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# MODAL LOGICS WITH THE MACINTOSH RULE*

## 1. INTRODUCTION

In various contexts, philosophers have occasionally propounded principles of the form $A \to \Box A$ or $\Diamond A \to A$ – theses to the effect that if a proposition is true then it is necessarily true, or that it is possibly so only if so. For example, in [6] Kripke asserts the logical truth of $a = b \to \Box(a = b)$ and $a \neq b \to \Box(a \neq b)$ (equivalent, as we shall see, to $\Diamond(a = b) \to a = b$) for certain kinds of terms $a$ and $b$. Recently, J. J. MacIntosh has discussed the role of such principles in proofs of the existence of God, where it is sometimes maintained that God exists if it is possible that He exists, or that God exists necessarily if He exists at all (see [9] and references therein). In conversations with us, MacIntosh raised the question under what conditions these are equivalent theses.

Of course a full answer to this question might depend on special properties that attach to propositions about identity and the existence of God. But for the equivalence generally of principles of the form $\Diamond A \to A$ and $A \to \Box A$, the presence of the following two rules of inference is obviously sufficient, and minimal:

$$\text{RMac.}\quad \frac{\Diamond A \to A}{A \to \Box A} \qquad \text{RMac}\Diamond.\quad \frac{A \to \Box A}{\Diamond A \to A}.$$

According to these rules, either both $\Diamond A \to A$ and $A \to \Box A$ are theorems, or else neither is, for any formula A.

In logics that afford the usual interdefinability of necessity and possibility –

$$\text{Df}\Diamond.\quad A \leftrightarrow \neg\Box\neg A \qquad \text{Df}\Box.\quad \Box A \leftrightarrow \neg\Diamond\neg A$$

– each of these rules is reversible: one holds if and only if the other does. For if $A \to \Box A$ is a theorem then so is $\neg\Box A \to \neg A$, and hence by Df$\Box$, also $\Diamond\neg A \to \neg A$. Supposing RMac holds, we conclude that $\neg A \to \Box\neg A$ is a theorem, from which it follows that $\neg\Box\neg A \to A$ is too. But by Df$\Diamond$ this means that $\Diamond A \to A$ is a theorem. Thus a logic has RMac$\Diamond$ if it has

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RMac, and the argument in the other direction is exactly similar. Because we deal always with logics in which $\Box$ and $\Diamond$ are interdefinable, we will refer to each of the rules above, indifferently, as the *MacIntosh rule.*

We should emphasize the difference between the MacIntosh rule, which says that the schemas

$$\mathrm{T_c}.\quad A \to \Box A \qquad \mathrm{T\Diamond_c}.\quad \Diamond A \to A$$

are *equivalent as theorems*, and the schema

$$(A \to \Box A) \leftrightarrow (\Diamond A \to A),$$

which says, if it is a theorem, that $A \to \Box A$ and $\Diamond A \to A$ are *logically equivalent*. This schema is truth-functionally equivalent to the conjunction of $\mathrm{T_c}$ and $\mathrm{T\Diamond_c}$. So given interdefinability, the presence of any of these three schemas implies the presence of the others, and of course the MacIntosh rule as well. This fact furnishes us with a simple example of what we shall call a MacIntosh logic, albeit a very strong one.

The present paper is a study of logics that have the MacIntosh rule. In Section 2 we provide some background on modal logics. In Section 3 we note some features of the MacIntosh rule, suggest a definition of a MacIntosh logic, and try to get a feeling for what logics have the rule and what theorems, if any, they all have in common. We ascertain some such theorems and succeed in identifying the smallest MacIntosh logic in terms of a pair of modal logics already familiar in the field. In Section 4 we consider what we call genuine MacIntosh logics, and in Section 5 we look a the possibilities of applying the MacIntosh rule in philosophical reasoning. In conclusion, in Section 6, we raise the question of what we call MacIntosh territory, and find that it remains *terra incognita* in at least one important respect.

## 2. PRELIMINARIES

We assume a propositional language in which all boolean connectives are available, along with operators for necessity and possibility. A *logic* is a set of formulas that contains all tautologies and is closed under tautological consequence. We do not insist that logics be closed under a rule of uniform substitution, so as to accommodate logics containing formulas – e.g. $P \to \Box P$ or $\Diamond P \to P$ – not all substitution instances of which are meant to be theorems. Logics in this more general sense may thus be regarded as *theories*, although we will not use this term.

A *normal* logic provides for replacement of logical equivalents, the interdefinability of $\Box$ and $\Diamond$, and the following schemas:

R. $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

N. $\Box\top$

The formula N may be replaced equivalently by a rule of necessitation:

$$\text{RN.} \quad \frac{A}{\Box A}$$

Let us note for future reference that normal logics are closed under the following rules of monotonicity:

$$\text{RM.} \quad \frac{A \rightarrow B}{\Box A \rightarrow \Box B} \qquad \text{RM}\Diamond. \quad \frac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}$$

Classes of logics weaker than normal can be distinguished. For example, with replacement and interdefinability guaranteed, a logic is said to be *regular* if it has R. Our focus in this paper is on normal logics, although several of the results stated below hold more generally for nonnormal logics.

In what follows we refer to a number of further modal schemas, the most important of which are:

D. $\Diamond\top$

$\bar{\text{D}}$. $\Box\bot$

T. $\Box A \rightarrow A$

T!. $\Box A \leftrightarrow A$

B. $A \rightarrow \Box\Diamond A$

4. $\Box A \rightarrow \Box\Box A$

5. $\Diamond A \rightarrow \Box\Diamond A$

The smallest normal logic is called *K*, and its normal extensions may be denoted (following Lemmon's idea, [7], p. 51) by suffixing names of schemas. For example, *KT*4 is the smallest normal logic containing T and 4. We prefer, however, to use more conventional designations wherever possible: *KD*, the basic normal system of deontic logic, is usually referred to simply as *D*; it has many formulations, most notably that using the schema $\Box A \to \Diamond A$ (in fact more commonly referred to as D) rather than $\Diamond\top$ as an axiom. *KT*, the "logique *t*" of Feys, is more commonly known as *T*. *KTB* is the so-called *Brouwersche* system (and B is known as the Brouwer schema). *KT*4 and *KT*5 are the well-known *S*4 and *S*5, respectively. Note that the latter is also, inter alia, *KTB*4, *KDB*4, and *KDB*5.

