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Gauge-invariant on-shell Z_2 in QED, QCD and the effective field theory of a static quark

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Abstract. We calculate the *on-shell* fermion wave-function renormalization constant Z_2 of a general gauge theory, to two loops, in D dimensions and in an arbitrary covariant gauge, and find it to be gauge-*invariant*. In QED this is consistent with the dimensionally regularized version of the Johnson-Zumino relation: $\mathrm{d}\log Z_2/\mathrm{d}a_0 = i(2\pi)^{-D} e_0^2 \int \mathrm{d}^D k/k^4 = 0$. In QCD it is, we believe, a new result, strongly suggestive of the cancellation of the gauge-dependent parts of non-abelian UV and IR anomalous dimensions to all orders. At the two-loop level, we find that the anomalous dimension γ_F of the fermion field in minimally subtracted QCD, with N_L light-quark flavours, differs from the corresponding anomalous dimension $\tilde{\gamma}_F$ of the effective field theory of a static quark by the gauge-invariant amount

$$\gamma_F - \tilde{\gamma}_F \equiv \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log\left(\frac{Z_2^{\mathrm{MS}}(\mu)}{\tilde{Z}_2^{\mathrm{MS}}(\mu)}\right) = 2\frac{\bar{\alpha}_s(\mu)}{\pi} + \left(\frac{41}{4} - \frac{11}{18} N_L\right)\frac{\bar{\alpha}_s^2(\mu)}{\pi^2} + O(\bar{\alpha}_s^3).$$

A complete description of two-loop on-shell renormalization of one-lepton QED, in D dimensions, is also given. More generally, we show that there is no need of integration in the two-loop calculation of on-shell two- and three-point functions.

1 Introduction

In a massive scalar field theory, the on-shell renormalization scheme is defined by identifying the wave-function renormalization constant with the constant Z in the LSZ [1] asymptotic relation of the bare Heisenberg field ϕ_0 to the in and out fields $\phi_{\mathrm{in,out}}$ which create correctly normalized initial and final physical states. In the sense of 'weak' convergence [2] one may write

$$\phi_0(x) \to \sqrt{Z}\,\phi_{\mathrm{in,out}}(x) \quad \text{as } x_0 \to \mp\infty.$$

The on-shell renormalization $\phi_0 = \sqrt{Z}\phi$ then ensures that S-matrix elements are given by on-shell limits of truncated (i.e. proper) renormalized Green functions [3]. In any other scheme, such as a minimal subtraction (MS) scheme with wave-function renormalization constant $Z^{\mathrm{MS}}(\mu)$, it is necessary to multiply a renormalized Green function by $(Z^{\mathrm{MS}}(\mu)/Z)^{N_E/2}$ to obtain the corresponding S-matrix element for a process with N_E external particles. In massive scalar field theory, such a correction factor has a finite perturbative expansion in terms of the renormalized mass and coupling, which is most easily found from the residue $Z/Z^{\mathrm{MS}}(\mu)$ of the renormalized propagator at $p^2 = M^2$, where M is the pole (i.e. physical) mass. This is because Z is the residue at the pole of the bare propagator [4]. Formally, one may regard Z as the probability for 'finding' the bare particle in the dressed one and use a dispersion relation [4] to show that $Z < 1$.

The situation in a gauge theory is rather different. If the ultraviolet (UV) infinities of the fermion propagator are removed by the MS renormalizations $\psi_0 = \sqrt{Z_2^{\mathrm{MS}}(\mu)}\,\psi$ and $m_0 = Z_m^{\mathrm{MS}}(\mu)\bar{m}(\mu)$ of the bare-fermion field and mass, the pole mass M has a finite perturbative expansion [5], but the residue at the pole does not, because of the 'infrared catastrophe' [6] of accumulating branch points of cuts with intermediate states consisting of one fermion and any number of gauge bosons. It is therefore straightforward to compute [7] the finite perturbative relation between the MS mass $\bar{m}(\mu)$ and the pole mass M, but much more problematic to give a meaningful expression for the factor $Z_2^{\mathrm{MS}}(\mu)/Z_2$, required to convert Green functions of the MS scheme into S-matrix elements, since it contains infrared (IR) singularities. In QED, these are cancelled by the Bloch-Nordsieck [8] mechanism of incoherently adding probabilities for low-energy photon emission to the probability given by the square of the S-matrix element, thereby obtaining finite answers to experimentally meaningful questions

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[9]. In QCD, however, this mechanism fails for certain [10] initial states.

In a previous paper [7] we have investigated the relation between MS and on-shell mass renormalization, by combining the results of three-loop MS mass renormalization [11] with our new results for the finite part of on-shell two-loop mass renormalization. The latter are commensurable with the former and turn out to dominate them numerically.

In this paper we use the same technique of simultaneous dimensional regularization of UV and IR singularities to calculate on-shell two-loop fermion wave-function renormalization, in an arbitrary covariant gauge of an arbitrary gauge theory in an arbitrary dimension $D \equiv 4-2\omega$, and show that it is gauge-invariant. In QED there exists an argument [12] why this should be the case. In the case of a non-abelian theory such as QCD, we know of no such general argument, but are encouraged by our two-loop result to believe that dimensional regularization renders Z_2 gauge-invariant to all orders, thereby respecting its formal probabilistic interpretation. We hope that a proof of this may eventually be forthcoming from non-abelian functional integration.

The utility of our result is demonstrated by deriving from it the two-loop anomalous dimension $\tilde{\gamma}_F$ of the field of a static quark, interacting with gluons and massless quarks in the effective field theory (EFT) obtained in the limit $M \to \infty$ [13–15]. The gauge invariance of Z_2 implies that the corresponding anomalous dimension γ_F of conventional QCD differs from $\tilde{\gamma}_F$ by a gauge-invariant amount, which is simply calculable from the Laurent expansion of Z_2. Confirmation of our result for $\tilde{\gamma}_F$ has recently been obtained by Broadhurst and Grozin [16], working exclusively within EFT.

The utility of our method is further demonstrated by obtaining, using only computer algebra, all the two-loop on-shell renormalization constants of one-lepton QED, in any dimension D, in terms of Γ functions and a single D-dimensional integral, $I(\omega)$, whose $D \to 4$ limit, $I(0) = \pi^2 \log 2 - \frac{3}{2}\zeta(3)$, was found by one of us [17]. More generally, on-shell two-loop three-point functions, such as that giving the $O(\alpha^2)$ corrections [18] to $g-2$, may be expressed in terms of Γ functions and $I(\omega)$, which may itself be expanded through $O(\omega^2)$ using exclusively algebraic methods [19].

