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Inversion of Real-Valued Functions and Applications

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Abstract: This work is devoted to a systematic study of the inversion of nondecreasing one variable extended real-valued functions. Its results are preparatory for a new duality theory for quasiconvex problem [6]. However the question arises in a variety of situations and as such deserves a separate treatment. Applications to topology, probability theory, monotone rearrangements, convex analysis are either pointed out or sketched.

Zusammenfassung: In dieser Arbeit wird systematisch die Umkehrung monoton nichtfallender Funktionen $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ studiert. Die Ergebnisse bilden die Grundlage für eine neue Dualitätstheorie quasikonvexer Probleme [6]. Da jedoch die Fragestellung bei einer ganzen Anzahl weiterer Situationen auftritt, verdient sie eine gesonderte Behandlung. Anwendungen in der Topologie, Wahrscheinlichkeitstheorie, monotonen Umordnungen und in der konvexen Analysis werden aufgezeigt und skizziert.

Key words: Quasi-inverse, epi-inverse, hypo-inverse, nondecreasing functions, quasiconvex functions, convex functions, duality, rearrangement, modulus of continuity.

This work is devoted to a systematic study of the invertibility of nondecreasing mappings from $\mathbb{R}$ to $\mathbb{R}$. This elementary problem arises in many questions; to our knowledge it has not been tackled in its full generality (however see [10] where a notion of quasi-inverse different from the one we use is introduced).

When $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing there is no problem for defining an inverse mapping from $f(\mathbb{R})$ into $\mathbb{R}$. When f is just nondecreasing ($r \leqslant r' \Rightarrow f(r) \leqslant f(r')$) several

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notions of inverse can be accepted. The best notion requires that the inverse g (called then the epi-hypo-inverse) satisfies the equivalence, for r, s in $\mathbb{R}$

$$(g(s) \leqslant r) \Leftrightarrow (s \leqslant f(r)).$$

This is a very restrictive condition and we have to relax it in order to be able to deal with situations in which this condition cannot be fulfilled. We get then the concept of quasi-inverse mappings: it is a symmetric notion sufficiently general for encompassing usual situations but uniqueness is not ensured. Refining it we get the former notion and intermediate concepts which appear to be useful. All these notions rely on the use of the epigraphs and the hypographs of functions. This is not a great surprise as we are interested in solving inequalities rather than equalities. Moreover the role of epigraphs in optimization theory and "unilateral analysis" (an expression coined by J.-J. Moreau) is known to be prominent.

Our first section is devoted to an exposition of these different notions. The set of quasi-inverses of a given nondecreasing function is characterized in section 2. We show in section 3 how quasi-inverses can be used for regularization processes of one variable functions. From this, one can pass to the regularization of some classes of functions of several variables. For instance, using the fact that a lower semicontinuous (l.s.c.) quasiconvex function on a locally convex topological vector space is a supremum of a family of functions of the form $g \circ h$ where $g : \mathbb{R} \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is l.s.c. and nondecreasing and h is a continuous linear functional (such a function is called a l.s.c. quasi-affine function) a formula can be given for computing the greatest l.s.c. quasiconvex function f^q minorizing a given function f. This formula readily yields duality results for quasiconvex optimization problems. Although this was the starting point of our study (see [1], [2], [4]) we do not consider this application here but refer to [6], as we believe that it deserves a separate treatment. Among several possible applications we point out the study of maximal monotone relations from $\mathbb{R}$ into $\mathbb{R}$ (section 2) and throw a glimpse at the theory of nondecreasing rearrangement (see [3], [5], [11] and their references). Other applications can be given in the geometry of Banach spaces and probability theory (c.f. [3]).

In the domain of one variable convex functions, quasi-inverses lead us to an extended Young formula (proposition 4.6) and to connect the left derivative of a closed proper convex function to the right derivative of its Fenchel conjugate (proposition 4.7). Quasi-inverses are also used in the proof of the perfect duality between the strict monotonicity of the one sided derivatives of a one variable finite convex function and the coincidence of the one-sided derivatives of its Fenchel conjugate (corollary 4.8).

1 Various Notions of Quasi-Inverses

In what follows we denote by $F = \bar{\mathbb{R}}^{\mathbb{R}}$ the set of all mappings from $\mathbb{R}$ into $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; we denote by G (resp. Q, resp. R) the subset of F consisting of nondecreasing functions (resp. nondecreasing and lower-semicontinuous (l.s.c.), resp. nondecreasing and upper-semicontinuous (u.s.c.) functions). The epigraph and the hypograph of $f \in F$ are defined respectively by

$$E(f) = \{(r, s) \in \mathbb{R}^2 : f(r) \leqslant s\}, H(f) = \{(r, s) \in \mathbb{R}^2 : s \leqslant f(r)\},$$

while the strict epigraph $E_s(f)$ and the strict hypograph $H_s(f)$ are defined similarly with the inequality replaced by the strict inequality.

Our basic concept of inversion is well suited for dealing with inequations: as it is not too restrictive it appears in many concrete instances.

1.1 Definition: Two elements f and g of F are said to be *quasi-inverses* if for any r and s in $\mathbb{R}$ one has the following implications

(a) $s < f(r) \Rightarrow g(s) \leqslant r$

(b) $g(s) < r \Rightarrow s \leqslant f(r)$.

This notion gives a symmetric role to f and g since (a) (resp. (b)) implies (a′) (resp. (b′)) and is in fact equivalent to it:

(a′) $r < g(s) \Rightarrow f(r) \leqslant s$

(b′) $f(r) < s \Rightarrow r \leqslant g(s)$.

In other terms, f and g are quasi-inverses iff

$$H_s(f) \subset E(g)^{-1}, \quad E_s(g)^{-1} \subset H(f),$$

iff

$$H_s(g)^{-1} \subset E(f), \quad E_s(f) \subset H(g)^{-1},$$

where, for a subset A of $\mathbb{R} \times \mathbb{R}$, A^{-1} denotes the set

$$A^{-1} = \{(y, x) \in \mathbb{R} \times \mathbb{R} : (x, y) \in A\},$$

a notation compatible with the inversion of relations (multifunctions), a relation being identified with its graph.

