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# Concave Gauge Functions and Applications

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*Abstract:* Many problems of optimization involve the minimization of an objective function on a convex cone. In this respect we define a concave gauge function which will be used in interior point methods.

Application are given in particular on the space of real symmetric matrices.

*Key Words:* Concave gauge function, concave barrier function, potential gauge function.

## 1 Introduction

People working in convex analysis and optimization are familiar with convex gauge functions. A convex gauge function is associated to a closed convex set $C$ verifying the following properties: $0 \in int(C)$ and $\lambda x \in C$ for all $x \in C$ and $\lambda \in [0, 1]$. In a similar way, a concave gauge function is also associated to a closed convex set $C$ but now this set verifies the two properties: $0 \notin C$ and $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 1$. The domain of a concave gauge function is the cone generated by $C$; a special case is when the value of this function goes to 0 when $x$ goes to the boundary of the cone, we speak then of a barrier gauge function. Barrier functions can be used in optimality problems where the objective function is to be minimized on the cone. Indeed, potential functions, in the same spirit that the Karmarkar's potential function, can be derived from barriers functions and be used for interior point methods.

The paper is organized as follows. In section 2, we first recall some results on sets and cones then, we introduce concave gauge functions and we give some of their properties. Section 3 deals with the subdifferentiability of these functions. We give some examples first in section 4 on the Euclidean space $\mathbb{R}^n$, next in section 5 on the space of real symmetric matrices and finally in section 6 on the Lebesgue space $L^2(\Omega)$ where $\Omega$ is a compact set of $\mathbb{R}^n$. In section 7, we define potential gauge functions and give conditions for which these functions are

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convex. Examples are given in section 8. In section 9, we study the quasiconvexity of potential gauge functions and finally, in section 10, we consider an example of a class of potential gauge functions on the space of real symmetric matrices.

Throughout this paper, we use the following notation. Given $E$ a reflexive Banach space, its topological dual $E'$ is also a reflexive Banach space for the norm

$$\|T\| = \sup\{|\langle T, x\rangle|: \|x\| \le 1\} \ , \quad T \in E'$$

where $\langle T, x\rangle$ is the value of $T$ at $x$.

Let $C$ be a subset of $E$. We denote by $int(C)$, $cl(C)$, $Bd(C)$, and $conv(C)$ the interior, the closure, the boundary and the convex hull of $C$ respectively. If $A$ is an $n \times n$ real matrix, $A^t$, $trace(A)$ and $det(A)$ are the transpose, the trace and the determinant of $A$ respectively. If $A$ is symmetric positive definite and $p \in \mathbb{R}$ we set $A^p = Q^t\, diag((\lambda_1^p, \ldots, \lambda_n^p)^t)Q$ where $\lambda_1, \ldots, \lambda_n$ are the $n$ positive eigenvalues of $A$ and $Q$ is an orthogonal matrix such that $A = Q^t\, diag((\lambda_1, \ldots, \lambda_n)^t)Q$. If $x \in \mathbb{R}^n$, $X = diag(x)$ is the real $n \times n$ matrix such that $X_{i,j} = 0$ if $i \neq j$ and $X_{i,i} = x_i$ for all $i \in \{1, \ldots, n\}$. Finally, we denote by the closure of some function $f$, the smallest upper semi continuous function which is greater than $f$.

## 2 Concave Gauge Functions and Duality

For $C, K \subset E$, define

$$C^{\oplus} = \{x^* \in E': \langle x^*, x\rangle \ge 1, \forall x \in C\}$$

and

$$K^+ = \{x^* \in E': \langle x^*, x\rangle \ge 0, \forall x \in K\} \ ;$$

then $C^{\oplus}$ is a closed convex set and $K^+$ is a closed convex cone. If $K$ is a nonempty closed convex cone then $K^{++} = K$.

Denote by $\mathscr{C}(\mathscr{C}')$, the class of nonempty closed convex sets $C$ of $E(E')$ such that

$$0 \notin C \quad \text{and} \quad C = \bigcup_{\lambda \ge 1} \lambda C \ .$$

Assume that $C \in \mathscr{C}$ and let $K$ be the closure of the cone generated by $C$, i.e., $K = cl\left(\bigcup_{\lambda > 0} \lambda C\right)$. Then $C^{\oplus} \in \mathscr{C}'$, $K^+$ is the closure of the cone generated by $C^{\oplus}$

and $C^{\oplus\oplus} = C$. For all these results on the duality between sets and cones see for instance Ruys-Weddepohl [17], Tind [18] and references herein.

Given $C \in \mathscr{C}$, define $\varphi_C$ by $\varphi_C(x) = \mathit{inf}\{\langle x^*, x\rangle : x^* \in C^\oplus\}$. Then $C = \{x \in E: \varphi_C(x) \geq 1\}$, and $\varphi_C$ is concave, upper semi continuous, positively homogeneous and nonnegative on its domain. Conversely, given $\varphi$ a function having these properties, set $C = \{x \in E: \varphi(x) \geq 1\}$ then $C \in \mathscr{C}$ and $\varphi = \varphi_C$.

Notice that $\mathit{dom}(\varphi) = \{x \in E: \varphi(x) > -\infty\} = \{x \in E: \varphi(x) \geq 0\}$ and is the closure of the convex cone generated by $C$ and,

$$\{x \in E: \varphi(x) > 0\} = \bigcup_{\lambda > 0} \lambda C \ .$$

We set

$$K = \mathit{dom}(\varphi) \quad \text{and} \quad \tilde{K} = \{x \in E: \varphi(x) > 0\} \ .$$

The function $\varphi$ is called a concave gauge function of $C$ and by extension of $K$; it verifies the following properties.

*Proposition 2.1:*

i) *$\varphi$ is the closure of the function $\psi(x) = \sup\{\lambda > 0: x \in \lambda C\}$*
ii) *Triangular inequality:*

$$\varphi(x + y) \geq \varphi(x) + \varphi(y) \ , \quad \textit{for all } x \in E \textit{ and for all } y \in E \ .$$

iii) *$\varphi$ is $K$-monotone, that means*

$$\varphi(x + y) \geq \varphi(x) \ , \quad \textit{for all } x \in E \textit{ and for all } y \in K \ ,$$

*and*

$$\varphi(x + y) > \varphi(x) \quad \textit{for all } x \in K \textit{ and for all } y \in \tilde{K} \ .$$

*Proof:*

i) Let $x \in \tilde{K}$, then the set $J = \{\lambda: \lambda \in (0 + \infty) \text{ and } x \in \lambda C\}$ is non empty. Since $C^{\oplus\oplus} = C$, then $\langle x^*, x\rangle \geq \lambda$ for all $x^* \in C^\oplus$ and for all $\lambda \in J$. Consequently $\psi(x) \leq \varphi(x)$.

Suppose for contradiction that $\psi(x) < \varphi(x)$ and let $\bar{\lambda} = \varphi(x)$, then $x \notin \bar{\lambda}C$. Let $I = \{tx: t \in [0, 1]\}$. Then $I$ is a convex compact set which does not intersect $\bar{\lambda}C$. Hence, according to separation theorems, there exist $a \in E' - \{0\}$ and $\alpha \in \mathbb{R}$ such that $\langle a, tx \rangle < \alpha < \langle \bar{\lambda}a, y \rangle$ for all $y \in C$ and for all $t \in [0, 1]$. Therefore, we deduce that $\alpha > 0$, $\bar{\lambda}\alpha^{-1}a \in C^{\oplus}$ and $\langle \bar{\lambda}\alpha^{-1}a, x \rangle < \bar{\lambda}$. Then

$$\bar{\lambda} = \varphi(x) = inf\{\langle x^*, x \rangle: x^* \in C^{\oplus}\} \leq \langle \bar{\lambda}\alpha^{-1}a, x \rangle < \bar{\lambda}$$

which is absurd.

ii) According to (i) we have

$$\langle x^*, x + y \rangle \geq \langle x^*, x \rangle + \overline{\varphi}(y) \quad \text{for all } x \in E, \text{for all } y \in E \text{ and for all } x^* \in C^{\oplus}$$

and so $\overline{\varphi}(x + y) \geq \overline{\varphi}(x) + \overline{\varphi}(y)$.

iii) is a consequence of (ii). □

Just after the submission of this paper, we have been aware of some works by Balan [1], [2], Calvaire [5] and Calvaire-Fitzpatrick [6] in mathematical physics. In particular Balan has introduced what he calls pseudo-superadditive norms and superadditive norms in order to characterize Minkowski-space time. The Minkowskian norm $\|\cdot\|_t$ is defined by

$$\|(t, x)\|_t = \sqrt{t^2 - \|x\|^2} \quad \text{if } (t, x) \in K$$

where $K = \{(t, x) \in \mathbb{R} \times \mathbb{R}^3: t^2 \geq \|x\|^2\}$. $K$ is called the light cone. Pseudo-superadditive and superadditive norms are closely related to concave gauge functions. Indeed, following Balan a real function $f$ is said to be a pseudo-superadditive norm on a convex cone $K$ if the following properties hold:

$$p(x) \geq 0 \quad \text{for all } x \in K \ ,$$

$$p(\lambda x) = \lambda p(x) \quad \text{for all } x \in K \quad \text{and} \quad \lambda \geq 0 \ ,$$

$$p(x + y) \geq p(x) + p(y) \quad \text{for all } x, y \in K \ .$$

It is said to be a superadditive norm if in addition:

$$p(x) = 0 \quad \text{if and only if } x = 0 \ .$$

Hence a pseudo-superadditive norm extended by upper semicontinuity to the closure of $K$ is a concave gauge function of this set.