The remainder of this paper is organized as follows.

In Sect. 2 we show how Z_2 and $Z_m \equiv m_0/M$ are reduced to integrals on the bare-mass shell, when all but one of the fermions are massless. The one-loop integrals are trivially evaluated. The two-loop integrals are related by recurrence relations to three general structures, of which only $I(\omega)$ is not reducible to Γ functions. Hence we obtain the Laurent expansions of Z_2 and Z_m as $\omega \to 0$, including important finite terms.

In Sect. 3 we evaluate the effects of non-trivial fermion mass ratios, since these are of importance in QED. Only when one has a finite mass ratio, such as M_e/M_μ, is it necessary to resort to Spence functions.

In Sect. 4 we derive the two-loop EFT anomalous dimension $\tilde{\gamma}_F$ from Z_2 and the known [20] two-loop QCD anomalous dimension γ_F.

In Sect. 5 we give all the two-loop on-shell renormalization constants of QED and indicate other QED calculations which are reduced to algebra by our method.

In Sect. 6 we summarize our findings and present conclusions.

2 Expansion in the bare coupling

We achieve the expansion, to $O(g_0^4)$ in the bare coupling, in four stages. First we determine which combinations of on-shell integrals enter the two-loop expansion of Z_2 and Z_m via the bare-fermion self energy $\Sigma(p)$ and its derivatives on the bare-mass shell, $p^2 = m_0^2$. Then we evaluate the one-loop terms and show that $Z_2 = Z_m + O(g_0^4)$. Next we evaluate the two-loop integrals in D dimensions, by computer algebra. Finally we give the Laurent expansions of Z_2 and Z_m as $D \to 4$.

Throughout this Sect. we assume that the fermion loop in the gauge boson propagator involves only the external fermion, of mass M, and (if desired) N_L massless fermions, so that we are evaluating integrals which depend only upon the dimension D and bare gauge parameter a_0. Non-trivial fermion mass ratios will be treated in Sect. 3. Coupling constant renormalization will be treated in Sects. 4 and 5, for QCD and QED respectively.

2.1 *Reduction to on-shell integrals*

Starting from the perturbative expansion of the bare self energy, $\Sigma(p)$, in terms of the bare coupling constant, g_0, the bare mass, m_0, and the bare gauge parameter, a_0, we calculate Z_2 by finding the residue, at the pole mass M, of the bare Feynman propagator

$$S_F(p) \equiv \frac{1}{\not{p} - m_0 - \Sigma(p)} = \frac{Z_2}{\not{p} - M} + (\text{terms regular at } p^2 = M^2) \tag{1}$$

in $D \equiv 4-2\omega$ dimensions. The essence of dimensional continuation is to regulate both ultraviolet and [21] infrared singularities by the introduction of a *single* dimensionless parameter, D, which formally preserves both the Lorentz invariance and the gauge invariance of the action, making no attempt to separate the resultant $\omega \to 0$ singularities into $1/\omega_{UV}$ and $1/\omega_{IR}$ terms. Whilst such a separation may be possible at the one-loop level, it is quite impractical at two loops, where the method of integration by parts [22] routinely introduces *extra* factors of $1/\omega$ in the process of reducing integrals to known forms. Computationally, the prescription is very well defined: one merely instructs a program like REDUCE [23] that $g^\mu_\mu = D$ and gives it a master formula, and/or a set of recurrence relations [7], sufficient to translate all possible terms encountered in momentum-space integrands, generated by the Feynman rules, into functions of D which correspond to the integrals.

We find it convenient to expand the bare self energy as

$$\Sigma(p) = \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} p^{2\omega}}\right]^n \cdot (m_0 A_n(m_0^2/p^2) + (\not{p} - m_0) B_n(m_0^2/p^2)) \tag{2}$$

where A_n and B_n are dimensionless functions of the dimension, D, the gauge parameter, a_0, and the dimensionless variable m_0^2/p^2. Note that the coupling constant has mass dimension ω, which has been cancelled by a power of the time-like momentum p, before taking the limit $p^2 \to m_0^2$. Then the coefficients of the expansions

$$\frac{m_0}{M} \equiv Z_m = 1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}}\right]^n M_n \tag{3}$$

$$(\not{p} - M) S_F(p)|_{\not{p}=M} \equiv Z_2 = 1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}}\right]^n F_n \tag{4}$$

are determined by combinations of A_n and B_n and their derivatives *on* the bare-mass shell. Specifically we find, by substitution of (2) in (1), that the following combinations are required at the two-loop level:

$$M_1 = -A_1$$
$$M_2 = -A_2 + A_1(A_1 + 2A_1' - B_1)$$
$$F_1 = B_1 - 2\omega A_1 - 2A_1'$$
$$F_2 = B_2 - 4\omega A_2 - 2A_2' + (2A_1' - B_1)^2 + 4A_1(A_1'' - B_1') + 2(1 + 2\omega) A_1(\omega A_1 + 3A_1') - 6\omega A_1 B_1$$

with all the A and B terms evaluated at $m_0^2/p^2 = 1$, for which the calculation of integrals is much simplified. To find the derivatives with respect to m_0^2/p^2, one has merely to differentiate diagrams one or two times with respect to the bare mass, before going on shell, thereby merely making zero-momentum insertions in internal fermion propagators.

2.2 One-loop result

From the one-loop integrals of Fig. 1a one easily obtains

$$A_1 = C_F \left(\frac{D-1}{D-3}\right) \Gamma(\omega)$$

$$A_1' = C_F \left(\frac{(D-1)(D-3) - a_0}{2(D-3)}\right) \Gamma(\omega)$$

$$A_1'' = C_F \left(\frac{(D-6)((D-1)(D-4) - 2a_0)}{4(D-5)}\right) \Gamma(\omega)$$

$$B_1 = -C_F \left(\frac{a_0}{D-3}\right) \Gamma(\omega)$$

$$B_1' = -C_F \left(\frac{(D-2) a_0}{2(D-3)}\right) \Gamma(\omega)$$

where $C_F = (N_C^2 - 1)/2N_C$ for a gauge group $SU(N_C)$. Hence we obtain the one-loop coefficients

$$M_1 = F_1 = -C_F \left(\frac{D-1}{D-3}\right) \Gamma(\omega) \tag{5}$$

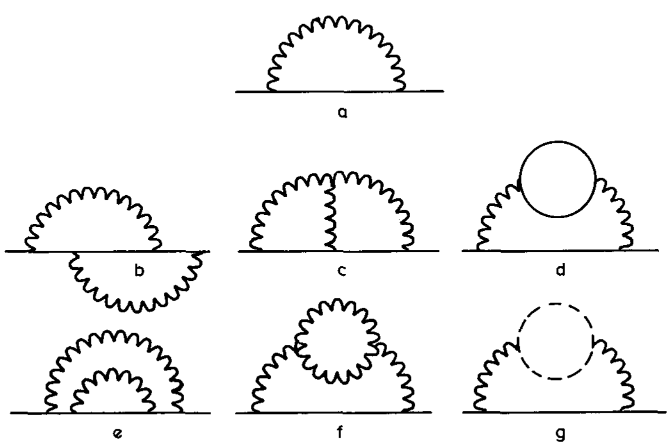

Fig. 1. Fermion self-energy diagrams, to two loops

showing that Z_m and Z_2 are gauge-invariant and *equal* at the one-loop level.

