension of S_n, since this dimension is equal to the number of solutions $(a_1,\ldots,a_l)\in\mathbb{N}^l$ of $a_1d_1+\ldots+a_ld_l=n$.

Now let P be a skew-invariant polynomial, then by (4.1) P is of the form $P=J\Pi$ where $J\in S_n$ for some n. If we apply the differential operator Δ from (2.2) on P the result will again be a skew-invariant homogeneous polynomial ($\Delta P\circ s=\Delta(P\circ s)$ for any isometry s of V), so $\Delta P=J'\Pi$ for some $J'\in S_{n-2}$, provided $n\geqq 2$. The case $n=1$ is not very interesting because $S_1=\{0\}$, and for the case $n=0$ we get, since $\Delta\Pi$ must be skew-invariant, $\Delta\Pi=0$ (so Π is the most simple example of a non-trivial skew-invariant spherical function).

We are now going to determine the dimension of the kernel of $\Delta: S_n\Pi\to S_{n-2}\Pi$, $n\geqq 2$.

For all types there is a basic invariant of degree 2 (uniquely determined up to a constant multiple), and we denote

$$J_1(\xi)=(\xi|\xi),\quad \text{i.e.}\quad J_1(\xi)=x_1^2+\ldots+x_l^2\,,$$

whenever $\varepsilon_1,\ldots,\varepsilon_l$ is an orthonormal basis of V and $\xi=x_1\varepsilon_1+\ldots+x_l\varepsilon_l$.

Now it is clear that for any homogeneous polynomial P we have

$$(\xi|DP)=\deg(P)P\,, \tag{4.6}$$

where $DP=\left(\frac{\partial P}{\partial x_1},\ldots,\frac{\partial P}{\partial x_l}\right)$ is the gradient of P in the usual sense of calculus. Since $DJ_1=2\xi$ we get immediately for any polynomial P

$$\Delta(J_1P)=[2l+4\deg(P)]P+J_1(\Delta P)\,. \tag{4.7}$$

If $\Delta P=0$ then it is easy to prove by induction that for all $n\geqq 1$ we have

$$\Delta J_1^nP=[2nl+4n(n-1)+4n\deg(P)]J_1^{n-1}P\,. \tag{4.8}$$

We now prove

(4.9) **Proposition.** *The map $\Delta: S_n\Pi\to S_{n-2}\Pi$ is surjective for all $n\geqq 2$.*

Proof. Whenever $S_{n-2}=\{0\}$ there is nothing to prove. Suppose now that we can find a basis $P_1,\ldots,P_s$ of S_{n-2} such that $\Delta J_1P_i\Pi=c_iP_i\Pi$, $1\leqq i\leqq s$, and $c_i>0$.

Then we can find such a basis for S_n too. For first of all $J_1P_1,\ldots,J_1P_s\in S_n$ are linearly independent. Furthermore let $Q_1\Pi,\ldots,Q_t\Pi$ be a basis of $\ker(S_n\Pi\to S_{n-2}\Pi)$, so $Q_1, \ldots, Q_t \in S_n$. Since $\langle J_1 P_1 \Pi, \ldots, J_1 P_s \Pi \rangle \cap \ker(S_n \Pi \to S_{n-2} \Pi) = \{0\}$ and $\Delta: S_n \Pi \to S_{n-2} \Pi$ is surjective, $J_1 P_1, \ldots, J_1 P_s, Q_1, \ldots, Q_t$ must be a basis of S_n. For this basis we have

$$\Delta J_1(J_1 P_i \Pi) = [2l + 4 \deg(J_1 P_i \Pi) + c_i] J_1 P_i \Pi, \qquad 1 \leqq i \leqq s,$$

by (4.7), where $2l + 4\deg(J_1 P_i \Pi) + c_i > 0$, and

$$\Delta J_1(Q_i \Pi) = [2l + 4 \deg(Q_i \Pi)] Q_i \Pi, \qquad 1 \leqq i \leqq t,$$

by (4.8), where $2l + 4\deg(Q_i \Pi) > 0$.

For $n = 2$ we can find such a basis since $\Delta J_1 \Pi = (2l + 4\deg(\Pi))\Pi$. When all d_i are even we are ready by induction. Otherwise, let d_i be the lowest odd degree and let J_i be a basis of S_{d_i} (it is easy to see from Table 1 that $\dim S_{d_i} = 1$). Then we must have $\Delta J_i \Pi = 0$, and therefore $\Delta(J_1 J_i \Pi) = [2l + 4\deg(J_i \Pi)] J_i \Pi$, so in this case we are ready by induction too.

As an immediate consequence of (4.9) we get

(4.10) **Corollary.** *The dimension of the space of skew-invariant spherical functions of degree* $r+n$ *is equal to* $(\dim S_n - \dim S_{n-2})$.

§ 5. Macdonald's Identities

In this section proofs are given of the formulas (8.9) and (8.13) of [9], and of one other identity, mentioned in Appendix I of [7]. First we take $L = L(R^\vee)$, $\langle\ ,\ \rangle = k(\ |\ )$ where k is defined by (3.2), and $\xi = \frac{1}{k}\varrho$. We are going to compare $\Theta(\tau, L, \Pi, \xi)$ with $\eta(\tau)^d$. We have

$$\begin{aligned}\Theta(\tau+1, L, \Pi, \xi) &= e^{2\pi i q(\xi)} \Theta(\tau, L, \Pi, \xi) \\ &= e^{\frac{\pi i d}{12}} \Theta(\tau, L, \Pi, \xi) \quad \text{by} \quad (3.9). \end{aligned} \tag{5.1}$$

Furthermore it follows from (2.14) and (3.2) that for all $\mu \in L^*/L$ which do not belong to the orbit of ξ in L^*/L we have $\Theta(\tau, L, \Pi, \mu) = 0$, and therefore we get

$$\Theta\left(-\frac{1}{\tau}, L, \Pi, \xi\right) = c\tau^{\frac{d}{2}} \Theta(\tau, L, \Pi, \xi),$$

where $c = \dfrac{i^{\frac{l}{2}}}{v(L)}(-i)^d \sum_{s \in W(R)} \det(s) e^{2\pi i \langle s\xi, \xi \rangle}$.

Now from (3.7) and (3.8) it is easy to see that

$$\lim_{\tau \to i\infty} \frac{\Theta(\tau, L, \Pi, \xi)}{\eta(\tau)^d} = \Pi(\xi) \neq 0\,; \tag{5.3}$$

we even know that $\Pi(\xi) = \prod_{\alpha \in R_+} (\xi|\alpha) > 0$, since $(\varrho|\alpha) > 0$ for any positive root α.