Some of the examples we give in this paper appear also in Calvaire [5]. Calvaire-Fitzpatrick [6] have used superadditive norms to characterize Lorenz spaces.

In the same manner, we define concave gauge functions on the dual space $E'$. Let $\varphi$ be a concave gauge function on $E$, then $C = \{x \in E: \varphi(x) \geq 1\} \in \mathscr{C}$ and $C^{\oplus} \in \mathscr{C}'$. We set $\varphi^{\oplus} = \varphi_{C^{\oplus}}$. Since $C^{\oplus\oplus} = C$, a thoroughly symmetric duality exists between $\varphi$ and $\varphi^{\oplus}$. In particular $\varphi^{\oplus\oplus} = \varphi$.

*Definition 2.1: Let $K$ be a closed convex cone in $E$ such that $int(K) \neq \varnothing$ and $\varphi$ a concave gauge function verifying the following properties,*

i) $dom(\varphi) = K$,
ii) $\varphi(x) = 0$ *for all* $x \in Bd(K)$,

*then $\varphi$ is called a concave barrier function of $K$.*

*Remark:* A concave gauge function is not necessarily a concave barrier function of its domain, take for instance:

$$\varphi(x) = \begin{cases} \sum_{i=1}^{n} x_i & \text{if } x \in [0, +\infty)^n , \\ -\infty & \text{if not} . \end{cases}$$

Then $dom\ \varphi = [0, +\infty)^n$, $\varphi$ is a concave gauge but not a concave barrier function of $[0, +\infty)^n$.

*Proposition 2.2: Let $\varphi$ be a concave gauge function on $E$ and $C = \{x \in E: \varphi(x) \geq 1\}$. Then*

i) $x^* \in C^{\oplus}$ *if and only if for all* $x \in dom(\varphi)$ $\langle x^*, x\rangle \geq \varphi(x)$,
ii) $(-\varphi)^*(x^*) = \delta(-x^*, C^{\oplus})$ *for all* $x^* \in E'$,
iii) *Let* $r \in (-\infty, 0) \cup (0, 1)$ *and* $s$ *be such that* $1/r + 1/s = 1$. *Then for all* $x^* \in E'$, *if* $r \in (0, 1)$

$$(-\varphi^r)^*(x^*) = \begin{cases} \dfrac{1-r}{r^s}(\varphi^{\oplus})^s(-x^*) & \text{if } \varphi^{\oplus}(-x^*) > 0 , \\ +\infty & \text{if not} \end{cases}$$

*and if* $r \in (-\infty, 0)$

$$(\varphi^r)^*(x^*) = \begin{cases} \dfrac{r-1}{(-r)^s}(\varphi^\oplus)^s(-x^*) & \text{if } -x^* \in K^+ \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

*Proof:*

i) By definition of $C^\oplus$

$$x^* \in C^\oplus \text{ if and only if } \langle x^*, x\rangle \geq \varphi(x) \text{ for all } x \text{ such that } \varphi(x) \geq 1$$

and by positive homogeneity

$$x^* \in C^\oplus \text{ if and only if } \langle x^*, x\rangle - \varphi(x) \geq 0 \quad \text{for all } x \in dom\ \varphi \ .$$

ii) By definition of the conjugate $(-\varphi)^*(x^*) = \sup\{\langle x^*, x\rangle + \varphi(x) \colon x \in dom\ \varphi\}$. Since $\varphi$ is positively homogeneous,

$$(-\varphi)^*(x^*) = \begin{cases} 0 & \text{if } \langle x^*, x\rangle + \varphi(x) \leq 0 \text{ for all } x \in dom\ \varphi \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

Hence

$$(-\varphi)^*(x^*) = \begin{cases} 0 & \text{if } -x^* \in C^\oplus \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

iii) Set $\varepsilon = \dfrac{r}{|r|}$. Then $(-\varepsilon\varphi^r)^*(x^*) = \sup\{\langle x^*, x\rangle + \varepsilon\varphi^r(x) \colon x \in dom\ \varphi\}$. Since $\varphi$ is positively homogeneous

$$(-\varepsilon\varphi^r)^*(x^*) = \sup\{k\langle x^*, x\rangle + \varepsilon k^r\varphi^r(x) \colon \varphi(x) \geq 0, k > 0\} \geq 0 \ .$$

It follows that $(-\varepsilon\varphi^r)^*(x^*) = +\infty$ if there exists $x \in dom\ \varphi$ such that $\langle x^*, x\rangle > 0$.
Assume that $x^* \in -K^+$ (i.e. $\langle x^*, x\rangle \leq 0$ for all $x \in dom\ \varphi$).
For $x \in E$ such that $\varphi(x) > 0$, define $\alpha(x) = \sup\{k\langle x^*, x\rangle + \varepsilon k^r\varphi^r(x) \colon k > 0\}$.
Assume that $r \in (0, 1)$. Then

$$\alpha(x) = \frac{1-r}{r^s}(\langle -x^*, x\rangle/\varphi(x))^s$$

and then

$$(-\varphi^r)^*(x^*) = \frac{1-r}{r^s}[\sup\{\langle -x^*, x\rangle/\varphi(x): \varphi(x) > 0\}]^s$$

$$= \frac{1-r}{r^s}(\varphi^{\oplus})^s(-x^*) \ .$$

Finally assume that $r \in (-\infty, 0)$. Then

$$\alpha(x) = \frac{r-1}{(-r)^s}(\langle -x^*, x\rangle/\varphi(x))^s$$

and then

$$(\varphi^r)^*(x^*) = \frac{r-1}{(-r)^s}[\sup\{\langle -x^*, x\rangle/\varphi(x): \varphi(x) > 0\}]^s$$

$$= \frac{r-1}{(-r)^s}(\varphi^{\oplus})^s(-x^*) \ . \qquad \square$$

Now, let $E$ and $G$ be two reflexive Banach spaces, let $K$ be a closed convex cone of $E$ and let $g: G \to E$.

*Definition 2.2: g is said to be $K$ – concave if for all $x^* \in K^+$ the function $x \mapsto \langle x^*, g(x)\rangle$ is concave. g is said to be K-usc if for all $x^* \in K^+$ the function $x \mapsto \langle x^*, g(x)\rangle$ is upper semi continuous.*

*Remark:* It is easy to see that $g$ is $K$ – *concave* if and only if for all $(x, y) \in G^2$ and for all $t \in [0, 1]$

$$g(tx + (1-t)y) - tg(x) - (1-t)g(y) \in K \ .$$

*Proposition 2.3: Let $\varphi$ be a concave gauge function such that dom $\varphi = K$.*

i) *If $g$ is $K -$ concave then $\varphi \circ g$ is concave on $G$.*
ii) *If $g$ is $K -$ usc then $\varphi \circ g$ is upper semi continuous on $G$.*

*Proof:*

i) Suppose $g$ $K -$ *concave*. Then, for all $t \in [0, 1]$ and $(x, y) \in G^2$ we have

$$g(tx + (1 - t)y) - tg(x) - (1 - t)g(y) \in K$$

Now $\varphi$ is concave and $K$-monotone therefore

$$\varphi(g(tx + (1 - t)y)) \geq \varphi(tg(x) + (1 - t)g(y)) \geq t\varphi(g(x)) + (1 - t)\varphi(g(y)) \ .$$

ii) Let $x \in G$ then $\varphi \circ g(x) = \inf\{\langle x^*, g(x)\rangle : \varphi^{\oplus}(x^*) \geq 1\}$. Since $g$ is $K - usc$ then $\varphi \circ g$ is upper semi continuous functions as the infimum of upper semi continuous functions. □

*Proposition 2.4: Let $\varphi$ be a concave gauge function on $\mathbb{R}^n$ such that dom $\varphi = [0, +\infty)^n$. Let $g_1, g_2, \ldots, g_n$ be n concave gauge functions defined respectively on $E_1, E_2, \ldots, E_n$ n reflexive Banach spaces.*
*Define $\psi$ by,*

$$\psi(x) = \varphi(g_1(x_1), g_2(x_2), \ldots, g_n(x_n)) \quad \text{for all } x = (x_1, x_2, \ldots, x_n) \in \prod E_i$$

