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A Vector Variational Inequality and Optimization Over an Efficient Set

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Abstract: Some relations are obtained between weak vector minimization, a vector variational inequality, and the optimization of a utility function over a set of efficient points.

Zusammenfassung: Es werden einige Beziehungen über schwache Vektorminimierung, einer vektoriellen Variationsgleichung und der Optimierung einer Nutzenfunktion über einer Menge effizienter Punkte hergeleitet.

1 Introduction

Multiobjective optimization problems arise in many applications. Usually, only efficient solutions need be considered as possible optima. One natural approach is to optimize a suitable utility function over the set of efficient solutions. This set is not generally convex, even when the given multiobjective problem is convex.

Philip [8] has considered some special multiobjective problems, and proposed an algorithm. More recently, Benson [1, 2] has discussed the optimization of a linear utility function over the set of efficient solutions, and has proposed an algorithm.

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2 Definitions and Preliminary Results

Let X be a real linear topological space, and let (Y, S) be a real topological linear space with a partial order $\geqslant$ induced by a pointed closed convex cone S, with nonempty interior int S; thus $y_1 \geqslant y_2 \Leftrightarrow y_1 - y_2 \in S$. Let $S_0 = S \backslash \{0\}$. Let $L(X, Y)$ be the space of continuous linear operators from X into Y. Let $C \subset X$ be a nonempty convex set. Let $f : C \to Y$ be a mapping, and let $G : C \to 2^Y$ be a point-to-set mapping. If $\phi \in X'$, the dual space of X, then ϕx denotes the evaluation of ϕ at $x \in X$.

Definition 1: The mapping $f : C \to Y$ is *S-convex* if

$$(\forall x_1, x_2 \in C, \forall \alpha \in (0, 1))\ \alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha) x_2) + S.$$

The point = to-set mapping $G : C \to 2^Y$ is S-convex if

$$(\forall x_1, x_2 \in C, \forall \alpha \in (0, 1))\ \alpha G(x_1) + (1 - \alpha) G(x_2) \subset G(\alpha x_1 + (1 - \alpha) x_2) + S.$$

Definition 2: Let $C \subset X$ be a nonempty set, $x_0 \in C$, and $G : C \to L(X, Y)$ a point-to-set mapping. A *generalized vector variational inequality* is the problem of finding a vector $x_0 \in C$ and a linear operator $A \in G(x_0)$ such that $(\forall x \in C)\ A(x - x_0) \notin -\text{int } S$.

Consider now a *multiobjective optimization problem:*

$$W\text{MIN } f(x) \text{ subject to } x \in C, \tag{1}$$

where $C \subset X$ is a nonempty set, $f : X \to Y$ is a mapping, and WMIN denotes weak minimum.

Definition 3: A point x_0 is a *weak minimum,* or weak *efficient point,* for problem (1) if $(\forall x \in C)\ f(x) - f(x_0) \notin -\text{int } S$. The set of all weak minimum points for (1) is denoted by C_E.

Definition 4: A linear operator $A \in L(X, Y)$ is a *weak subgradient* of $f : C \to Y$ (where $C \subset X$) at $x_0 \in C$ if

$$(\forall x \in C)\ f(x) - f(x_0) - A(x - x_0) \notin -\text{int } S.$$

The *weak subdifferential* of f at x_0 is the set $\partial_W f(x_0)$ set of all weak subgradients of f at x_0. A linear operator $A \in L(X, Y)$ is a *strong subgradient* of $f : C \to Y$ at $x_0 \in C$ if $(\forall\, x \in C)\; f(x) - f(x_0) - A(x - x_0) \in S$. The set of all strong subgradients of f at x_0 is denoted by $\partial_S f(x_0)$.

Since S is a pointed cone with nonempty interior,

$$(\forall\, x_0 \in C)\; \partial_S f(x_0) \subset \partial_W f(x_0).$$

Definition 5: [7] A topological vector space Y, partially ordered by a convex cone S, is *order-complete* if every subset A which has an upper bound b in terms of the ordering (that is, $(\forall\, y \in A)\; b - y \in S$) then has a supremum $b^\wedge$ (that is, there exists $b^\wedge \in Y$ such that $b^\wedge$ is an upper bound to A, and each upper bound b to A satisfies $b^\wedge - b \in S$).