In particular (5.3) implies that $\Theta(\tau, L, \boldsymbol{\Pi}, \xi)$ does not vanish identically, hence we can choose $\tau_0 \in \mathscr{H}$ such that $\Theta(\tau_0, L, \boldsymbol{\Pi}, \xi) \neq 0$. Then we have

$$\Theta\left(-\frac{1}{\tau_0}, L, \boldsymbol{\Pi}, \xi\right) = c\tau_0^{\frac{d}{2}} \Theta(\tau_0, L, \boldsymbol{\Pi}, \xi),$$

and

$$\Theta(\tau_0, L, \boldsymbol{\Pi}, \xi) = (-i)^{-d} c \tau_0^{-\frac{d}{2}} \Theta\left(-\frac{1}{\tau_0}, L, \boldsymbol{\Pi}, \xi\right),$$

and therefore $c^2 = (-i)^d$. Then we get

$$\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)^2 = (-i\tau)^d \Theta(\tau, L, \boldsymbol{\Pi}, \xi)^2,$$

and with (2.8), (2.9), (2.10), and (5.1) this shows that $\dfrac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)^2}{\eta(\tau)^{2d}}$ is a holomorphic function, invariant under the action of the full modular group Γ. By (5.3) this quotient is bounded, hence constant, and again by (5.3) we must have

$$\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\eta(\tau)^d} = \prod_{\alpha \in R_+} (\xi \mid \alpha).$$

So we have proved

(5.4) **Theorem** (Macdonald). $\displaystyle\sum_{\lambda \in L(R^\vee)} \prod_{\alpha \in R_+} \frac{(k\lambda + \varrho \mid \alpha)}{(\varrho \mid \alpha)} \; e^{2\pi i \tau \Phi_R(k\lambda + \varrho, k\lambda + \varrho)} = \eta(\tau)^d.$

From (2.9) we can now determine the constant c, and we get $c = (-i)^{\frac{d}{2}}$. Hence

(5.5) **Corollary.** $\displaystyle\sum_{s \in W(R)} \det(s) e^{2\pi i \frac{(s\varrho \mid \varrho)}{k}} = i^r k^{\frac{l}{2}} (\det((a_i^\vee \mid \alpha_j^\vee)))^{\frac{1}{2}}.$

Next we take $L = L(R)$, $\langle \, , \rangle = h(\mid)$, where h is the Coxeter number of R, and $\xi = \frac{1}{h}\varrho$. We want to prove formula (8.13) of [7], so we want to compare $\Theta(\tau, L, \boldsymbol{\Pi}, \xi)$ with $\left(\prod_{i=1}^{l} \eta\left(\frac{(\alpha_i \mid \alpha_i)}{2}\tau\right)\right)^{h+1}$; let us call this last function $\chi(\tau)$.

Let p denote $\frac{(\beta \mid \beta)}{(\alpha \mid \alpha)}$ where β is a long root and α a short one, so $p = 1$ if R is of type A_l, D_l, E_6, E_7 or E_8, $p = 2$ if R is of type B_l, C_l or F_4 and $p = 3$ if R is of type G_2. From our normalisation of $(\mid)$ (which was suggested by W. L. J. van der Kallen) it follows that

$$pL(R^\vee) \subset L = L(R) \subset L(R^\vee). \tag{5.6}$$

Moreover it is easy to check that h is divisible by p. When $p = 1$, i.e. when all the roots of R have the same length, we have $L(R) = L(R^\vee)$, $k = h$, $\chi(\tau) = \eta(\tau)^d$, and in this case we get (5.4) again. In the cases that $p = 2$ or 3 the function $\chi(\tau)$ is a product of powers of $\eta(\tau)$ and $\eta(2\tau)$, or $\eta(\tau)$ and $\eta(3\tau)$ respectively and in those cases we should not consider the transformations from the full modular group. Instead, we are going to consider the subgroup $\Gamma_0(p)$. First of all we have

$$\left.\begin{aligned}\Theta(\tau+1,L,\boldsymbol{\Pi},\xi)&=e^{2\pi i q(\xi)}\Theta(\tau,L,\boldsymbol{\Pi},\xi)=e^{2\pi i\frac{(\varrho|\varrho)}{2h}}\Theta(\tau,L,\boldsymbol{\Pi},\xi)\,,\\ \chi(\tau+1)&=e^{2\pi i\frac{(\varrho|\varrho)}{2h}}\chi(\tau)\quad\text{by (2.8) and (3.10)}\,.\end{aligned}\right\}\tag{5.7}$$

As for the transformation $\tau\mapsto\dfrac{-\tau}{p\tau-1}$ we have

$$\begin{aligned}\Theta\left(\frac{-\tau}{p\tau-1},L,\boldsymbol{\Pi},\xi\right)&=\frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{p\tau-1}{\tau}\right)^{\frac{d}{2}}\sum_{\mu\in L^*}\prod_{\alpha\in R_+}(\mu|\alpha)e^{2\pi i\left(\frac{p\tau-1}{\tau}\right)q(\mu)+2\pi i\langle\mu,\xi\rangle}\\ &=\frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{p\tau-1}{\tau}\right)^{\frac{d}{2}}\sum_{\mu\in L^*/L(R^\vee)}\sum_{\lambda\in L(R^\vee)}\prod_{\alpha\in R_+}(\mu+\lambda|\alpha)\\ &\quad e^{-\frac{2\pi i}{\tau}q(\mu+\lambda)+2\pi i[pq(\mu+\lambda)+\langle\mu+\lambda,\xi\rangle]}\end{aligned}\tag{5.8}$$

Now we have $pq(\mu+\lambda)=pq(\mu)+pq(\lambda)+p\langle\mu,\lambda\rangle$

$$\begin{aligned}&=pq(\mu)+\frac{1}{2}\frac{h}{p}(p\lambda|p\lambda)+\langle\mu,p\lambda\rangle\\ &\equiv pq(\mu)(\operatorname{mod}\mathbb{Z})\ \text{ by (5.6)}\,.\end{aligned}$$

Besides $\langle\mu+\lambda,\xi\rangle\equiv\langle\mu,\xi\rangle\,(\operatorname{mod}\mathbb{Z})$ for all $\lambda\in L(R^\vee)$, since $\xi\in L(R^\vee)^*$.