We shall show below that any $f \in G$ has at least one quasi-inverse (in general an infinity), and we shall characterize all the quasi-inverses of f. It will be shown that if g is a quasi-inverse of f then its lower semicontinuous hull g^L and its upper semicontinuous hull g_U are also quasi-inverses of f. Let us observe that supposing $f \in G$ is not an artificial restriction.

1.2 Lemma: If f has a quasi-inverse g then f is nondecreasing: $f \in G$.

Proof: Suppose we can find r, r' in $\mathbb{R}$ with $r < r'$, $f(r') < f(r)$. Taking $s \in \mathbb{R}$ with $f(r') < s < f(r)$ it follows from (a) that $g(s) \leqslant r < r'$, hence, by (b) $s \leqslant f(r')$, a contradiction with the choice of s. □

It can be useful to extend definition 1.1 to mappings $f : I \to \overline{\mathbb{R}}$, $g : J \to \overline{\mathbb{R}}$ defined on open intervals I, J: one takes $(r, s) \in I \times J$ in definition 1.1. In particular in example 1.5 below we take $I = J = \mathbb{P}$, the set of positive real numbers.

1.3 Proposition: Let $f : I \to \overline{\mathbb{R}}$, $g : J \to \overline{\mathbb{R}}$ be quasi-inverse mappings, where $I = (\alpha_I, \omega_I)$, $J = (\alpha_J, \omega_J)$ are nonempty open intervals of $\mathbb{R}$. When $f(I) \subset \bar{J}$ the following properties are equivalent:

(i) $\sup g(J) < \omega_I$ (resp. $\inf g(J) > \alpha_I$)

(ii) there exists $\bar{r} \in I$ with $f(\bar{r}) = \omega_J$ (resp. $f(\bar{r}) = \alpha_J$).

Proof: Suppose $\sup g(J) < \omega_I$. Let $\bar{r} \in I$ with $\sup g(J) < \bar{r} < \omega_I$. As $g(s) < \bar{r}$ for each $s \in J$ we have $s \leqslant f(\bar{r})$ for each $s \in J$ hence $f(\bar{r}) = \omega_J$.

Conversely if $f(\bar{r}) = \omega_J$ for some $\bar{r} \in I$, then for each $s \in J$ we have $g(s) \leqslant \bar{r}$ since otherwise we would have $g(s) > \bar{r}$, $s \geqslant f(\bar{r})$, a contradiction with $s < \omega_J$. The proof of the bracketted assertions is similar. □

1.4 Corollary: Let $f : \mathbb{R} \to \overline{\mathbb{R}}, g : \mathbb{R} \to \overline{\mathbb{R}}$ be quasi-inverse mappings. Then the following assertions are equivalent:

(i) f is finite valued;

(ii) $\sup g = +\infty, \inf g = -\infty$.

1.5 Example: Given two metric spaces (X, d_X), (Y, d_Y) and a mapping $f : X \to Y$, its *modulus of uniform continuity* $\mu : \mathbb{P} \to \overline{\mathbb{R}}_+ = [0, +\infty]$ is given by

$$\mu(r) = \sup \{d_Y(f(x), f(x')) : d_X(x, x') \leqslant r\}, \quad r \in \mathbb{P} = (0, +\infty);$$

f is uniformly continuous iff $\inf_{\mathbb{P}} \mu = 0$ iff $\lim_{r \to 0+} \mu(r) = 0$ (obviously μ is nondecreasing). Let $\delta : \mathbb{P} \to \overline{\mathbb{R}}_+$ be any quasi-inverse of μ. Then proposition 1.3 shows that $\inf_{\mathbb{P}} \mu = 0$ iff $\delta(\epsilon) > 0$ for each $\epsilon \in \mathbb{P}$. The use of a mapping $\delta : \mathbb{P} \to \overline{\mathbb{R}}_+$ such that $\delta(\epsilon) > 0$ for each $\epsilon > 0$ and

$$d_X(x, x') < \delta(\epsilon) \Rightarrow d_Y(f(x), f(x')) \leqslant \epsilon$$

is quite familiar. Note that the choice of δ contains some arbitrariness while μ is intrinsically tied to f. We will see that two optimal quasi-inverses μ^e, μ^h of μ can be defined so that in fact two canonical mapping $\delta_e = \mu^e$ and $\delta_h = \mu^h$ can be attached to f in an intrinsic way. Moreover it will be shown that a slight modification of the definition of the modulus of uniform continuity yields a more handable notion. Namely, setting

$$\hat{\mu}(r) = \sup \{d_Y(f(x), f(x')) : d_X(x, x') < r\}$$

and

$$\hat{\delta}(s) = \sup \{t : \hat{\mu}(t) \leqslant s\}$$

it will be shown that one has the convenient equivalence

$$r \leqslant \hat{\delta}(s) \Leftrightarrow \hat{\mu}(r) \leqslant s.$$

Here

$$\hat{\delta}(\epsilon) = \inf \{r \in \mathbb{P} : \exists\, (x, x') \in X^2, d_X(x, x') = r, d_Y(f(x), f(x')) > \epsilon\}.$$

1.6 Example: Let E be a normed vector space with topological dual space E' and let $F : E \to E'$ be a *firmly monotone relation,* i.e. a relation such that for some non-decreasing *firm* function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e., such that $(t_n) \to 0$ whenever the sequence $(\gamma(t_n))$ converges to 0 or, as $\gamma \in G$, such that $\gamma(t) > 0$ for $t > 0$) one has

$$\langle x' - y', x - y \rangle \geqslant \gamma(\|x - y\|) \quad \|x - y\|$$

for any $x, y \in \operatorname{dom} F$, $x' \in F(x), y' \in F(y)$. Then one has

$$\|x' - y'\| \geqslant \gamma(\|x - y\|)$$

so that $F^{-1} : F(E) \to E$ is a well defined mapping and is uniformly continuous since for any quasi-inverse $\delta : \mathbb{P} \to \overline{\mathbb{R}}_+$ of $\gamma | \mathbb{P}$ one has

$$x', y' \in F(E), \|x' - y'\| < \epsilon \Rightarrow \|F^{-1}(x') - F^{-1}(y')\| \leqslant \delta(\epsilon),$$

while proposition 1.3 shows that $\lim_{\epsilon \to 0+} \delta(\epsilon) = \inf_{\epsilon > 0} \delta(\epsilon) = 0$ as γ is firm. As the notion of quasi-inverse is rather weak it will be useful to consider a stronger inversion property. Its drawback lies in its lack of symetry.