Remark: From Def. 5, a similar statement holds, replacing upper bound by lower bound, and supremum by infimum. It is well known that $\boldsymbol{R}^n$, with an order cone S having exactly n generators, is thus order-complete; but that $C(I)$ (the space of continuous functions on an interval I, with the uniform norm) is not order-complete.

3 Existence of Subgradients

In this section, the existence is proved of weak and strong subgradients. In order to prove the existence of strong subgradients, a generalized Hahn-Banach extension theorem is introduced. This result (Theorem 2) is generalized from Giles' "Hahn-Banach dominated extension theorem" for functionals [6].

Lemma 1: Let $C \subset X$ be a convex set, with $\operatorname{int} C \neq \phi$; let $f : C \to Y$ be an S-convex mapping, continuous at some point $x_0 \in \operatorname{int} C$; let $\operatorname{int} S \neq \phi$. Then the set

$$\operatorname{epi} f := \{(x, y) \in X \times Y : x \in C,\, y - f(x) \in S\}$$

is convex, and $\operatorname{int} \operatorname{epi} f \neq \phi$.

Proof: Let $(x_1, y_1), (x_2, y_2) \in \text{epi} f$, and let $0 < \alpha < 1$.
Let $(x, y) = \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)$. Since f is S-convex,

$$y \geqslant_S \alpha f(x_1) + (1 - \alpha) f(x_2) \geqslant_S f(\alpha x_1 + (1 - \alpha) x_2) = f(x).$$

Thus epi f is convex. Choose $y_0 \in Y$ such that $y_0 - f(x_0) \in \text{int}\, S$, thus $y_0 - F(x_0) + 2M \subset S$ for some neighbourhood M of 0 in Y. Since $x_0 \in \text{int}\, C$ and f is continuous at x_0, there is some neighbourhood N of 0 in X, such that $x_0 + N \subset C$, and $f(x_0 + M) \subset f(x_0) + M$. Hence $(x_0 + N, y_0 + M) \subset \text{epi} f$; thus int epi f is nonempty. □

Theorem 1: Let $C \subset X$ be convex, with int $C \neq \phi$; let the cone S be pointed, with int $S \neq \phi$; let $F : C \to Y$ be an S-convex mapping, continuous at $x_0 \in \text{int}\, C$. Then there exists a weak subgradient B of F at x_0, satisfying the further condition that $Bz \notin -\text{int}\, S \Leftrightarrow Bz \in S$. (Continuity of F need not be assumed in finite dimensions.)

Proof: Let $D = C - \{x_0\}$, and $g(z) = F(x_0 + z) - F(x_0)$. Then $0 \in \text{int}\, D$, $g(0) = 0$, and g is S-convex and continuous at 0. Let $K = \{(z, y) \in D \times Y : y - g(z) \in \text{int}\, S\}$. By Lemma 1, K is a nonempty convex set. Since $(0, 0) \notin K$, by the separation theorem for convex sets, there exists nonzero $(-\rho, \sigma) \in X' \times Y'$ (the dual space of $X \times Y$) such that $(\forall\, (z, y) \in K) - \rho z + \sigma y \geqslant 0$. If $\rho = 0$ then $(\forall\, y \in Y)\ \sigma y \geqslant 0$, contradicting $(-\rho, \sigma) \neq (0, 0)$; hence $\rho \neq 0$. If $\sigma = 0$ then $(\forall\, z \in D) - \rho z \geqslant 0$; this, with $0 \in \text{int}\, D$, shows that $\rho = 0$, contradicting $(-\rho, \sigma) \neq (0, 0)$; hence $\sigma \neq 0$. Since $f(z)$ is a limit of points $y \in K$, $(\forall\, z \in D) - \rho z + \sigma g(z) \geqslant 0$.

