ZOR – Methods and Models of Operations Research (1991) 35:175–184

Super Efficiency in Convex Vector Optimization

By J.M. Borwein[1] and D.M. Zhuang[2]

Abstract: We establish a Lagrange Multiplier Theorem for super efficiency in convex vector optimization and express super efficient solutions as saddle points of appropriate Lagrangian functions. An example is given to show that the boundedness of the base of the ordering cone is essential for the existence of super efficient points.

Key words: Super efficiency, convex vector optimization, Lagrange Multiplier Theorem, scalarization

In our previous paper [B-Z1], we introduced the concept of *super efficiency*, a new kind of proper efficiency. Super efficiency refines the notions of efficiency and other kinds of proper efficiency, and provides concise (and equivalent) scalar characterizations and duality results when the underlying decision problem is convex. In this paper, we continue our study of super efficiency. We establish a Lagrange Multiplier Theorem for super efficiency in convex settings and express super efficient points as saddle points of an appropriate Lagrangian function. Similar developments for other kinds of optimality notions in vector optimization theory can be found in, for example, [Benson 1], [Borwein 1], [D-Sa 1], [Hurwicz 1], [K-T 1] and in many other papers.

For the convenience of the reader, we first recall the definition and basic properties of super efficiency. The reader is referred to our previous paper for details. The preliminary materials on vector optimization theory, in particular, notions of various efficiency and proper efficiency are also discussed in the paper [B-Z 1]. Excellent reference books and survey papers on infinite dimensional vector optimization theory and applications are [Jahn 1], [D-St 1], and [Hurwicz 1]. See also [Borwein 1].

---

[1] J.M. Borwein, Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5. Research is supported partially by NSERC.
[2] D.M. Zhuang, Department of Mathematics and Computer Studies, Mount St. Vincent University, Halifax, Nova Scotia, Canada, B3M 2J6. Research is supported partially by NSERC and Mount St. Vincent University grant.

0340–9422/91/3/175–184 $2.50 © 1991 Physica-Verlag, Heidelberg

Definition ([B-Z1]): Let X be a real normed linear space. We say that x is a *super efficient point* of a non-empty subset C of X with respect to the convex ordering cone S, written $x \in SE(C,S)$, if there is a real number $M>0$ such that

$$MB \supset K \cap (B-S) \tag{1}$$

where $K := cl[\text{cone}(C-x)]$, the closure of the cone generated by the set $(C-x)$, and B is the closed unit ball of X.

A super efficient point can also be expressed explicitly in terms of the norm. We observe that, $x \in SE(C,S)$ if and only if for each c in C, y in X and $c-x \leq_S y$, then

$$\|c-x\| \leq M\|y\|$$

with an uniform constant M (depending only on x, not on y or c) [B-Z1].

Super efficiency has a very simple description in a normed lattice. In this setting, $x \in SE(C,S)$ is equivalent to the existence of some uniform constant $M>0$ with

$$\|c-x\| \leq M\|(c-x)^+\| \tag{2}$$

for all $c \in C$.

When the set C is convex, our definition of super efficiency has a concise dual form. We can prove that (1) in the definition of super efficiency is equivalent to

$$X^* = K^+ - S^+ = (C-x)^+ - S^+ \ , \tag{3}$$

where X^* is the norm dual of X [B-Z1].

With this duality, we can characterize a super efficient point as an optimal solution of a scalar minimization problem:

Theorem 2 (The scalarization theorem [B-Z1]): Let X be a normed space. If the convex pointed ordering cone S has a closed bounded base Θ and C is convex then x is in $SE(C,S)$ if and only if there is ϕ in the norm-interior of S^+, denoted by $\phi \in \text{int}(S^+)$, such that

$$\phi(C-x) \geq 0 \ .$$

Our next theorem says that every bounded closed set in a Banach space has super efficient points provided the ordering cone has a bounded base.