The logics *KT!* and $K\bar{D}$ are more rarely encountered. The former is called the *Trivial* logic, the latter the *Verum*. Notice that $K\bar{D}$ can also be characterized as the smallest logic in which every formula of the form $\Box A$ is a theorem. Because it is so to speak the antithesis of *D*, we will refer to the *Verum* logic here as $\bar{D}$.

For the semantics of normal logics, we use *models* $\mathcal{M} = \langle U, R, V \rangle$, where *U* is a set of points (or "possible worlds"), *R* is a binary relation in *U*, and (the valuation) *V* assigns subsets of *U* to atomic formulas. A model is *finite* if *U* is. An *R-chain* in a model is a sequence of points each of which is *R*-related to the next if there is one (this allows also for singleton sequences). A model is *generated* if it contains a point (a generator) from which every other point can be reached by an *R*-chain. Truth conditions for formulas at a point *x* in a model $\mathcal{M}$ are standard for boolean combinations. For atomic formulas P and modal formulas $\Box A$ and $\Diamond A$, the conditions are as follows:

$$\mathcal{M} \models_x \mathrm{P} \text{ iff } x \in V(\mathrm{P})$$

$$\mathcal{M} \models_x \Box \mathrm{A} \text{ iff } \forall y \in U(xRy \Rightarrow \mathcal{M} \models_y \mathrm{A})$$

$$\mathcal{M} \models_x \Diamond \mathrm{A} \text{ iff } \exists y \in U(xRy \,\&\, \mathcal{M} \models_y \mathrm{A})$$

A formula is said to be *valid in a class of models* just in case it is true at every point in every model in the class. Every class of models determines a logic – viz. the set of sentences valid in the class – and every logic is determined by one or more classes of models (trivially, since each logic is determined alone by its so-called canonical model). For example, *T* is determined by the class of models that are (more precisely, whose relations are) reflexive, and *S*4 is determined by the class of models that are reflexive and transitive.

As the logics *D* and $\bar{D}$ figure prominently in what follows, we record here some further information about them. First, theoremhood in these logics can be characterized alternatively in terms of deducibility in $K$:

$$A \in D \text{ if and only if } \{\Box^n \Diamond \top : n \geqslant 0\} \vdash_K A$$

$$A \in \bar{D} \text{ if and only if } \{\Box^n \bot : n \geqslant 0\} \vdash_K A$$

The second characterization is in fact equivalent to

$$A \in \bar{D} \text{ if and only if } \Box \bot \rightarrow A \in K$$

– since $\Box \bot \rightarrow \Box^n \bot$ is a theorem of $K$ for every $n > 0$.

Secondly, $D$ is determined by the class of serial (and also finite serial) models, i.e. those in which $\forall x \in U \exists y \in U(xRy)$, whereas $\bar{D}$ is determined by the class of models with an empty accessibility relation – or even by the class of models consisting of one isolated point.

For more information on the topics covered above, see e.g. [2], [3], [5], [7], and [10].

## 3. MACINTOSH LOGICS

We begin with the observation that N is present in every logic with the MacIntosh rule. For, since every logic contains $\Diamond\top \rightarrow \top$, a logic that has the MacIntosh rule contains $\top \rightarrow \Box\top$, and hence $\Box\top$. It follows that every regular logic with the MacIntosh rule is normal as well. This suggests the definition of a *MacIntosh logic* as a normal logic that has the MacIntosh rule.

Notice that the intersection of any two – indeed, of any class of – MacIntosh logics is a MacIntosh logic. For $\Diamond A \rightarrow A$ belongs to such an intersection only if it belongs to the intersectors, and so, by the rule, $A \rightarrow \Box A$ belongs to the intersectors and hence the intersection.

What logics are MacIntosh logics? We begin to get an idea if we note that RMac and RMac$\Diamond$ are special cases of the following rules, respectively:

$$\text{RX.} \quad \frac{\Diamond A \rightarrow B}{A \rightarrow \Box B} \qquad \text{RX}\Diamond. \quad \frac{A \rightarrow \Box B}{\Diamond A \rightarrow B}$$

Like RMac and RMac$\Diamond$, each of the rules RX and RX$\Diamond$ is reversible. Thus any logic that has either of these rules also has the MacIntosh rule. The rules RX and RX$\Diamond$ are equivalent in normal logics to the presence of the Brouwer schema (see [3], p. 136; cf. [4], p. 58). So any normal logic containing B – e.g. the *Brouwersche* system, *S*5, and the *Trivial* logic – is a MacIntosh logic.

On the other hand, again for example, *K*4, *KD*4, *S*4, *K*5, and *KD*5 are not MacIntosh logics, since for each of these either 4 or 5 is a theorem, but not both. This is of interest because it shows that a sublogic of a MacIntosh logic need not be a MacIntosh logic.

There are many other MacIntosh logics. For a large class of them, consider the following generalization of the Brouwer schema:

$$\mathrm{B}^n.\quad \mathrm{A} \to \Box\Diamond^n\mathrm{A}$$

Thus $\mathrm{B}^0$ is $\mathrm{T_c}$ and $\mathrm{B}^1$ is B itself.