As there is no non-abelian coupling at this order, the *one-loop* gauge invariance of Z_2 may be obtained directly from the dimensionally regularized version of the QED Johnson-Zumino identity [12, 24]

$$\mathrm{d}\log Z_2/\mathrm{d}a_0 = i(2\pi)^{-D} e_0^2 \int \mathrm{d}^D k/k^4 = 0 \tag{6}$$

which derives from an earlier analysis by Landau and Khalatnikov [25] of the transformation of Green functions under covariant gauge transformations. Note that Z_2 is therefore gauge-invariant to all orders in QED. We are not aware of a nonabelian generalization of (6) that would ensure the gauge invariance of Z_2 to all orders in QCD.

Whilst the one-loop gauge invariance of Z_2 is to be expected from QED, we have no explanation of the remarkable coincidence

$$Z_2 = Z_m + O(g_0^4) \tag{7}$$

which means that, to leading order, the mass term $\bar{\psi}_0 m_0 \psi_0$, in the bare Lagrangian density, is renormalized by a factor $Z_2 Z_m$ which is the *square* of the factor Z_2 by which the kinetic energy term $\bar{\psi}_0 i\not{\partial} \psi_0$ is renormalized. We shall show that this 'virial' relationship does not persist at two loops, where it is replaced by a simple relation between the contributions to Z_2 and Z_m with three-fermion intermediate states.

There is a rather instructive consistency check on (7), provided by conventional MS renormalization. With $\bar{a}$ and $\bar{\alpha}_s$ representing the gauge parameter and coupling renormalized at scale μ in the MS scheme, the anomalous dimensions [26]

$$\gamma_F(\bar{a}, \bar{\alpha}_s) \equiv \frac{\mathrm{d}\log Z_2^{MS}(\mu)}{\mathrm{d}\log\mu} = \frac{\bar{a} C_F \bar{\alpha}_s}{2\pi} + O(\bar{\alpha}_s^2) \tag{8}$$

$$\gamma_m(\bar{\alpha}_s) \equiv \frac{\mathrm{d}\log Z_m^{MS}(\mu)}{\mathrm{d}\log\mu} = \frac{3 C_F \bar{\alpha}_s}{2\pi} + O(\bar{\alpha}_s^2) \tag{9}$$

are indeed equal at the one-loop level in precisely that gauge for which there is no [9] infrared catastrophe, namely the Yennie gauge [27] with $\bar{a} = 3$.

It was shown by Abrikosov [6] that in QED the electron propagator has a one-loop *infrared* anomalous dimension $\tilde{\gamma}_F=(a-3)\alpha/2\pi$. Other authors [28] verified that this result is spin-independent. Recently it has become possible to give a precise definition [16] to $\tilde{\gamma}_F$ in EFT, in analogy with (8), namely

$$\tilde{\gamma}_F(\bar{a},\bar{\alpha}_s)\equiv\frac{d\log\tilde{Z}_2^{MS}(\mu)}{d\log\mu}=\frac{(\bar{a}-3)C_F\bar{\alpha}_s}{2\pi}+O(\bar{\alpha}_s^2) \qquad (10)$$

where $\tilde{Z}_2^{MS}(\mu)$ gives the minimal subtractions which regularize the fermion propagator in the effective field theory [13, 14] of a static fermion, obtained as $M\to\infty$. In EFT it is trivial [16] to obtain the one-loop Abrikosov result (10) by repeating the one-loop self-energy calculation of Eichten and Hill [14] in an arbitrary covariant gauge. The coincidence of the one-loop $1/\omega$ singularities in (7) may thus be written as

$$\gamma_F-\tilde{\gamma}_F=\gamma_m+O(\bar{\alpha}_s^2). \qquad (11)$$

The relation of $\gamma_F-\tilde{\gamma}_F$ to the gauge-invariant $1/\omega$ singularity of our on-shell Z_2 becomes apparent when one compares S-matrix elements of QCD and EFT. With N_E external heavy fermions, these differ from truncated MS-renormalized Green functions by factors of $(Z_2^{MS}(\mu)/Z_2)^{N_E/2}$ and $(\tilde{Z}_2^{MS}(\mu)/\tilde{Z}_2)^{N_E/2}$ respectively. Now the Green functions are finite, by construction, and the S-matrix elements of the two theories can differ, at most, by finite radiative corrections which vanish as $M\to\infty$ and hence $\bar{\alpha}_s(M)\to 0$. Moreover $\tilde{Z}_2=1$, since there can be no on-shell wave-function renormalization in dimensionally regularized EFT, as all the integrals contributing to the on-shell self energy are scale free. It follows that all singularities must cancel in the finite ratio $R(\mu)\equiv(Z_2^{MS}(\mu)/Z_2)/\tilde{Z}_2^{MS}(\mu)$ and hence that a knowledge of Z_2 suffices to determine the *difference* of QCD and EFT anomalous dimensions. In Sect. 4 we shall verify that this is indeed the case at the two-loop level (provided one neglects heavy-quark loops in QCD, since these are discarded *ab initio* in EFT).

It is thus apparent that the gauge invariance of Z_2 guarantees the gauge invariance of the difference (11) of the anomalous dimensions of QCD and EFT. It was long ago remarked [24] that to leading order Z_2 has no ultraviolet divergence in the Landau gauge and no infrared divergence in the Yennie gauge. Dimensional regularization assigns Z_2 a unique gauge-invariant value in QED and (to two loops, at least) in QCD. Since this unique value provides an important link between QCD and EFT, its calculation becomes of practical as well as theoretical interest.