Therefore (5.8) takes the following form

$$\begin{aligned}&\Theta\left(\frac{-\tau}{p\tau-1},L,\boldsymbol{\Pi},\xi\right)\\ &=\frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{p\tau-1}{\tau}\right)^{\frac{d}{2}}\sum_{\mu\in L^*/L(R^\vee)}e^{2\pi i[pq(\mu)+\langle\mu,\xi\rangle]}\Theta\left(-\frac{1}{\tau},L(R^\vee),\boldsymbol{\Pi},\mu\right).\end{aligned}$$

Then by (2.13), (2.14), and (3.6) we get

$$\begin{aligned}&\Theta\left(\frac{-\tau}{p\tau-1},L,\boldsymbol{\Pi},\xi\right)\\ &=\frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{p\tau-1}{\tau}\right)^{\frac{d}{2}}e^{2\pi i p q\left(\frac{1}{h}\sigma\right)}\left(\sum_{s\in W(R)}\det(s)e^{2\pi i\left\langle s\frac{1}{h}\sigma,\xi\right\rangle}\right)\\ &\quad\cdot\Theta\left(-\frac{1}{\tau},L(R^\vee),\boldsymbol{\Pi},\frac{1}{h}\sigma\right)\\ &=(-i)^d(p\tau-1)^{\frac{d}{2}}e^{2\pi i p q\left(\frac{1}{h}\sigma\right)}\Theta(\tau,L,\boldsymbol{\Pi},\xi)\,.\end{aligned}\tag{5.9}$$

On the other hand we have

$$\eta\left(\frac{(\alpha_j|\alpha_j)}{2}\,\frac{-\tau}{p\tau-1}\right)=(-i)(p\tau-1)^{\frac{1}{2}}e^{\frac{2\pi i}{24}\frac{2p}{(\alpha_j|\alpha_j)}}\eta\left(\frac{(\alpha_j|\alpha_j)}{2}\tau\right),$$

by (2.8) and (2.9), and therefore

$$\chi\left(\frac{-\tau}{p\tau-1}\right)=(-i)^d(p\tau-1)^{\frac{d}{2}}e^{2\pi i p q\left(\frac{1}{h}\sigma\right)}\chi(\tau), \tag{5.10}$$

by (3.11) and the fact that $l(h+1)=d$.

From (5.7), (5.9), and (5.10) we see that $\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}$ is invariant under all transformations from $\Gamma_0(p)$. Again we have to take care of the parabolic vertices, which were determined in (2.12). From the definition of $\Theta(\tau, L, \boldsymbol{\Pi}, \xi)$, (3.7) and (3.8) we get easily that

$$\lim_{\tau\to i\infty}\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}=\prod_{\alpha\in R_+}(\xi|\alpha). \tag{5.11}$$

Furthermore we can write

$$\lim_{\tau\to 0}\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}=\lim_{\tau\to i\infty}\frac{\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)}{\chi\left(-\frac{1}{\tau}\right)},$$

and since $\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)=c\tau^{\frac{d}{2}}\Theta\left(\tau, L(R^\vee), \boldsymbol{\Pi}, \frac{1}{h}\sigma\right)$ for some constant c, and $\chi\left(-\frac{1}{\tau}\right)=c'\tau^{\frac{d}{2}}\left(\prod_{j=1}^{l}\eta\left(\frac{2}{(\alpha_j|\alpha_j)}\tau\right)\right)^{h+1}$ for some constant $c'\neq 0$, we get by (3.7), (3.8), and (3.11) that $\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}$ is regular at $\tau=0$ too $\left[\text{as for (3.8) (ii): } \frac{1}{h}\sigma\in F(R)\right]$. Hence this quotient must be constant and by (5.11) equal to $\prod_{\alpha\in R_+}(\xi|\alpha)$. So we have proved

(5.12) **Theorem** (Macdonald).

$$\sum_{\lambda\in L(R)}\prod_{\alpha\in R_+}\frac{(h\lambda+\varrho|\alpha)}{(\varrho|\alpha)}\, e^{2\pi i\tau\frac{(h\lambda+\varrho|h\lambda+\varrho)}{2h}}=\left(\prod_{j=1}^{l}\eta\left(\frac{(\alpha_j|\alpha_j)}{2}\tau\right)\right)^{h+1}.$$

For the transformation $\tau\mapsto -\frac{1}{\tau}$ we get

$$\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)=c'\tau^{\frac{d}{2}}\prod_{\alpha\in R_+}(\xi|\alpha)\left(\prod_{j=1}^{l}\eta\left(\frac{2}{(\alpha_j|\alpha_j)}\tau\right)\right)^{h+1}.$$

On the other hand we have

$$\begin{aligned}\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)&=c\tau^{\frac{d}{2}}\Theta\left(\tau, L(R^\vee), \boldsymbol{\Pi}, \frac{1}{h}\sigma\right)\\ &=c\tau^{\frac{d}{2}}\prod_{\alpha\in R_+}\left(\frac{1}{h}\sigma|\alpha^\vee\right)\left(\prod_{j=1}^{l}\eta\left(\frac{(\alpha_j^\vee|\alpha_j^\vee)}{2}\tau\right)\right)^{h+1} \qquad \text{by (5.12)}\\ &=c\tau^{\frac{d}{2}}\prod_{\alpha\in R_+}\left(\frac{1}{h}\sigma|\alpha^\vee\right)\left(\prod_{j=1}^{l}\eta\left(\frac{2}{(\alpha_j|\alpha_j)}\tau\right)\right)^{h+1}.\end{aligned}$$

Hence $c' \prod_{\alpha\in R_+} (\xi|\alpha) = c \prod_{\alpha\in R_+} \left(\frac{1}{h}\sigma|\alpha^\vee\right)$.

Now $c = \frac{i^{\frac{l}{2}}}{v(L)}(-i)^d \sum_{s\in W(R)} \det(s) e^{2\pi i\langle s\frac{1}{h}\sigma,\xi\rangle}$, and $c' = (-i)^{\frac{d}{2}}\left(\prod_{j=1}^{l} \frac{2}{(\alpha_j|\alpha_j)}\right)^{\frac{h+1}{2}}$.

Therefore we get the following

(5.13) **Corollary.**

$$\sum_{s\in W(R)} \det(s) e^{2\pi i\frac{(s\sigma|\varrho)}{h}} = i^r h^{\frac{l}{2}}\left(\prod_{\alpha\in R_+} \frac{(\varrho|\alpha)}{(\sigma|\alpha^\vee)}\right)\left(\prod_{j=1}^{l}\frac{2}{(\alpha_j|\alpha_j)}\right)^{\frac{h+1}{2}} (\det((\alpha_i|\alpha_j)))^{\frac{1}{2}}.$$

If $p=1$ this is just (5.5) again.

Now let R be of type $B_l (l\geqq 2)$ and let $(\ |\ )$ be normalized as in (3.12).

Let $\langle\ ,\ \rangle = (2l+1)(\ |\ )$, $\varrho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$, $\sigma = \frac{1}{2}\sum_{\alpha\in R_+}\alpha^\vee$, $\xi = \frac{1}{2l+1}\varrho$. We take $L = L(R)$.

We want to compare $\Theta(\tau, L, \Pi, \xi)$ with $\chi(\tau) = \left(\frac{\eta(\tau)^{2l+3}}{\eta(2\tau)^2}\right)^l$. In this special case the situation is different from the one considered so far. Besides the fact that the normalisation of $(\ |\ )$ is not the same as before, we have neither $q(L)\subset\mathbb{Z}$ nor $\xi\in L^*$. However, we still can use formula (2.5), and since $2q(L)\subset\mathbb{Z}$ as well as $2\xi\in L^*$, we will consider what happens under the transformations $\tau\mapsto\tau+2$ and $\tau\mapsto\frac{-\tau}{2\tau-1}$ which generate $\Gamma(2)$.

