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Shock wave interaction with area changes in ducts

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Abstract. Whitham's approximation for handling shock wave propagation in area changes (reductions) in a duct was checked in comparison with a numerical solution. Also the Whitham approximation for shock wave propagation from a constant cross-sectional duct to a duct of a smaller cross-sectional area was studied and compared with a numerical solution. It was found that for modest incident shock Mach numbers and modest area reductions the Whitham approximation provided a fair solution for the shock Mach number and for the post-shock pressure. For higher shock Mach numbers and/or area reductions, large discrepancies exit between the approximate and exact solutions. A wider range of applicability of the Whitham approximation is found for the monotonical area reduction case; it is quite narrow for the passage of a shock wave from a wider to a narrower duct case. In addition, the effect of the extent of the area change region on the time required for reaching a quasi-steady flow was studied. It was shown that the longer the area change segment is, the longer it takes to reach a quasi-steady flow.

Key words: Duct flow, Quasi one-dimensional flow, Shock propagation, Whitham's theory

1. Introduction and theoretical background

When a shock wave propagating in a constant cross-sectional duct encounters an area change, it will experience changes in its strength. These changes, an increase or a decrease, in the pressure jump across the shock wave front (and in all other flow properties) depend on whether the cross-sectional area of the duct is decreasing or increasing. The ability to calculate accurately the post-shock flow properties when the incident shock wave, in a duct, experiences area changes is of great importance since such flows appear in many engineering problems. For example, in the exhaust system of internal combustion engines; shock wave propagation in channels like coal mines or defense trenches; jet engines, etc.

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The solution for the post shock flow field will be available upon solving the conservation equations for mass, momentum, and energy, which, for an inviscid non-conductive one-dimensional flow, are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = -\rho u \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}, \quad (1)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2 + P\right) = -\rho u^2 \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}, \quad (2)$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(ue + uP) = -u(e + P)\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} \quad (3)$$

where the variables ρ, u, P, e, x, t, and A denote the density ,flow velocity, static pressure, total energy per unit volume $\frac{P}{\gamma - 1} + \frac{1}{2}\rho u^2$, distance, time, and the local cross-sectional area of the duct, respectively.

Using the presently available numerical schemes and computers, the solution of

(1)–(3) can easily be achieved. A few examples will be shown subsequently. Thirty years ago, solving equations (1)–(3) was beyond reach, and therefore, one had to use approximations. One of the better approximate solutions proposed for handling shock wave propagation in ducts

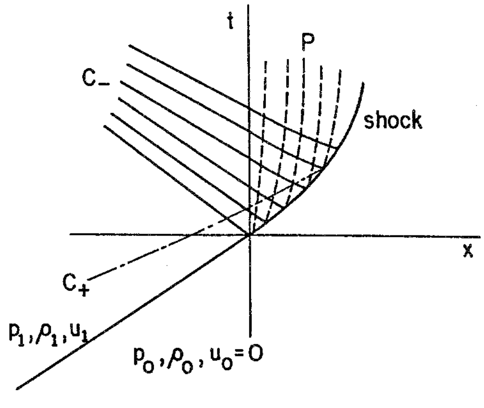

Fig. 1. Schematic description, in the (x, t) plan, of a shock wave entering an area change segment in a duct

with area change, is that proposed by Whitham (1974). His approximation (Whitham 1974) is essentially as follows:

The envisioned initial conditions are a uniform post-shock flow in a uniform cross-sectional duct, the shock being positioned at the entrance to a smoothly converging segment of the duct; the pre-shock initial condition is quiescent gas. Let the shock propagate in the positive x direction so that C_+ characteristics in the (x,t) plane reach the shock while carrying the uniform value of the Riemann invariant R_+ corresponding to the post-shock initial conditions, and the C_- characteristics starting at the shock trajectory carry the Riemann invariant R_- corresponding to the flow at the post-shock side of the shock front. A schematic description of these waves/characteristics is given in Fig. 1. By neglecting the interaction of the C_+ and C_- characteristics in the (x,t) region adjacent to the shock trajectory (i.e., the interaction leading to a non-uniform R_+ arriving at the shock), the compatibility relation along C_+ reduces to a single ordinary differential equation for the shock Mach number M as a function of the duct cross-section $A(x)$. This ordinary differential equation is referred to by Whitham as the "Area Rule". For a perfect gas we have

$$-\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}M}=\frac{M}{M^2-1}\lambda(M) \tag{4}$$

where $\lambda(M)=\left(1+\frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right)\left(1+2\mu+\frac{1}{M^2}\right)$ and $\mu^2=\frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)}$.