A continuous linear mapping $B : X \to Y$ is a weak subgradient of g at 0 if $(\forall\, z \in D)$ $g(z) - Bz \notin -\text{int}\, S$. If, for some $x \in D$, this does not hold then, since $0 \neq \sigma \in Y'$, $\sigma[g(z) - Bz] < 0$. If B is chosen to satisfy $(\forall\, z \in D)\ \sigma Bz + \rho z$ then $\sigma g(z) - \rho z < 0$ for some $z \in D$, giving a contradiction. Hence any B satisfying $\sigma B = \rho$ gives a weak subgradient to g, and therefore to F.

One special case chooses $Bz = (\rho z) y_0$ for some fixed $y_0 \in Y$; then $\sigma Bz = \rho z$ provided that also $\sigma y_0 = 1$. For this choice of B, taking $y_0 \in \text{int}\, S$ and using int $S \cap (-S) = \phi$, there follows the special property

$$Bz \notin -\text{int}\, S \Leftrightarrow (\rho z) y_0 \notin -\text{int}\, S \Leftrightarrow \rho z \geqslant 0 \Leftrightarrow Bz \in S.$$ □

Theorem 2: Let X be a real linear topological space, and let (Y, S) be a real order-complete linear topological space, with order cone S. Let $C \subset X$ be convex, with int $C \neq \phi$. Let the mapping $F : C \to Y$ be S-convex, and let X_0 be a proper subspace of X, with $X_0 \cap \text{cor}\, C \neq \phi$. Let $h : X_0 \to Y$ be a continuous affine mapping such that

$h(x) \leqslant F(x)$ (in terms of S), for each $x \in X_0 \cap C$. Then there exists a continuous affine mapping $k : X \to Y$ such that $k(x) = h(x)$ for all $x \in X_0 \cap C$, and $k(x) \leqslant F(x)$ (in terms of S), for all $x \in C$.

Proof: For notational convenience, adjoin an upper bound element, ∞, to (Y, S). The theorem holds trivially if $C = X_0$. If $C \neq X_0$, suppose that h has been extended to a continuous affine mapping $h^{\wedge} : X^{\wedge} \to Y$, where $X^{\wedge} \supset X_0$ is a subspace, with $X^{\wedge} \cap \operatorname{cor} C \neq \phi$. If $X^{\wedge} \neq X$, choose $u \in (X \backslash X^{\wedge}) \cap C$, and set $X_1 = \{x + \kappa u : x \in X^{\wedge}, \kappa \in R\}$. If $x_1 \in X_1 \cap \operatorname{cor} C$, and $|\mu|$ is sufficiently small, then $x_1 \pm \mu u \in X_0 \cap \operatorname{cor} C$. Define $F^*(x) = F(x)$ for $x \in C$, and $F^*(x) = \infty$ otherwise. When $x_1 - \mu u, x_2 + \lambda u \in X_1 \cap C$, for $\lambda, \mu > 0$, set $\alpha = \lambda/(\lambda + \mu)$, $\beta = 1 - \alpha$; then

$$\begin{aligned}
\alpha h^{\wedge}(x_2) + \beta h^{\wedge}(x_1) &= h^{\wedge}(\alpha x_2 + \beta x_1) \\
&\leqslant F^*(\alpha x_2 + \beta x_1) \\
&= F^*(\alpha(x_2 - \mu u) + \beta(x_1 + \lambda u)) \\
&\leqslant \alpha F^*(x_2 - \mu u) + \beta F^*(x_1 + \lambda u) \\
&= \alpha F(x_2 - \mu u) + \beta F(x_1 + \lambda u).
\end{aligned}$$

Then, by rearrangement,

$$[F(x_1 + \lambda u) - h^{\wedge}(x_1)]/\lambda \geqslant [h^{\wedge}(x_2) - F(x_2 - \mu u)]/\mu,$$

which holds whenever $x_1 + \lambda u$, $x_2 - \mu u \in X_1 \cap C$ and $\lambda, \mu > 0$, and thus for all sufficiently small positive μ and λ. Since Y is an *order-complete* linear space, there exists

$$y^{\#} \equiv \inf \{[F(x_1 + \lambda u) - h^{\wedge}(x_1)]/\lambda : x_1 + \lambda u \in X_1 \cap C, \lambda > 0\} \in Y.$$