1.7 Definition: An element g is said to be an *epi-inverse* of $f \in G$ if assertions (a) and ($\bar{\text{b}}$) below hold true:

($\bar{\text{b}}$) $g(s) \leqslant r \Rightarrow s \leqslant f(r)$.

Thus g is an epi-inverse of f iff

$$H_s(f)^{-1} \subset E(g) \subset H(f)^{-1}.$$

Similarly, g is called an *hypo-inverse* of f when

$$E_s(f)^{-1} \subset H(g) \subset E(f)^{-1}$$

i.e. when (b) and ($\bar{a}$) hold true, with

($\bar{a}$) $r \leqslant g(s) \Rightarrow f(r) \leqslant s.$

These two notions are clearly different. Let us observe that when $g(s)$ is finite ($\bar{a}$) (resp. ($\bar{b}$)) is equivalent to $f(g(s)) \leqslant s$ (resp. $f(g(s)) \geqslant s$). Therefore when $g : \mathbb{R} \to \mathbb{R}$ is both an epi-inverse of f and an hypo-inverse of f it is a right inverse of f. It will be more useful to consider another combination of the preceding concepts.

1.8 Definition: An element g of G is said to be an *epi-hypo-inverse* of $f \in G$ if g is an epi-inverse of f and f is an hypo-inverse of g. Then f is said to be an *hypo-epi-inverse* of g.

Then we have simultaneously ($\bar{a}$) in which the roles of f and g are interchanged and ($\bar{b}$) so that

$$(g(s) \leqslant r) \Leftrightarrow (s \leqslant f(r))$$

or

$$E(g) = H(f)^{-1}.$$

This formula shows that if $f \in G$ has an epi-hypo-inverse, then it can have only one. Let us observe that a function $f \in G$ may have several epi-inverses. For instance, if f is constant with value 0, then any $g \in F$ with $g(s) = -\infty$ for $s < 0$, $g(s) = +\infty$ for $s > 0$ is an epi-inverse of f. However we do have a converse uniqueness result.

1.9 Lemma: If g is an epi-inverse of $f_1 \in G$ and $f_2 \in G$ then $f_1 = f_2$.

Proof: As for $i = 1, 2$ we have $H_s(f_i) \subset E(g)^{-1} \subset H(f_i)$, we get for any $r \in \mathbb{R}$

$$f_i(r) = \sup \{s : (r, s) \in H(f_i)\} = \sup \{s : (r, s) \in H_s(f_i)\}$$

hence

$$f_i(r) = \sup \{s : (r, s) \in E(g)^{-1}\}. \qquad \square$$

The following result is an obvious necessary condition for g to be an epi-hypo-inverse of some $f \in G$ as for each $r \in \mathbb{R}$ $g^{-1}(]-\infty, r]) =]-\infty, f(r)]$.

1.10 Lemma: If g is an epi-hypo-inverse then g is l.s.c.

Similarly, we see that any hypo-epi-inverse is u.s.c.

The versatility of the notion of quasi-inverses can be tested with the following result which will be used in [6].

1.11 Lemma: Let $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ be two families of elements of G such that for each $i \in I$ f_i and g_i are quasi-inverses. Then $f := \inf_{i\in I} f_i$ and $g := \sup_{i\in I} g_i$ are quasi-inverse. Moreover when f_i is the hypo-epi-inverse of g_i for each $i \in I$, then f is the hypo-epi-inverse of g.

This follows from the following relations

$$H_s(f) \subset \bigcap_{i\in I} H_s(f_i), \quad E(g) = \bigcap_{i\in I} E(g_i)$$

$$H(f) = \bigcap_{i\in I} H(f_i), \quad E_s(g) \subset \bigcap_{i\in I} E_s(g_i).$$

2 Characterization of Quasi-Inverses

Up to now epigraphs and hypographs have occupied the stage while graphs remainded in the shadow. Nevertheless graphs are likely to be of crucial importance for defining generalized inverses. As the inverse M^{-1} of a relation (or multifunction) $M : X \to Y$

given by

$$M^{-1} = \{(y, x) \in Y \times X : (x, y) \in M\},$$

(a relation being identified with its graph) is a relation even when M is a mapping we are led to consider monotone relations rather than non-decreasing functions. Furthermore, the domain of M^{-1} being the image of M, it is advisable to "complete M by vertical segments" so that its image becomes as large as possible. More precisely, any monotone relation $A : \mathbb{R} \to \mathbb{R}$ (i.e. any relation such that for $(r, s) \in A$, $(r', s') \in A$ one has $s' \geqslant s$ whenever $r' > r$) is contained in a maximal monotone relation (in fact a largest monotone relation containing A) $M : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, where $M(-\infty) = [-\infty, \inf A(\mathbb{R})]$, $M(+\infty) = [\sup A(\mathbb{R}), +\infty]$, and for $r \in \mathbb{R}$

$$M(r) = [\sup \bigcup_{r' < r} A(r') \cup \{-\infty\},\ \inf \bigcup_{r'' > r} A(r'') \cup \{+\infty\}].$$

Then $M^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, also is maximal monotone and its restriction $M^{-1} \cap \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ again is maximal monotone as a relation from $\mathbb{R}$ to $\mathbb{R}$.

Now with any relation $N : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ with domain $\mathbb{R}$ are associated the two functions e_N and h_N given by

$$e_N(r) := \inf N(r)$$

$$h_N(r) := \sup N(r);$$

we have $e_N \leqslant h_N$ and when N is monotone e_N and h_N are nondecreasing. Moreover, when N is maximal monotone its graph is obviously closed so that for any $r \in \overline{\mathbb{R}}$

$$N(r) = [e_N(r), h_N(r)]$$

$$e_N(r) = \sup \{s' : \exists\, r' < r, (r', s') \in N\}$$

$$h_N(r) = \inf \{s'' : \exists\, r'' > r, (r'', s'') \in N\},$$

e_N is l.s.c. and h_N is u.s.c. When one takes $N = M^{-1}$, where M is the maximal monotone relation containing the graph of a given $g \in G$ one gets two quasi-inverses of g which appear to be of fundamental importance. Let us give first a direct naive definition which can be phrased for any mapping in F.