*then $\psi$ is a concave gauge function on $\prod E_i$ and $\psi^{\oplus}$ is given by*

$$\psi^{\oplus}(x^*) = \varphi^{\oplus}(g_1^{\oplus}(x_1^{\oplus}), g_2^{\oplus}(x_2^*), \ldots, g_n^{\oplus}(x_n^*)) \quad \text{for all } x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \prod E_i'$$

*where $E_1', E_2', \ldots, E_n'$ are respectively the topological dual spaces of $E_1, E_2, \ldots, E_n$.*

*Proof:* Let $K = [0, +\infty)^n$ and $g$ be defined by $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))$. Then $\varphi \circ g$ is positively homogeneous and positive on its domain which is equal to $\prod$ *dom* $g_i$. Since $g$ is $K -$ concave and $K - usc$, $\varphi \circ g$ is a concave gauge function on $\prod E_i$. Let $x^* = (x_1^*, \ldots, x_n^*) \in$ *dom* $\psi^{\oplus}$, then

$$\psi^{\oplus}(x^*) = \inf\{\textstyle\sum \langle x_i^*, x_i\rangle : \varphi(g_1(x_1), \ldots, g_n(x_n)) \geq 1\}$$

$$= \inf\{\textstyle\sum \langle x_i^*, x_i\rangle : \varphi(y) \geq 1, g_i(x_i) \geq y_i \text{ for all } i\} \ ,$$

and by duality

$$\psi^{\oplus}(x^*) = \sup_{(y,z)\in K^2} \{\sum y_i z_i + \sum \inf\{\langle x_i^*, x_i\rangle + g_i(x_i): x_i \in dom\, g_i\}\}$$

$$= \sup\{\sum y_i z_i - z_i(-g)_i^*(-x_i^*/z_i): z \in int(K) \text{ and } y \in K\}$$

$$= \sup\{\sum y_i z_i: g_i^{\oplus}(x_i^*) \geq z_i \text{ for all } i \text{ and } \varphi(y) \geq 1\}$$

$$= \sup\{\sum y_i g_i^{\oplus}(x_i^*): \varphi(y) \geq 1\}$$

$$= \varphi^{\oplus}(g_1^{\oplus}(x_1^*), \ldots, g_n^{\oplus}(x_n^*)) \ .$$

□

## 3 Subdifferential of Concave Gauge Functions

Let us recall that the subdifferential of a concave function $\phi$ is defined as

$$\partial\phi(x) = \{x^*: \phi(x) + \langle x^*, y - x\rangle \geq \phi(y), \text{ for all } y \in E\} \ .$$

Then we have the following result:

*Theorem 3.1: Let $\varphi$ be a concave gauge function and $C = \{x \in E: \varphi(x) \geq 1\}$. Then*

i) *$x^* \in \partial\varphi(x)$ if and only if $\langle x^*, x\rangle = \varphi(x)$ and $x^* \in C^{\oplus}$,*
ii) *$\varphi(x)\varphi^{\oplus}(x^*) \leq \langle x^*, x\rangle$, for all $x \in dom\ \varphi$, $x^* \in dom\ \varphi^{\oplus}$,*
iii) *Assume that $\langle x^*, x\rangle > 0$ then the three following statements are equivalent.*

$$\varphi(x)\varphi^{\oplus}(x^*) = \langle x^*, x\rangle \ ,$$

$$x/\varphi(x) \in \partial\varphi^{\oplus}(x^*) \ ,$$

$$x^*/\varphi^{\oplus}(x^*) \in \partial\varphi(x) \ .$$

*Proof:*

i) Assume that $x^* \in \partial\varphi(x)$, then for all $y \in E$

$$\varphi(y) \leq \varphi(x) + \langle x^*, y - x\rangle \ .$$

Set first $y = 0$ and next $y = 2x$ then $\varphi(x) \geq \langle x^*, x\rangle$ and $\varphi(x) \leq \langle x^*, x\rangle$. Hence $\varphi(x) = \langle x^*, x\rangle$ and $\varphi(y) \leq \langle x^*, y\rangle$ for all $y \in E$. It follows that $x^* \in C^\oplus$. The converse part is immediate.

ii) Let $x \in dom\ \varphi$ and $x^* \in dom\ \varphi^\oplus$, then $\langle x^*, x\rangle \geq 0$. Thus the inequality holds when $\varphi^\oplus(x^*) = 0$.

Suppose now $\varphi^\oplus(x^*) > 0$. Since $\varphi(x) = \inf\{\langle y^*, x\rangle : \varphi^\oplus(y^*) \geq 1\}$, then $\varphi(x) \leq \left\langle \dfrac{x^*}{\varphi^\oplus(x^*)}, x \right\rangle$ and the result follows.

iii) is a direct consequence of (i). □

## 4 Application to the Euclidean Space

In the different examples of this section, we shall frequently use the following result of P. Newman [16], Crouzeix [7].

*Proposition 4.1: Let $f$ be a quasiconvex positively homogeneous function on $\mathbb{R}^n$, let us denote by $D$ the domain of $f$, and assume it to be non empty. If $f$ is lower semi continuous at every point of $D$ and if one of the two following conditions is true, then $f$ is convex.*

i) $f(x) \geq 0$ *for every* $x$.
ii) $f(x) < 0$ *for every* $x$ *belonging to* $ri(D)$.

*Remark 4.1:* The result remains true for any reflexive Banach space $E$ by replacing condition (ii) by the following statement: "$cl(\{x \in E: f(x) > 0\})$ is dense in $dom(f)$".

*Example 4.1:* Let $a \in (0, +\infty)^n$ and let $\psi$ be defined as follows

$$\psi(x) = \begin{cases} (\prod x_i^{a_i})^{1/\alpha} & \text{if } x \in [0, +\infty)^n \ , \\ -\infty & \text{otherwise} \end{cases}$$

where $\alpha = \sum a_i$.

*Proposition 4.2:*

i) $\psi$ *is a concave barrier function on* $[0, +\infty)^n$,

ii) $\psi^{\oplus}(x) = \frac{\sum a_i}{\psi(a)} \psi(x)$, *for all* $x \in \mathbb{R}^n$,

iii) $\prod x_i^{y_i} \leq \left( \frac{\langle x, y \rangle}{\sum y_i} \right)^{\sum y_i}$ *for all* $(x, y) \in (0, +\infty)^n$, (4.1)

*the equality holds, if and only if there exists a real* $\lambda > 0$ *such that* $x = \lambda y$.

*Proof:*

i) Clearly $\psi$ is upper semi continuous, positively homogeneous on its domain $[0, \infty)^n$, positive on $(0 + \infty)^n$ and satisfies $\psi(x) = 0$ for all $x \in Bd([0, +\infty)^n)$. Furthermore the function $\zeta$ defined by

$$\zeta(x) = \begin{cases} (1/\sum a_j) \sum a_i ln\, x_i & \text{if } x \in (0, +\infty)^n , \\ -\infty & \text{if not} \end{cases}$$

is concave. Hence $\psi$ is quasiconcave and then concave according to proposition 4.1.

ii) Let $x^* \in (0, +\infty)^n$. Then

$$\psi^{\oplus}(x^*) = \inf\{\sum x_i x_i^* : \zeta(x) \geq 0\} .$$

we are faced with a classical convex minimisation problem. A point $x$ is an optimal solution if and only if there is $\lambda \in [0, +\infty)$ such that

$$\begin{cases} x_i^* - \lambda a_i / x_i = 0 & \text{for all } i \in \{1, \ldots, n\} , \\ \lambda \sum a_i\, ln\, x_i = 0 . \end{cases}$$

Hence $x_i = \lambda a_i / x_i^*$ and $\lambda = (1/\psi(a))\psi(x^*)$ and finally for all $x^* \in (0, +\infty)^n$

$$\psi^{\oplus}(x^*) = (\sum a_i / \psi(a))\psi(x^*) .$$

By continuity, we extend the result on $[0, +\infty)^n$; outside we know that $\psi^{\oplus}(x^*) = -\infty$.

iii) The last result which appears here as a direct consequence of theorem 3.1 is a well known result, see for instance theorem 3 in Gaffke and Krafft [11]. □

*Example 4.2:* For $p \in (-\infty, 0) \cup (0, 1)$ let $\xi_p$ be defined by if $p \in (0, 1)$

$$\xi_p(x) = \begin{cases} (\sum x_i^p)^{1/p} & \text{if } x \in K \text{ and } x \neq 0 \ , \\ 0 & \text{if } x = 0 \ , \\ -\infty & \text{otherwise} \end{cases}$$

and for $p < 0$

$$\xi_p(x) = \begin{cases} (\sum x_i^p)^{1/p} & \text{if } x \in int(K) \ , \\ 0 & \text{if } x \in Bd(K) \ , \\ -\infty & \text{otherwise} \ . \end{cases}$$

*Proposition 4.3: Set $q$ be such that $1/p + 1/q = 1$. Then*

i) *$\xi_p$ is a concave gauge function,*
ii) *$\xi_p^{\oplus} = \xi_q$,*
iii) *$(\sum x_i^p)^{1/p}(\sum y_i^q)^{1/q} \leq \sum x_i y_i$ for all $(x, y) \in (0, +\infty)^{2n}$.*

*Proof:* First assume that $p \in (0, 1)$, then $\xi_p$ is positively homogeneous, upper semi-continuous and positive on the interior of its domain which is equal to $[0, +\infty)^n$. Since $\xi_p^p$ is concave, $\xi_p$ is quasiconcave and then, according to proposition 4.1, concave. Hence $\xi_p$ is a concave gauge function.