Hence, when $x_1 + \lambda u$, $x_2 - \mu u \in X_1 \cap C$ and $\lambda, \mu > 0$,

$$[F(x_1 + \lambda u) - h^{\wedge}(x)]/\lambda \geqslant y^{\#} \geqslant [h^{\wedge}(x) - F(x_1 - \mu u)]/\mu.$$

Then an extension of h to X_1 is defined by

$$(\forall\, x \in X\hat{}\,,\ \forall\, \lambda \in R)\ h^{\#}(x + \lambda u) = h\hat{}(x) + \lambda y^{\#},$$

and then $h^{\#}(x) \leqslant F(x)$ for all $x \in X_1 \cap C$. Since $h\hat{}$ is continuous, so is $h^{\#}$.

If $X \neq X_1$, h may similarly be extended to a subspace of one higher dimension. The extension to $k : X \to Y$ follows, by an application of Zorn's lemma, as in the usual proof of the Hahn-Banach extension theorem. □

The following result is proved similarly:

Theorem 3: Let X be a real linear topological space; let (Y, S) be an order-complete real partially ordered linear topological space; let $C \subset X$ be convex, with nonempty interior, and let $x_0 \in \operatorname{int} C$; let $F : X \to Y$ be a S-convex mapping. Then there exists a continuous affine mapping $h : X \to Y$ such that $h(x_0) = F(x_0)$, and $(\forall\, x \in C)\ h(x) \leqslant F(x)$.

Theorem 4: Let X, Y, S, C, x_0 and F be as in Theorem 2. Then there exists a strong subgradient of F at $x_0 \in \operatorname{int} C$. If S is a pointed cone, then there also exists a weak subgradient of F at x_0.

Proof: Let $x_0 \in \operatorname{int} C$. By Theorem 2, there exists a continuous affine mapping $h : X \to Y$ such that $h(x_0) = F(x_0)$, and $(\forall\, x \in C)\ h(x) \leqslant F(x)$. Then, for some $A \in L(X, Y)$,

$$(\forall\, x \in X)\ h(x) = h(x_0) + A(x - x_0),$$

and

$$(\forall\, x \in C)\ h(x_0) + A(x - x_0) \leqslant F(x).$$

Therefore $F(x) - F(x_0) - A(x - x_0) \in S$ for all $x \in C$. When S is pointed, a weak subgradient exists since $(\forall\, x \in C)\ \partial_S f(x) \subset \partial_W f(x)$. □

4 Equivalence of the Weak Minimization Problem (1) and a Vector Variational Inequality

In this section, the weak minimization problem (1) is shown to be equivalent, under some restrictions, to a generalized vector variational inequality.

Consider the following *generalized variational inequality:* Given nonempty $C \subset X$ and S-convex $f : C \to Y$, find $x_0 \in C$ and $B \in \partial_W f(x_0)$ such that

$$(\forall\, x \in C)\; B(x - x_0) \notin -\text{int}\, S. \tag{2}$$

Theorem 5: Let $C \subset X$ be convex with $\text{int}\, C \neq \phi$; let the convex cone S be pointed, and let $\text{int}\, S \neq \phi$; let $f : C \to Y$ be S-convex, and continuous at a point $x_0 \in \text{int}\, C$. If x_0 is a weak minimum of the multiobjective optimization problem (1), then x_0 solves the generalized variational inequality (2). Conversely, if (x_0, A) solves (2), where also $A \in \partial_S f(x_0)$, then x_0 is a weak minimum of (1).