2.1 Definition: Given $f \in F$ one sets for $s \in \mathbb{R}$

$$f^e(s) = \inf \{r \in \mathbb{R} : s \leqslant f(r)\}$$

$$f^h(s) = \inf \{r \in \mathbb{R} : s < f(r)\}.$$

Obviously one has $f^e \leqslant f^h$. Moreover for $f \in G$ one also has

$$f^e(s) = \sup \{t \in \mathbb{R} : f(t) < s\}$$

$$f^h(s) = \sup \{t \in \mathbb{R} : f(t) \leqslant s\},$$

since we have partitions of $\mathbb{R}$ into two intervals; furthermore for $f \in G$ and $s \in f(\mathbb{R})$ one has

$$f^e(s) = \inf f^{-1}(s), \quad f^h(s) = \sup f^{-1}(s).$$

Taking $N = M^{-1}$, where M is the maximal monotone relation containing the graph of f, i.e. the maximal monotone relation N containing $A = f^{-1}$ what precedes shows that for each $s \in \mathbb{R}$

$$e_N(s) = \sup \{t \in \overline{\mathbb{R}} : \exists\, s' \in \mathbb{R}, s' < s, (s', t) \in A\}$$

$$= \sup \{t \in \mathbb{R} : \exists\, s' \in \overline{\mathbb{R}}, s' < s, s' = f(t)\} = f^e(s),$$

and similarly $h_N(s) = f^h(s)$.

The following result whose proof is elementary describes the main properties of f^e and f^h.

2.2 Proposition:

a) For any $f \in G$, f^e and f^h are quasi-inverses of f. More precisely, f is an epi-inverse of f^h and an hypo-inverse of f^e.

b) An element g of G is a quasi-inverse of f iff $f^e \leqslant g \leqslant f^h$.

c) For each $s \in \mathbb{R}$ one has $\sup_{s'<s} f^h(s') \leqslant f^e(s) \leqslant f^h(s) \leqslant \inf_{s''>s} f^e(s'')$.

d) A quasi-inverse g of f coincides with f^e iff g is l.s.c.

e) A quasi-inverse g of f coincides with f^h iff g is u.s.c.

Proof: Let us observe that implications (a) and (b) of definition 1.1 can be rewritten

$$(a) \Leftrightarrow g(s) \leqslant \inf \{r : s < f(r)\} := f^h(s)$$

$$(b) \Leftrightarrow (b') \Leftrightarrow g(s) \geqslant \sup \{r : f(r) < s\} := f^e(s)$$

so that f and g are quasi-inverses iff $f^e \leqslant g \leqslant f^h$.

The inclusion $H(f) \subset E(f^e)^{-1}$ (resp. $E(f) \subset H(g)^{-1}$) follows from the first (resp. second) characterization of f^e (resp. f^h). This proves assertions (a) and (b). Assertion c) follows from the facts observed before definition 2.1 or from a direct argument: for any $s' < s$ we have $f^h(s') \leqslant f^e(s)$ since for any $r \in \mathbb{R}$ with $s \leqslant f(r)$ and any $t \in \mathbb{R}$ with $f(t) \leqslant s'$ we have $f(t) < f(r)$ hence $t \leqslant r$.

Assertion c) shows that f^e (resp. f^h) is left (resp. right) continuous hence l.s.c. (resp. u.s.c.).

Now if g is a l.s.c. quasi-inverse of f, for any $s \in \mathbb{R}$ we have $f^e(s) \leqslant g(s) = \sup_{s'<s} g(s') \leqslant \sup_{s'<s} f^h(s') \leqslant f^e(s)$, and $g = f^e$. The proof of assertion e) is similar. □

As an obvious consequence of proposition 2.2 we have that $f \in G$ has a continuous quasi-inverse iff it has a unique quasi-inverse.

The idea of the following statement has been suggested to us by R. Correa.

2.3 Corollary: Let I be an open interval of $\mathbb{R}$ and let $f : I \to \overline{\mathbb{R}}$ be nondecreasing. Then f has a continuous quasi-inverse iff f is (strictly) increasing on $D = f^{-1}(\mathbb{R})$.

The proof relies on Proposition 2.2d), e) and on the following obvious but useful observation.

2.4 Lemma: For any $f \in G$, $s \in \mathbb{R}$, one has

$$]f^e(s), f^h(s)[\subset f^{-1}(s) \subset [f^e(s), f^h(s)].$$

2.5 Remark: The conclusion of corollary 2.3 fails when D is replaced by I, as shown by the following example: $I = \mathbb{R}$, $f(t) = -\infty$ for $t \leqslant 0$, $f(t) = +\infty$ for $t > 0$ so that f is not strictly increasing but has as a continuous quasi-inverse the constant mapping with value 0. □

Proposition 2.2.a) and Lemma 1.10 show that f^h cannot be the hypo-inverse of f (and f the epi-hypo-inverse of f^h) unless f is l.s.c. In fact this condition is also sufficient.

2.6 Proposition: For any $f \in G$, f^h is the hypo-inverse of f iff f is l.s.c.

Proof: Let us suppose f is l.s.c. As we already know that $E(f) \subset H(f^h)^{-1}$ we only have to prove the opposite inclusion. Let $(r, s) \in H_s(f)$, the complement of $E(f)$. As $s < f(r)$ and f is l.s.c. we can find $r' < r$ such that $s < f(t)$ for each $t \in [r', r]$. As f is nondecreasing $f(t) \leqslant s$ can occur only for $t < r'$ (since $s < f(r')$), and we get $f^h(s) \leqslant r' < r$ or $(r, s) \in E_s(f^h)^{-1}$, the complement of $H(f^h)^{-1}$. □

Let us present an easy variant of the preceding result; we give a direct proof although it is a consequence of lemmas 1.9 and 1.10.

2.7 Proposition: If f is an epi-inverse of some $g \in G$ then $f^h = g$.
If f is an hypo-inverse of some $g \in G$ then $f^e = g$.

Proof: It suffices to prove the first assertion, the second one being similar.
As $H_s(g)^{-1} \subset E(f) \subset H(g)^{-1}$ we have

$$r < g(s) \Rightarrow f(r) \leqslant s \Rightarrow r \leqslant g(s)$$

hence

$$f^h(s) := \sup \{t \in \mathbb{R} : f(t) \leqslant s\} = g(s).$$ □

3 Inversion and Regularization

In this section we intend to show that the notions of generalized inverses we introduced can be useful for regularization of functions in F. Let us recall the following notion and introduce some notations and terminology.