2.3 Two-loop result

We now need the two-loop integrals contributing to

$$M_2=-A_2+(D-2)A_1^2 \qquad (12)$$

$$F_2=B_2-4\omega A_2-2A_2'+\left(\frac{D^2-7D+8+a_0}{D-5}\right)A_1^2. \qquad (13)$$

In [7] we gave the exact result for the two-loop term A_2, required in (12). It involved four colour factors and the three terms

$$R_1=\Gamma^2(\omega),\qquad R_2=\frac{\omega\Gamma^2(-\omega)\Gamma(-4\omega)\Gamma(2\omega)\Gamma(\omega)}{\Gamma(-2\omega)\Gamma(-3\omega)},$$
$$R_3=I(\omega) \qquad (14)$$

which derive from the three irreducible integrals to which all other on-shell two-loop integrals may eventually be reduced by the method of integration by parts [7]. The last of these is the D-dimensional (Minkowski space) integral

$$I(\omega)\equiv\frac{(p^2)^{5-D}}{\pi^D}\cdot\iint\frac{d^Dk\,d^Dl}{(k^2+2p\cdot k)k^2(l^2+2p\cdot l)l^2((k+l)^2+2p\cdot(k+l))} \qquad (15)$$
$$=\pi^2\log 2-\tfrac{3}{2}\zeta(3)+O(\omega) \qquad (16)$$

whose 4-dimensional value was obtained in [17].

In this paper, we find it convenient to work with the colour factors

$$C_1=C_F(C_A-2C_F),\qquad C_2=C_F^2,\qquad C_3=2T_FN_LC_F,$$
$$C_4=2T_FC_F \qquad (17)$$

where $C_A=N_C$ and $T_F=\frac{1}{2}$ for a gauge group SU(N_C) and N_L is the number of light fermions contributing to Fig. 1d, here taken to be massless. Note that Fig. 1b, e give gauge-dependent contributions proportional to the colour factors C_1 and C_2, respectively, whilst the light- and heavy-quark loops in Fig. 1d give gauge-invariant contributions proportional to C_3 and C_4, respectively. The nonabelian couplings in Fig. 1c, f and the ghost loop in Fig. 1g give gauge-dependent contributions proportional to $C_FC_A=C_1+2C_2$. In the case of one-lepton QED, one sets $C_A=N_L=0$ and $C_F=T_F=1$.

In terms of the structures (14) and (17) the two-loop coefficient M_2 of Z_m in (3) is given by Table 1, which lists the non-vanishing coefficients M_{ij} of the matrix coupling the colour and integral structures in

$$M_2=\sum_{i=1}^{4}\sum_{j=1}^{3}C_iM_{ij}R_j. \qquad (18)$$

For the $O(g_0^4)$ corrections to Z_2 we need to calculate new two-loop terms, namely the B_2 and A_2' terms of (13). These may be obtained by the methods of [7], albeit with considerably greater effort, needed to extend the recurrence relations to deal with terms which are generated by the doubling of fermion propagators in A_2'. We have evaluated them for any dimension, D, and gauge parameter, a_0, but the results are too bulky to reproduce here. What concerns us is the combination (13), which turns out to be gauge-invariant, thanks to remarkable cancellations, between diagrams, of terms linear and quadratic in a_0 in several (colour factor × integral) structures, each of which involves complicated rational functions of D, of which Table 1 is indicative. Since we are

Table 1. Non-vanishing coefficients M_{ij} of $C_i R_j$ in (18)

$$M_{11}=\frac{3(5D^3-58D^2+180D-152)}{2(3D-8)(3D-10)(D-3)}$$

$$M_{12}=\frac{4(4D^3-41D^2+122D-104)}{3(3D-8)(3D-10)(D-3)}$$

$$M_{13}=-\frac{4(D^2-7D+8)(D-3)(D-6)}{(3D-8)(3D-10)}$$

$$M_{21}=\frac{D^2-8D+13}{(D-3)^2}$$

$$M_{22}=-\frac{8(2D^3-21D^2+68D-71)(D-2)}{3(3D-8)(3D-10)(D-3)^2}$$

$$M_{32}=-\frac{16(D-2)}{3(3D-8)(3D-10)}$$

$$M_{41}=-\frac{12(D^3-12D^2+50D-68)}{(3D-8)(3D-10)(D-3)(D-6)}$$

$$M_{42}=\frac{16(D^3-7D^2+6D+16)}{3(3D-8)(3D-10)(D-3)(D-6)}$$

$$M_{43}=-\frac{8(D^3-7D^2+6D+16)(D-4)}{(3D-8)(3D-10)(D-6)}$$

Table 2. Non-vanishing coefficients F_{ij} of $C_i R_j$ in (19)

$$F_{11}=-\frac{3D^5-61D^4+469D^3-1679D^2+2756D-1648}{8(D-3)^2(D-5)^2}$$

$$F_{12}=-\frac{2D^5-29D^4+148D^3-321D^2+268D-60}{3(3D-10)(D-3)^2(D-5)}$$

$$F_{21}=-\frac{D^3-12D^2+37D-36}{4(D-3)^2}$$

$$F_{22}=\frac{2(2D^3-17D^2+42D-29)(D-2)}{3(3D-10)(D-3)^2}$$

$$F_{32}=\frac{4(D-2)}{3(3D-10)}$$

$$F_{41}=-\frac{2(D^2-8D+11)(D-4)}{(D-2)(D-3)(D-5)(D-7)}$$

now highly sensitive to the three-gluon coupling of a non-abelian gauge theory, we regard the two-loop gauge invariance of Z_2 as a strong indication of its gauge invariance to *all* orders. It should however be remarked that we are not yet sensitive to the four-gluon coupling.

To present our result compactly, we exploit another interesting feature, namely that the combination

$$F_2-(1+D/4)M_2=\sum_{i=1}^{4}\sum_{j=1}^{2}C_iF_{ij}R_j \tag{19}$$

does not involve the integral (15). As in the case of the one-loop relationship $F_1=M_1$, we lack an argument as to why such a simplification should occur. It involves matching cancellations in each of Fig. 1b, d and these are apparent only after extensive use of the recurrence relations of [7]. Based on these two instances, one is tempted to speculate that at L loops there is always a linear combination of F_L and M_L in which there is no net contribution from intermediate states with the maximum number of massive fermions, namely $2L-1$. The proof of such a conjecture might be easier to find in old-fashioned, time-ordered perturbation theory.