Theorem 3 ([B-Z 1]): Let X be a Banach space and let the convex pointed ordering cone S have a closed bounded base. Then every bounded closed set C possesses super efficient points.

We offer an example showing that the boundedness of the base of the ordering cone is essential for the existence of super efficient points.

Example 4: A norm compact convex subset of Hilbert space lying in a norm-compact order interval but which has no super efficient point.

Let $X = l_2(\mathbb{N})$, $S = l_2^+(\mathbb{N})$, the non-negative orthant of X, and

$$C := \{x \in l_2(\mathbb{N}) \mid \Sigma n^2 x_n^2 \leq 1\} \ .$$

It is clear that C is convex and closed. Note that if $x = (x_1, x_2, \ldots)$ is in C, then $|x_n| \leq 1/n$ for $n = 1, 2, \ldots$.

Define a linear operator $T: l_2(\mathbb{N}) \to l_2(\mathbb{N})$ by

$$(Tx)_n = x_n/n \ .$$

Then T is a compact operator and $C = T(B)$, hence C is compact.

Let x_0 be in C and let

$$\phi \in (C - x_0)^+ - S^+ \ .$$

Then there is ψ in $(C - x_0)^+$ such that $\phi \leq_{S^+} \psi$. Now ψ in $(C - x_0)^+$ implies that for each c in C, $\psi(c - x_0) \geq 0$. As $C = T(B)$, $x_0 = Tb_0$ for some b_0 in B and $c = Tb$ for b in B. Hence $\psi(Tb - Tb_0) \geq 0$, or

$$T^* \psi(b - b_0) \geq 0$$

for all b in B. Thus, either $\psi = 0$ or $\|b_0\| = 1$. This implies that

$$T^* \psi = -b_0 \|T^* \psi\| = t(-b_0)$$

where $t := \|T^* \psi\| \geq 0$. Let $b_0 = (b_1, b_2, \ldots)$, $\phi = (\phi_1, \phi_2, \ldots)$ and $\psi = (\psi_1, \psi_2, \ldots)$ so that for $n = 1, 2, \ldots$,

$$\phi_n \leq \psi_n = t(-n b_n) \ . \tag{4}$$

(i) If $t=0$ or $b_n=0$ for some n, then $\phi_n \leq 0$ and

$$(C-x_0)^+ - S^+ \neq X^* .$$

(ii) If $|b_n|>0$ for all n, then for each k in $\mathbb{N}$, select $n_k<n_{k+1}$ with

$$n_k|b_{n_k}|<\frac{1}{k2^k}$$

as is possible because by (4), $(n|b_n|)$ is in c_0 – the normed linear space of all sequences of real numbers converging to zero and normed by the supremum norm. Set

$$\lambda_n := \begin{cases} kn_k|b_{n_k}| & \text{if} \quad n \in \{n_k\}_{k=1}^{\infty} \\ 0 & \text{else} . \end{cases}$$

Then $\lambda := (\lambda_n) \in l_1(\mathbb{N}) \subset l_2(\mathbb{N})$ and we claim $\lambda \notin (C-x_0)^+ - S^+$. Otherwise, (4) implies that

$$kn_k|b_{n_k}| = \lambda_{n_k} \leq tn_k|b_{n_k}|$$

which, as $|b_{n_k}|>0$, means $k \leq t$ for all k in $\mathbb{N}$. This is impossible as t is fixed. Therefore for any x_0 in C,

$$(C-x_0)^+ - S^+ \neq X^* .$$

By the duality form of super efficiency (3), $SE(C,S)$ is empty. However, one observes that the set of efficient points of C is

$$-S \cap \{x \mid \|(nx_n)\| = 1\}$$

and one can also show that the set of all proper efficient points of C, in the sense of Borwein, is norm dense in the set of efficient points of C. (See, for example, [Zhuang 1].)