3.1 Definition: Given a subclass K of F the K-lower hull of $f \in F$ is

$$f^K := \sup K(f) \quad \text{where} \quad K(f) = \{k \in K : k \leqslant f\}.$$

Similarly the K-upper hull of f is given by

$$f_K := \inf \{k \in K : k \geqslant f\}.$$

When K is stable under suprema (i.e. $\sup L \in K$ for each subfamily L of K) then $f^K \in K$ and f^K is the greatest element of K minorizing f. More generally, introducing $S(K)$, the smallest subclass of F containing K and stable under suprema, we get that $f^K \in S(K)$ and f^K is the greatest element of $S(K)$ minorizing f. Moreover $S(K) = \{f^K : f \in F\}$. For instance when K is the set L of l.s.c. functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$ we have $S(L) = L$ and for each $f \in F$ f^L is given by

$$f^L(r) = \liminf_{s \to r} f(s)$$

or the equality $E(f^L) = \operatorname{cl} E(f)$, where $\operatorname{cl} A$ *denotes the closure* of the subset A of $\mathbb{R}^2$. Similarly if U denotes the set of upper semi-continuous (u.s.c.) functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$

$$f_U(r) = \limsup_{s \to r} f(s) = -(-f)^L(r).$$

When $f \in G$ the preceding formulas can be simplified into

$$f^L(r) = \sup \{f(t) : t < r\}, f_U(r) = \inf \{f(s) : s > r\}.$$

Several other cases of interest are described in the two following lemmas.

3.2 Lemma: The G-(lower) hull f^G of $f \in F$ is given by

$$f^G(r) = \inf \{f(s) : s \geqslant r\}.$$

Proof: Obviously, the right hand side defines a nondecreasing function g of r and $g \leqslant f$. Moreover, for any $h \in G$ with $h \leqslant f$ and any $s \geqslant r$ we have $h(r) \leqslant h(s) \leqslant f(s)$, hence $h(r) \leqslant g(r)$. Therefore $g = f^G$. □

3.3 Lemma: The Q-hull f^Q of $f \in F$ is given by $f^Q = f^{GL} = f^{LG}$. Moreover $f^Q = ((f^G)_U)^L$:

$$f^Q(r) = \sup_{t<r} \inf_{s \geqslant t} f(s) = \sup_{t<r} \inf_{s>t} f(s).$$

In particular, for $g \in G$, $g^Q = g^L$.

Proof: As $Q(f) \subset G(f)$ and $Q(f) \subset L(f)$ we have $f^Q \leqslant f^G$ and $f^Q \leqslant f^L$. As $f^Q \in L$ and $f^Q \in G$ we deduce from these inequalities that $f^Q \leqslant f^{GL}, f^Q \leqslant f^{LG}$. Now the formula giving g^L for $g \in G$ shows that $g^L \in G$ when $g \in G$ so that $(f^G)^L \in Q$ hence $f^{GL} \leqslant f^Q$ and $f^{GL} = f^Q$ by what precedes. Similarly, one easily sees that h^G is l.s.c. whenever h is l.s.c. so that $f^{LG} \in Q$ and the definition of f^Q yields $f^Q = f^{LQ}$.

The equality $f^Q(r) = \sup_{t<r} \inf_{s \geqslant t} f(s)$ is just a rewriting of the relation $f^Q(r) = f^{GL}(r)$. In order to obtain the last equality let us set $h(t) = \inf_{s>t} f(s)$ so that $f^G \leqslant h$ and $f^Q = f^{GL} \leqslant h^L$. Now for each $t < r$ we have $h(t) \leqslant f(r)$ hence

$$h^L(r) = \sup_{t<r} h(t) \leqslant f(r)$$

and $h^L = h^Q \leqslant f^Q$ so that $h^L = f^Q$. □

The following results point out the links between hulls and quasi-inverses.

3.4 Proposition: Let $f \in G$ and let $g \in G$ be any quasi-inverse of f. Then g^e is the l.s.c. hull f^L of f. In particular $f^{ee} = f^L = f^{he}$.

Proof: We have to show that for any $r \in \mathbb{R}$

$$g^e(r) := \inf \{s : r \leqslant g(s)\} = \sup \{f(t) : t < r\} := f^L(r)$$

For any $s \in \mathbb{R}$ such that $r \leqslant g(s)$ and any $t < r$ we have $t < g(s)$ hence $f(t) \leqslant s$ as f is a quasi-inverse of g, so that $f^L(r) \leqslant g^e(r)$. Now for any $q < g^e(r) = \sup \{t : g(t) < r\}$ we can find $t > q$ with $g(t) < r$. Then, as g is a quasi-inverse of f, for any $r' \in]g(t), r[$ we have $t \leqslant f(r')$, hence $q < f(r')$ so that $q < f^L(r)$. Thus $g^e(r) = f^L(r)$. □

3.5 Proposition: Let $f \in G$ and let $g \in G$ be any quasi-inverse of f. Then g^h is the u.s.c. (upper) hull f_U of f. In particular $f^{eh} = f_U = f^{hh}$.

The proof is similar to the preceding one. □

3.6 Remark: Proposition 2.6 is a consequence of proposition 3.4 since for $f \in Q$ $f = f^{he}$ so that by proposition 2.2a) f^h is an hypo-inverse of $f^{he} = f$ and f is an epi-inverse of f^h.

In a similar way we get from proposition 3.5 that for $f \in G \cap U$ f^e is an epi-hypo-inverse of f since $f = f^{eh}$. The following result is used in [6].

3.7 Proposition: For each $f \in G$, $(f^L)^h = f^h$. If g is a quasi-inverse of f then g_U is the hypo-epi-inverse of f^L.