THEOREM 3.1. (1) *$KB^n$ is a MacIntosh logic, for every $n \geqslant 0$.* (2) *Indeed, every normal extension of $KB^n$ is a MacIntosh logic, for every $n \geqslant 0$.*

*Proof.* For $n \geqslant 0$, Let $L$ be a normal logic that extends $KB^n$. Assume that $L$ contains $\Diamond\mathrm{A} \to \mathrm{A}$, for some formula A. Then the presence of the rule RM$\Diamond$ guarantees that $\Diamond^{i+1}\mathrm{A} \to \Diamond^i\mathrm{A}$ is in $L$ for any $i > 0$. Consequently, so is $\Diamond^n\mathrm{A} \to \mathrm{A}$, and hence, by RM, $\Box\Diamond^n\mathrm{A} \to \Box\mathrm{A}$. The fact that $L$ extends $KB^n$ then gives us $\mathrm{A} \to \Box\mathrm{A}$, as we wanted. ■

We remarked earlier that the intersection of any class of MacIntosh logics is itself such a logic. So Theorem 3.1 yields the following:

COROLLARY 3.2. *$\bigcap\{KB^n : n \geqslant 0\}$ is a MacIntosh logic.*

In order to explain the interrelations among normal $KB^n$-logics, and to convince oneself that they really form a large and nontrivial family, it is well to turn to semantics. From one of Lemmon and Scott's general completeness results ([7], pp. 58ff.), it follows that $KB^n$ is determined by the following condition, which we might call *n-symmetry*:

$$xRy \Rightarrow yR^nx$$

THEOREM 3.3. *$KB^m \subseteq KB^n$ if and only if $m \equiv n \pmod{n+1}$, for every $m, n \geqslant 0$.*

*Proof. Only-if* part: Assume that $KB^m \subseteq KB^n$, for $m, n \geqslant 0$. We define a model $\mathscr{M} = \langle U, R, V \rangle$ as follows: $U$ is the set of integers *modulo* $n + 1$, $R$ is the binary relation on $U$ given by the condition

$$iRj \text{ iff } i + 1 \equiv j \,(\text{mod } n + 1),$$

and $V$ is any valuation such that for a certain atomic formula P,

$$V(\text{P}) = \{0\}.$$

In other words, in $\mathscr{M}$ each integer $i$ is $R$-related to all and only its "successors" $i + 1$ within the modulus $(n + 1)$, and the atomic formula P is true at 0 and only at 0. Note that this model is $n$-symmetric: if $iRj$ then $jR^n i$. Hence it validates $KB^n$ and, by our assumption, $KB^m$ as well. In particular, it validates the formula $\text{P} \to \Box\Diamond^m\text{P}$. Thus, since $\mathscr{M} \models_0 \text{P}$, we have that $\mathscr{M} \models_0 \Box\Diamond^m\text{P}$ and so $\mathscr{M} \models_1 \Diamond^m\text{P}$ since $0R1$. Evidently, since P is true at 0 alone, $1R^m0$. Again since $0R1$, $1R^n0$ by $n$-symmetry. Therefore $m \equiv n$ (mod $n + 1$), as we wished to show.

*If* part: For the argument here it is important to observe that an $n$-symmetric relation is also $k(n + 1) + n$ – symmetric for every $k \geqslant 0$. To see this, suppose that $xRy$. Then by $n$-symmetry, $yR^nx$. Thus there are routes from $x$ via $y$ back to $x$ of any multiple of $(n + 1)$ $R$-steps, and so $x$ can always be reached from $y$ in $k(n + 1) + n$ steps – i.e. $yR^{k(n+1)+n}x$ – for every $k \geqslant 0$. Assume now, for $m, n \geqslant 0$, that $m \equiv n$ (mod $n + 1$), i.e. that $m = k(n + 1) + n$ for some $k \geqslant 0$. To show that $KB^m \subseteq KB^n$, it is sufficient to show that $KB^n$ contains $\text{B}^m$, and for this it will be enough (via the Lemmon and Scott result) to show that $\text{B}^m$ is validated by any $n$-symmetric model. So let $\mathscr{M} = \langle U, R, V \rangle$ be such a model and suppose that $\mathscr{M} \models_x \text{A}$, for some point $x$ in $U$. To show that $\mathscr{M} \models_x \Box\Diamond^m\text{A}$, we suppose that $xRy$ and argue that $\mathscr{M} \models_y \Diamond^m\text{A}$. By $n$-symmetry, $yR^nx$. Hence also $yR^{k(n+1)+n}x$ for every $k \geqslant 0$. Therefore, by our initial assumption, $yR^mx$. So $\mathscr{M} \models_y \Diamond^m\text{A}$. ■

The following corollary to Theorem 3.3 tells us that there are infinitely many MacIntosh logics:

COROLLARY 3.4. *If* $m \neq n$, *then* $KB^m \neq KB^n$, *for every* $m, n \geqslant 0$.

*Proof.* Suppose, for $m, n \geqslant 0$, that $m \neq n$, or, without loss of generality, $m < n$. Then $m \not\equiv n$ (mod $n + 1$). By Theorem 3.3, $KB^m \not\subseteq KB^n$ and hence $KB^m \neq KB^n$. ■

The notion of $n$-symmetry admits of an obvious generalization. Let us say that a relation $R$ is $*$-*symmetric* ("star-symmetric") if it satisfies the condition

$$xRy \Rightarrow yR^*x$$

– where $R^*$ is the ancestral of $R$ (i.e. $R^*$ holds just in case $R^n$ holds for some $n \geqslant 0$). This condition is of course different from symmetry. However, as is readily checked, it is equivalent to the condition that the ancestral be symmetric:

$$xR^*y \Rightarrow yR^*x$$

The following result yields a new proof of the first part of Theorem 3.1, but not the second; it will also be used below.

THEOREM 3.5. *Any class of* $^*$*-symmetric models determines a MacIntosh logic.*

*Proof.* Let **C** be a class of $^*$-symmetric models. It will be enough to show that the logic $L(\mathbf{C})$ determined by **C** is a MacIntosh logic. Suppose that $L(\mathbf{C})$ contains $\Diamond \mathrm{A} \rightarrow \mathrm{A}$, for some formula A. Let $x$ be a point in a model $\mathscr{M}$ in **C**. Assume that $\mathscr{M} \models_x \mathrm{A}$. If we can show that $\mathscr{M} \models_x \Box \mathrm{A}$, then we are entitled to assert that $\mathrm{A} \rightarrow \Box \mathrm{A}$ is a theorem of $L(\mathbf{C})$ and hence that $L(\mathbf{C})$ is a MacIntosh logic.