Hence, without a bounded base for the ordering cone there need not be super efficient points even for a very well behaved norm-compact set. ■

Let X be a linear space, Y, Z be normed spaces partially ordered by convex and pointed cones S and P respectively. Let f: $X \to Y$ and g: $X \to Z$ be vector-valued. We consider the following vector minimization problem:

$$\text{(VMP)} \quad \min_S \{f(x) \mid g(x) \leq_P 0,\ x \in C\} \ .$$

Definition 5: We say that x_0 is a *super efficient solution* of (VMP) if x_0 is a super efficient point of feasible set F with respect to the partial ordering cone S, that is, $x_0 \in SE(f(F)+S, S)$, where

$$F := \{x \in C \mid g(x) \leq_P 0\} \ .$$

The Lagrange Multiplier Theorem for a constrained optimization problem with a real-valued objective function asserts that under certain regularity assumptions, one can find a continuous linear operator T from Z to Y such that an optimal solution x_0 of the constrained problem is also an optimal solution of the unconstrained problem:

$$\min \{f(x) + T[g(x)] \mid x \in C\}$$

and

$$T[g(x_0)] = 0 \ .$$

(See, for example, [Luenberger 1] or [Jahn 1].)

Motivated by this, our next theorem shows that super efficient solutions of (VMP) are exactly the super efficient solutions for some unconstrained vector optimization problem.

Theorem 6: Let X be a linear space, Y, Z be normed spaces partially ordered by convex and pointed cones S and P respectively. Assume further that S has a bounded base and P is closed and has non-empty norm-interior. Let f: $X \to Y$ and g: $X \to Z$ be convex with respect to the respective partial orderings, $C \subset X$ be convex and suppose there is an $x^{\sim}$ in C such that

$$g(x^{\sim}) \in \text{int}\,(P) \ .$$

Then x_0 is a super efficient solution of (VMP) if and only if there is a continuous linear operator T from Z to Y such that $T(P) \subset S$, and $T[g(x_0)] = 0$, and x_0 is in the feasible set and is a super efficient solution of the unconstrained vector optimization problem:

$$\min_S \{f(x) + T[g(x)] \,|\, x \in C\} \ .$$

Proof: Assume first that x_0 is a super efficient solution of (VMP). Since g is convex with respect to P and C is convex, the feasible set

$$F := \{x \in C \,|\, g(x) \leq_P 0\}$$

is convex. Moreover, f is convex with respect to S, S has a bounded base and hence S^+ has non-empty norm-interior ([Jameson 1], p. 122). By the scalarization theorem (Theorem 2), there is $\phi \in \mathrm{int}\,(S^+)$ such that

$$\phi[f(x_0)] \leq \phi[f(x)] \quad \text{for all} \quad x \in F \ .$$

Apply the standard Lagrange Multiplier Theorem [Luenberger 1], we can find $\lambda \in P^+$ such that $\lambda[g(x_0)] = 0$ and

$$\phi[f(x_0)] \leq \phi[f(x)] + \lambda[g(x)] \quad \text{for all} \quad x \in C \ . \tag{5}$$

Choose $s \in S$ with $\phi(s) = 1$. Let $T: Z \to Y$ be defined as

$$T(z) = \lambda(z)s$$

then $T(P) = \lambda(P)s \subset S$ as $\lambda(p) \geq 0$ for all $p \in P$. T is a continuous linear operator and

$$T[g(x_0)] = \lambda[g(x_0)] \cdot s = 0 \cdot s = 0 \ . \tag{6}$$

Now, by (5) and (6),

$$\phi[f(x_0) + Tg(x_0)] = \phi[f(x_0)] \leq \phi[f(x)] + \lambda[g(x)]\phi(s)$$

$$= \phi[f(x) + Tg(x)] \quad \text{for all} \quad x \in C \ .$$

Since $\phi \in \text{int}(S^+)$, by the scalarization theorem again, the above implies that x_0 is a super efficient solution of the unconstrained problem