It can be easily seen that both $\Theta(\tau, L, \Pi, \xi)$ and $\chi(\tau)$ are transformed in the same way under $\tau\mapsto\tau+2$.

So let us consider $\tau\mapsto\frac{-\tau}{2\tau-1}$. We have

$$\Theta\left(\frac{-\tau}{2\tau-1}, L, \Pi, \xi\right)$$

$$= \frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{2\tau-1}{\tau}\right)^{\frac{d}{2}} \sum_{\mu\in L^*}\prod_{\alpha\in R_+}(\mu|\alpha) e^{2\pi i\left(\frac{2\tau-1}{\tau}\right)q(\mu)+2\pi i\langle\mu,\xi\rangle}$$

$$= \frac{i^{\frac{l}{2}}}{v(L)}(-i)^d\left(\frac{2\tau-1}{\tau}\right)^{\frac{d}{2}} \sum_{\mu\in L^*/L(R^\vee)} e^{2\pi i[\langle\mu,\mu\rangle+\langle\mu,\xi\rangle]}\,\Theta\left(-\frac{1}{\tau}, L(R^\vee), \Pi, \mu\right).$$

By (3.12) we need only to consider the orbits of $\mu_1 = \frac{1}{2l+1}\sigma$ and

$$\mu_2 = \frac{1}{2l+1}(\sigma+\omega_1).$$

We have $e^{2\pi i\langle\mu_1,\mu_1\rangle}=e^{2\pi i\langle\mu_2,\mu_2\rangle}$ by (3.13), and therefore we get

$$\left.\begin{aligned}\Theta\left(\frac{-\tau}{2\tau-1}, L, \boldsymbol{\Pi}, \xi\right)&=(2\tau-1)^{\frac{d}{2}}e^{2\pi i\frac{2l(2l+2)}{24}}\Theta(\tau, L, \boldsymbol{\Pi}, \xi),\\ \chi\left(\frac{-\tau}{2\tau-1}\right)&=(2\tau-1)^{\frac{d}{2}}e^{2\pi i\frac{2l(2l+2)}{24}}\chi(\tau).\end{aligned}\right\} \tag{5.14}$$

We conclude that $\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}$ is invariant under all transformations from $\Gamma(2)$. We have to take care of the parabolic vertices of $\Gamma(2)$ which were determined in (2.12). It is obvious that $\xi\in F(R^\vee)$, and therefore we get by (3.7):

$$\lim_{\tau\to i\infty}\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}=\prod_{\alpha\in R_+}(\xi|\alpha). \tag{5.15}$$

As before we can write $\lim_{\tau\to 0}\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}=\lim_{\tau\to i\infty}\frac{\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)}{\chi\left(-\frac{1}{\tau}\right)}$.

We have

$$\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)=c_1\tau^{\frac{d}{2}}\Theta(\tau, L(R^\vee), \boldsymbol{\Pi}, \mu_1)+c_2\tau^{\frac{d}{2}}\Theta(\tau, L(R^\vee), \boldsymbol{\Pi}, \mu_2),$$

for some constants c_1, c_2. Since $\mu_1, \mu_2\in F(R)$ we find by (3.7) that both

$$\frac{\tau^{\frac{d}{2}}\Theta(\tau, L(R^\vee), \boldsymbol{\Pi}, \mu_1)}{\chi\left(-\frac{1}{\tau}\right)} \text{ and } \frac{\tau^{\frac{d}{2}}\Theta(\tau, L(R^\vee), \boldsymbol{\Pi}, \mu_2)}{\chi\left(-\frac{1}{\tau}\right)}$$

are regular at $\tau=i\infty$, and therefore $\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}$ is regular at $\tau=0$.

In the same way we get that $\frac{\Theta(\tau, L, \boldsymbol{\Pi}, \xi)}{\chi(\tau)}$ is regular at $\tau=1$ too. Hence this quotient must be constant, and by (5.15) equal to $\prod_{\alpha\in R_+}(\xi|\alpha)$. So we have proved:

(5.16) **Theorem** (Macdonald). *If R is of type $B_l (l\geq 2)$ then*

$$\sum_{\lambda\in L(R)}\prod_{\alpha\in R_+}\frac{((2l+1)\lambda+\varrho|\alpha)}{(\varrho|\alpha)}e^{2\pi i\tau\frac{((2l+1)\lambda+\varrho|(2l+1)\lambda+\varrho)}{2(2l+1)}}=\left(\frac{\eta(\tau)^{2l+3}}{\eta(2\tau)^2}\right)^l.$$

Finally we consider the case that R is of type E_8 again. We take $L=L(R^\vee)$, $\langle,\rangle=31(\ |\ )$ and $\xi=\frac{1}{31}\varrho$. We know by (3.9) that $\frac{(\varrho|\varrho)}{2\times 30}=\frac{248}{24}$, hence $\frac{(\varrho|\varrho)}{2\times 31}=10$. So we get $\Theta(\tau+1, L, \boldsymbol{\Pi}, \xi)=\Theta(\tau, L, \boldsymbol{\Pi}, \xi)$. By (3.4) we find that

$$\Theta\left(-\frac{1}{\tau}, L, \boldsymbol{\Pi}, \xi\right)=c\tau^{124}\Theta(\tau, L, \boldsymbol{\Pi}, \xi),$$

where $c=\frac{1}{v(L)}\sum_{s\in W(R)}\det(s)e^{2\pi i\frac{(s\varrho|\varrho)}{31}}$. By a calculation as used in the proof of (5.4) we get the following formula

$$\sum_{\lambda\in L(R^{\vee})}\prod_{\alpha\in R_+}\frac{(31\lambda+\varrho|\alpha)}{(\varrho|\alpha)}e^{2\pi i\tau\frac{(31\lambda+\varrho|31\lambda+\varrho)}{62}}=\eta(\tau)^{240}\mathscr{E}_2(\tau). \tag{5.17}$$

§6. Some Other Identities

In this section we will study $\Theta(\tau, L, P, \xi)$ where $L=L(R^{\vee})$, $q(v)=\frac{1}{2}k(v|v)$ where k is the number defined in (3.2), and $\xi=\frac{1}{k}\varrho$. The polynomial P is a skew-invariant spherical function, and as we have seen, P can be written as $P=J\Pi$ for some invariant polynomial J. We define

$$\Phi(\tau, J)=\frac{\Theta(\tau, L, P, \xi)}{\eta(\tau)^d} \qquad (\xi \text{ and the lattice } L \text{ are fixed}).$$

Then we have the following transformation formulas for this function $\Phi(\tau, J)$:

$$\Phi(\tau+1, J)=\Phi(\tau, J), \tag{6.1}$$

by (2.3) and (3.9), and

$$\Phi\left(-\frac{1}{\tau}, J\right)=(-1)^{\deg(J)}\tau^{\deg(J)}\Phi(\tau, J). \tag{6.2}$$