Proof: As by proposition 2.6 and 3.5 $g_U = f^h$ and $(f^L)^h$ is the hypo-epi-inverse of f^L it suffices to prove that $(f^L)^h = f^h$. Since $f^L \leqslant f$ we have $f^h \leqslant (f^L)^h$. Now given $s \in \mathbb{R}$, for each $q < (f^L)^h(s)$ we can find $t > q$ with $f^L(t) \leqslant s$. As $f^L(t) = \sup \{f(t') : t' < t\}$, we get $f(q) \leqslant s$ and $f^h(s) \geqslant q$. Thus $f^h(s) = (f^L)^h(s)$. □

3.8 Corollary: Let $f \in G$ and let $g \in G$ be any quasi-inverse of f. Then $f^h = g^{eh}$. In particular for any $f \in G$ one has $f^h = f^{heh} = f^{eeh}$.

Proof: The first assertion follows from propositions 3.4 and 3.7 as $f^L = g^e$ and $f^h = (f^L)^h$. The second is then a consequence of proposition 2.2a). □

A similar stabilization property holds true for $f^e : f^e = f^{ehe} = f^{hhe}$. Let us note the following observation.

3.9 Proposition: For any $f, g \in G$, the following properties are equivalent:

a) f and g have the same l.s.c. hull: $f^L = g^L$;

b) f and g have the same u.s.c. hull: $f_U = g_U$;

c) f and g generate the same maximal monotone relation in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

Proof: The equivalence between a) and b) follows from lemma 3.3. In fact, for a) ⇒ b) we have

$$f_U = (f^L)_U = (g^L)_U = g_U$$

and for b) ⇒ a)

$$f^L = (f_U)^L = (g_U)^L = g^L.$$

The equivalence between a) and b) on one hand and c) on the other hand is due to the fact that the maximal monotone relation M in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ generated by $k \in G$ is given by $M(-\infty) = [-\infty, \inf k^L], M(+\infty) = [\sup k_U, +\infty], M(t) = [k^L(t), k_U(t)]$ for any $t \in \mathbb{R}$. □

3.10 Remark: When f and g are not identically $+\infty$ neither $-\infty$, the assertions of proposition 3.9 are also equivalent to the fact that f and g generate the same maximal monotone relation in $\mathbb{R} \times \mathbb{R}$.

4 Applications

A Duality Schemes

The following example abstracts an approach of duality theory for quasiconvex problems due to J.-P. Crouzeix.

Let X be a set and let f, g be two mappings from X into $\overline{\mathbb{R}}$. Given $r \in \mathbb{R}$ we define the *strict slice* $S(f, r)$ of f at level r as

$$S(f, r) = \{x \in X : f(x) < r\}$$

while the *slice* (or trench) of f at level r, $T(f, r)$ is obtained by replacing the strict inequality by inequality. Then one can set

$$g_f(r) = \sup \{g(x) : x \in T(f, r)\}.$$

When X is a topological vector space and g is a continuous linear functional $g_f(r)$ appears as the support function of the trench $T(f, r)$ of f. When f is quasiconvex and l.s.c., i.e. $T(f, r)$ is closed convex for each $r \in \mathbb{R}$, then $\{g_f(.) : g \in X^*\}$ characterizes the family $\{T(f, r) : r \in \mathbb{R}\}$, hence f. This explains the importance of the function g_f. Here we do not make any assumption of this type. Along with g_f we define

$$f_g^-(r) = \inf \{f(x) : g(x) \geqslant r\}$$
$$= -(-f)_{-g}(-r).$$

4.1 Lemma ([2] Prop. 1, 2, 3): The functions g_f and f_g^- are nondecreasing and are quasi-inverses.

The proof is easy and similar to the proof of the following result in which we use the following variants of the preceding functions:

$$\hat{g}_f(s) = \sup \{g(x) : x \in S(f, s)\}$$

$$\hat{f}_g^-(r) = \inf \{f(x) : g(x) > r\} = -(-\hat{f})_{-g}(-r).$$

4.2 Proposition: The function $\hat{f}_g^-$ is the hypo-epi-inverse of $\hat{g}_f$.

This follows from the following equivalences for each $(r, s) \in \mathbb{R}^2$:

$$[\hat{f}_g^-(r) \geqslant s] \Leftrightarrow [\forall\, x \in X, g(x) > r \Rightarrow f(x) \geqslant s]$$
$$\Leftrightarrow [\forall\, x \in X, f(x) < s \Rightarrow g(x) \leqslant r] \Leftrightarrow [\hat{g}_f(s) \leqslant r]. \qquad \square$$

Let us observe that for $X = \mathbb{R}$, $f \in G$, $g = I$, the identity mapping of $\mathbb{R}$, we have $g_f = f^h$, $\hat{g}_f = f^e$ while $f_g^- = f$, $\hat{f}_g^- = f_U$. Thus the present situation encompasses the framework of the preceding sections. However the results of these sections can be used here. In particular [2] corol. 4 is a consequence of our proposition 2.4b) while [2] prop. 5 follows from our proposition 3.4. Other results can also be deduced from the previous sections. Observing that

$$S(f, r) = \bigcup_{t<r} T(f, t),$$

$$\{x : g(x) > r\} = \bigcup_{s>r} \{x : g(x) \geq s\},$$

we have that

$$\hat{g}_f(r) = \sup_{t<r} g_f(t) = (g_f)^Q(r)$$

$$\hat{f}_g^-(r) = \inf_{s>r} f_g^-(s) = (f_g^-)_U(r)$$

Using lemma 3.3 we can conclude:

4.3 Proposition: The mappings f_g^- and $\hat{f}_g^-$ have the same Q-hull. Moreover $\hat{g}_f = (g_f)^Q$.

This result enables one to give new duality relations for quasiconvex problems. See [6] for details.

What precedes can be applied in other situations. For instance one can take for X a metric space and, given an element w of X, one can set $g(x) = -d(x, w)$. Then

$$g_f(r) = -d(w, T(f, r)), \quad \hat{g}_f(r) = -d(w, S(f, r))$$

while

$$f_g^-(r) = \inf \{f(x) : x \in D(w, -r), \quad \hat{f}_g^-(r) = \inf \{f(x) : x \in B(w, -r)\}$$

where $B(w, -r)$ (resp. $D(w, -r)$) is the open (resp. closed) ball with center w and radius $-r$ and $d(w, A) = \inf \{d(w, a) : a \in A\}$. Using similar techniques as in [6] one can show that the lower semicontinuous hull f^L of f can be written as

$$f^L = \sup_{i \in I} h_i \circ (-d(w_i, .))$$

where $(w_i)_{i \in I}$ is a family of points of X and $(h_i)_{i \in I}$ a family of nondecreasing l.s.c. function from $\mathbb{R}$ into $\overline{\mathbb{R}}$.