Let $x^* \in (0, +\infty)^n$ and set $C = \{x: \xi_p(x) \geq 1\}$. Then $\xi_p^{\oplus}(x^*) = \inf\{\langle x^*, x\rangle: x \in C\}$. Since the function $x \mapsto \langle x^*, x\rangle$ is inf-compact on $C$, there exists $y \in C$ such that

$$\xi_p^{\oplus}(x^*) = \langle x^*, y\rangle \ .$$

According to Kuhn-Tucker theorem, there exists $\delta \in [0, +\infty)$ such that

$$x^* - \delta(\sum y_i^p)^{1/p-1} Y^{p-1} e = 0 \tag{1}$$

and

$$\delta(\xi_p(y) - 1) = 0$$

with $e = (1, 1, \ldots, 1)^t$ and $Y = diag(y)$. Since $x^* \neq 0$, we get $\delta \neq 0$, thus

$$\xi_p(y) = 1 \ . \tag{2}$$

Set $\lambda = \delta(\sum y_i^p)^{1/p-1}$. Then (1) implies $y = \lambda^{-1/(p-1)} X^{*\,1/(p-1)} e$ with $X^* = diag(x^*)$. So by replacing $y$ by the expression above, we obtain the following equality

$$\xi_p^{\oplus}(x^*) = \lambda$$

and from (2) we get

$$1 = \sum y^p = \lambda^{-p/(p-1)} \sum x_i^{p/(p-1)} \ ,$$

that means $\lambda = (\sum x_i^{*q})^{1/q}$ with $1/p + 1/q = 1$. So

$$\xi_p^{\oplus}(x^*) = \xi_q(x^*) \quad \text{for all } x^* \in (0, +\infty)^n$$

and (ii) follows by continuity.

The case when $p < 0$ is obtained by duality.

(iii) is a direct consequence of theorem 3.1. □

*Remark:* $\xi_p$ is a concave barrier function of $K$ if and only if $p \in (-\infty, 0)$.

*Example 4.3:* Let us now consider $\xi_{-\infty}$ defined by

$$\xi_{-\infty} = \begin{cases} min_i\, x_i & \text{if } x \in K \ , \\ -\infty & \text{otherwise} \ . \end{cases}$$

*Proposition 4.4: $\xi_{-\infty}$ is a concave gauge function and $\xi_{-\infty}^{\oplus} = \xi_1$, where*

$$\xi_1(x^*) = \sum x_i^* \quad \textit{for all } x^* \in [0, +\infty)^n \ .$$

*Proof:* $\xi_{-\infty}$ is a concave gauge function as being the infimum of concave gauge functions.

Let $x^* \in [0, +\infty)^n$ then the following optimisation problem

$$\xi^{\oplus}_{-\infty}(x^*) = \inf\{\langle x^*, x\rangle : \xi(x) \geq 1\}$$

$$= \inf\{\langle x^*, x\rangle : x_i \geq 1 \ \forall i \in \{1, \ldots, n\}\} \ .$$

admits for a unique optimal solution (1, 1, ... 1), thus, $\xi^{\oplus}_{-\infty}(x^*) = \sum x_i^*$. □

## 5 Application to the Space of Real Symmetric Matrices

Set $E$ be the space of $n \times n$ symmetric matrices and $\mathscr{P}$ be the cone of the $n \times n$ symmetric positive semi definite matrices. As usual, we define

$$\langle A, B\rangle = trace(AB) \quad \text{for all } A, B \in E \ .$$

Given $A \in E$, with eigenvalues $d_1 \leq d_2 \leq \cdots \leq d_n$, we set $d(A)$ be the vector of $\mathbb{R}^n$ whose components are $d_1, d_2, \ldots, d_n$.

*Theorem 5.1: Let $\xi: \mathbb{R}^n \to \overline{\mathbb{R}}$ be such that $\xi(\prod x) = \xi(x)$ for any permutation matrix $\prod$. We define $f: E \to \overline{\mathbb{R}}$ by $f(A) = \xi(d(A))$.*

i) *If $\xi$ is a concave gauge function on $\mathbb{R}^n$, then $f$ is a concave gauge function on $E$ and*

$$f^{\oplus}(B) = \xi^{\oplus}(d(B)) \quad \textit{for all } B \in E \ .$$

ii) *If $\xi$ is a convex gauge function on $\mathbb{R}^n$, then $f$ is a convex gauge function on $E$ and*

$$f^{o}(B) = \xi^{o}(d(B)) \quad \textit{for all } B \in E \ .$$

*Remark:* It was proved by J. M. Ball [3] that if $\xi$ is convex on $[0, +\infty)^n$, $f$ is convex (see also Marques and Moreau [13] for a generalization).

*Proof:*

i) Clearly $\xi^{\oplus}(\xi^{o})$ satisfies the condition on permutation matrices, therefore $f^{\oplus}$ is well defined.

Let $A \in E$ and set

$$g(A) = \inf\{trace(AB): B \in E, \xi^{\oplus}(d(B)) \geq 1\} .$$

Then

$$g(A) = \inf\{trace(P^tDPB): B \in E, \xi^{\oplus}(d(B)) \geq 1\}$$

$$= \inf\{trace(DB): B \in E, \xi^{\oplus}(d(B)) \geq 1\}$$

$$= \inf\{trace(Q^tDQ\Delta): Q \in \mathcal{O}_n, \Delta = diag(\delta), \delta \in \mathbb{R}^n \text{ and } \xi^{\oplus}(\delta) \geq 1\}$$

where $\mathcal{O}_n$ is the set of the orthogonal matrices, $D = diag(d(A))$ and $P \in O_n$ are such that $A = P^tDP$.

Notice that for all $Q \in E$ $trace(Q^tDQ\Delta) = \langle R(Q)d(A), \delta\rangle$ with $R(Q) = (q_{i,j}^2)$.

Then $g(A) = \inf\{\xi(R(Q)d(A)): Q \in \mathcal{O}_n\}$ and then $g(A) \leq \xi(d(A))$. Now $R(Q)$ is a doubly stochastic matrix for all $Q \in \mathcal{O}_n$, therefore, using the following theorem (Birkhoff [4], Von Neumann [16])

"*A doubly stochastic matrix is a convex combination of permutation matrices*" there exist $t_1, t_2, \ldots, t_m$ $m$ positive reals and $\prod_1, \prod_2, \ldots, \prod_m$ $m$ permutation matrices such that

$$\sum t_i = 1 \quad \text{and} \quad R(Q) = \sum t_i \prod_i .$$

Since $\xi$ is concave we get

$$\xi(R(Q)d(A)) \geq \sum t_i \xi(\prod_i d(A)) = \xi(d(A)) .$$

Hence $g(A) = f(A)$. By duality we obtain

$$\inf\{trace(AB): B \in E, \xi(d(B)) \geq 1\} = \xi^{\oplus}(d(A))$$

and the result follows.

ii) Set $g(A) = \sup\{trace(AB): B \in E, \xi^{o}(d(B)) \leq 1\}$. Then in the same manner as above one has on the one hand

$$g(A) = \sup\{\xi(R(Q)d(A)) \colon q \in \mathcal{O}_n\} \geq \xi(d(A))$$

and on the other hand, using again the Birkhoff-Von Neumann theorem,

$$g(A) = \xi(d(A)) \ .$$

Hence by duality we get

$$\xi^o(d(A)) = \sup\{trace(AB) \colon B \in E,\ \xi(d(B)) \leq 1\}$$

and (ii) holds. □

*Example 5.1:* Let $\varphi$ be defined by

$$\varphi(X) = \begin{cases} (det\ X)^{1/n} & \text{if } X \in \mathscr{P}_n \ , \\ -\infty & \text{otherwise} \ . \end{cases}$$

Then $\varphi(X) = \xi(d(X))$ where $\xi$ is the concave gauge function defined in example 4.1. Hence we have the following result.

*Proposition 5.1:*

i) *$\varphi$ is a concave gauge function,*
ii) $\varphi^{\oplus}(X) = \begin{cases} n(det\ X)^{1/n} & \textit{if } X \in \mathscr{P}_n \\ -\infty & \textit{otherwise.} \end{cases}$
iii) *$(det\ X)^{1/n} \leq (1/n)trace(X)$ for all $X \in \mathscr{P}_n$ and the equality holds, if and only if, there exists a real $\lambda > 0$ such that $X = \lambda I_n$,*
iv) *$det\ X \leq c^n$ for all $X \in \mathscr{P}_n$ with $c = \sup_i |A_{i,i}|$.*
v) *Let $\mathscr{M}_n$ the space of the real $n \times n$ matrices; then we have*

$$|det\ A| \leq n^{n/2}c^n \quad \textit{for all } A \in \mathscr{M}_n$$

*where $c = \sup_{i,j} |A_{i,j}|$.*

*Proof:* The proofs of (i) and (ii) are obtained by using theorem 5.1.
(iii) and (iv) are direct consequences of theorem 3.1 and proposition 4.2. Finally (v) is obtained from (iv) by setting $B = AA^t$.