The solution to this ordinary differential equation, with A_0 and M_0 being the initial constant, cross-sectional area and shock Mach number, respectively, is

$$\frac{A}{A_0}=f(M) \tag{5}$$

where $f(M)=\exp\left(-\int_{M_0}^{M}\frac{m\lambda(m)}{m^2-1}dm\right)$.

Equations (4)–(5) offer a direct relation between the shock Mach number and the local duct cross-sectional area. Once the shock Mach number is known the post-shock flow properties can easily be calculated from the Rankine-Hugoniot relations.

It is of interest to check the error involved in using Whitham's approximation for two different cases: first, for a duct in which, at a given distance, changes monotonically the cross-sectional area; second, for the case of two ducts of different cross-sections which are connected to a section of monotonically changing area. In a numerical solution the effect of the extent of the area change segment will also be investigated.

2. Results and discussion

The propagation of a shock wave in a duct having the geometry shown in Fig. 2a is studied. An approximate solution for $M(x)$ is obtained by solving (4) and (5) for $1 < A_u/A < 4$ and $1.25 \leq M_0 \leq 4.0$. An exact solution for $M(x)$ can be obtained by a numerical solution of (1) to (3). To reach this end the Random Choice Method (RCM) is adopted; this is a first order accuracy solution whose technical details are found in Glimm (1965), Chorin (1976), Sod (1977), and Colella (1982). In Fig. 3 the percentage difference in $M(x)$, between the RCM solution and those obtained by using the Whitham area rule, is shown as a function of $A_u/A(x)$. The incident shock Mach number, M_0, appears as a parameter. It is apparent from Fig. 3 that for small values of A_u/A and/or M_0 the difference is relatively small and the Whitham area rule can safely be used, at least for engineering purposes. For example, when $A_u/A \leq 2$ and $M_0 \leq 1.5$ the error associated with using the Whitham area rule is not more than 3 %. It will reach about 9 % for $A_u/A = 4$ and $M_0 = 4$. In Fig. 4 the percentage difference in the flow static pressure, between the RCM solution and the Whitham approximation, is shown as a function of A_u/A; M_0 appears as a parameter. It is apparent that errors in estimating the post-shock flow static pressure are higher than those observed in $M(x)$. Now, for $A_u/A = 2$ and $M_0 = 1.5$ the error associated with the usage of the Whitham approximation is about 6.5 % and for $A_u/A = 4$ and $M_0 = 4$ it reaches 17 %.

Another configuration which appears in many engineering problems is shown in Fig. 2b. In this configuration the incident shock wave is propagating from a uniform