Thanks to the relative simplicity of combination (19) and to gauge invariance, we are able to give a complete account of two-loop on-shell fermion mass and wave-function renormalization, in any dimension D, by complementing Table 1 with Table 2. In comparison to individual results for the contribution of a particular diagram to one of the relevant terms $\{A_2, A_2', B_2\}$, the full D-dimensional results of Tables 1 and 2 are rather compact.

2.4 *Laurent expansion as $D\to 4$*

We now perform Laurent expansions in ω, obtaining the following two-loop results, in terms of the *bare* coupling:

$$Z_{\mathrm{m}}=1-\left(\frac{\alpha_0}{\pi M^{2\omega}}\right)C_{\mathrm{F}}\left\{\frac{3}{4\omega}+1+\left(\tfrac{3}{8}\zeta(2)+2\right)\omega+O(\omega^2)\right\}+\left(\frac{\alpha_0}{\pi M^{2\omega}}\right)^2\cdot\sum_{i=1}^{4}C_i\{M_2^i/\omega^2+M_1^i/\omega+M_0^i+O(\omega)\}+O(\alpha_0^3) \tag{20}$$

$$Z_2=1-\left(\frac{\alpha_0}{\pi M^{2\omega}}\right)C_{\mathrm{F}}\left\{\frac{3}{4\omega}+1+\left(\tfrac{3}{8}\zeta(2)+2\right)\omega+O(\omega^2)\right\}+\left(\frac{\alpha_0}{\pi M^{2\omega}}\right)^2\cdot\sum_{i=1}^{4}C_i\{F_2^i/\omega^2+F_1^i/\omega+F_0^i+O(\omega)\}+O(\alpha_0^3) \tag{21}$$

where $\alpha_0\equiv(g_0^2/4\pi)(4\pi/e^{\gamma})^{\omega}$ and the two-loop coefficients M_n^i and F_n^i, associated with the colour factors (17), are given in Table 3. Note that it is necessary to retain the one-loop $O(\omega)$ terms, since these generate finite contributions after coupling constant renormalization.

In [7], we used (20) to derive the relation between the pole mass and the three-loop MS mass. In Sects. 4 and 5 we apply (21) to wave-function renormalization in different schemes of coupling constant renormaliza-

Table 3. Coefficients M_n^i and F_n^i of C_i/ω^n in (20) and (21)

n	2	1	0
M_n^1	$-\frac{11}{32}$	$-\frac{91}{64}$	$\frac{5}{32}\zeta(2)-\frac{1}{4}I(0)-\frac{605}{128}$
M_n^2	$-\frac{13}{32}$	$-\frac{137}{64}$	$-\frac{41}{32}\zeta(2)-\frac{1011}{128}$
M_n^3	$\frac{1}{16}$	$\frac{7}{32}$	$\frac{5}{16}\zeta(2)+\frac{45}{64}$
M_n^4	$\frac{1}{16}$	$\frac{7}{32}$	$-\frac{7}{16}\zeta(2)+\frac{69}{64}$
F_n^1	$-\frac{11}{32}$	$-\frac{101}{64}$	$\frac{49}{32}\zeta(2)-\frac{1}{2}I(0)-\frac{803}{128}$
F_n^2	$-\frac{13}{32}$	$-\frac{151}{64}$	$-\frac{49}{32}\zeta(2)-\frac{1173}{128}$
F_n^3	$\frac{1}{16}$	$\frac{9}{32}$	$\frac{5}{16}\zeta(2)+\frac{59}{64}$
F_n^4	$\frac{1}{8}$	$\frac{19}{96}$	$-\frac{7}{8}\zeta(2)+\frac{1139}{576}$

tion, namely minimal subtraction of the QCD coupling, and on-shell charge renormalization in QED. But first we calculate the form of the contribution of Fig. 1d when the internal fermion is neither massless, nor of the external mass M, since this is clearly of some consequence in QED, where the effects of one of the leptons $\{e, \mu, \tau\}$ on the other two need investigation.

3 Radiative effects of non-trivial fermion mass ratios

The effect on (20) of finite internal fermion mass $M_i \neq M$ in Fig. 1d was computed, in terms of dilogarithms, in [7]. The same dilogarithms suffice to express the corresponding effect on (21), but in the case of wave-function renormalization they result from a finite integral over the fermion contribution to the gauge-boson propagator subtracted at *zero* momentum. This is because we must separate out infrared singularities present in Z_2, but absent from Z_m. We find that the $O(g_0^4)$ contribution to Z_2, of a single internal fermion of mass $M_i = rM$, is of the form

$$\Delta Z_2 = \left(\frac{\alpha_0}{\pi M^{2\omega}}\right)^2 C_4 \left(\frac{1}{8\omega^2} + \frac{19 - 24\log r}{96\omega} + \tfrac{1}{4}\log^2 r - \tfrac{11}{24}\log r + \tfrac{1}{8}\zeta(2) + \tfrac{59}{192} + \bar{\Delta}(r) + O(\omega)\right) \tag{22}$$

where

$$\bar{\Delta}(r) = \frac{1}{8}\int_0^1 \frac{\mathrm{d}y}{y}(2+y)(1-y)\,\bar{\Pi}(r^2(1-y)/y^2)$$

$$\bar{\Pi}(z) = 2(1-2z)\sqrt{1+4z}\ \operatorname{arccoth}\ \sqrt{1+4z} + 4z - \tfrac{5}{3}$$

and the bars are to distinguish $\bar{\Delta}$ and $\bar{\Pi}$ from the related but different functions Δ and Π involved in the corresponding analysis [7] of Z_m. Note the presence of a mass-dependent singular term, $\omega^{-1}\log r$, in (22). In Sect. 5 we will show that this is removed by on-shell charge renormalization of QED.

It remains to reduce $\bar{\Delta}(r)$ to the dilogarithms [7]

$$\begin{aligned} L_\pm(r) &\equiv \int_0^1 \mathrm{d}x \left(\frac{\log x - \log r}{x \pm r}\right) \\ &= \tfrac{1}{2}\log^2 r + (\tfrac{1}{2} \mp \tfrac{3}{2})\zeta(2) - L_\pm(1/r) \\ &= \log r \log\left(\frac{r}{r \pm 1}\right) + \mathrm{Li}_2(\mp 1/r) \qquad \text{for } r \geq 1 \end{aligned}$$

where $\mathrm{Li}_p(x) = \sum_{n=1}^{\infty} x^n/n^p$ for $p > 1 \geq |x|$. An intricate calculation yields

$$\begin{aligned} \bar{\Delta}(r) &= \tfrac{1}{8}(r+1)(6r^3 - r^2 + r + 2)L_+(r) \\ &\quad + \tfrac{1}{8}(r-1)(6r^3 + r^2 + r - 2)L_-(r) \\ &\quad + \tfrac{19}{24}\log r + \tfrac{229}{288} + (\tfrac{1}{2}\log r + \tfrac{7}{8})r^2 \\ &= \sum_{n=1}^{\infty}(-2G(n)\log r + G'(n))r^{-2n} \qquad \text{for } r \geq 1 \end{aligned} \tag{23}$$

where $G(n) = 3(n^2-1)/4n(n+2)(2n+1)(2n+3)$ and $G'(n)$ is its derivative.