Proof: If x_0 is a weak minimum of (1), then $(\forall\, x \in C)\; f(x) - f(x_0) \notin -\text{int}\, S$. Then, by definition of $\partial_W f(x_0)$ (see Def. 3 and Def. 4), $0 \in \partial_W f(x_0)$. So (2) is satisfied with the given x_0, and $B = 0$. Conversely, let (x_0, B) solve (2), where $x_0 \in \text{int}\, C$ and $B \in \partial_S f(x_0)$. Let $W := Y \backslash (-\text{int}\, S)$. Then

$$(\forall\, x \in C)\; B(x - x_0) \in W;$$

and

$$(\forall\, x \in C)\; f(x) - f(x_0) - B(x - x_0) \in S,$$

since B is, by hypothesis, a *strong* subgradient. Combining these,

$$(\forall\, x\; C)\; f(x) - f(x_0) \in W + S \subset W;$$

thus x_0 is a weak minimum of (1). □

Remarks: If (x_0, B) solves (2), but $B \in \partial_W f(x_0) \backslash \partial_S f(x_0)$, then x_0 is *not* necessarily a weak minimum for (1). The above proof does not extend, since $W + W$ is not contained in W.

If $B \in \partial_W f(x_0)$ and $(\forall\, x \in C)\; B(x - x_0) \in S$ then, by a similar proof, x_0 is a weak minimum of (1).

If x_0 is a weak minimum of (1), then usually $0 \in \partial_S f(x_0)$ does *not* hold, except when the constraint $x_0 \in C$ is inactive, thus when $x_0 \in \operatorname{int} C$. However, suppose now also that C is convex, and f is (Fréchet or linear Gâteaux) differentiable at x_0 (as well as S-convex). A weak minimum of (1) at x_0 implies that $(\forall\, x \in C)\; f(x) - f(x_0) \in W := Y \backslash (-\operatorname{int} S)$. Let $x_0 + v \in C$; since C is convex, $(\forall\, \alpha \in (0, 1))\; x_0 + \alpha v \in C$. Hence

$$(\forall\, \alpha \in (0, 1))\; [f(x_0 + \alpha v) - f(x)]/\alpha \in W.$$

Since W is closed, and f is differentiable at x_0, it follows that $f'(x_0)(x - x_0) \in W$. Thus x_0 and $f'(x_0)$ satisfy the generalized variational inequality (2), and $f'(x_0) \in \partial_S f(x_0)$.

Theorem 6: Let $C \subset X$ be convex; let $f : C \to Y$ be S-convex and linearly Gâteaux differentiable at x_0. Then $x_0 \in C$ is a weak minimum of (1) if and only iff x_0 and $f'(x_0)$ solves (2).

Proof: See the above remark, together with the proof of Theorem 5 for the converse.□

5 Optimization Over an Efficient Set

Given $C \subset C, f : C \to Y$, and $\phi : X \to \boldsymbol{R}$, consider the following problems:

(I): *Multiobjective optimization:* WMIN $f(x)$ subject to $x \in C$. Denote the set of weak minimum (= efficient) points for this problem by $\boldsymbol{E}$.

(II): *Generalized vector variational inequality:* Find $x_0 \in C$ and $A \in \partial_W f(x_0)$ such that $(\forall\, x \in C)\; A(x - x_0) \notin -\operatorname{int} S$. Denote the set of optima for this problem by V.

(III): *Optimization over an efficient set:* Minimize $\phi(x)$ subject to $x \in \boldsymbol{E}$.

(IV): *Linearized problem:* WMIN $f'(x_0)(x - x_0)$ subject to $x \in C$.

(V) $WMIN\,[\phi(x), f'(x_0)((x - x_0)]$ subject to $x \in C$.

(VI): Minimize $\phi(x)$ subject to $x \in V$.

(VII): Minimize $\phi(x)$ subject to $(\forall\, x \in C)\; f'(x_0)(x - x_0) \in W \equiv Y \backslash (-\text{int}\, S)$.

For (IV), (V) and (VII), f is assumed (linearly Gâteaux) differentiable. The weak minimization is with respect to the cone S, or $R_+ \times S$ for (V).

Theorem 7: Let $C \subset X$ be closed convex, let int S be nonempty, and let $f: X \to Y$ be S-convex and (Fréchet or linearly Gâteaux) differentiable, with derivative $f'(x_0)$ at $x_0 \in X$. Then:

(a) Problem (I) is equivalent to problem (IV); if also f is (linearly Gâteaux) differentiable at x_0, then (I) is equivalent to (II).

(b) If f is (linearly Gâteaux) differentiable at x_0, then (III) is equivalent to (VI).