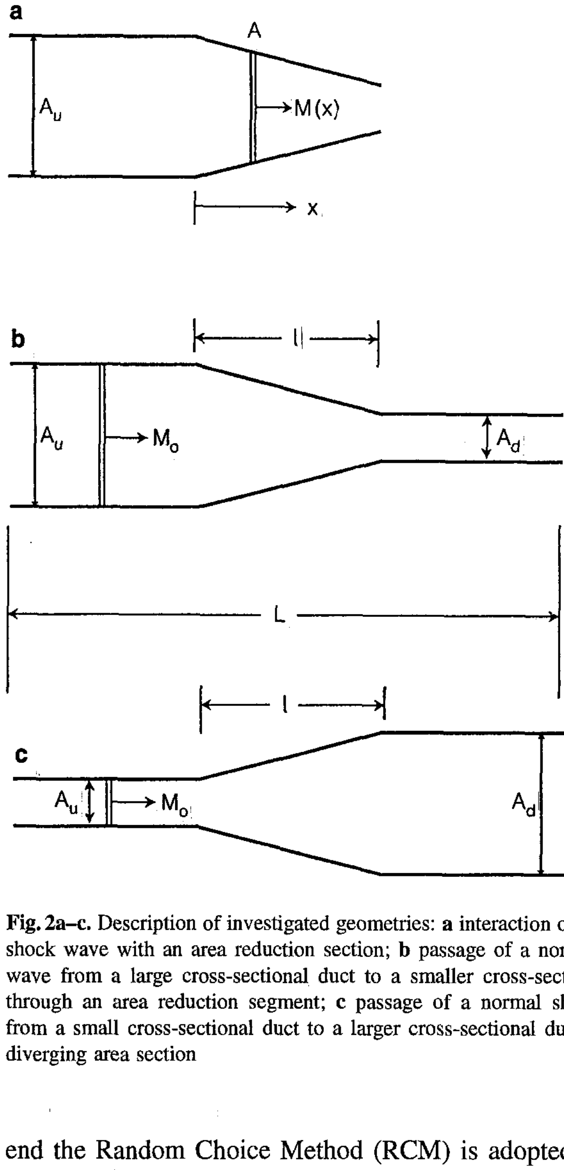

Fig. 2a–c. Description of investigated geometries: **a** interaction of a normal shock wave with an area reduction section; **b** passage of a normal shock wave from a large cross-sectional duct to a smaller cross-sectional duct through an area reduction segment; **c** passage of a normal shock wave from a small cross-sectional duct to a larger cross-sectional duct through diverging area section

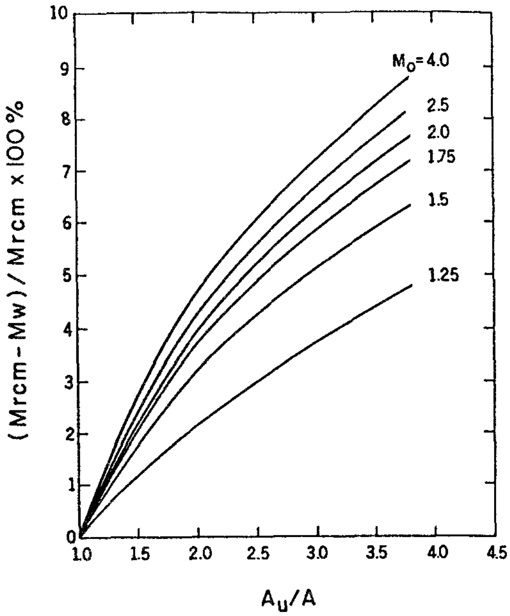

Fig. 3. The percentage difference in the shock Mach number between the RCM solution and that obtained by the Whitham area rule versus the duct area ratio

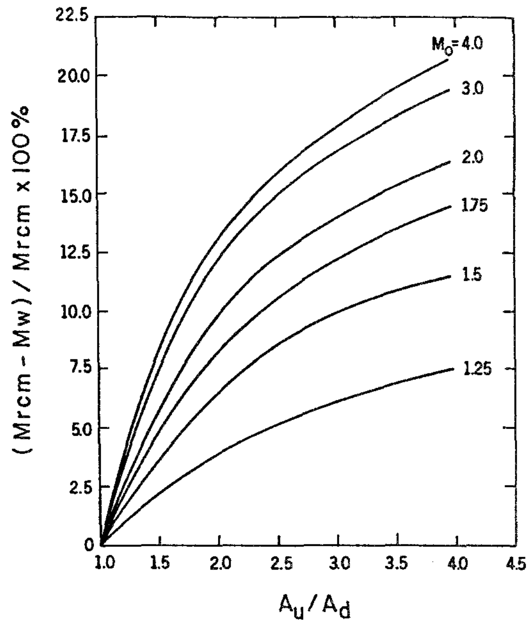

Fig. 5. The percentage difference in the shock Mach number, at the smaller cross-sectional duct, between the RCM solution and that obtained by the Whitham area rule versus the duct area ratio

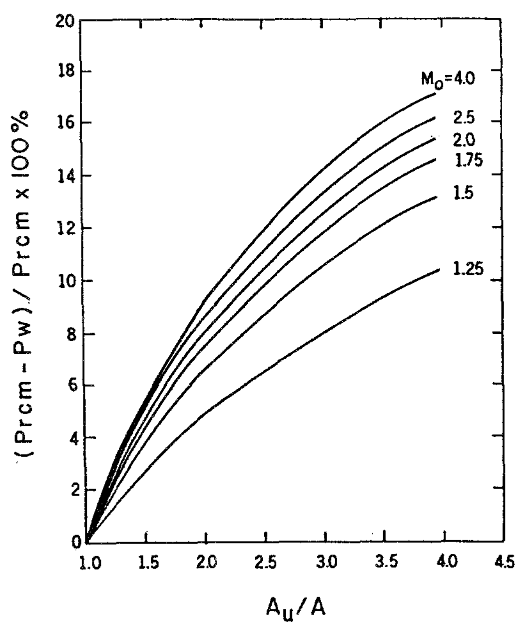

Fig. 4. The percentage difference in the flow static pressure between the RCM solution and that obtained by the Whitham area rule versus the duct area ratio