A check on this result is provided by setting $r=1$. We find that $\bar{\Delta}(1) = \frac{481}{288} - \zeta(2)$, giving a contribution (22) which, with $r=1$, agrees with the C_4 term of (21). This agreement between a long algebraic calculation and a difficult analytical evaluation gives us considerable confidence in each. The limiting behaviours of $\bar{\Delta}(r)$ at large and small mass ratios r are as follows:

$$\bar{\Delta}(r) = \tfrac{1}{30}r^{-2} + O(r^{-4}\log r) \tag{24}$$

$$\bar{\Delta}(r) = \tfrac{1}{4}\log^2 r + \tfrac{19}{24}\log r + \tfrac{1}{4}\zeta(2) + \tfrac{229}{288} + O(r) \tag{25}$$

in marked contrast to the corresonding term $\Delta(r)$ in

$$\Delta Z_m = \left(\frac{\alpha_0}{\pi M^{2\omega}}\right)^2 C_4 \left(\frac{1}{16\omega^2} + \frac{7}{32\omega} + \frac{5}{16}\zeta(2) + \frac{45}{64} - \Delta(r) + O(\omega)\right) \tag{26}$$

which is given exactly by [7]

$$\Delta(r) = -\tfrac{1}{4}(r+1)(r^3+1)L_+(r) - \tfrac{1}{4}(r-1)(r^3-1)L_-(r) + \tfrac{1}{4}\log^2 r + \tfrac{1}{4}\zeta(2) - (\tfrac{1}{4}\log r + \tfrac{3}{8})r^2 \tag{27}$$

and has the limiting behaviours

$$\Delta(r) = \tfrac{1}{4}\log^2 r + \tfrac{13}{24}\log r + \tfrac{1}{4}\zeta(2) + \tfrac{151}{288} + O(r^{-2}\log r) \tag{28}$$

$$\Delta(r) = \tfrac{3}{4}\zeta(2)r + O(r^2) \tag{29}$$

with $\Delta(1) = \frac{3}{4}\zeta(2) - \frac{3}{8}$.

To summarize thus far: in Sect. 2 we found the contribution of N_L massless fermions and the fermion of mass M to Fig. 1d in any dimension D, whilst in this section we deal with internal fermions of any finite mass, but must resort to dilogarithms to find their contributions as $D \to 4$. This complication does not affect the proof of the gauge invariance of Z_2 to two loops in all dimensions, since Fig. 1d is separately gauge-invariant, for any fermion mass ratio. It is, however, apparent from (22, 25) that for Z_2 (unlike Z_m) one must decide *ab initio* whether one treats light quarks as massless: there is clearly no way of obtaining the massless quark contributions from those of finite-mass quarks, since the vanishing of r in (22) produces infrared mass singularities, which were dimensionally regularized in Sect. 2. Despite this complication, we have sufficient equations to handle all mass cases and may now proceed to renormalize the coupling.

4 MS coupling renormalization in QCD and EFT

In QCD, unlike QED, one cannot renormalize the coupling merely by calculating the wave-function renormalization of the gauge boson on its $q^2=0$ mass shell: that is the really significant consequence of the nonabelian structure. Our perturbative analysis suggests that the on-shell infrared problems of quarks and leptons are rather similar and equally gauge-invariant, after dimensionally regularized on-shell fermion wave-function renormalization. But gluons are decidedly different from photons, even in perturbation theory. This, we suggest, is the *real* infrared problem of QCD: gluon confinement. If so, there is hope of devising an intermediate scheme, in which one dares to approach the perturbative heavy-quark mass shell, but requires substantial virtualities of gluons and light quarks, which dress the heavy quark as a decent hadron. As stressed in a recent review by Bjorken [29], the formal limit $M\to\infty$ of EFT [13, 14] provides a well-defined starting point for such an attempt.

4.1 Derivation of $\tilde{\gamma}_{\rm F}$ from Z_2

To relate our result for Z_2 to EFT, we renormalize the coupling in the MS scheme, with $N_{\rm L}$ light quarks:

$$\frac{g_0^2}{4\pi}=\left(\frac{\mu^2 e^{\gamma}}{4\pi}\right)^{\omega}\bar{\alpha}_{\rm s}(\mu)\left(1-\frac{\bar{\alpha}_{\rm s}(\mu)}{\pi\omega}\left(\tfrac{11}{12}C_{\rm A}-\tfrac{1}{3}T_{\rm F}N_{\rm L}\right)+O(\bar{\alpha}_{\rm s}^2)\right) \quad (30)$$

where μ is an arbitrary mass scale, introduced to make $\bar{\alpha}_{\rm s}$ dimensionless, and the power of $(e^{\gamma}/4\pi)$ suppresses needless factors of $(\log 4\pi-\gamma)$ in the $\omega\to 0$ limit of (20, 21). It is important to realize that (30) applies for *all* $D=4-2\omega$; not just as $\omega\to 0$. There are no further terms in the Laurent expansion, otherwise the renormalization would not be minimal. Note also that we do not include the effect of the heavy-quark loop in (30), since that is discarded in EFT.

After MS coupling renormalization, the Laurent expansion (21) may conveniently be decomposed as

$$Z_2=Z_2^{\rm L}+Z_2^{\rm H}+O(\bar{\alpha}_{\rm s}^3) \quad (31)$$

where

$$Z_2^{\rm L}=1-\left(\frac{\alpha_{\rm s}(M)}{\pi}\right)C_{\rm F}\left\{\frac{3}{4\omega}+1+\left(\tfrac{3}{8}\zeta(2)+2\right)\omega+O(\omega^2)\right\}$$
$$+\left(\frac{\alpha_{\rm s}(M)}{\pi}\right)^2\sum_{i=1}^{3}C_i\{\bar{F}_2^i/\omega^2+\bar{F}_1^i/\omega+\bar{F}_0^i+O(\omega)\} \quad (32)$$

is the contribution of light quarks and gluons, whilst

$$Z_2^{\rm H}=\left(\frac{\alpha_{\rm s}(M)}{\pi}\right)^2 C_4\left(\frac{1}{8\omega^2}+\frac{19}{96\omega}-\frac{7}{8}\zeta(2)+\frac{1139}{576}+O(\omega)\right) \quad (33)$$

is the contribution of the heavy quark itself, which is unaffected by coupling renormalization and will play no role in establishing the link with EFT.