(c) If x_0 is an optimum for (III), then x_0 is an optimum for (V). (So any necessary conditions for an optimum of (V) hold also for (III).)

(d) Problem (III) is equivalent to problem (VII).

Proof: (a): If x_0 is a weak minimum for (I), then $(\forall\, x \in C)\; f(x) - f(x_0) \in W$, where $W \equiv Y \backslash (-\text{int}\, S)$. If C is convex and $x_0 + v \in C$ then $(\forall\, \alpha \in (0, 1))\; x = x_0 + \alpha v \in C$, hence $[f(x_0 + \alpha v) - f(x)]/\alpha \in W$; since W is closed, and f is differentiable, $f'(x_0)v \in W$; thus x_0 is a weak minimum for (IV). Conversely, if x_0 is NOT a weak minimum of (I), then $(\exists\, u \in C)\; f(u) - f(x_0) \in -\text{int}\, S$; since f is S-convex, $(\exists\, u \in C,\ \exists\, s \in S)\; f'(x_0)(u - x_0) + s \in -\text{int}\, S$, thus $f'(x_0)(u - x_0) \in -\text{int}\, S$, so that x_0 is not a weak minimum for (IV). The equivalence of (I) and (II) follows from Theorem 6.

(b): By Theorem 6, $E = V$.

(c) and (d): Let x_0 be a minimum for (III). Then $(\forall\, x \in C,\ \forall\, u \in C)\; \phi(x) \geqslant \phi(x_0)$, $f(u) - f(x_0) \in W \equiv Y \backslash (-\text{int}\, S)$. By a similar argument to that in (a), it follows that $(\forall\, x \in C,\ \forall\, u \in C)\; \phi(x) \geqslant \phi(x_0),\ f'(x_0)(u - x_0) \in W$. Thus x_0 is a weak minimum for (VII). Conversely, if x_0 is a weak minimum for (VII), then $(\forall\, x \in C,\ \forall\, u \in C)\; \phi(x) \geqslant \phi(x_0),\ f'(x_0)(u - x_0) \in W$. Since f is S-convex, $(\forall\, x \in C,\ \forall\, u \in C)\; \phi(x) \geqslant \phi(x_0)$, $f(u) - f(x_0) \in S + W \subset W$. Thus x_0 is a weak minimum for (III). Now, if x_0 is a weak minimum for (VII), then $(\forall\, x \in C,\ \forall\, u \in C)\; \phi(x) \geqslant \phi(x_0), f'(x_0)(u - x_0) \in W$, which is equivalent to $(\forall\, x \in C,\ \forall\, u \in C)\; [\phi(x) - \phi(x_0),\ f'(x_0)(u - x_0)] \notin -\text{int}\, [R_+ \times S]$, which states that x_0 is a weak minimum for (V). □

6 Kuhn-Tucker Necessary Conditions for Optimization Over an Efficient Set

Theorem 8: Kuhn-Tucker necessary conditions for the point x_0 to be an optimum for problem (III) are:

$$\alpha^T \phi'(x_0) + \zeta^T f'(x_0) \in N_C(x_0), \quad \alpha \in R_+, \ \zeta \in S^*, \ (\alpha, \xi) \neq (0, 0),$$

where $N_C(x_0)$ denotes the normal cone to C at x_0.

Proof: From Theorem 7(c), Kuhn-Tucker conditions for (V) are also necessary for (III). The usual Kuhn-Tucker theorem for weak vector minimization (see [4]) then applies. □

Consider now the constraint $-g(x) \in T$, where T is a closed convex cone, and g is a differentiable vector function. Replacing $x \in C$ by $-g(x) \in T$, and assuming a constraint qualification holds for this constraint at x_0, necessary Kuhn-Tucker conditions for $x = z$ to be a weak minimum of $f(x)$, subject to $-g(x) \in T$, are:

$$(Q): \quad \tau^T f'(z) + \rho^T g'(z) = 0, \quad \tau \in S^*, \ \rho \in T^*, \ \tau^T e = 1, \ -g(z) \in T, \ \rho^T g(z) = 0,$$

where e is any constant vector in int S, so that $\tau^T e = 1$ ensures that $\tau \neq 0$. (If $S = R^p_+$ then $e = (1, 1, \ldots, 1)^T$ may be chosen.) Denote now by K the set of weak minima for $f(x)$, subject to $-g(x) \in T$.