So take any $y$ such that $xRy$. It is sufficient to show that $\mathscr{M} \models_y \mathrm{A}$. By $^*$-symmetry, $yR^nx$ for some $n \geqslant 0$. This means that there exist $n+1$ elements, which we may represent by the numbers $0, \ldots, n$, such that $y = 0$, $x = n$, and, for each $i < n$, $iR(i+1)$. We assert that, for all $i \leqslant n$, $\mathscr{M} \models_i \mathrm{A}$ and prove this by backward induction. The claim is certainly true for $i = n$. Suppose that $\mathscr{M} \models_{i+1} \mathrm{A}$, where $0 \leqslant i < n$. Since $iR(i+1)$, $\mathscr{M} \models_i \Diamond \mathrm{A}$. But $\Diamond \mathrm{A} \rightarrow \mathrm{A}$ is valid in **C**; so $\mathscr{M} \models_i \mathrm{A}$. This completes the induction. It follows that $\mathscr{M} \models_0 \mathrm{A}$, i.e. $\mathscr{M} \models_y \mathrm{A}$. ■

In [2], Bull and Segerberg show that every frame for $\Box(\mathrm{A} \rightarrow \Box \mathrm{A}) \rightarrow (\Diamond \mathrm{A} \rightarrow \mathrm{A})$ is $n$-symmetric, for some $n \geqslant 0$ – i.e., is $^*$-symmetric. Of course it is easy to see that in a normal logic the schema yields the MacIntosh rule.

Since the class of MacIntosh logics is closed under intersection, there is a smallest such logic. What is it? It is clear that it must be included in

$\cap\{KB^n : n \geqslant 0\}$. On the other hand, it is stronger than $K$, for $K$ is not a MacIntosh logic: $\Diamond\top \to \Box\Diamond\top$ is not a theorem of $K$, even though $\Diamond\Diamond\top \to \Diamond\top$ is. In other words, every MacIntosh logic – and hence the smallest – contains $\Diamond\top \to \Box\Diamond\top$. As a little reflection shows, this formula is equivalent in a normal logic to $\Box\Diamond\top$.

Let $D\bar{D}$ be the smallest normal extension of $K$ that contains this formula, $\Box\Diamond\top$. Here our nomenclature does *not* mean $KD\bar{D}$, which of course is the inconsistent logic. Rather it anticipates our identification of this logic; for it turns out that $D\bar{D}$ is the intersection – the synthesis as it were – of $D$ and $\bar{D}$:

THEOREM 3.6. $D\bar{D} = D \cap \bar{D}$.

*Proof.* Note that $D\bar{D}$ is included in $D \cap \bar{D}$, since $\Box\Diamond\top$ is in each of $D$ and $\bar{D}$. For the reverse inclusion, we offer two arguments. The first uses the alternative characterizations of theoremhood for $D$ and $\bar{D}$ noted earlier. Suppose that A is in $D \cap \bar{D}$. Since A is then a theorem of $D$, we have that $\{\Box^n\Diamond\top : n \geqslant 0\} \vdash_K$ A. So from the fact that $D\bar{D}$ contains $\Box^n\Diamond\top$ for every $n > 0$, it follows that $\Diamond\top \to$ A is in $DD$. On the other hand, since A is a theorem of $\bar{D}$, we have, as remarked earlier, that $\Box\bot \to$ A is a theorem of $K$ – and hence also of $D\bar{D}$. Therefore A is a theorem of $D\bar{D}$. The second argument for this inclusion is semantic and relies on the fact that $D\bar{D}$ is determined by the class of models $\mathcal{M} = \langle U, R, V\rangle$ in which for every $x$ and $y$ in $U$ it holds that

$$xRy \Rightarrow \exists z \in U(yRz)$$

– i.e. in which every point is $R$-isolated or is part of an endless $R$-chain. (This is perhaps easier to see when one considers $\Diamond\top \to \Box\Diamond\top$.) Our argument uses the fact that $D\bar{D}$ is determined moreover by the class of generated models that satisfy this condition. In each of these either there is just one point, the $R$-isolated generator, or a generator begins endless $R$-chains. We reason contrapositively, using the determination results for $D$ and $\bar{D}$ noted earlier. Assume that A is not a theorem of $D\bar{D}$. Then it is false at a generator of a model satisfying the condition above. If the generator is isolated, then we have a model of $\bar{D}$, and so A is not a theorem of this logic. If the generator commences endless $R$-chains, then the model is serial and hence models $D$, and so A is not a theorem of this logic. In either case, A is not a theorem of $D \cap \bar{D}$. ■

It turns out that $D\bar{D}$ is exactly what we are after, the smallest MacIntosh logic. We know that every MacIntosh logic – including the smallest – contains $\Box\Diamond\top$. So $D\bar{D}$ will be the smallest MacIntosh logic if it is a MacIntosh logic. But then $D\bar{D}$ is a MacIntosh logic if $D$ is, since $\bar{D}$ is one for sure. Thus we need to establish that $D$ is a MacIntosh logic. We do so by showing that $D$ is determined by a class of $^*$-symmetric models and applying Theorem 3.5.

We begin by defining a certain construction. Where $\mathscr{M} = \langle U, R, V\rangle$ is a model, let us say that an $R$-chain $(x_0, \ldots, x_n)$ in $\mathscr{M}$ is of length $n$. (Note that length measures the number of links or $R$-steps, not the number of elements, in the chain.) Where $\mathbf{x}$ is an $R$-chain $(x_0, \ldots, x_n)$, we write $\mathrm{lng}(\mathbf{x})$ for the length, $n$, of $\mathbf{x}$ and $\mathbf{x}^\#$ for the last point, $x_n$, of $\mathbf{x}$. For each $n \geqslant 0$, we define the model $\mathscr{M}_n = \langle U_n, R_n, V_n\rangle$ in terms of $R$-chains from $\mathscr{M}$. First, $U_n$ is the set of $R$-chains of length at most $n$. Next, $\mathbf{x}R_n\mathbf{y}$ is defined to hold between an $R$-chain $\mathbf{x}$ – say $(x_0, \ldots, x_i)$ – and an $R$-chain $\mathbf{y}$ if and only if:

$$\text{either } \mathrm{lng}(\mathbf{x}) < n \quad \text{and} \quad \mathbf{y} = (x_0, \ldots, x_i, \mathbf{y}^\#),$$
$$\text{or } \mathrm{lng}(\mathbf{x}) = n \quad \text{and} \quad \mathbf{y} = (x_0)$$

That is to say, in $\mathscr{M}_n$ an $R$-chain of length less than $n$ is related to every $R$-chain that extends it by a single point, and an $R$-chain of maximum length, $n$, is simply related by fiat to the $R$-chain consisting solely of its first point. Finally, we define $V_n$ so that chains in $\mathscr{M}_n$ agree with their last points in $\mathscr{M}$ on all atomic formulas: $V_n(\mathrm{P}) = \{\mathbf{x} \in U_n : \mathbf{x}^\# \in V(\mathrm{P})\}$, for every atomic formula P.