The coefficients $\bar{F}_n^i$ of Table 4 are obtained from the corresponding coefficients in Table 3, taking into account the renormalization of the one-loop term by (30). They uniquely determine the minimal subtractions in

$$\frac{Z_2^{\rm MS}(\mu)}{\tilde{Z}_2^{\rm MS}(\mu)}=1-3C_{\rm F}\left(\frac{\bar{\alpha}_{\rm s}(\mu)}{4\pi\omega}\right)+C_{\rm F}\left(\frac{\bar{\alpha}_{\rm s}(\mu)}{4\pi\omega)}\right)^2$$
$$\cdot\{\tfrac{11}{2}C_{\rm A}+\tfrac{9}{2}C_{\rm F}-2T_{\rm F}N_{\rm L}$$
$$-\left(\tfrac{127}{12}C_{\rm A}-\tfrac{3}{4}C_{\rm F}-\tfrac{11}{3}T_{\rm F}N_{\rm L}\right)\omega\}+O(\bar{\alpha}_{\rm s}^3) \quad (34)$$

Table 4. MS-renormalized coefficients $\bar{F}_n^i$ of C_i/ω^n in (32)

n	2	1	0
$\bar{F}_n^1$	$\frac{11}{32}$	$-\frac{127}{192}$	$\frac{15}{8}\zeta(2)-\frac{1}{2}I(0)-\frac{1705}{384}$
$\bar{F}_n^2$	$\frac{31}{32}$	$-\frac{101}{192}$	$-\frac{27}{32}\zeta(2)-\frac{2111}{384}$
$\bar{F}_n^3$	$-\frac{1}{16}$	$\frac{11}{96}$	$\frac{1}{4}\zeta(2)+\frac{113}{192}$

by the requirement that $R(\mu)\equiv(Z_2^{\rm MS}(\mu)/Z_2^{\rm L})/\tilde{Z}_2^{\rm MS}(\mu)$ be finite as $\omega\to 0$. Note that $Z_2^{\rm MS}(\mu)/\tilde{Z}_2^{\rm MS}(\mu)$ is *not* obtained by mere subtraction of the singularities in $Z_2^{\rm L}$, but rather by the requirement that (34) have a minimal structure such that when divided by the non-minimal $\tilde{Z}_2^{\rm L}$ the result, $R(\mu)$, is finite. The finiteness of $R(\mu)$ then ensures that a ratio of QCD and EFT S-matrix elements is finite, given that the corresponding ratio of renormalized Green functions is finite and that there is no on-shell wave-function renormalization in dimensionally regularized EFT.

A strong check on (34) is provided by calculating the difference of the anomalous dimensions (8) and (10), using the D-dimensional beta function

$$\frac{d\log\bar{\alpha}_{\rm s}(\mu)}{d\log\mu}=-2\omega-2\frac{\bar{\alpha}_{\rm s}(\mu)}{\pi}\left(\tfrac{11}{12}C_{\rm A}-\tfrac{1}{3}T_{\rm F}N_{\rm L}\right)+O(\bar{\alpha}_{\rm s}^2)$$

which gives the finite result

$$\gamma_{\rm F}-\tilde{\gamma}_{\rm F}=\frac{3C_{\rm F}\bar{\alpha}_{\rm s}}{2\pi}+\left(\tfrac{127}{12}C_{\rm A}-\tfrac{3}{4}C_{\rm F}-\tfrac{11}{3}T_{\rm F}N_{\rm L}\right)\frac{C_{\rm F}\bar{\alpha}_{\rm s}^2}{4\pi^2}+O(\bar{\alpha}_{\rm s}^3) \quad (35)$$

$$=2\frac{\bar{\alpha}_{\rm s}}{\pi}+\left(\tfrac{41}{4}-\tfrac{11}{18}N_{\rm L}\right)\frac{\bar{\alpha}_{\rm s}^2}{\pi^2}+O(\bar{\alpha}_{\rm s}^3)\qquad \text{for SU(3).} \quad (36)$$

Combining (35) with the known [20] two-loop QCD anomalous dimension

$$\gamma_{\rm F}=\frac{\bar{a}C_{\rm F}\bar{\alpha}_{\rm s}}{2\pi}+\left\{\left(\frac{\bar{a}^2}{32}+\frac{\bar{a}}{4}+\frac{25}{32}\right)C_{\rm A}-\frac{3}{16}C_{\rm F}\right.$$
$$\left.-\frac{1}{4}T_{\rm F}N_{\rm L}\right\}\frac{C_{\rm F}\bar{\alpha}_{\rm s}^2}{\pi^2}+O(\bar{\alpha}_{\rm s}^3) \quad (37)$$

we obtain the EFT result

$$\tilde{\gamma}_{\rm F}=\frac{(\bar{a}-3)C_{\rm F}\bar{\alpha}_{\rm s}}{2\pi}+\left\{\left(\frac{\bar{a}^2}{32}+\frac{\bar{a}}{4}-\frac{179}{96}\right)C_{\rm A}\right.$$
$$\left.+\frac{2}{3}T_{\rm F}N_{\rm L}\right\}\frac{C_{\rm F}\bar{\alpha}_{\rm s}^2}{\pi^2}+O(\bar{\alpha}_{\rm s}^3) \quad (38)$$

which has recently been verified by Broadhurst and Grozin [16], working entirely within EFT. Note that the effective field theory obtained by taking the electron mass to infinity in pure QED corresponds to $C_{\rm A}=N_{\rm L}=0$ and hence has no anomalous dimension at two loops in the (renormalized) Yennie gauge, which was chosen for precisely that reason in [27]. By contrast, the EFT of a static quark is not greatly simplified by choosing the Yennie gauge, since there is still an anomalous dimension at the two-loop level.