$$\min_S \{f(x) + T[g(x)] \mid x \in C\} .$$

Conversely, we assume that x_0 is feasible and is a super efficient solution of the unconstrained minimization problem:

$$\min_S \{f(x) + T[g(x)] \mid x \in C\}$$

where $T: Z \to Y$ is a continuous linear operator satisfying

$$T(P) \subset S , \quad T[g(x_0)] = 0 .$$

By the scalarization theorem, there is $\phi \in \text{int}(S^+)$ such that

$$\phi[f(x_0) + Tg(x_0)] \leq \phi[f(x) + Tg(x)] \quad \text{for all} \quad x \in C .$$

Now for all feasible x, we have

$$\phi[f(x_0)] \leq \phi[f(x)] + \phi[Tg(x)] \leq \phi[f(x)] ,$$

because $\phi[Tg(x)] \leq 0$ for all x in F. This implies that x_0 is a super efficient solution of (VMP), by the scalarization theorem.

Finally, we present a theorem which allows us to express a super efficient solution of (VMP) as a saddle points of an appropriate Lagrangian function.

Theorem 7: Let X be a linear space, Y, Z be normed spaces partially ordered by convex and pointed cones S and P respectively. Assume further that S has a bounded base and P has non-empty norm-interior. Let $f: X \to Y$ and $g: X \to Z$ be convex with respect to the appropriate partial orderings, $C \subset X$ be convex and suppose there is an $x^{\sim}$ in C such that

$$g(x^{\sim}) \in \text{int}(P) .$$

Then x_0 is a super efficient solution of (VMP) if and only if there is $\phi \in$norm-int(S^+) and $\psi_0 \in P^+$ such that (x_0, ψ_0) is a saddle point of the Lagrangian functions $L: C \times P^+ \to \mathbb{R}$ defined by

$$L(x, \psi) := \phi[f(x)] + \psi[g(x)] \ .$$

Proof: Since x_0 is a super efficient solution of (VMP), by Theorem 6, there is a continuous linear operator T from Z to Y such that x_0 is a super efficient solution of the unconstrained optimization problem:

$$\min_S \{f(x) + T[g(x)] \mid x \in C\}$$

with

$$T(P) \subset S \ , \quad T[g(x_0)] = 0 \ .$$

By the scalarization theorem, there is $\phi \in \operatorname{int}(S^+)$ such that

$$\phi[f(x_0) + Tg(x_0)] \leq \phi[f(x) + Tg(x)] \quad \text{for all} \quad x \in C \ .$$

Let $\phi \circ T = \psi_0$, we have $\psi_0 \in P^+$, $\psi_0[g(x_0)] = 0$, and

$$\phi[f(x_0)] + \psi_0[g(x_0)] \leq \phi[f(x)] + \psi_0[g(x)] \quad \text{for all} \quad x \text{ in } C \ .$$

That is

$$L(x_0, \psi_0) \leq L(x, \psi_0) \quad \text{for all} \quad x \text{ in } C \ . \tag{7}$$

Note also that since x_0 is a super efficient solution of (VMP), x_0 is feasible, that is $g(x_0) \leq_P 0$, we have

$$\psi[g(x_0)] \leq 0 \quad \text{for all} \quad \psi \in P^+ \ .$$

Hence

$$\phi[f(x_0)] + \psi[g(x_0)] \leq \phi[f(x_0)] + \psi_0[g(x_0)] \quad \text{for all} \quad \psi \in P^+$$

because $\psi_0[g(x_0)] = 0$. Therefore,

$$L(x_0, \psi) \leq L(x_0, \psi_0) \quad \text{for all} \quad \psi \in P^+ \ . \tag{8}$$

Combining (7) and (8), we have

$$L(x_0, \psi) \leq L(x_0, \psi_0) \leq L(x, \psi_0) \quad \text{for all} \quad (x, \psi) \text{ in } C \times P^+ \ .$$

Thus, (x_0, ψ_0) is a saddle point of the Lagrangian function $L(x, \psi)$.