*Example 5.2:* Set $f_p(A) = \xi_p(d(A))$ for all $A \in E$ and $p \in [-\infty, 0) \cup (0, 1]$, where $\xi_p$ is the function defined in example 4.2 and 4.3.

Notice that $f_p(A) = (trace(A^p))^{1/p}$ for all $A \in int(\mathscr{P})$.

Using theorem 5.1 and proposition 4.3 we obtain the following result.

*Proposition 5.2: Let $p \in (-\infty, 0) \cup [0, 1]$ and $q$ be such that $1/q + 1/q = 1$. Then*

i) *$f_p$ is a concave gauge function,*
ii) $f_p^{\oplus} = f_q^{\oplus}$,
iii) *Minkowsky inequality:*

$$(trace(A^p))^{1/p}(trace(B^q))^{1/q} \leq trace(AB)$$

*for all $A, B \in int(\mathscr{P})$.*

*Remark:* The concavity of $f_p$ and the above Minkowsky inequality have been proved by Gaffke and Krafft [11].

*Corollary 5.1: $(trace(C))^2 \leq trace(A^{-1})trace(CAC)$ for all $A \in \mathscr{P}$ and $C \in E$.*

*Proof:* Set $B = C^2$, $p = -1$, $q = 1/2$ and use the above Minkowsky inequality. □

## 6 Example of Concave Gauge Functions on Infinite Dimensional Spaces

Let $\Omega$ be a compact set of $\mathbb{R}^n$. We consider $E = L^2(\Omega)$.

For $p \in (-\infty, 0) \cup (0, 1)$, let $\zeta_p$ be defined by
if $p \in (0, 1)$

$$\zeta_p(f) = \begin{cases} \left(\int_\Omega f^p\right)^{1/p} & \text{if } f \geq 0 \ , \\ -\infty & \text{otherwise} \end{cases}$$

and if $p < 0$,

$$\zeta_p(f) = \begin{cases} \left(\int_\Omega f^p\right)^{1/p} & \text{if } f \geq 0 \text{ and } 1/f \in L^{-p}(\Omega) \ , \\ 0 & \text{if } f \geq 0 \text{ and } 1/f \notin L^{-p}(\Omega) \ , \\ -\infty & \text{otherwise} \ . \end{cases}$$

We recall that $L^p(\Omega) \subset L^2(\Omega)$ when $p \in (0, 1)$.

*Proposition 6.1: Set q be such that $1/p + 1/q = 1$. Then.*

i) *$\zeta_p$ is a concave gauge function,*
ii) *$\zeta_p^\oplus = \zeta_q$,*
iii) *suppose that $p < 0$, then for all $f$, $g \in L^2(\Omega)$ such that $1/|f| \in L^{-p}(\Omega)$, $|g| \in L^q(\Omega)$*

$$\left(\int_\Omega |f|^p\right)^{1/p} \left(\int_\Omega |g|^q\right)^{1/q} \leq \int_\Omega |fg| \ .$$

*Proof:* First assume that $p \in (0, 1)$, then $\zeta_p$ is positively homogeneous and positive on its domain. Let us prove now that $\zeta_p$ is upper semi continuous, i.e, the set

$$A = \left\{f : f \in L^2(\Omega), f \geq 0, \left(\int_\Omega f^p\right)^{1/p} \geq 1\right\}$$

is closed in $L^2(\Omega)$.

Let $f_n$ be a sequence of $A$ and $f \in L^2(\Omega)$ such that $lim_{n\to+\infty}(\int_\Omega |f_n - f|^2)^{1/2} = 0$. Since $(\int_\Omega f^p)^{1/p} \leq k(\int_\Omega f^2)^{1/2}$ where $k = (mes(\Omega))^{2-p/2p}$ we get $lim_{n\to+\infty}(\int_\Omega |f_n - f|^p)^{1/p} = 0$ and then $lim_{n\to+\infty}(\int_\Omega f_n^p)^{1/p} = (\int_\Omega f^p)^{1/p} \geq 1$, i.e, $f \in A$. Hence $\zeta_p$ is upper semi continuous.

$\zeta_p$ is quasiconcave because $\zeta_p^p$ is concave. But $cl(\{f \in L^p(\Omega) : f > 0\}) = dom(\zeta) = \{f \in L^p(\Omega) : f \geq 0\}$, so $\zeta_p$ is concave according to remark 4.1. Hence $\zeta_p$ is a concave gauge function.

Let $f \in L^2(\Omega)$. Then

$$\zeta_p^\oplus(f) = \inf\left\{\left(\int_\Omega fg : g \in L^2(\Omega), g \geq 0, \int_\Omega g^p \geq 1\right\} \ .$$

Hence, by duality,

$$\zeta_p^{\oplus}(f) = \sup_{\lambda>0}\left\{\lambda + \inf\left\{\int_{\Omega}(fg - \lambda g^p) : g \in L^2(\Omega), g \geq 0\right\}\right\} .$$

Set $\alpha(\lambda, f) = \inf\{\int_\Omega (fg - \lambda g^p): g \in L^2(\Omega), g \geq 0\}$ and suppose that $f \geq 0$. Then if $1/f \in L^{-q}(\Omega)$ we get $\alpha(\lambda, f) = (1/qp^{q/p})\lambda^{-q/p}\int_\Omega f^q$ and then $\zeta_p^{\oplus}(f) = (\int_\Omega f^q)^{1/q}$, else $\alpha(\lambda, f) = -\infty$ and then $\zeta_p^{\oplus}(f) = 0$.

Suppose now that $f$ is such that $mes(B(f)) \neq \varnothing$ where $B(f) = \{x \in \Omega: f(x) < 0\}$.

Set

$$g_n(x) = \begin{cases} 0 & \text{if } x \notin B(f) , \\ n/(mes(B(f)))^{1/q} & \text{if } x \in B(f) . \end{cases}$$

Then $\int_\Omega g_n f = n \int_\Omega f$ goes to $-\infty$ when $n$ goes to $+\infty$. That means $\zeta_p^{\oplus}(f) = -\infty$. So $\zeta_p^{\oplus} = \zeta_q$

Now the case $p < 0$ is obtained by duality.

(iii) is a direct consequence of theorem 3.1. □

## 7 Potential Gauge Functions and Convexity

Let $K$ be a closed convex cone of $E$ such that $cl(\tilde{K}) = K$ and $\varphi$ be a concave gauge function such that $dom(\varphi) = K$. Given $(p, q) \in (0, \infty)^2$, we define $f$ as follows,

$$f(x, t) = t^p/(\varphi(x))^q \quad \text{for all } x \in \tilde{K} \quad \text{and} \quad t \in (0, \infty)$$

extended by lower semi-continuity to the boundary of $K \times [0, \infty)$. Then $f$ is called a potential gauge function of $K$ associated to $\varphi$. We are interested in the convexity of $f$.

Let $e^* \in \tilde{K}^+$ and set $\Omega_{e^*} = \{x \in E: x \in K, \langle e^*, x\rangle = 1\}$.

*Theorem 7.1:*

i) *Assume that $p = q$. Then $f$ is convex on $\Omega_{e^*} \times [0, +\infty)$ if and only if for all $x^* \in E'$ the function $\theta_p(x^*, \mu) = (\varphi^{\oplus}(x^* + \mu e^*))^p$ is convex in $\mu$ on the set $I_{e^*}(x^*) = \{\mu \in \mathbb{R}: x^* + \mu e^* \in K^+\}$.*

ii) *Assume that* $p > q$, *set* $r = q/p$ *and* $s$ *such that* $1/r + 1/s = 1$. *Then* $f$ *is convex on* $\Omega_{e^*} \times [0, +\infty)$ *if and only if for all* $x^* \in E'$ *the function*

$$G(x^*, \lambda) = \inf\left\{\mu + \frac{1-r}{r^s}(\varphi^{\oplus}(x^* + \mu e^*))^s \lambda^{1/p-q}: \mu \in I_{e^*}(x^*)\right\}$$

*is concave in* $\lambda$ *on* $(0, +\infty)$.

The proof is based on the following result of Crouzeix [7],

*Theorem 7.2: Let* $f$ *be a quasiconvex function which is lower semi continuous at every point of its domain D. Then* $f$ *is convex if and only if the function*

$$F(x^*, \mu) = \sup\{\langle x^*, x\rangle : f(x) \leq \mu\}$$

*is concave in* $\mu$.