It should be clear that whenever $\mathscr{M}$ is serial $\mathscr{M}_n$ will be both serial and $^*$-symmetric, and that one of the models is finite if and only if the other is.

By the *degree* of a formula A, in symbols deg(A), is meant the maximum number of nested modal operators in it. The following claim can now be proved:

LEMMA 3.7. *For every formula A, if* $\mathbf{x}$ *is an element of* $U_n$ *such that* $\deg(\mathrm{A}) + \mathit{lng}(\mathbf{x}) \leqslant n$, *then*

$$\mathscr{M}_n \models_{\mathbf{x}} \mathrm{A} \text{ if and only if } \mathscr{M} \models_{\mathbf{x}^\#} \mathrm{A}$$

*Proof.* By induction on A. As usual in proofs of this kind, the interest is solely in the modal part of the inductive step. Let A be a formula and $\mathbf{x}$ an element in $U_n$ such that $\deg(\Box A) + \operatorname{lng}(\mathbf{x}) \leqslant n$. The inductive hypothesis is that the claim to be proved holds for A. Note that $\operatorname{lng}(\mathbf{x}) \leqslant n - 1$. Thus there is some $i \leqslant n - 1$ and some $x_0, \ldots, x_i$ in $U$ such that $\mathbf{x} = (x_0, \ldots, x_i)$. By our convention, $\mathbf{x}^\# = x_i$.

First suppose that $\mathscr{M}_n \models_{\mathbf{x}} \Box A$. Then:

$$(1)\ \forall \mathbf{y} \in U_n(\mathbf{x}R_n\mathbf{y} \Rightarrow \mathscr{M}_n \models_{\mathbf{y}} A)$$

Take any $u$ in $U$ such that $\mathbf{x}^\# Ru$. Since $i \leqslant n - 1$, it follows that the sequence $\mathbf{y} = (x_0, \ldots, x_i, u)$ is in $U_n$. By definition, $\mathbf{x}R_n\mathbf{y}$. Hence by (1), $M_n \models_{\mathbf{y}} A$. Since $\deg(A) + \operatorname{lng}(\mathbf{y}) = \deg(\Box A) + \operatorname{lng}(\mathbf{x}) \leqslant n$, the inductive hypothesis applies. So $\mathscr{M} \models_{\mathbf{y}^\#} A$; that is, $\mathscr{M} \models_u A$. Consequently, $\mathscr{M} \models_{\mathbf{x}^\#} \Box A$.

For the reverse, suppose that $\mathscr{M} \models_{\mathbf{x}^\#} \Box A$. Since $\mathbf{x}^\# = x_i$, this implies:

$$(2)\ \forall u \in U_n(x_iRu \Rightarrow \mathscr{M} \models_u A)$$

Take any sequence $\mathbf{y}$ in $U_n$ such that $\mathbf{x}R_n\mathbf{y}$. Since $i \leqslant n - 1$, it follows that $\mathbf{y} = (x_0, \ldots, x_i, u)$, for some $u$ in $U$ such that $x_iRu$. By (2), $\mathscr{M} \models_u A$, i.e. $\mathscr{M} \models_{\mathbf{y}^\#} A$. Again the inductive hypothesis applies; so $\mathscr{M}_n \models_{\mathbf{y}} A$. Consequently, $\mathscr{M}_n \models_{\mathbf{x}} \Box A$. ■

THEOREM 3.8. *D is a MacIntosh logic.*

*Proof.* Every nontheorem of *D* has a serial countermodel. Our construction shows that for each of those models there exists a countermodel (for the nontheorem in question) that is *-symmetric as well as serial. It follows that *D* is determined by a class of *-symmetric models. Therefore, by Theorem 3.5, *D* is a MacIntosh logic. ■

This result is interesting in its own right and also because it shows that an extension of a MacIntosh logic need not be a MacIntosh logic (e.g. *S*4 is an extension of *D*). In this respect the MacIntosh rule differs from the rule RX, which, as remarked earlier, is present in a normal logic just in case the logic extends *KB*. But to us the main interest of Theorem 3.8 derives from the following consequence, as explained above:

COROLLARY 3.9. *$D\bar{D}$ is a MacIntosh logic, indeed the smallest MacIntosh logic.*

Yet another outcome of the proof of Lemma 3.7 is this:

COROLLARY 3.10. $D\bar{D} = \bigcap\{KB^n\colon n \geqslant 0\}$.

*Proof.* Every nontheorem of $D$ is false in a finite serial model. In this case the countermodel mentioned in the proof of theorem 3.8 is finite and so is $n$-symmetric, for some $n \geqslant 0$. ■

The logic $D\bar{D}$ has numerous other identities, of which we shall state just this one:

THEOREM 3.11. *$D\bar{D} = D \cap KB^n$, for every $n \geqslant 0$.*

*Proof.* We have already seen that $D$ contains $\Box\Diamond\top$. Note then that, for each $n \geqslant 0$, $KB^n$ also contains this formula. For otherwise, for some such $n$, $\Diamond\Box\bot$ holds somewhere in an $n$-symmetric model, which means that $\Box\bot$ holds at some point $R$-related to a point where, absurdly, $\bot$ is true. So $D\bar{D}$ is included in $D \cap KB^n$, for every $n \geqslant 0$. For the other direction, we reason as in the proof of Theorem 3.6. Suppose A to be a nontheorem of $D\bar{D}$. Then A is false at a generator in a model of $D\bar{D}$. Again, either this point is $R$-isolated or it commences endless $R$-chains. In the former case, the model is $n$-symmetric for every $n \geqslant 0$, and so validates $KB^n$ for every such $n$; in the latter case, as before, it is a model of $D$. In either case, A fails to be a theorem of $D \cap KB^n$, for any $n \geqslant 0$.