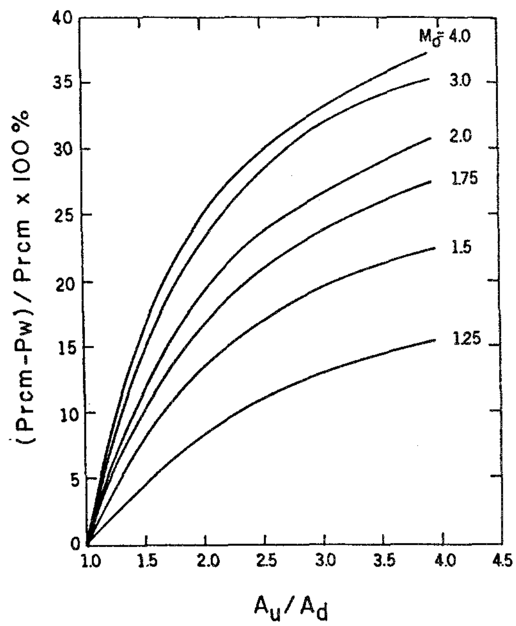

Fig. 6. The percentage difference in the flow static pressure, at the smaller cross-sectional duct, between the RCM solution and that obtained by the Whitham area rule versus the duct area ratio

cross-sectional duct to a smaller constant cross-sectional duct through an area change region. This problem was also solved, first by using the Whitham area rule and thereafter by the RCM. For the transition zone the following expression was used for the duct cross-section:

$$A(x) = A_u \exp\left[\ln\left(\frac{A_d}{A_u}\right)^{\frac{1}{2}}\left(1 - \cos\frac{\pi x}{l}\right)\right] \tag{6}$$

where A_u is the duct cross-section at $x = 0$ and A_d is the duct cross-section at $x = l$. The l is length of the area change region.

The difference between the RCM result and that obtained using the Whitham area rule for the shock Mach number is shown in Fig. 5 and for pressure in Fig. 6; both are in percent. It is apparent that the error associated with using the Whitham area rule is larger in the present case as compared with that obtained for similar conditions while using the geometry shown in Fig. 2a. This should be expected since