B Applications to One Variable Convex Functions

Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a non-decreasing function, finite at some $a \in \mathbb{R}$. It is well-known (see [7]–[9] for instance) that F given by

$$F(t) = \int_a^t f(r)dr, \qquad t \in \mathbb{R}$$

is a closed proper convex function with left and right derivatives F'_-, F'_+ respectively which coincide with the l.s.c. hull f^L of f and the u.s.c. hull f_U of f respectively. Moreover the subdifferential of F at t, that is

$$\partial F(t) = \{u \in \mathbb{R} : \forall\ s \in \mathbb{R}, F(s) - F(t) \geqslant u(s-t)\}$$

is given by

$$\partial F(t) = [F'_-(t), F'_+(t)] \cap \mathbb{R} = [f^L(t), f_U(t)] \cap \mathbb{R}.$$

Let $g : \mathbb{R} \to \overline{\mathbb{R}}$ be a quasi-inverse of f, finite at some $b \in \mathbb{R}$ and let G be the convex function given by

$$G(u) = \int_b^u g(s)ds.$$

Using our observations in section 2 we see that ∂G is the inverse multifunction of ∂F so that G differs from the conjugate F^* of F given by

$$F^*(u) = \sup\ \{u \cdot r - F(r) : r \in \mathbb{R}\}$$

by an additive constant c. We intend to give an explicit calculation of this constant in terms of f and g. This will result from the following lemma.

4.4 Lemma: For each real number u such that $g(u)$ is finite one has

$$F^*(u) = ug(u) - F(g(u))$$

This formula is akin to the definition of the Legendre transform of F but here no differentiability assumption on F is made and f and g are not supposed to be bijective.

Proof: In the definition of $F^*(u)$ the supremum is attained at some $t \in \mathbb{R}$ iff $u \in \partial F(t)$, i.e., using the preceding formula for $\partial F(t)$, iff

$$f^L(t) \leqslant u \leqslant f_U(t)$$

or, by proposition 3.7, iff

$$g^L(u) \leqslant t \leqslant g_U(u).$$

This occurs with $t = g(u)$ when $g(u)$ is finite, hence the formula holds true. □

Taking $u = b$ we get that $c := F^*(b) - G(b) = F^*(b) = bg(b) - F(g(b))$, so that, exchanging the roles of F and G and noting that $F = F^{**} = G^* - c$, we get the following calculation:

4.5 Lemma: There exists a constant $c \in \mathbb{R}$ such that $F^* = G + c$. Moreover

$$c = bg(b) - F(g(b)) = af(a) - G(f(a)).$$

In particular, when $b = f(a)$ we get $c = ab$.

The following result generalizes the classical Young formula [12] (see also [13] Th. 130.1 for another extension of this formula we recapture here) as f is not supposed to be injective.

4.6 Proposition: Let f and g be two quasi-inverse mappings from $\mathbb{R}$ into $\overline{\mathbb{R}}$ finite at a and $b = f(a)$ respectively. Then for each $t \in \mathbb{R}$ such that $f(t)$ is finite one has

$$\int_a^t f(r)dr = tf(t) - af(a) - \int_{f(a)}^{f(t)} g(s)ds.$$

Proof: Exchanging the role of f and g in lemma 4.4 we get

$$G^*(t) + G(f(t)) = tf(t)$$

$$G^*(a) + G(f(a)) = af(a).$$

Substracting and using the fact that $F(t) = G^*(t) + c$ with $c = F(a) - G^*(a)$ we get the result. □

When $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, with $f(0) = 0$, taking $g = f^{-1}$ we obtain the Young formula : for each $t \in \mathbb{R}_+$

$$\int_0^t f(r)dr = tf(t) - \int_0^{f(t)} f^{-1}(s)ds.$$

The following statement which incorporates the previous discussion, taking into account the symmetry of the notion of quasi-inverses, was suggested to us by a referee.

4.7 Proposition: Let f, g be non-decreasing extended real-valued functions on $\mathbb{R}$ and let $a, b \in \mathbb{R}$ with $f(a) = b$, $g(b) = a$. Let F, G be given by

$$F(r) = \int_a^r f(t)dt, \qquad G(s) = \int_b^s g(t)dt.$$

Then the following assertions are equivalent:

(i) f, g are quasi-inverses;

(ii) f is a selection of ∂G^*;

(iii) g is a selection of ∂F^*;

(iv) $\partial F = \partial G^*$;

(v) $\partial G = \partial F^*$;

(vi) $F^* = G + ab$;

(vii) $G^* = F + ab$;

(viii) $F(r) + G(f(r)) = rf(r) - ab$ for $r \in f^{-1}(\mathbb{R})$;

(ix) $G(s) + F(g(s)) = sg(s) - ab$ for $s \in g^{-1}(\mathbb{R})$.

Another generalization of the Legendre transform formula is as follows.

4.8 Proposition: Let $F = \mathbb{R} \to \overline{\mathbb{R}}$ be a closed proper convex function and let $G = F^*$. Then the left derivative G'_- of G is the epi-hypo-inverse of the right derivative F'_+ of F: for each $(s, t) \in \mathbb{R}^2$

$$G'_-(s) \leqslant t \Leftrightarrow s \leqslant F'_+(t).$$

Here, as in [8] th. 24.2 the derivatives of F and G are extended by $-\infty$ and $+\infty$ on the left (resp. on the right) of dom F and dom G respectively. Proposition 4.7 implies that $G'_- = (F'_+)^e$, $G'_+ = (F'_-)^h$, a fact proved in [14] Prop. A2 when F is a convex function, nondecreasing on $\mathbb{R}_+$ such that there exists $x_0 > 0$ with $F'_+(x_0) \in (0, +\infty)$.

Proof: This follows from the fact that F can be written $F(t) = F(a) + \int_a^t f(r)dr$ for some non-decreasing f ([8] th. 24.2); choosing $f = F'_+ = f_U$ proposition 3.7 yields the result. □

The following corollary could be deduced from [8] th. 26.3 as its assertion (a) is equivalent to the property that F is essentially strictly convex and its assertion (b) is equivalent to essential smoothness of G. With one variable functions a direct proof as here avoids these general concepts.