Consider now the problem:

$$(H): \quad \text{Minimize } \varphi(z) \text{ subject to } z \in K,$$

Assume now also that f is S-convex, and g is T-convex; then the necessary conditions (Q) for $x \in K$ are also sufficient. (Note that the hypotheses could be reduced; it suffices if f is S-pseudoconvex and g is T-quasiconvex.) Under these assumptions, problem (H) is equivalent to minimizing $\varphi(z)$ subject to constraints (Q). A Lagrangian function for this latter problem is:

$$\varphi(z) + \theta[\tau^T f'(z) + \rho^T g'(z)] + \lambda^T g(z) - \tau^T \sigma - \rho^T \omega + \beta[\tau^T e - 1] + \delta \rho^T g(z).$$

Consequently, Kuhn-Tucker necessary conditions (assuming a constraint qualifications) for z_0 to minimize $\varphi(z)$ over the efficient set K are that Lagrange multipliers $\lambda \in T^*$, $\theta \in \boldsymbol{R}$, $\sigma \in S$, $\omega \in T$, $\beta \in \boldsymbol{R}$, $\delta \in \boldsymbol{R}$ exist, satisfying the conditions:

$$\varphi'(z_0) + \theta[\tau^T f''(z_0) + \rho^T g''(z_0)] + \lambda^T g'(z_0) + \delta\rho^T g(z_0) = 0,$$

$$\theta f'(z_0) - \sigma + \beta e = 0;$$

$$\theta g'(z_0) - \omega + \delta g(z_0) = 0;$$

$$\lambda^T g(z) = 0;$$

$$\tau^T \sigma = 0;$$

$$\rho^T \omega = 0.$$

Consider, in particular, the *multilinear* (linear multiobjective) problem:

(L): $\quad WMIN\, Mx$ subject to $Ax - b \leqslant 0$.

Here M is an $r \times n$ matrix, and A is an $m \times n$ matrix; $x \in \boldsymbol{R}^n$. Denote by $\boldsymbol{E}_L$ the set of weak minima for problem (L), minimizing with respect to the cone $\boldsymbol{R}^r_+$. Then

$$x \in \boldsymbol{E}_L \Leftrightarrow [Ax - b \leqslant 0, \tau^T M + \rho^T M + \rho^T A = 0, \tau \in \boldsymbol{R}^r_+, \rho \in \boldsymbol{R}^m_+, \tau^T e = 1,$$
$$\rho^T(Ax - b) = 0].$$

The requirement $\tau^T e = 1$ ensures that $\tau \neq 0$. If $c^T x$ is another (real) objective function, then the problem of minimizing $c^T x$ over $x \in \boldsymbol{E}_L$ is equivalent to the problem:

Minimize $c^T x$ subject to

$$Ax - b \leqslant 0,\ \tau^T M + \rho^T A = 0,\ \tau \in \boldsymbol{R}^r_+,\ \rho \in \boldsymbol{R}^m_+,\ \tau^T e = 1,\ \rho^T(Ax - b) = 0.$$

This last problem fails to be a linear program, because of the complementary slackness constraint $\rho^T(Ax - b) = 0$. The minimization is with respect to the variables x, τ, ρ.

Kuhn-Tucker necessary conditions for a minimum are that Lagrange multipliers $\lambda \in R_+^m$, $\theta \in R$, $\sigma \in R_+^r$, $m \in R_+^m$, $\beta \in R$, $\delta \in R$ exist, satisfying the constraints:

$$c^T + \lambda^T A = 0,\ \theta M - \sigma + \beta e = 0,\ \theta A - \omega + \gamma(Az_0 - b) = 0,\ \lambda^T(Az_0 - b) = 0,$$

$$\tau^T \sigma = 0,\ \rho^T \omega = 0.$$

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