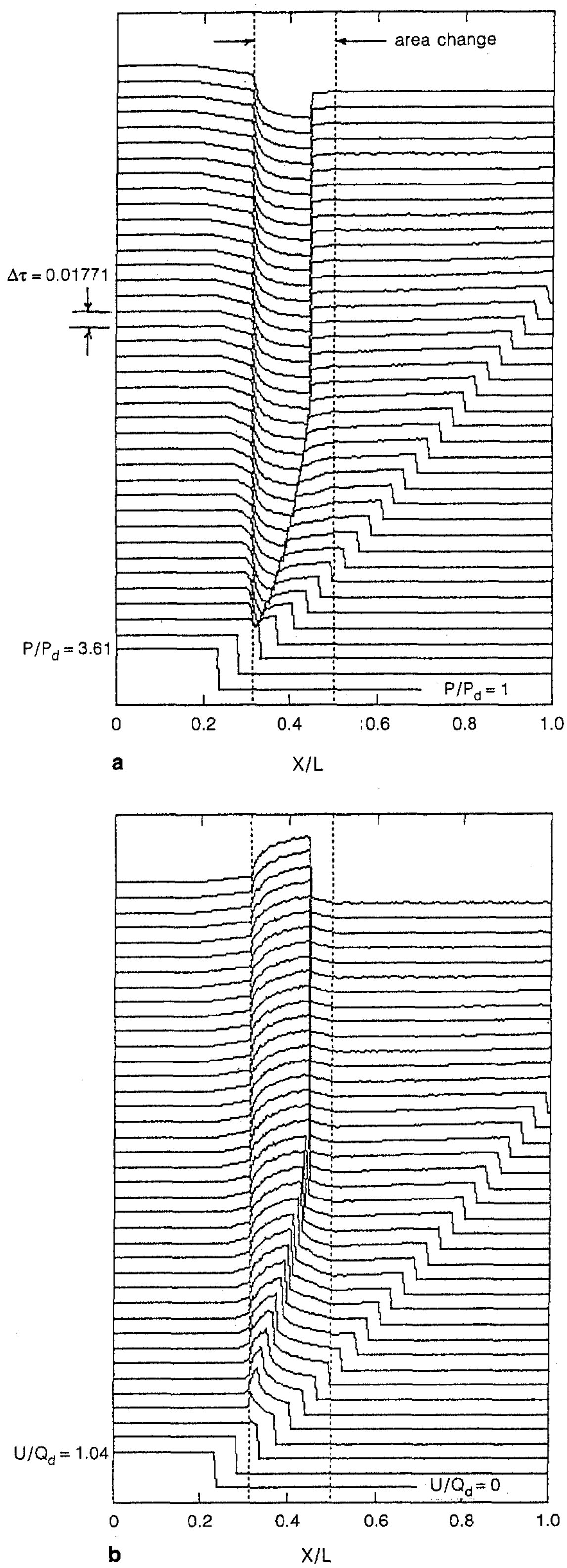

Fig. 7a,b. Spatial distributions of **a** pressure and **b** flow velocity for the interaction of a normal shock wave (M_0 = 1.8) with diverging area (A_u/A_d = 0.2). $\Delta\tau$ = 0.01771 and l/L = 0.1875.

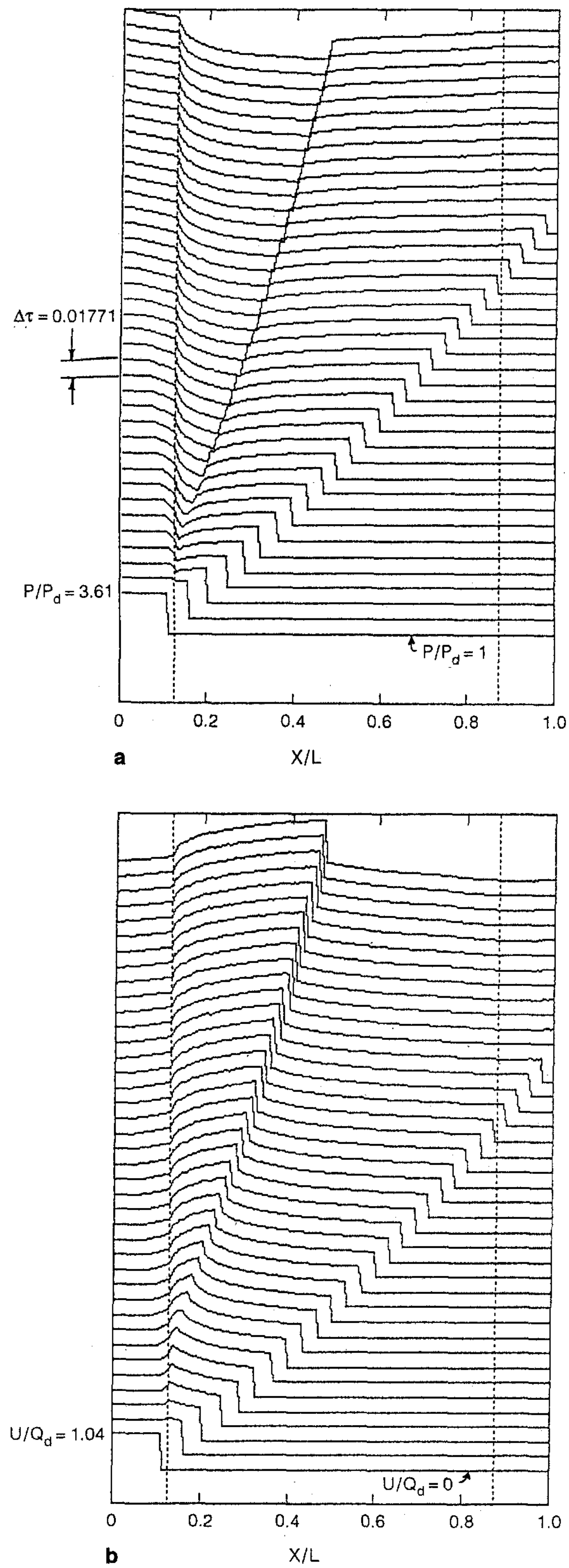

Fig. 8a,b. Spatial distributions of **a** pressure and **b** flow velocity for the interaction of a normal shock wave (M_0 = 1.8) with diverging area (A_u/A_d = 0.2). $\Delta\tau$ = 0.01771 and l/L = 0.75.