The construction of $\mathcal{M}_n$ in the proof of Lemma 3.7 can be adapted to show that $T$ is a MacIntosh logic. In outline, this extended proof assumes that the given model $\mathcal{M}$ is reflexive (and so is a model for $T$), the relations in $\mathcal{M}_n$ are "reflexivized" by the addition of all pairs $\langle \mathbf{x}, \mathbf{x} \rangle$ where $\mathbf{x}$ is in $U_n$, and the original conditions for $\mathbf{x}R_n\mathbf{y}$ are stipulated only for the case in which $\mathbf{x} \neq \mathbf{y}$. The result is clearly a $^*$-symmetric model. The subsequent induction is then modified to take into account the reflexive possibilities. Given this, we argue that $T$ is a MacIntosh logic analogously as for $D$: Every nontheorem of $T$ is rejected by a reflexive model, and our construction shows that the nontheorem also fails in a reflexive *-symmetric model. So $T$ is determined by a class of *-symmetric models and hence, by Theorem 3.5, is a MacIntosh logic. Let us state this formally for the record:

THEOREM 3.12. *$T$ is a MacIntosh logic.*

We shall return to the construction in the proof of Lemma 3.7, and its extension to the reflexive case, in the context of applying some of our results.

## 4. GENUINE MACINTOSH LOGICS

Which MacIntosh logics contain one of, and hence both, $\Diamond A \rightarrow A$ and $A \rightarrow \Box A$, for some formula A, but neither $\neg\Diamond A$ nor $\Box A$? Which MacIntosh logics are, as we shall say, *genuine*?

It is easy to see that $S5$ is genuine, since for any atomic formula P it contains $\Diamond\Box P \rightarrow \Box P$ and $\Box P \rightarrow \Box\Box P$, whereas neither $\neg\Diamond\Box P$ nor $\Box\Box P$ is a theorem. Indeed, it can be shown that except for $\bar{D}$ every consistent MacIntosh logic extending $K45$ has this property and so is genuine. On the other hand, it is evident that neither $\bar{D}$ nor the inconsistent logic is genuine. What about other MacIntosh logics?

We make a little progress when we consider the so-called "rule of disjunction" (see [3], pp. 181–182, and [7], pp. 44–46). A logic has this rule if whenever it contains an $n$-termed disjunction of the form $\Box A_1 \vee \ldots \vee \Box A_n$ it also contains $A_i$ for some $i \leqslant n$.

THEOREM 4.1. *No MacIntosh logic that has the rule of disjunction is genuine.*

*Proof.* Suppose we have a MacIntosh logic that has the rule of disjunction. If the logic contains $\Diamond A \rightarrow A$ and hence $A \rightarrow \Box A$, for some formula A, it contains $\Diamond A \rightarrow \Box A$ as well, and thus, equivalently, $\Box\neg A \vee \Box A$. By the rule of disjunction, either $\neg A$ or A is a theorem. Because the logic is normal, it follows by the rule of necessitation that either $\Box\neg A$ or $\Box A$ is a theorem – i.e. that one of $\neg\Diamond A$ and $\Box A$ is. Therefore the logic is not genuine. ∎

The logics $D$ and $T$ have the rule of disjunction. So we may conclude from Theorem 4.1 that neither of them is a genuine MacIntosh logic. The result for $D$ leads to one for $D\bar{D}$: Suppose $D\bar{D}$ contains $A \rightarrow \Box A$ and hence $\Diamond A \rightarrow A$. Then these formulas are in $D$. Because $D$ is not genuine, it contains $\neg\Diamond A$ or $\Box A$. But both $\neg\Diamond A$ and $\Box A$ are in $\bar{D}$. So $D \cap \bar{D}$, i.e. $D\bar{D}$, contains $\neg\Diamond A$ or $\Box A$.

The idea of this reasoning for $D$ and $T$ was suggested to us by Max Cresswell. That for $D\bar{D}$ was inspired by Tim Williamson, after he pointed out that $D$, $T$, and $D\bar{D}$ obey his "rule of margins" and are therefore not genuine MacIntosh logics.

A logic has the rule of margins just in case it contains either A or $\neg A$ whenever it contains $A \rightarrow \Box A$ or, equally well, $\Diamond A \rightarrow A$; see [11]. Notice that normal logics with this rule also have the MacIntosh rule.

THEOREM 4.2 *No MacIntosh logic that has the rule of margins is genuine.*

*Proof.* The argument resembles that for Theorem 4.1. Suppose $A \rightarrow \Box A$ is a theorem of a MacIntosh logic with the rule of margins. Then $\neg A$ is a theorem or A is. By normality, $\neg\Diamond A$ or $\Box A$ is a theorem, and so the logic is not genuine. ■

Williamson has also shown (in [11]) that the rule of margins is present in $KDB$ and $KTB$ (i.e. the *Brouwersche* system). So neither of these MacIntosh logics is genuine, and he has subsequently (in [12]) proved that, indeed, for $n > 0$ none of the MacIntosh logics $KB^n$ is genuine. ($KB^0$ is $KT_c$, an extension of $K45$.)