This last formula is an immediate consequence of (2.4), (3.2), and (5.5). The function $\Phi(\tau, J)$ is holomorphic; it is holomorphic at $\tau=i\infty$ too; by (3.7), (3.8), and (3.9) we have

$$\lim_{\tau\to i\infty}\Phi(\tau, J)=J(\xi)\Pi(\xi). \tag{6.3}$$

From (6.2) it is easy to see that whenever $\deg(J)$ is odd, which can only occur if R is of type $A_l (l\geqq 2)$, $D_l (l$ odd) or E_6, we must have $\Phi(\tau, J)=0$, and so we get

(6.4) **Lemma.** *If $P=J\Pi$ is a skew-invariant spherical function such that* $\deg(J)$ *is odd, then we have $\Theta(\tau, L, P, \xi)=0$.*

From now on we assume that $\deg(J)$ is even. It follows from (6.1), (6.2), and (6.3) that $\Phi(\tau, J)$ is a modular form for Γ of weight $\deg(J)$.

We conclude from (2.11) that for low degrees of J, i.e. $\deg(J)=4, 6, 8$ or 10, $\Phi(\tau, J)$ must be a multiple of an Eisenstein series. We will consider the cases $\deg(J)=4$ or 6 and after that we will deal with a few special cases where J is of higher degree. We will make explicit calculations and to that end we will use the following descriptions of the various root systems.

Let $\mathbb{E}_l$ be a Euclidean space of dimension l, and let $\varepsilon_1,\dots,\varepsilon_l$ be an orthonormal basis of $\mathbb{E}_l$. For all root systems we take $J_1(x_1,\dots,x_l)=x_1^2+\dots+x_l^2$ as in Section 4, the basic invariant of degree 2.

(6.5) Type $A_l (l \geqq 1)$. Let $V \subset \mathbb{E}_{l+1}$ be the subspace of vectors $\sum_{j=1}^{l+1} x_j \varepsilon_j$ such that $\sum_{j=1}^{l+1} x_j = 0$. Then a set of positive roots is given by $\{\varepsilon_i - \varepsilon_j | 1 \leqq i < j \leqq l+1\}$. A set of algebraically independent invariant polynomials is given by $J_1, J_2, \ldots, J_l$ where J_i is defined by $J_i(x_1, \ldots, x_l) = \dfrac{1}{(l-i+1)!\,i!} \sum_{\phi \in \mathfrak{S}_{l+1}} x_{\phi(1)} \ldots x_{\phi(i)}$ for $2 \leqq i \leqq l$, and where $x_{l+1} = -(x_1 + \ldots + x_l)$. We write the elements of V as $x_1\varepsilon_1 + \ldots + x_l\varepsilon_l + \left(-\sum_{i=1}^{l} x_i\right)\varepsilon_{l+1} = x_1(\varepsilon_1 - \varepsilon_{l+1}) + \ldots + x_l(\varepsilon_l - \varepsilon_{l+1})$.

The Gramian matrix of the basis $\varepsilon_1 - \varepsilon_{l+1}, \ldots, \varepsilon_l - \varepsilon_{l+1}$ of V is

$$\begin{pmatrix} 2 & 1 & & 1 \\ 1 & 2 & & \\ & & \ddots & \\ & & \ddots & 1 \\ 1 & & 1 & 2 \end{pmatrix}. \text{ Its inverse is } \frac{1}{l+1}\begin{pmatrix} l & -1 & & -1 \\ -1 & l & & \\ & & \ddots & \\ & & \ddots & -1 \\ -1 & & -1 & l \end{pmatrix},$$

and therefore Δ takes the form

$$\Delta = \frac{l}{l+1} \sum_{i=1}^{l} \frac{\partial^2}{\partial x_i^2} - \frac{2}{l+1} \sum_{1 \leqq i < j \leqq l} \frac{\partial^2}{\partial x_i \partial x_j}.$$

(6.6) Type $B_l (l \geqq 2)$. $V = \mathbb{E}_l$, and a set of positive roots is given by

$$\{\sqrt{2}(\varepsilon_i \pm \varepsilon_j) | 1 \leqq i < j \leqq l\} \cup \{\sqrt{2}\varepsilon_i | 1 \leqq i \leqq l\}$$

(we recall our normalisation). A set of basic polynomial invariants is given by $J_1, J_2, \ldots, J_l$, where J_i is defined by $J_i(x_1, \ldots, x_l) = \dfrac{1}{(l-i+1)!\,i!} \sum_{\phi \in \mathfrak{S}_l} x_{\phi(1)}^2 \ldots x_{\phi(i)}^2$ for $2 \leqq i \leqq l$. We write the elements of V as $x_1\varepsilon_1 + \ldots + x_l\varepsilon_l$ and then Δ is

$$\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_l^2}.$$

(6.7) Type $C_l (l \geqq 2)$. $V = \mathbb{E}_l$. In this case we have as a set of positive roots $\{\varepsilon_i \pm \varepsilon_j | 1 \leqq i < j \leqq l\} \cup \{2\varepsilon_i | 1 \leqq i \leqq l\}$. We take as a set of polynomial invariants $J_1, \ldots, J_l$, defined in (6.6), and Δ takes the same form as in (6.6) too.

(6.8) Type $D_l (l \geqq 3)$. $V = \mathbb{E}_l$, and a set of positive roots is given by

$$\{\varepsilon_i \pm \varepsilon_j | 1 \leqq i < j \leqq l\}.$$

As a set of polynomial invariants we take $J_1, J_2, \ldots, J_l$, where $J_2, \ldots, J_{l-1}$ are the polynomials defined in (6.6), and $J_l(x_1, \ldots, x_l) = x_1 \ldots x_l$. Again $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \ldots + \dfrac{\partial^2}{\partial x_l^2}$.

(6.9) Type F_4. $V = \mathbb{E}_4$, and a set of positive roots is given by $\sqrt{2}\varepsilon_i$, $1 \leqq i \leqq 4$, $\sqrt{2}(\varepsilon_i \pm \varepsilon_j)$, $1 \leqq i < j \leqq 4$, $\frac{1}{2}\sqrt{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. In this case we do not know a complete set of basic polynomial invariants[1], but since we will consider polynomials of degree 6 only in this case we tried to find a basic invariant of degree 6. We did so following a method of Flatto (see [3]). We got

$$J_2(x_1,x_2,x_3,x_4)=16(x_1^6+x_2^6+x_3^6+x_4^6)+5(x_1^2+x_2^2+x_3^2+x_4^2)^3 \\ -20(x_1^2+x_2^2+x_3^2+x_4^2)(x_1^4+x_2^4+x_3^4+x_4^4)\,.$$

(J_2 is constructed such that J_2 is invariant, and $\Delta J_2=0$) $\Delta=\dfrac{\partial^2}{\partial x_1^2}+\ldots+\dfrac{\partial^2}{\partial x_4^2}$.