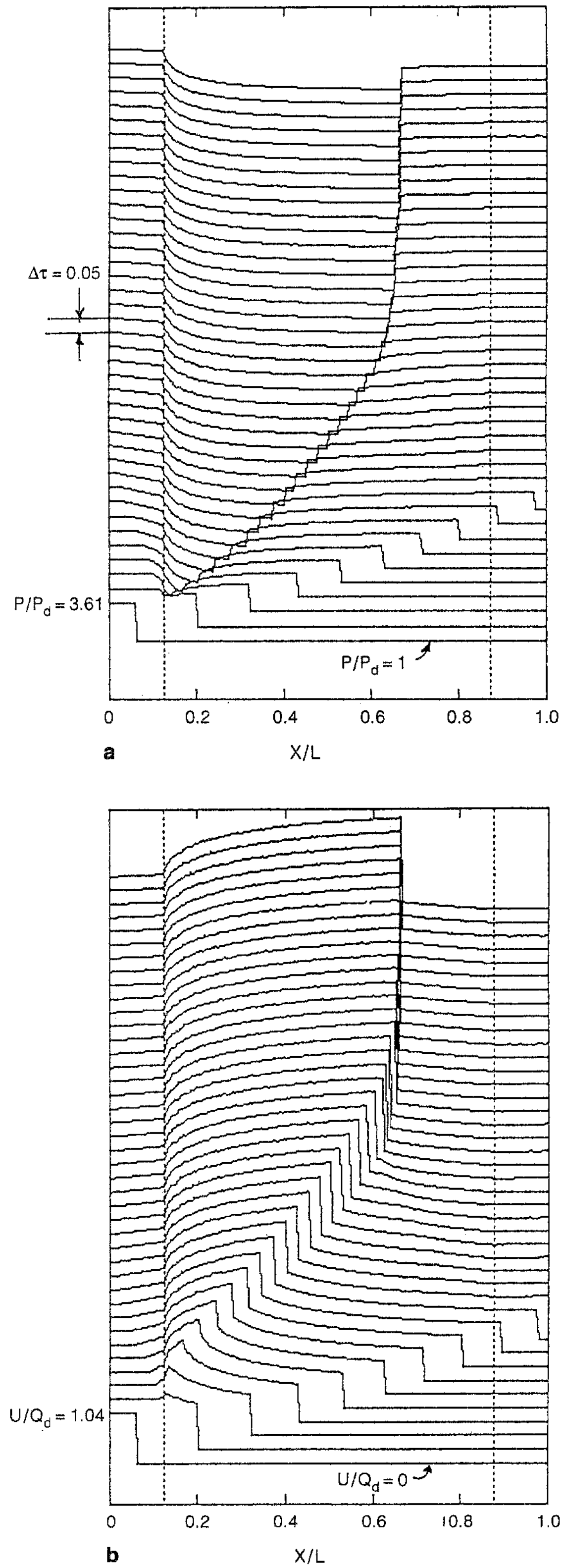

Fig. 9a,b. Spatial distributions of **a** pressure and **b** flow velocity for the interaction of a normal shock wave (M_0 = 1.8) with diverging area (A_u/A_d = 0.2). $\Delta\tau$ = 0.05 and l/L = 0.75.

the RCM solves for the transient flow resulting from the shock wave interaction with the involved area changes. The Whitham area rule ignores the initial interaction of the shock wave which emerges from the area change zone into the smaller cross-sectional duct A_d. For A_u/A_d = 4 and M_0 = 4 the approximate solution is almost 38 % off from the RCM solution. It is only 5 % off for A_u/A_d = 1.5 and M_0 = 1.25; see Fig. 6.

From the foregoing discussion it is clear that the Whitham area rule may safely be used for estimating the flow behind a normal shock wave propagating in a duct in which a monotonic smooth area reduction exists, and/or ducts of different constant cross-sections connected to an area reduction zone, provided that the area reduction ratio (A_u/A or A_u/A_d) and the incident shock Mach number (M_0) are relatively small, i.e., $M_0 \leq 2$ and $A_u/A \leq 2$ (the error in P is less that 8 %) or $M_0 \leq 2$ and $A_u/A_d \leq 1.25$ (again, the error in P is less than 8 %).