4.8 Corollary: Let $F: \mathbb{R} \to \overline{\mathbb{R}}$ be a closed convex function; let $G = F^*$ and let $f: \mathbb{R} \to \overline{\mathbb{R}}$ be such that $F'_- \leqslant f \leqslant F'_+$. The following assertions are equivalent:

(a) f is (strictly) increasing on the domain of F;

(b) $G'_- = G'_+$.

Proof: Using proposition 4.7 and the characterization of quasi-inverses (proposition 2.2) we have that $g : \mathbb{R} \to \overline{\mathbb{R}}$ is a quasi-inverse of f iff $G'_- \leqslant g \leqslant G'_+$.

Now as the domain of F and $D = f^{-1}(\mathbb{R})$ have the same interior, proposition 2.3 shows that assertion (a) is equivalent to the fact $f : \mathbb{R} \to \overline{\mathbb{R}}$ has a unique (continuous) quasi-inverse. By what precedes, uniqueness of the quasi-inverse is equivalent to $G'_- = G'_+$. □

4.9 Proposition: Let $F : \mathbb{R} \to \mathbb{R}$ be a convex function and let f be a non-decreasing function such that $F'_- \leqslant f \leqslant F'_+$. The following assertions are equivalent:

(a) $G = F^*$ is finite valued;

(b) $\lim_{r \to +\infty} f(r) = +\infty, \quad \lim_{r \to -\infty} f(r) = -\infty.$

(c) $\lim_{|t| \to +\infty} F(t)/|t| = +\infty.$

Proof: (a) ⇔ (b) By propositions 4.7 and 2.2.b we have G'_- is a quasi-inverse of f. Now, G is finite valued if and only if G'_- is finite valued.

Then corollary 1.4 yields the equivalence (a) ⇔ (b).

(b) ⇒ (c) For each $A \in \mathbb{R}$ we can find $b \in \mathbb{R}$ such that $f(b) \geqslant A$. Then for $t \geqslant b$ we have

$$\frac{F(t)}{t} = \frac{F(b)}{t} + \frac{1}{t} \int_b^t f(r)dr \geqslant \frac{F(b)}{t} + \frac{1}{t}(t - b)A$$

hence $\liminf_{t \to \infty} \frac{F(t)}{t} \geqslant A$ so that $\lim_{t \to \infty} \frac{F(t)}{t} = +\infty$. The proof of $\lim_{t \to -\infty} \frac{F(t)}{t} = -\infty$ is similar.

(c) ⇒ (a) In the definition of $G(s) = \sup_{r \in \mathbb{R}} (rs - F(r))$ the supremum is attained since $r \mapsto rs - F(r)$ is u.s.c. and

$$\lim_{|r| \to \infty} (rs - F(r)) = \lim_{|r| \to \infty} |r| \left(\frac{r}{|r|} s - \frac{F(r)}{|r|} \right) = \lim_{|r| \to \infty} -|r| \frac{F(r)}{|r|} = -\infty.$$

Thus $G(s) = r_s s - F(r_s)$ for some $r_s \in \mathbb{R}$ and G is finite valued. □

C Application to Rearrangements

Let $(T, \mathcal{T}, \mu)$ be a measured space and let $u : T \to \bar{\mathbb{R}}$ be a measurable function. The distribution function f_u associated with u is the function $f_u : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_u(r) = \mu(S_u(r)) \quad \text{with} \quad S_u(r) = u^{-1}((-\infty, r)).$$

The nondecreasing rearrangement of u is the function $\hat{u} : \mathbb{R} \to \bar{\mathbb{R}}$ given by

$$\hat{u}(s) = \inf \{r \in \mathbb{R} : f_u(r) > s\} = f_u^h(s).$$

In fact usually $\hat{u}$ is restricted to $\hat{T} =]0, m[$, with $m = \mu(T)$ or defined on $[0, m]$ by $\hat{u}(0) = f^h(0)$, $\hat{u}(m) = \text{ess sup } u$, but this is inessential for what concerns the measurability properties of $\hat{u}$. The correspondence $u \mapsto \hat{u}$ has many interesting properties (see [5] and its references for instance). In particular for any $p \in [1, \infty]$ it is a nonexpansive mapping from $L_p(T)$ into $L_p(\hat{T})$ and for any borelian function $F : \mathbb{R} \to \mathbb{R}$ bounded below by an integrable mapping one has

$$\int_T F(u(t)) d\mu(t) = \int_{\hat{T}} F(\hat{u}(\hat{t})) d\lambda(\hat{t})$$

where λ is the Lebesgue measure on $\hat{T}$. As an illustration of our methods let us present the two following properties.

4.10 Properties: The nondecreasing rearrangement $\hat{u}$ of u is the hypo-epi-inverse of f_u:

$$(r \leqslant \hat{u}(s)) \Leftrightarrow (\mu(S_u(r)) \leqslant s)$$

Proof: This follows from proposition 2.6 since, by the very definition of a measure, f_u is l.s.c. □

4.11 Proposition: For any $s \in \mathbb{R}$ one has

$$\hat{u}(s) \geqslant \inf \{\sup u(S) : S \in \mathcal{C}, \mu(S) \geqslant s\}$$

Equality holds exept, at most, on a countable set.

When $(T, \mathcal{T}, \mu)$ is a nonatomic measured space one can write

$$\hat{u}(s) = \inf \{\sup u(S) : S \in \mathcal{C}, \mu(S) = s\} \text{ a.e. } s \in \mathbb{R}.$$

Proof: Let $v(s)$ denote the right hand side of the preceding inequality. Using proposition 2.2.b) it suffices to prove that v is a quasi-inverse of f_u, owing to the fact that two quasi-inverses of f_u can differ only on a countable set.

Now v is easily seen to be nondecreasing and $v(f_u(r)) \leqslant r$ for each $r \in \mathbb{R}$ so that $v(s) \leqslant r$ whenever $s \leqslant f_u(r)$. Now if $v(s) < r$ we can find $S \in \mathcal{C}$ with $\mu(S) \geqslant s$, sup $u(S) < r$ so that $S \subset S_u(r), f_u(r) = \mu(S_u(r)) \geqslant \mu(S) \geqslant s$. □

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