Actually, this result was stated for $E = \mathbb{R}^n$ the Euclidean space, but it can be seen with the help of the proof, that it remains true for any reflexive Banach space.

*Proof of Theorem 7.1:* Since $\varphi$ is a concave gauge function and $p \geq q$, $\varphi^{q/p}$ is a concave function on $K$ and then $t - \varphi^{q/p}(x)$ is a convex function in $(x, t)$. Therefore $f$ is quasiconvex on $K \times [0, +\infty)$.

Let $(x^*, t^*) \in E' \times \mathbb{R}$ and $\lambda \in \mathbb{R}$. Set

$$F(x^*, t^*, \lambda) = \sup\{\langle x^*, x\rangle + t^*t: f(x, t) \leq \lambda, \langle e^*, x\rangle = 1, x \in \tilde{K}, t \geq 0\}$$

If $\lambda \leq 0$, $F(x^*, t^*, \lambda) = -\infty$.

Assume now that $\lambda > 0$. Then

$$F(x^*, t^*, \lambda) = \sup\{\langle x^*, x\rangle + t^*t: t - \lambda^{1/p}\varphi^r(x) \leq 0, \langle e^*, x\rangle = 1, x \in \tilde{K}, t \geq 0\}\ .$$

If $t^* \leq 0$

$$F(x^*, t^*, \lambda) = \sup\{\langle x^*, x\rangle : \langle e^*, x\rangle = 1, x \in \tilde{K}\}$$

then $F(x^*, t^*, \lambda)$ is concave on $(-\infty, +\infty)$.

Assume now $t^* > 0$. Notice that $F(t^*x^*, t^*, \lambda) = t^*F(x^*, 1, \lambda)$, hence it is enough to prove that $F(x^*, 1, \lambda)$ is concave in $\lambda$.

Now for $\lambda > 0$,

$$F(x^*, 1, \lambda) = \sup\{\langle x^*, x\rangle + \lambda^{1/p}\varphi^r(x): \langle e^*, x\rangle = 1 \text{ and } x \in \tilde{K}\} ,$$

therefore by duality,

$$F(x^*, 1, \lambda) = \inf\{\sup\{\langle x^*, x\rangle + \lambda^{1/p}\varphi^r(x) - \mu\langle e^*, x\rangle + \mu: x \in \tilde{K}\}: \mu \in \mathbb{R}\}$$

$$\text{i.e.}\quad F(x^*, 1, \lambda) = \inf\left\{\mu + \lambda^{1/p}(-\varphi^r)^*\left(\frac{x^* - \mu e^*}{\lambda^{1/p}}\right): \mu \in \mathbb{R}\right\} \qquad (6.1).$$

If $p = q$ i.e $r = 1$ then (proposition 2.3(ii))

$$\begin{aligned} F(x^*, 1, \lambda) &= \inf\left\{\mu \in \mathbb{R}: \varphi^{\oplus}\left(\frac{-x^* + \mu e^*}{\lambda^{1/p}}\right) \geq 1,\ -x^* + \mu e^* \in K^+\right\} \\ &= \inf\{\mu \in \mathbb{R}: \varphi^{\oplus}(-x^* + \mu e^*)^p \geq \lambda,\ -x^* + \mu e^* \in K^+\} \\ &= \inf\{\mu \in \mathbb{R}: \theta_p(-x^*, \mu) \geq \lambda,\ -x^* + \mu e^* \in K^+\} . \end{aligned}$$

Now $\{\mu \in \mathbb{R}: -x^* + \mu e^* \in K^+\}$ is a closed convex cone of $\mathbb{R}$, therefore, there exists $\mu_0$ such that $\{\mu \in \mathbb{R}: -x^* + \mu e^* \in K^+\} = [\mu_0, +\infty)$. Since $e^* \in \tilde{K}^+$, then from the monotonicity of $\varphi^{\oplus}$ we deduce that $\theta_p(-x^*, \mu)$ is a strictly increasing function in $\mu$ and then one to one from $[\mu_0, +\infty)$ to $[\theta_p(-x^*, \mu_0), +\infty)$. Set $\theta_p^{-1}(-x^*, \mu)$ the inverse function of $\theta_p(-x^*, \mu)$ in $\mu$ then

$$F(x^*, 1, \lambda) = \begin{cases} \theta_p^{-1}(-x^*, \lambda) & \text{if } \lambda \in [\theta_p(-x^*, \mu_0), +\infty) \\ \mu_0 & \text{otherwise} . \end{cases}$$

Now $\theta_p(-x^*, \mu)$ is a convex function if and only if $\theta_p^{-1}(-x^*, \mu_0)$ is a concave function and then the result follows.

Suppose now that $p > q$, i.e $r \in (0, 1)$ then (proposition 2.3(iii)) implies

$$\begin{aligned} F(x^*, 1, \lambda) &= \inf\left\{\mu + \frac{1-r}{r^s}(\varphi^{\oplus}(-x^* + \mu e^*))^s\lambda^{1/p-q}: \mu \in I_{e^*}(-x^*)\right\} \\ &= G(-x^*, \lambda) . \end{aligned}$$

□

Notice that a necessary condition for the convexity of $f$ is that $p \geq 1$.

## 8 Examples of Potential Gauge Functions on the Positive Orthant of $\mathbb{R}^n$

In this section $E = \mathbb{R}^n$, $K = [0, +\infty)^n$, $e \in int(K)$ and $p \in (0, +\infty)$.

*Theorem 8.1: Let $a \in int(K)$ and let $f$ be defined by*

$$f(x, t) = t^p / \prod x_i^{pa_i/\sum a_i} \text{ if } (x, t) \in \Omega_e \times [0, +\infty) .$$

*Then $f$ is convex on its domain if and only if $p \geq \sum a_i / min(a_i)$.*

*Proof:* We apply theorem 7.1(i) to the function $\varphi$ defined in example 4.1. For any $x \in \mathbb{R}^n$ we are faced with the convexity in $\lambda$ of the function

$$\theta(x, \lambda) = (\sum a_i/\psi_a(a))(\prod (x_i + \lambda e_i)^{a_i})^{p/\Sigma a_i} .$$

Using the second order characterization of convexity we must have

$$p/\sum a_i \geq \sup_y \{\sum a_i y_i^2/(\sum a_i y_i)^2 : y \in int(K)\}$$

i.e $p \geq \sum a_i / min(a_i)$. □

*Remark:* In the case where $a_i = 1$ for all $i = 1, \ldots, n$ the result has been established by Crouzeix, Ferland and Schaible [8].

We consider now for $r \in [-\infty, 0)$ and $p \in (0, +\infty)$ the function

$$f(x, t) = (t/\xi_r(x))^p \quad \text{if } (x, t) \in \Omega_e \times [0, +\infty)$$

where $\xi_r$ is the function defined in example 4.2. Then we have

*Theorem 8.2:*

i) *If $r \in (-\infty, 0)$ and $n \geq 2$ then $f$ is not convex on its domain.*
ii) *Set $M = max_i\, e_i$ and $m = min_i\, e_i$. If $r \in (0, 1)$ then $f$ is convex on its domain whenever $p \geq \dfrac{1}{1-r}\left[\left(\dfrac{M}{m}\right)^r - r\right]$.*
iii) *If $r = -\infty$ then $f$ is convex on its domain if and only if $p \geq 1$.*

*Proof:* for $r \in (-\infty, 0) \cup (0, 1)$, theorem 7.1(i) leads to analyse the convexity of the function

$$\theta_p(x, \lambda) = (\sum (x_i + \lambda e_i)^s)^{p/s}$$

with $1/s + 1/r = 1$.

Using the second order characterization of convexity we must have

$$p - s/(1-s) \geq \sup\{g(y): y \in int(K)\} = \alpha$$

with $g(y) = \sum e_i^2 y_i^{s-2} \sum y_i^s / (\sum e_i y_i^{s-1})^2$.

i) Assume $r \in (-\infty, 0)$. We shall prove that

$$\sup\{g(y): y \in int(K)\} = +\infty \ .$$

For this, set $y(t) = ((1-t), t, t, \ldots, t)^t$ for all $t \in (0, 1)$. Since $s \in (0, 1)$ we get

$$\lim_{t \to 0} g(y(t)) = (\textstyle\sum_2^n e_i^2 / (\sum_2^n e_i)^2) \lim_{t \to 0} t^{-s} = +\infty \ .$$

Thus, for all $r \in (-\infty, 0)$ and for all $p \in (0, +\infty)$, the function $f(x, t) = t^p / \xi_r^p(x)$ is not convex.

ii) Assume now $r \in (0, 1)$. Set $z_i = 1/y_i$, $i \in \{1, \ldots, n\}$ and $s' = -s$. Then

$$\alpha = \sup\{\textstyle\sum e_i^2 z_i^{2+s'} \sum z_i^{s'} : \sum e_i z_i^{1+s'} = 1, z \in [0, +\infty)^n\} \ .$$

But $z_i \leq e_i^{-1/(1+s')}$, $i = 1$ to $n$ whenever $\sum e_i z_i^{1+s'} = 1$, $z \in [0, +\infty)^n\}$; it follows that

$$\alpha \leq \sum e_i^{s'/(1+s')} \sum e_i^{-s'/(1+s')} \leq \left(\frac{M}{m}\right)^{s'/(1+s')} .$$

But $\dfrac{s'}{1+s'} = r$. The result then follows.

iii) Suppose that $r = -\infty$ then (theorem 7.1(i)) the result is equivalent to the convexity in $\lambda$ of the function

$$\theta_p(x, \lambda) = (\sum (x_i + \lambda e_i))^p$$

i.e $p \geq 1$. □

## 9 Quasiconvexity of Potential Gauge Functions

In this section we use the concept of convexity index given first by Debreu and Koopmans [10] and reformulated by Crouzeix and Lindberg [9], for studying conditions which imply the quasiconvexity of the function

$$f(x) = \sum_{i=1}^{n} f_i(x_i) \ , \quad x_i \in X_i \ ,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $X_i$ is a finite dimensional open convex set and $f_i$ a real-valued non constant function on $X_i$ for all $i = 1, 2, \ldots, n$.