The interaction of shock and/or rarefaction waves with area changes (reduction or enlargement) in ducts was studied by Greatrix and Gottlieb (1982), Gottlieb and Igra (1983), and Igra and Gottlieb (1985). In their work they have shown that the flow in the area change segment of the duct is initially nonstationary. The required time for the nonstationary flow to become quasi-steady, and establish the predicted steady flow wave pattern, was shown to be dependent upon the incident shock wave strength (P_2/P_1) and the area ratio (A_u/A_d). It is of interest to check how changes in the length of the area change region, (l), for a given incident shock wave strength and area ratio, affects the required time for establishing a quasi-steady flow in the area-change region. For reaching this goal one of the cases covered by Greatrix and Gottlieb (1982) was repeated, this time with two different lengths. The case to be solved, using the RCM, is the interaction of a shock wave (M_0 = 1.8) with an area enlargement (A_u/A_d = 0.2); see the geometry shown in Fig. 2c. The first solution, shown in Fig. 7 is similar to that obtained by Greatrix and Gottlieb (1982); in the present case l/L = 0.1875 where L is the total length of ducts. The obtained results for pressure and velocity are shown in the form of separate sets of spatial distributions at successive time levels for nondimensional pressure P/P_1 and flow velocity u/a_d. a_d stands for the speed of sound. Each successive distribution is displaced upward slightly from the previous one, both for clarity and to produce the effect of a time-distance diagram. The nondimensional time interval between adjacent distributions is given by $\Delta\tau = a_d\Delta t/l$, and the nondimensional value of $\Delta\tau$ for each case is given in the figure caption. It is apparent from Fig. 7 that the expected stationary shock wave, in the area transition region, is formed after a period of time equal to about $20\Delta\tau$. The total computation time, covered in Fig. 7, is $38\Delta\tau$. When the length of the area change segment was changed to l/L = 0.75, all other initial and boundary conditions remained unchanged, no stationary shock wave in the area change segment is found; see Fig. 8. Now the position of the shock wave is changing throughout the investigated time ($38\Delta\tau$) indicating that the expected quasi-stationary flow has not been reached, as yet. When computations were repeated, for the same initial and boundary conditions but for longer times, then the expected quasi-stationary flow conditions and wave pattern were reached; see Fig. 9. In the present case, ($l/L = 0.75$) it takes almost four times longer to establish the expected quasi-stationary wave pattern, i.e., a stationary, normal, shock wave in the area change region of the duct.

The fact that an increase in the extent of the area change region has a marked effect on the time required for establishing a quasi-stationary flow should have been expected. This is so since the term $\mathrm{d}A/\mathrm{d}x$ has direct effect on all flow properties, see (1)–(3). In the present work the variations of A with changes in x are given by (6). Therefore,

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\pi}{2l}\ln\frac{A_d}{A_u}\sin\frac{\pi x}{l}. \tag{7}$$

It is apparent from (7) that changes in l will have a pronounced effect on $(\mathrm{d}A/\mathrm{d}x)/A$ which, in turn, affects the conservation equations, (1)–(3). Another way to explain the obtained numerical results is by transforming the governing equations (1)–(3) into a nondimentional form where x is proportional to l and the time is proportional to l/a_d. Therefore, the transition time is proportional to the length of the area change segment, l. It should be noted however that in the two numerical results (Figs. 7 and 9) the ratio l/L was not the same. Therefore, the numerical results shown in Fig. 7 are only approximately replicated by those of Fig. 9 since in the later case non-reflecting boundary conditions (inflow as well as outflow) were used to represent a long uniform duct on either side of the area change segment. It should also be noted that, strictly speaking, the RCM solution is not an accurate solution to the considering problem since we are dealing with a two-dimensional flow. The RCM provides a good approximation to this two-dimensional flow problem (shock wave interaction with area changes in ducts). How good is this approximation is a subject for a separate investigation.

3. Conclusions

It was shown that the interaction of shock waves of moderate strength with a relatively small area change (area ratio) could safely be studied using the Whitham approximation. Specifically, the shock Mach number and all the flow properties inside the area transition segment, or in the constant area duct behind the area transition segment could be evaluated. This would not be the case when the interaction of strong shock waves with large area changes (large area ratios) is studied using the Whitham area rule. In such a case very large errors are expected and one should use an accurate, numerical, solution. It was also shown that the extent of the area transition segment has a marked effect on the time elapsed until the expected quasi-steady flow is reached. The longer is the area transition segment, the longer it takes to establish a quasi-steady flow.

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