(6.10) Type E_6. In this case we use a rather unusual description, given by Frame (see [4]; it may be readily seen that the reflection group considered there is the Weyl group of type E_6). We take $V=\mathbb{E}_6$. A full set of roots is given by

$$\pm\frac{1}{3}\sqrt{6}\left(\cos\frac{2\pi a}{3}\varepsilon_1+\sin\frac{2\pi a}{3}\varepsilon_2+\cos\frac{2\pi b}{3}\varepsilon_3+\sin\frac{2\pi b}{3}\varepsilon_4\right.$$

$$\left.+\cos\frac{2\pi c}{3}\varepsilon_5+\sin\frac{2\pi c}{3}\varepsilon_6\right),\qquad 1\leqq a,b,c\leqq 3\,,$$

$$\pm\left(-\sin\frac{2\pi a}{3}\varepsilon_1+\cos\frac{2\pi a}{3}\varepsilon_2\right),\qquad 1\leqq a\leqq 3\,,$$

$$\pm\left(-\sin\frac{2\pi b}{3}\varepsilon_3+\cos\frac{2\pi b}{3}\varepsilon_4\right),\qquad 1\leqq b\leqq 3\,,$$

$$\pm\left(-\sin\frac{2\pi c}{3}\varepsilon_5+\cos\frac{2\pi c}{3}\varepsilon_6\right),\qquad 1\leqq c\leqq 3\,.$$

We take as positive roots those roots whose scalar product with the vector $6\varepsilon_1+5\varepsilon_2+4\varepsilon_3+3\varepsilon_4+2\varepsilon_5+\varepsilon_6$ is positive. We write the elements of V as $x_1\varepsilon_1+y_1\varepsilon_2+x_2\varepsilon_3+y_2\varepsilon_4+x_3\varepsilon_5+y_3\varepsilon_6$, and furthermore $p_i=x_i^2+y_i^2$, $1\leqq i\leqq 3$, $q_i=\frac{1}{3}x_i^3-x_iy_i^2$, $1\leqq i\leqq 3$. Then a complete set of basic invariants in terms of p_i and q_i is given by Frame. We will only use J_1 and J_2:

$$J_1=p_1+p_2+p_3\,.$$

$$J_2=p_1^2(p_2+p_3)+p_2^2(p_1+p_3)+p_3^2(p_1+p_2)-3p_1p_2p_3+q_1^2+q_2^2+q_3^2 \\ -5\{q_1(q_2+q_3)+q_2(q_1+q_3)+q_3(q_1+q_2)\}\,.$$

$$\Delta=\sum_{i=1}^{3}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}\right).$$

(6.11) Type E_7. Let $V\subset\mathbb{E}_8$ be the subspace of vectors $\sum_{i=1}^{8}x_i\varepsilon_i$ such that $x_7+x_8=0$. A set of positive roots is given by $\pm\varepsilon_i+\varepsilon_j$, $1\leqq i<j\leqq 6$, $\varepsilon_8-\varepsilon_7$, $\frac{1}{2}\left(\varepsilon_8-\varepsilon_7+\sum_{i=1}^{6}(-1)^{\nu(i)}\varepsilon_i\right)$ with $\sum_{i=1}^{6}\nu(i)$ odd. Again we do not know a complete set of basic invariants, but we only need an invariant of degree 6. If we put $J_2(\xi)=$

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[1] Quite recently Professor Coxeter informed me that a complete set can be found on p. 179 of his Regular Complex Polytopes (Cambridge, 1974)

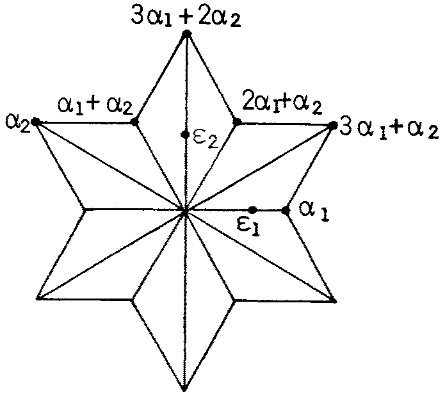

Fig. 1

$\sum_{\alpha\in R_+} (\xi|\alpha)^6$ we get an invariant of degree 6, which appears to be linearly independent of J_1^3, and so it must be algebraically independent of J_1 too, so J_2 is a basic invariant of degree 6. We write the elements of V as

$$x_1\varepsilon_1+\ldots+x_7\varepsilon_7+(-x_7)\varepsilon_8=x_1\varepsilon_1+\ldots+x_6\varepsilon_6+x_7(\varepsilon_7-\varepsilon_8).$$

The Gramian matrix of the basis $\varepsilon_1,\ldots,\varepsilon_6,\varepsilon_7-\varepsilon_8$ of V is

$$\begin{pmatrix}1&&&&&&\\&\cdot&&&&&\\&&\cdot&&&&\\&&&\cdot&&&\\&&&&\cdot&&\\&&&&&1&\\&&&&&&2\end{pmatrix},\quad \text{and its inverse is}\quad \begin{pmatrix}1&&&&&&\\&\cdot&&&&&\\&&\cdot&&&&\\&&&\cdot&&&\\&&&&\cdot&&\\&&&&&1&\\&&&&&&\frac{1}{2}\end{pmatrix},$$

and therefore Δ takes the form $\Delta=\dfrac{\partial^2}{\partial x_1^2}+\ldots+\dfrac{\partial^2}{\partial x_6^2}+\frac{1}{2}\dfrac{\partial^2}{\partial x_7^2}$.

(6.12) Type G_2. Let $V=\mathbb{E}_2$. A set of positive roots is given in Figure 1.

Here $\alpha_1=\sqrt{2}\varepsilon_1$ and $\alpha_2=-\frac{3}{2}\sqrt{2}\varepsilon_1+\frac{1}{2}\sqrt{6}\varepsilon_2$. As in (6.9) we can construct a basic invariant of degree 6, and we get $J_2(x_1,x_2)=x_1^6-x_2^6-15(x_1^4x_2^2-x_1^2x_2^4)$. In this case $\Delta=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$.

The case that R is of type E_8 is omitted since $\deg(J)=4$ or 6 cannot occur when R is of this type.