## 5. PRACTICALITIES

Given the fickle character of the MacIntosh rule with respect to which logics it inhabits, one may well ask to what practical uses it may be put in connection with theses such as $a = b \rightarrow \Box(a = b)$ and "God exists if it is possible that God exists". To focus the remarks that follow, let us distinguish a certain atomic formula P, which we may think of as interpreting $a = b$, "God exists", or some other such sentence. The formulas we are interested in, then, are:

$$\mathrm{P}_{+}.\quad \mathrm{P} \to \Box\mathrm{P} \qquad \mathrm{P}^{+}.\quad \Diamond\mathrm{P} \to \mathrm{P}$$

To adopt one of these formulas as a theorem of a modal logic is not of course to say that the logic contains the formula, but rather to *add* the formula to the logic. Where $L$ is a MacIntosh logic, let $L_{+}$ be the least normal extension of $L$ containing $\mathrm{P}_{+}$, and similarly for $L^{+}$ and $\mathrm{P}^{+}$. Then let $L+$ be the least normal extension of $L$ that contains both $\mathrm{P}_{+}$ and $\mathrm{P}^{+}$. Evidently, $L+ = L_{+}{}^{+} = L^{+}{}_{+}$. What are the results of these additions?

Consider any of the MacIntosh logics $KB^{n}$. As we have seen (Theorem 3.1), the MacIntosh rule is present in every normal extension of $KB^{n}$. Hence $KB^{n}_{+}$ and $KB^{n+}$ have the rule and so are identical with $KB^{n}+$. In this way a normal *KB* logic provides a setting for proofs of the existence of God: One first establishes $\mathrm{P} \to \Box\mathrm{P}$ and hence $\Diamond\mathrm{P} \to \mathrm{P}$, then argues (presumably independently) for $\Diamond\mathrm{P}$, and concludes with P. Compare the reasoning, e.g., in [8], p. 161, where P means $a = b$.

Of course in this context there is no need to restrict oneself to the MacIntosh rule, since the stronger rules RX and RX$\Diamond$ are available, as remarked earlier, in any normal logic containing B. This is probably a good place to point out that if truth, and not theoremhood, is at issue, one has the option of arguing for the truth of $\Box(\mathrm{P} \to \Box\mathrm{P})$ and then moving to the truth of $\Diamond\mathrm{P} \to \mathrm{P}$ via the following theorem of *KB*:

$$\Box(\mathrm{A} \to \Box\mathrm{B}) \to (\Diamond\mathrm{A} \to \mathrm{B})$$

(This schema is equivalent as a theorem to the Brouwer schema; see [3], p. 136.) Compare the reasoning, e.g., in [1], pp. 43–44, where P means "God exists". In this situation, one could as well appeal to an instance of the schema:

$$\Box(\mathrm{A} \to \Box\mathrm{A}) \to (\Diamond\mathrm{A} \to \mathrm{A})$$

But note the necessity here of the initial $\Box$; without it the logic is at least as strong as $KT_{c}$. Recall the remarks about $\mathrm{T}_{c}$ and $\mathrm{T}\Diamond_{c}$ at the beginning of the paper.

Indeed, unless one is prepared to adopt a modal logic as strong as *S*5, one may wish not to employ a logic containing the Brouwer schema in philosophical argumentation involving modalities, i.e. one may wish not to presume that whatever is so is necessarily possibly so. Is there an alternative? Can one make do with modest extensions of such relatively innocuous modal logics as *D* and *T*, where the presumption is only that whatever is necessary is possible or that whatever is necessary is so? The answer is not so clear.

To begin with, unlike $KB_+$, $KB^+$, and their relatives, none of $D_+$, $D^+$, $T_+$, and $T^+$ is a MacIntosh logic. To see this in the case of $D_+$ and $D^+$, we begin with the observation that these logics are determined by the classes of serial models $\mathcal{M} = \langle U, R, V \rangle$ satisfying, respectively, the following conditions:

$$(\Rightarrow) \quad xRy \Rightarrow (x \in V(\mathrm{P}) \Rightarrow y \in V(\mathrm{P}))$$

$$(\Leftarrow) \quad xRy \Rightarrow (y \in V(\mathrm{P}) \Rightarrow x \in V(\mathrm{P}))$$

But where $x$ and $y$ are distinct, $U = \{x, y\}$, $R = \{\langle x, y \rangle, \langle y, y \rangle\}$ and $V(\mathrm{P}) = \{y\}$, we have a serial countermodel for $\mathrm{P}^+$ that satisfies $(\Rightarrow)$. Thus, $D_+$ does not have the MacIntosh rule. Likewise, where $\mathcal{M}$ is as above except that $V(\mathrm{P}) = \{x\}$ we have a serial countermodel for $\mathrm{P}_+$ that satisfies $(\Leftarrow)$ – which shows that $D^+$ is not a MacIntosh logic. The logics $T_+$ and $T^+$ are determined by the classes of reflexive models satisfying the respective conditions above. For these cases, adding $\langle x, x \rangle$ to $R$ in the countermodels shows that these logics also fail to have the MacIntosh rule.

It might be argued that someone who accepted the principles of *D* or *T* – and hence accepted the MacIntosh rule – could without qualm accept the logics *MacD* and *MacT* obtained respectively from (say for the sake of definiteness) $D_+$ and $T_+$ by closing them under the MacIntosh rule. Such logics would thus possess principles otherwise less controversial than those found in $KB^n$ systems.

But there is no difference between adding the MacIntosh rule to $D_+$ and $T_+$, on the one hand, and simply adding $\mathrm{P}^+$ to them, on the other. This emerges as a corollary to the following:

THEOREM 5.1. (1) *$D+$ is a MacIntosh logic.* (2) *$T+$ is a MacIntosh logic.*

*Proof.* We concentrate on part (1). Clearly by Theorem 3.5 it is enough to show that $D+$ is determined by a class of *-symmetric models.

Let $\mathcal{M}$ be a serial model that satisfies conditions ($\Leftarrow$) and ($\Rightarrow$). We construct the model $\mathcal{M}_n$ as for Lemma 3.7. Then $\mathcal{M}_n$ is serial and *-symmetric, and we need to show that it satisfies ($\Leftarrow$) and ($\Rightarrow$). Accordingly, let $\mathbf{x} = (x_0, \ldots, x_i)$ be a point in $U_n$, where $i \leqslant n$. Note that $\mathbf{x}^{\#} = x_i$.