Let $E$ be a reflexive Banach space and $C$ a non-empty convex subset of $E$. Let $f$ be a real-valued function on $C$ and $r_\lambda$ defined on $C$ by $r_\lambda(x) = e^{-\lambda f(x)}$; then, following Crouzeix-Lindberg [9], the convexity index $c(f)$ of $f$ is defined as follows

"if there exists $\mu < 0$ such that $r_\mu$ is not convex, then

$$c(f) = \sup\{\lambda\colon \lambda < 0, r_\lambda \text{ is convex}\}$$

if not

$$c(f) = \sup\{\lambda\colon \lambda \geq 0, r_\lambda \text{ is concave}\}" \ .$$

*Theorem 9.1: Let $F$ and $G$ two reflexive Banach spaces. Assume that $X$ and $Y$ are non-empty open convex subsets of $F$ and $G$ respectively, $f$ and $g$ are non constant real-valued functions on $X$ and $Y$ respectively.*

*We consider the function $s$ on $X \times Y$ defined by*

$$s(x, y) = f(x) + g(y)$$

*then $s$ is quasiconvex if and only if $c(f) + c(g) \geq 0$.*

The result was given by Crouzeix-Lindberg [9] in the finite dimensional case, but it can be seen with the help of the proof that it remains true when $F$ and $G$ are reflexive Banach spaces.

Let $K$ be a closed convex cone of $E$ such that $cl(\tilde{K}) = K$ and $cl(\tilde{K}^+) = K^+$. Let $\alpha \in (0, +\infty)$ and one considers the function $f(x, t) = t/\varphi^\alpha(x)$ on $\tilde{K} \times (0, +\infty)$.

Let $e \in int(K^+)$, we set

$$\tilde{\Omega}_e = \{x \in E\colon x \in \tilde{K} \text{ and } \langle e, x\rangle = 1\} .$$

*Theorem 9.2: $f$ is quasiconvex on $\tilde{\Omega}_e \times (0, +\infty)$ if and only if the function $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$.*

*Proof:* Set $h(t) = \alpha^{-1}\, lnt$ and $g(x) = -ln\, \varphi(x)$ then $f$ is quasiconvex if and only if the function $h(t) + g(x)$ is quasiconvex, i.e (theorem 9.1),

$$c(h) + c(g) \geq 0 . \tag{1}$$

Now $e^{-\lambda h(t)} = t^{-\lambda\alpha^{-1}}$ is convex for all $\lambda \leq -\alpha < 0$, therefore $c(h) = -\alpha$. So the condition (1) means that $c(g) \geq \alpha > 0$, i.e, $e^{(-\alpha)g(x)} = \varphi^\alpha(x)$ is concave on $\tilde{\Omega}_e$. □

*Proposition 9.1: Given $\alpha$ a positive real and $e \in \tilde{K}^+$, $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if the function*

$$\theta(x^*, r) = \sup\{\mu + r^{1/\alpha}\varphi^\oplus(x^* - \mu e)\colon \mu \in \mathbb{R}\}$$

*is convex in $r$ on the set $[0, +\infty)$ for all $x^* \in E$.*

*Proof:* According to theorem 7.2, $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if the function

$$\psi(x^*, r) = \inf\{\langle x^*, x\rangle\colon x \in \tilde{K}, \langle e, x\rangle = 1, \varphi^\alpha(x) \geq r\}$$

is convex in $r$ for all $x^* \in E$.

If $r \in (-\infty, 0]$ then $\psi(x^*, r)$ is a constant function.

If $r \in (0, +\infty)$ then

$$\psi(x^*, r) = \inf\{\langle x^*, x\rangle\colon x \in \tilde{K}, \langle e, x\rangle = 1, \varphi(x) \geq r^{1/\alpha}\} .$$

By duality

$$\psi(x^*, r)$$

$$= \sup_{(\mu,\lambda)\in\mathbb{R}\times[0,+\infty)} \{\inf\{\langle x^*, x\rangle - \lambda\varphi(x) + \lambda r^{1/\alpha} + \mu - \mu\langle e, x\rangle : x \in \tilde{K}\}$$

$$= \sup_{\substack{(\mu,\lambda)\in\mathbb{R}\times(0,+\infty)\\ x^*-\mu e\in K^+}} \left\{\mu + \lambda r^{1/\alpha} - \lambda \sup\left\{\left\langle \frac{-x^* + \mu e}{\lambda}, x\right\rangle + \varphi(x) : x \in \tilde{K}\right\}\right.$$

$$= \sup\left\{\mu + \lambda r^{1/\alpha} - \lambda(-\varphi)^*\left(\frac{-x^* + \mu e}{\lambda}\right) : \mu \in \mathbb{R}, \lambda \in (0, +\infty), x - \mu e \in K^+\right\}$$

$$= \sup\{\mu + \lambda r^{1/\alpha} : \mu \in \mathbb{R}, \lambda \in (0, +\infty), \varphi^{\oplus}(x^* - \mu e) \geq \lambda\}$$

$$= \sup\{\mu + r^{1/\alpha}\varphi^{\oplus}(x^* - \mu e) : \mu \in \mathbb{R}\}$$

$$= \theta(x^*, r)\ . \qquad \square$$

Now using proposition 9.1 and theorem 7.2 we obtain the following result.

*Corollary 9.1: Given $p$, $q \in (0, +\infty)$, the function $t^p/\varphi^q(x)$ is convex on $\tilde{\Omega}_e \times [0, +\infty)$ if and only if the following conditions hold.*

i) *The function $\theta(x^*, r) = \sup\{\mu + r^{p/q}\varphi^{\oplus}(x^* - \mu e) : \mu \in \mathbb{R}\}$ is convex on $(0, +\infty)$ in $r$,*
ii) *the function $F(x^*, t^*, \lambda) = \sup\{tt^* - \theta(-x^*, t/\lambda^{1/p}) : t \in (0, +\infty)\}$ is concave in $\lambda$ on the set $(0, +\infty)$.*

*Example 9.1:* Let $\xi_p$ the function defined in examples 4.2 and 4.3. We set $e = (1, 1, \ldots, 1)^t$ then we have the following result.

*Proposition 9.2: Assume that $p \in [-\infty, 0)$ then $\xi_p^\alpha$ is concave on the set $\tilde{\Omega}_e$ if and only if $\alpha \in [0, 1]$.*

*Proof:* First assume $p \in (-\infty, 0)$. Clearly $\xi_p^\alpha$ is concave on $\tilde{\Omega}_e$ for all $\alpha \in [0, 1]$. Suppose now $\alpha > 1$. Let $x \in \tilde{\Omega}_e$ and set $\sigma = \sum x_i^p$, then the hessian of $\xi_p^\alpha$ in $x$ is

$$\nabla^2 \xi_p^\alpha(x) = \alpha(p - 1)\sigma^{(\alpha/p)-1} X^{(p/2)-1}\left(I - \frac{\alpha - p}{1 - p}\sigma^{-1}bb^t\right)X^{(p/2)-1}$$

with $b = X^{p/2}e$, $X = diag(x)$ and $I$ the identity matrix. Notice that $\|b\|^2 = \sigma$.

$\xi_p^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if the matrix $X^{(p/2)-1}\left(I - \frac{\alpha - p}{1 - p}\sigma^{-1}bb^t\right)X^{p/2-1}$ is positive semi-definite on $\tilde{\Omega}_e$, i.e the matrix

$$H = \begin{bmatrix} I - \frac{\alpha - p}{1 - p}\sigma^{-1}bb^t & X^{1-(p/2)}e \\ (X^{1-(p/2)}e)^t & 0 \end{bmatrix}$$

has at most one negative eigenvalue.