Let us now consider the functions $\Phi(\tau,J)$ where $\deg(J)=4$. By (4.10) we know that $\ker(S_4\Pi\to S_2\Pi)\neq 0$ if and only if 4 is degree of a basic invariant. We get the following

(6.13) **Theorem.** *If R is of type $A_l(l \geqq 3)$, $B_l(l \geqq 2)$, $C_l(l \geqq 2)$ or $D_l(l \geqq 5)$, there exists an invariant polynomial J of degree 4 such that*

(i) *$J\Pi$ is a spherical function,*

(ii) $$\sum_{\lambda \in L(R^v)} \frac{J(\lambda+\xi)\Pi(\lambda+\xi)}{J(\xi)\Pi(\xi)} e^{2\pi i \tau \Phi_R(k\lambda+\varrho, k\lambda+\varrho)} = \eta(\tau)^d \mathscr{E}_2(\tau).$$

In detail:

$$J = J_1^2 - \frac{8(l+1)(l^2+2l+2)}{(l-1)(l-2)(l+2)} J_3 \quad \text{if } R \text{ is of type } A_l(l \geqq 3),$$

$$= J_1^2 - \frac{4l^2+2l+4}{(2l-1)(l-1)} J_2 \quad \text{if } R \text{ is of type } B_l \text{ or } C_l(l \geqq 2),$$

$$= J_1^2 - \frac{4l^2-2l+4}{(l-1)(2l-3)} J_2 \quad \text{if } R \text{ is of type } D_l(l \geqq 5).$$

Proof. First of all we have $\Delta J_1^2\Pi = (4l+8+8\deg(\Pi))J_1\Pi$ by (4.8). Then we know that we must have $\Delta J_3\Pi = aJ_1\Pi$ if R is of type A_l and $\Delta J_2\Pi = aJ_1\Pi$ in the other cases, for some constant a. By using the description of Δ given in (6.5), (6.6), (6.7), and (6.8), a straigthforward calculation (for instance by looking at the highest powers of x_1 that can occur) shows that

$$a = \tfrac{1}{2}(l-1)(l-2)(l+2)/(l+1) \quad \text{if } R \text{ is of type } A_l,$$

$$= 2(2l-1)(l-1) \quad \text{if } R \text{ is of type } B_l \text{ or } C_l,$$

$$= 2(l-1)(2l-3) \quad \text{if } R \text{ is of type } D_l.$$

From this it is easy to see that the polynomials J, stated in (6.13), are such that $J\Pi$ generates $\ker(S_4\Pi \to S_2\Pi)$. Then from (6.1), (6.2), and (6.3) we see that $\Phi(\tau, J) = J(\xi)\Pi(\xi)\mathscr{E}_2(\tau)$. So it only remains to be checked that $J(\xi) \neq 0$. By calculation we find

$$J(\varrho) = -\frac{1}{360} l(l+1)^2(l+3)(l+4) \quad \text{if } R \text{ is of type } A_l,$$

$$= -\frac{1}{90} l(2l+1)(l+1)(2l-7)(2l+3) \quad \text{if } R \text{ is of type } B_l,$$

$$= -\frac{1}{180} l(l+1)(2l+1)(2l+3)(l+4) \quad \text{if } R \text{ is of type } C_l,$$

$$= -\frac{1}{180} l(2l-1)(l-4)(2l+1)(l+1) \quad \text{if } R \text{ is of type } D_l.$$

So we see that $J(\varrho) \neq 0$ (we assumed $l > 4$ if R is of type D_l), hence $J(\xi) \neq 0$ because J is a homogeneous polynomial, which proves the theorem.

Remark. If R is of type D_4, the polynomial J constructed above is such that $J(\xi) = 0$. However, in this case $\ker(S_4\Pi \to S_2\Pi)$ is 2-dimensional; we have $\Delta J_4\Pi = 0$

[we recall that $J_4(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4$], and so $J_4\Pi$ is another generator of $\ker(S_4\Pi \to S_2\Pi)$. But we also have $J_4(\xi)=0$, and therefore we get $\Theta(\tau, L, J\Pi, \xi)=0$ for all $J\Pi \in \ker(S_4\Pi \to S_2\Pi)$.

Next let us consider the functions $\Phi(\tau, J)$ where $\deg(J)=6$. Analogous to (6.13) we can prove

(6.14) **Theorem.** *If R is of type $A_l (l \geqq 2)$, $B_l (l \geqq 3)$, $C_l (l \geqq 3)$, $D_l (l \geqq 4)$, E_6, E_7, F_4 or G_2, there exists an invariant polynomial of degree 6 such that*

(i) *$J\Pi$ is a spherical function,*

(ii) $$\sum_{\lambda \in L(R^\vee)} \frac{J(\lambda+\xi)\Pi(\lambda+\xi)}{J(\xi)\Pi(\xi)} e^{2\pi i \tau \Phi_R(k\lambda+\varrho, k\lambda+\varrho)} = \eta(\tau)^d \mathscr{E}_3(\tau).$$

In detail:

$J = J_1^3 - 216 J_2^2$ if R is of type A_2,

$= J_1^3 - \dfrac{6(l+1)(l^2+2l+4)(l^2+2l+8)}{l(l-1)(3l+2)} J_2^2 - \dfrac{24(l+1)(l^2+2l+4)}{l(l-1)(3l+2)} J_1 J_3$ if R is of type $A_l (l \geqq 3)$,

$= J_1^3 - \dfrac{3(2l^2+l+4)}{(l-1)(2l-1)} J_1 J_2 + \dfrac{3(2l^2+l+4)(2l^2+l+8)}{(l-1)(l-2)(2l-1)(2l-3)} J_3$ if R is of type B_l or $C_l (l \geqq 3)$,

$= J_1^3 - \dfrac{6l^2-3l+12}{(l-1)(2l-3)} J_1 J_2 + \dfrac{(6l^2-3l+12)(2l^2-l+8)}{(l-1)(l-2)(2l-3)(2l-5)} J_3$ if R is of type $D_l (l \geqq 4)$,

$= J_1^3 - \dfrac{137}{1512} J_2$ if R is of type E_7,

$= J_1^3 - \dfrac{246}{35} J_2$ if R is of type E_6,

$= J_2$ if R is of type F_4,

$= J_2$ if R is of type G_2.

Remark. Whenever R is not of type $A_l (l \geqq 5)$ or D_6, the polynomial J is uniquely determined up to a constant factor. If R is of type $A_l (l \geqq 5)$ or D_6, then

$$\ker(S_6\Pi \to S_4\Pi)$$

is 2-dimensional. We can choose $J'\Pi \in \ker(S_6\Pi \to S_4\Pi)$, J' linearly independent of J, such that $J'(\varrho)=0$, and consequently $\Theta(\tau, L, J'\Pi, \xi)=0$, so to get the identity (6.14) (ii), we can add to J any multiple of J'.