We assume that $\mathbf{x}R_n\mathbf{y}$ and argue by cases, according as $\mathrm{lng}(\mathbf{x}) < n$ or $\mathrm{lng}(\mathbf{x}) = n$. In the first case, $\mathbf{y} = (x_0, \ldots, x_i, \mathbf{y}^{\#})$. For ($\Rightarrow$), suppose that $\mathbf{x}$ is in $V_n(\mathrm{P})$ – to show that $\mathbf{y}$ is too. Then $\mathbf{x}^{\#}$ is in $V(\mathrm{P})$, and hence by ($\Rightarrow$) in $\mathcal{M}$:

$$\forall y \in U(\mathbf{x}^{\#}Ry \Rightarrow y \in V(\mathrm{P}))$$

But $\mathbf{x}^{\#}R\mathbf{y}^{\#}$. So $\mathbf{y}^{\#}$ is in $V(\mathrm{P})$ and thus $\mathbf{y}$ is in $V_n(\mathrm{P})$. The reasoning for ($\Leftarrow$) is parallel. If $\mathbf{y}$ is in $V_n(\mathrm{P})$ then $\mathbf{y}^{\#}$ is in $V(\mathrm{P})$, and hence by ($\Leftarrow$) in $\mathcal{M}$:

$$\forall x \in U(xR\mathbf{y}^{\#} \Rightarrow x \in V(\mathrm{P}))$$

But again $\mathbf{x}^{\#}R\mathbf{y}^{\#}$, so that $\mathbf{x}^{\#}$ is in $V(\mathrm{P})$ and $\mathbf{x}$ is in $V_n(\mathrm{P})$.

In the other case, $\mathbf{y} = (x_0)$. Note that here $\mathbf{x}^{\#} = x_i = x_n$, and $\mathbf{y}^{\#} = x_0$. For ($\Rightarrow$), suppose once more that $\mathbf{x}$ is in $V_n(\mathrm{P})$. So $x_n$ is in $V(\mathrm{P})$. This is the basis of an induction, backward, to show that $x_j$ is in $V(\mathrm{P})$ for each $j$ from $n$ through 0. Suppose for $0 < j \leqslant n$ that $x_j$ is in $V(\mathrm{P})$. Because $x_{j-1}Rx_j$, it follows by ($\Leftarrow$) in $\mathcal{M}$ that $x_{j-1}$ is in $V(\mathrm{P})$. So $x_0$ is in $V(\mathrm{P})$. Thus $(x_0)$ is in $V_n(\mathrm{P})$, which is to say that $\mathbf{y}$ is in $V_n(\mathrm{P})$. Finally, for ($\Leftarrow$), suppose that $\mathbf{y}$ is in $V_n(\mathrm{P})$. This is the basis of an induction, this time forward, to show that $x_j$ is in $V(\mathrm{P})$ for each $j$ from 0 through $n$. Suppose for $0 \leqslant j < n$ that $x_j$ is in $V(\mathrm{P})$. Because $x_jRx_{j+1}$ and ($\Rightarrow$) holds in $\mathcal{M}$, $x_{j+1}$ is in $V(\mathrm{P})$. So $x_n$ is in $V(\mathrm{P})$, which is to say that $\mathbf{x}^{\#}$ is. Consequently $\mathbf{x}$ is in $V_n(\mathrm{P})$.

By Lemma 3.7, it follows that every nontheorem of $D+$ is false in a *-symmetric model of this logic. Therefore $D+$ is a MacIntosh logic.

The reasoning for $T+$ is similar and uses the "reflexivization" of $\mathcal{M}_n$ described for Theorem 3.12. ∎

COROLLARY 5.2. (1) $D+ = MacD$. (2) $T+ = MacT$.

*Proof.* Clearly $D+ \subseteq MacD$ and $T+ \subseteq MacT$. By Theorem 5.1, $D+$ has every axiom and rule of inference that $MacD$ does, and similarly for $T+$ and $MacT$. So the inclusions are identities. ∎

Thus there is no difference between adding the MacIntosh rule and adding $\Diamond P \rightarrow P$ to $D_+$ or $T_+$, and it would seem that the argument – an *ad hominem* argument, be it noted – for adopting the rule loses cogency. But it may be that addition of further theses to $D+$ or $T+$ would bring to the fore considerations in favor of, or against, accepting the MacIntosh rule.

The general question of when rules may be carried forward in forming extensions of modal logics is vexing and deserves investigation on its own.

Let us close this section with the observation that $D+$, $T+$, and all the logics $KB^n+$ are genuine. To see this, it is enough to show that neither $\neg\Diamond P$ nor $\Box P$ is a theorem of these logics. Let $\mathcal{M} = \langle U, R, V\rangle$ be a model such that $U = \{x\}$, and $R = \{\langle x, x\rangle\}$. Note that no matter what $V(P)$ is, $\mathcal{M}$ will be reflexive, hence serial, and $n$-symmetric, and it will satisfy both $(\Rightarrow)$ and $(\Leftarrow)$. So it will be a model of $D+$, $T+$, and $KB_n+$. When $V(P) = \{x\}$, $\neg\Diamond P$ is false at $x$. So the logics do not contain $\neg\Diamond P$. When $V(P) = \emptyset$, $\Box P$ is false at $x$. So this formula is also outside all the logics.

## 6. CONCLUSION

Having gained some idea of what MacIntosh logics there are, we conclude this paper with a remark about the totality of them. Let the *territory* of a rule or condition be the class of all modal logics that have the rule or satisfy the condition. What is MacIntosh territory, the class of all normal logics with the MacIntosh rule, like? What is its structure?

The class of all normal logics is a lattice under the operations $\cdot$ and $+$, defined by:

$$L_1 \cdot L_2 = L_1 \cap L_2$$

$$L_1 + L_2 = \cap\{L : L_1 \subseteq L \,\&\, L_2 \subseteq L\}$$

We know that $L_1 \cdot L_2$ is a MacIntosh logic if $L_1$ and $L_2$ are. Thus if $L_1 + L_2$ is a MacIntosh logic whenever $L_1$ and $L_2$ are, it follows that MacIntosh territory is also a lattice under the operations $+$ and $\cdot$. Whether or not this is so is a question we had not been able to answer, but here again Timothy Williamson has enlightened us. He has proved in [12] that the answer is negative.

## NOTE

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