Now recall some results on inertias of real symmetric matrices. Let $M$ be a real symmetric $q \times q$ matrix. The inertia of $M$ is the triple

$$In(M) = (\pi(M), \nu(M), \delta(M))$$

where $\pi(M)$, $\nu(M)$ and $\delta(M)$ are respectively the numbers of positive, negative and zero eigenvalues. Clearly $\pi(M) + \nu(M) + \nu(M) + \delta(M) = q$.

Now assume that $A$ is a $r \times r$ matrix, $B$ a $r \times q$ matrix, $C$ a $q \times q$ matrix. Assume in addition that $A$ and $C$ are symmetric and $A$ is non-singular. Let $M$ be the symmetric $(r + q) \times (r + q)$ matrix

$$M = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} .$$

The matrix $C - B^tA^{-1}B$ is called the Schur complement of $A$ in $M$ and is denoted by $M/A$. The following formula was given by Haynworth [12]

$$In(M) = In(A) + In(M/A) .$$

It follows that

$$In(H) = In\left(I - \frac{\alpha - p}{1 - p}\sigma^{-1}bb^t\right) + In\left(-\sum x_i^{2-p} + \frac{\alpha - p}{\alpha - 1}\sigma^{-1}\right)$$

clearly, the inertia of $\left(I - \frac{\alpha - p}{1 - p}\sigma^{-1}bb^t\right)$ is $(n - 1, 1, 0)$. Then $\xi_p^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if

$$\frac{\alpha - p}{\alpha - 1} \geq \sup\{\sum x_i^{2-p}\sum x_i^p : x \in (0, +\infty)^n, \Sigma x_i = 1\} .$$

Set $x(t) = (1 - t, t/n - 1, \ldots, t/n - 1)^t$ for all $t \in (0, 1)$, then since $p < 0$ one has

$$\lim_{t \to 0} \sum x_i^{2-p} \sum y_i^p = \lim_{t \to 0} (1 - t)^p + (n - 1)^{1-p} t^p = +\infty \ .$$

So for all $\alpha > 1$, the function $\xi_p^\alpha$ is not concave on $\tilde{\Omega}_e$.

Suppose now $p = -\infty$. According to proposition 9.1 the function $\xi_p^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if the function

$$\theta(x, r) = \sup\{\mu + r^{1/\alpha}(\sum x_i - n\mu) \mid \mu \le min_i(x_i)\}$$

is convex on $[0, +\infty)$ in $r$ for all $x \in \mathbb{R}^n$.

Now

$$\theta(x, r) = \begin{cases} r^{1/\alpha}(\sum x_i - min_i(x_i)n) + min_i(x_i) & \text{if } r \in [0, 1/n^\alpha] \ , \\ +\infty & \text{otherwise} \ , \end{cases}$$

and the result follows. □

*Example 9.3:* Let $a \in (0, +\infty)^n$ be such that $\sum a_i = 1$ and $\varphi$ the function defined by $\varphi(x) = \prod x_i^{a_i}$ for all $x \in (0, +\infty)^n$. Let $\alpha \in \mathbb{R}$, then we have the following result.

*Proposition 9.2: The function $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if $\alpha \in [0, 1/(1 - min_i(a_i))]$.*

*Proof:* Clearly (example 4.1) $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ for all $\alpha \in [0, 1]$. Suppose now $\alpha > 1$. Let $x \in \tilde{\Omega}_e$, then the hessian of $\varphi^\alpha$ in $x$ is

$$\nabla^2 \varphi^\alpha(x) = -\alpha\varphi^\alpha(x) X^{-1} A^{1/2}(I - \alpha bb^t) A^{1/2} X^{-1}$$

with $X = diag(x)$, $A = diag(a)$, $b = A^{1/2}e$ and $I$ the identity matrix.

$\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if the matrix

$$H = \begin{pmatrix} I - \alpha bb^t & Xb \\ (Xb)^t & 0 \end{pmatrix}$$

has at most one negative eigenvalue. Now $In(H) = (n-1, 1, 0) + In\left(-\sum a_i x_i^2 + \frac{\alpha}{\alpha - 1}(\sum a_i x_i)^2\right)$, therefore $\varphi^\alpha$ is concave on $\tilde{\Omega}_e$ if and only if

$$\frac{\alpha}{\alpha - 1} \geq \sup\{\sum a_i x_i^2 : x \in (0, +\infty)^n, \sum a_i x_i = 1\} = 1/min_i(a_i)$$

and the result follows.

## 10 Potential Gauge Functions on the Space of Real Symmetric Matrices

Set $E$ the space of real $n \times n$ symmetric matrices. For all $A \in E$, $d(A)$ is the vector of eigenvalues of $A$ defined as in section 5.

Let $K$ be a closed convex cone of $\mathbb{R}^n$ such that $int(K) \neq \emptyset$ and $int(K^+) \neq \emptyset$. Set

$$L = \{A \in E : d(A) \in K\} ,$$

clearly, $L$ is a closed convex cone of $E$ such that $int(L) \neq \emptyset$ and $int(L^+) \neq \emptyset$.

Let $B \in int(L^+)$, we set

$$\tilde{W}_B = \{A \in E : A \in int(K) \text{ and } trace(BA) = 1\}$$

and

$$\tilde{\Omega}_{d(B)} = \{x \in \mathbb{R}^n : x \in int(K) \text{ and } \langle d(B), x\rangle = 1\} .$$

Notice that $d(B) \in int(K^+)$. Let $p, q \in (0, +\infty)$, one has the following result.

*Theorem 10.1: Let $\varphi$ be a concave barrier function of $K$ such that $\varphi(\prod x) = \varphi(x)$ for any permutation matrix $\prod$. We define $g: E \to [-\infty, +\infty]$ by $g(A) = \varphi(d(A))$. Then*

i) *$g$ is a concave barrier function of $L$,*
ii) *if the function $f(x, t) = t^p/\varphi^q(x)$ is quasiconvex(convex) on $\tilde{\Omega}_{d(B)} \times (0, +\infty)$ then the function $g(A, t) = t^p/g^q(A)$ is quasiconvex(convex) on $\tilde{W}_B \times (0, +\infty)$.*

*Proof:*

i) It follows straightforwardly from theorem 5.1 that $g$ is a concave barrier function of $L$.
ii) Suppose that $f(x, t)$ is a quasiconvex function on $\tilde{\Omega}_{d(B)} \times (0, +\infty)$ then (theorem 9.2) $\varphi^{p/q}$ is concave on $\tilde{\Omega}_{d(B)}$ and then (theorem 1 Marques-Moreau [13]) $g^{p/q}$ is concave on $\tilde{W}_B$. Thus (theorem 9.2) $h(A, t)$ is quasi-convex on $\tilde{W}_B$ by $(0, +\infty)$.

Suppose now $f(x, t)$ convex on $\tilde{\Omega}_{d(B)} \times (0, +\infty)$. Set on the one hand

$$F(x^*, t^*, \lambda) = \sup\{\langle x^*, x\rangle + t^*t : f(x, t) \leq \lambda, (x, t) \in \tilde{\Omega}_{d(B)} \times (0, +\infty)\}$$

for all $(x^*, t^*) \in \mathbb{R}^n \times \mathbb{R}$ and on the other hand

$$H(A^*, t^*, \lambda) = \sup\{\langle A^*, A\rangle + t^*t : h(A, t) \leq \lambda, (A, t) \in \tilde{W}_B \times (0, +\infty)\}$$

for all $(A^*, t^*) \in E \times \mathbb{R}$. Let us prove that

$$H(A^*, t^*, \lambda) = F(d(A^*), t^*, \lambda) .$$

Notice that $F(x^*, t^*, \lambda)$ is convex in $x^*$. and $F(\prod x^*, t^*, \lambda) = F(x^*, t^*, \lambda)$ for any permutation matrix $\prod$.

It follows that

$$H(A^*, t^*, \lambda) = \sup\{F(R(Q)d(A^*), t^*, \lambda) : Q \in \mathcal{O}_n\}$$

where $\mathcal{O}_n$ and $R(Q)$ are defined as in section 5. Then

$$H(A^*, t^*, \lambda) \geq F(d(A^*), t^*, \lambda) .$$

Now (Birkhoff-Von Neuman's theorem) for all $Q \in \mathcal{O}_n$ there exist $s_1, s_2, \ldots, s_m \in [0, +\infty)$ and $\prod_1, \prod_2, \ldots, \prod_m$ permutation matrices such that

$$\sum s_i = 1 \quad \text{and} \quad R(Q) = \sum s_i \prod_i .$$

therefore

$$F(R(Q)d(A^*), t^*, \lambda) \leq \sum s_i F(\prod_i d(A^*), t^*, \lambda) = F(d(A^*), t^*, \lambda)$$

and then

$$H(A^*, t^*, \lambda) = F(d(A^*), t^*, \lambda) \ .$$

Since $f(x, t)$ is convex on $\tilde{\Omega}_{d(B)} \times (0, +\infty)$ we obtain (theorem 7.1) $F(x^*, t^*, \lambda)$ concave in $\lambda$ and then the result follows from the above inequality and theorem 7.1. □

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