The space $\mathscr{M}_{12}$ is the first one in which there appear two linearly independent modular forms for Γ [see (2.11)]. Now we will first consider the functions $\Phi(\tau, J)$, where $\deg(J)=12$, in the cases that R is of rank 2. By (4.10) it is easy to see that $\ker(S_{12}\Pi \to S_{10}\Pi)$ is 1-dimensional. As generators we find $J\Pi$, where

$$\begin{aligned} J &= J_1^6 - 648 J_1^3 J_2^2 + 46656 J_2^4 && \text{if } R \text{ is of type } A_2\,, \\ &= J_1^6 - 40 J_1^4 J_2 + 384 J_1^2 J_2^2 - 1024 J_2^3 && \text{if } R \text{ is of type } B_2\,, \\ &= J_1^6 - 4 J_2^2 && \text{if } R \text{ is of type } G_2\,. \end{aligned}$$

Now a basis for $\mathscr{M}_{12}$ is given by $\mathscr{E}_6$ and η^{24}, and one might hope that $J(\xi)=0$, for in that case $\Phi(\tau, J)$ would be a multiple of η^{24}. However, $J(\xi)$ turns out to be non-zero, and therefore $\Phi(\tau, J)$ must be a linear combination of $\mathscr{E}_6$ and η^{24}. It appears that there is exactly one $\lambda \in L(R^\vee)$ such that $q(\lambda+\xi) = 1 + q(\xi)$; we find that

$$\begin{aligned} \lambda &= -\varepsilon_1 + \varepsilon_2 && \text{if } R \text{ is of type } A_2\,, \\ &= -\tfrac{1}{2}\sqrt{2}\varepsilon_1 - \tfrac{1}{2}\sqrt{2}\varepsilon_2 && \text{if } R \text{ is of type } B_2\,, \\ &= -\tfrac{1}{3}\sqrt{6}\varepsilon_2 && \text{if } R \text{ is of type } G_2\,. \end{aligned}$$

Then we get respectively

$$\begin{aligned} \sum_{\lambda \in L(R^\vee)} \frac{J(\lambda+\xi)\Pi(\lambda+\xi)}{J(\xi)\Pi(\xi)} e^{2\pi i \tau \Phi_R(k\lambda+\varrho, k\lambda+\varrho)} &= e^{2\pi i\tau\cdot\frac{1}{3}} - 32768 e^{2\pi i\tau\cdot\frac{4}{3}} + \dots, \\ &= e^{2\pi i\tau\cdot\frac{5}{12}} + \frac{7286170}{527} e^{2\pi i\tau\cdot\frac{17}{12}} + \dots, \\ &= e^{2\pi i\tau\cdot\frac{7}{12}} + \frac{17824702}{703} e^{2\pi i\tau\cdot\frac{19}{12}} + \dots. \end{aligned}$$

Now $\mathscr{E}_6(\tau) = 1 + \dfrac{65520}{691} e^{2\pi i\tau} + \dots$ (see [6], p. 53, where the wrong number 54600 must be replaced by 65520, as can easily be seen from the short table on p. 52), and by a simple calculation we find that

$$\sum_{\lambda \in L(R^\vee)} \frac{J(\lambda+\xi)\Pi(\lambda+\xi)}{J(\xi)\Pi(\xi)} e^{2\pi i \tau \Phi_R(k\lambda+\varrho, k\lambda+\varrho)} = \eta(\tau)^d (\mathscr{E}_6(\tau) + a\eta(\tau)^{24})\,, \tag{6.15}$$

where

$$\begin{aligned} a &= -\frac{22702680}{691}\,, && d=8 \text{ if } R \text{ is of type } A_2\,, \\ &= \frac{5003856000}{364157}\,, && d=10 \text{ if } R \text{ is of type } B_2\,, \\ &= \frac{12277609344}{485773}\,, && d=14 \text{ if } R \text{ is of type } G_2\,. \end{aligned}$$

Of course, whenever $\dim\ker(S_{12}\Pi \to S_{10}\Pi) \geqq 2$ we can choose a spherical function $J\Pi$, where $\deg(J)=12$, and such that $J(\xi)=0$, and then $\Phi(\tau, J)$ must be a multiple of $\eta(\tau)^{24}$. We consider the cases that R is of type A_3 or B_3. First let R be of type A_3. The 7-dimensional space S_{12} is generated by J_1^6, $J_1^4 J_3$, $J_1^3 J_2^2$, $J_1^2 J_3^2$, $J_1 J_2^2 J_3$, J_2^4, J_3^3.

Now it is easy to determine generators for the 2-dimensional kernel, but we even want to have $J(\xi)=0$, and then we must take

$$J=207J_1^6+2752J_1^4J_3-131080J_1^3J_2^2-886016J_1^2J_3^2+4599936J_1J_2^2J_3 \\ +8186640J_2^4+15769600J_3^3 .$$

It depends on the second term in the series $\Theta(\tau, L, J\Pi, \xi)$ whether or not this function vanishes identically. It appears that there is exactly one $\lambda \in L(R^\vee)$ such that $q(\lambda+\xi)=1+q(\xi)$; it is easy to see that $\lambda=-\varepsilon_1+\varepsilon_4$, and $J(\lambda+\xi)\Pi(\lambda+\xi)\neq 0$, so we get

$$\frac{-1}{2^{14}\cdot 3^3\cdot 5\cdot 11\cdot 863}\sum_{\lambda\in L(R^\vee)} J(4\lambda+\varrho)\Pi(4\lambda+\varrho)e^{2\pi i\tau\Phi_R(4\lambda+\varrho,4\lambda+\varrho)}=\eta(\tau)^{39} . \tag{6.16}$$

If R is of type B_3 we can determine $J\Pi\in\ker(S_{12}\Pi\to S_{10}\Pi)$ such that $J(\xi)=0$ in the same way. Here we get

$$J=578995J_1^6-10769307J_1^4J_2+32305340J_1^3J_3+58782790J_1^2J_2^2 \\ -238404790J_1J_2J_3-92139125J_2^3+1309160489J_3^2 .$$

Again there is exactly one $\lambda\in L(R^\vee)$ such that $q(\lambda+\xi)=1+q(\xi)$, in this case $\lambda=-\frac{1}{2}\sqrt{2}(\varepsilon_1+\varepsilon_2)$, and again $J(\lambda+\xi)\Pi(\lambda+\xi)\neq 0$, so we get in this case

$$\frac{-1}{2^{21}\cdot 3^3\cdot 5^4\cdot 11^2\cdot 13\cdot 17^2\cdot 887}\sum_{\lambda\in L(R^\vee)} J(10\lambda+\varrho)\Pi(10\lambda+\varrho)e^{2\pi i\tau\Phi_R(10\lambda+\varrho,10\lambda+\varrho)} \\ =\eta(\tau)^{45} . \tag{6.17}$$

Remark 1. An identity for $\eta(\tau)^{45}$ appears in another way too, when we take R of type D_5 in (5.4).

Remark 2. If $\dim\ker(S_{12}\Pi\to S_{10}\Pi)\geqq 2$, which happens if R is of type $A_l(l\geqq 3)$, $B_l(l\geqq 3)$, $C_l(l\geqq 3)$, $D_l(l\geqq 3)$, E_6, E_7, F_4, we can find $J\in S_{12}$ such that $\Theta(\tau, L, J\Pi, \xi)= c\cdot\eta(\tau)^{d+24}$. The problem is that we do not know whether or not $c=0$.

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