Conversely, if $\psi_0 \in P^+$, and (x_0, ψ_0) is a saddle point of the Lagrangian function $L(x, \psi)$, then

$$L(x_0, \psi) \leq L(x_0, \psi_0) \leq L(x, \psi_0) \quad \text{for all} \quad (x, \psi) \text{ in } C \times P^+ \ .$$

From this we see that, for all $\psi \in P^+$,

$$\phi[f(x_0)] + \psi[g(x_0)] \leq \phi[f(x_0)] + \psi_0[g(x_0)] \tag{9}$$

and, for all x in C,

$$\phi[f(x_0)] + \psi_0[g(x_0)] \leq \phi[f(x)] + \psi_0[g(x)] \ . \tag{10}$$

The inequality (9) implies that

$$\psi[g(x_0)] \leq \psi_0[g(x_0)] \quad \text{for all} \quad \psi \in P^+ \ .$$

Since P^+ is a cone, this implies that

$$\psi_0[g(x_0)] = 0 \ .$$

Now (10) implies that

$$\phi[f(x_0)] \leq \phi[f(x)] + \psi_0[g(x)] \leq \phi[f(x)] \quad \text{for all} \quad x \text{ in } F \ , \tag{11}$$

because $\psi_0 \in P^+$ and hence $\psi_0[g(x)] \leq 0$.

Note that since ϕ is assumed to be in int (S^+), (11) implies that x_0 is a super efficient solution of (VMP).

References

[Benson 1] Benson HP (1979) An improved definition of proper efficiency for vector maximization with respect to cones. J Math Anal Appl 71:232 – 241

[Borwein 1] Borwein JM (1977) Proper efficient points for maximizations with respect to cones. SIAM J Control Optim 15:57 – 63

[Borwein 2] Borwein JM (1980) The geometry of Pareto optimality. Math Operationsforsch Statist 11:235 – 248

[Borwein 3] Borwein JM (1987) Convex cones, minimality notions and consequences. In: Jahn J, Krabs W (eds) Recent Advances and Historical Development of Vector Optimization. Springer, New York, pp 64 – 73

(B-Z 1] Borwein JM, Zhuang D (to appear) Super efficiency in vector optimization

[D-St 1] Dauer JP, Stadler W (1986) A survey of vector optimization in infinite dimensional spaces, part II. JOTA 51

[D-Sa 1] Dauer JP, Saleh OA (to appear) A characterization of proper minimal points as solutions of sublinear optimization problems

[D-G 1] Dauer JP, Gallagher RJ (to appear) Positive proper efficient points and related cone results in vector optimization theory

[Holmes 1] Holmes RB (1975) Geometric functional analysis and its applications. Springer, New York

[Hurwicz 1] Hurwicz L (1958) Programming in Linear Spaces. In: Arrow, Hurwicz, Uzawa (eds) Studies in Linear and Nonlinear Programming. Standford Univ Press, Standford, California, pp 38 – 102

[Jameson 1] Jameson G (1970) Ordered linear spaces. Springer, Lecture Notes in Mathmatics No 141, Berlin

[Jahn 1] Jahn J (1986) Mathematical vector optimization in partially ordered inear spaces. Peter Lang, Frankfurt am Main

[Jahn 2] Jahn J (1984) Scalarization in vector optimization. Math Prog 29:203 – 218

[K-T 1] Kuhn HW, Tucker AW (1951) Nonlinear programming. In: Neyman J (ed) Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley, pp 481 – 492

[Luenberger 1] Luenberger DG (1969) Optimization by vector space methods. Wiley, New York

[Peressini 1] Peressini AL (1967) Ordered topological vector spaces. Harper & Row, New York

[Zhuang 1] Zhuang D (to appear) Density results of proper efficiencies