4.2 *Renormalization-group improvement*

We can integrate (36), using the one- and two-loop terms of the beta function [26]

$$2\omega+\frac{\mathrm{d}\log\bar{\alpha}_s}{\mathrm{d}\log\mu}=-2\sum_{n=1}^{\infty}b_n\left(\frac{\bar{\alpha}_s}{4\pi}\right)^n$$

with $b_1=11-\frac{2}{3}N_L$ and $b_2=102-\frac{38}{3}N_L$. Writing (36) in the similar form

$$\frac{\mathrm{d}\log(Z_2^{\mathrm{MS}}/\tilde{Z}_2^{\mathrm{MS}})}{\mathrm{d}\log\mu}=2\sum_{n=1}^{\infty}e_n\left(\frac{\bar{\alpha}_s}{4\pi}\right)^n$$

with $e_1=4$ and $e_2=82-\frac{44}{9}N_L$, we readily obtain the 4-dimensional, two-loop, renormalization-group improved result

$$R(\mu)\equiv\frac{1}{Z_2^L}\frac{Z_2^{\mathrm{MS}}(\mu)}{\tilde{Z}_2^{\mathrm{MS}}(\mu)}\approx R(M)\left(\frac{\bar{\alpha}_s(M)}{\bar{\alpha}_s(\mu)}\right)^{4/b_1}\left(\frac{1+E_2\bar{\alpha}_s(M)/\pi}{1+E_2\bar{\alpha}_s(\mu)/\pi}\right) \tag{39}$$

$$E_2\equiv\frac{e_2b_1-e_1b_2}{4b_1^2}=\frac{175}{162}\ \text{or}\ \frac{4253}{3750}\quad\text{for } N_L=3\ \text{or}\ 4 \tag{40}$$

for the finite ratio (39) of the factors which convert MS-renormalized Green functions to S-matrix elements in QCD and EFT. Moreover, the finite part of (32) determines the integration constant $R(M)$ in (39) to be

$$R(M)=1+\tfrac{4}{3}\bar{\alpha}_s(M)/\pi+K_2\bar{\alpha}_s^2(M)/\pi^2+O(\bar{\alpha}_s^3) \tag{41}$$

$$K_2=\tfrac{2}{9}\pi^2\log 2-\tfrac{1}{3}\zeta(3)+\tfrac{7}{6}\zeta(2)+\tfrac{4663}{288}-(\tfrac{1}{3}\zeta(2)+\tfrac{113}{144})N_L\approx 19.23-1.33\,N_L. \tag{42}$$

Thus, from the gauge-invariant, renormalization-group invariant, on-shell quantity (32) we have derived the renormalization-group improved, two-loop expression (39) for the scale-dependence of the ratio of two gauge-dependent artifacts of the MS scheme and also the boundary condition (41) to the level commensurate with three-loop MS renormalization.

It is clear that on-shell wave-function renormalization corresponds in many respects with on-shell mass [7] renormalization: each is gauge-invariant; each determines a gauge-invariant anomalous dimension; the finite parts of each at two loops are needed to relate off-shell results of the MS scheme at three loops to physical quantities; these finite parts are large. For comparison with (42), note that the corresponding coefficient of $\bar{\alpha}_s^2(M)/\pi^2$ in $Z_m^{\mathrm{MS}}(M)/Z_m=M/\bar{m}(M)$ is $K=16.11-1.04\,N_L$ [7].

Finally, we remark on the relation between the leading behaviour of (32) and the one-loop EFT anomalous dimension $\tilde{\gamma}_J$ of heavy-light $\bar{Q}\gamma_\mu(\gamma_5)q$ currents, apparent in the logarithms of [15] and elucidated in [14]. From our point of view, it is best obtained from the gauge-invariant one-loop dimensionally regularized singularity of (32), associated with an on-shell fermion:

$$\tilde{\gamma}_J=\tfrac{1}{2}(\tilde{\gamma}_F-\gamma_F)+O(\bar{\alpha}_s^2)=-\bar{\alpha}_s/\pi+O(\bar{\alpha}_s^2)=-\tfrac{1}{2}\gamma_m+O(\bar{\alpha}_s^2). \tag{43}$$

In the Landau gauge, one may blame it all on the static-quark field, since the coupling and the light-quark field are regular:

$$\tilde{\gamma}_J=\tfrac{1}{2}\tilde{\gamma}_F(\bar{a}=0)+O(\bar{\alpha}_s^2).$$

In the Yennie gauge, the static-quark field is regular, but the divergence of the coupling has the opposite sign to that of the light-quark field and twice its magnitude, since the light-light $\bar{q}\gamma_\mu(\gamma_5)q$ current is conserved:

$$\tilde{\gamma}_J=(\tfrac{1}{2}-1)\gamma_F(\bar{a}=3)+O(\bar{\alpha}_s^2).$$

The $O(\bar{\alpha}_s^2)$ corrections to the relations between the anomalous dimensions of (43) are studied in detail in [16], in an arbitrary covariant gauge.

5 Complete on-shell two-loop renormalization of QED

We achieve this in three stages. First we give exact results, in D dimensions, for all the two-loop renormalization constants of 'pure' QED, uncomplicated by electroweak effects or the existence of μ and τ. In other words, we effect the two-loop on-shell renormalization of the $U(1)$ gauge theory of a single fermion in D dimensions. Then we give the Laurent expansions of the renormalization constants, including finite parts. Finally we indicate how these are modified by the addition of other leptons. We take no account of the existence of weak interactions.

5.1 *D-dimensional QED, without integration*

There is only one more independent renormalization constant to determine in QED: the on-shell photon wave-function renormalization constant Z_3, which also determines the charge renormalization $e_0^2=e_R^2/Z_3$, thanks to the Ward identity [30] $Z_1=Z_2$.

In comparison with Z_m and Z_2, we find it rather easy to calculate $Z_3=1/(1+\Pi(0))$, to two loops, from the bare-photon self energy $\Pi(q^2)$ at $q^2=0$. One has merely to operate on self-energy diagrams with $(\partial^2/\partial q_\alpha\partial q_\beta)$ and then set the external momentum q to zero. This results in a series of bubble diagrams, with four insertions of gamma matrices, which add up to give $\Pi(0)$ times the constant tensor

$$(\partial^2/\partial q_\alpha\partial q_\beta)(q_\mu q_\nu-q^2g_{\mu\nu})=g_{\mu\alpha}g_{\nu\beta}+g_{\nu\alpha}g_{\mu\beta}-2g_{\mu\nu}g_{\alpha\beta}.$$

The one-loop integrals give a multiple of $\Gamma(\omega)$, along with obvious powers of π, e_0 and m_0. Very conveniently, every two-loop integral [7] gives a rational function of D times $\Gamma^2(\omega)$. It is thus a simple matter of book-keeping to obtain the two-loop expansion in terms of the bare quantities and then use the one-loop renormalization of e_0 and m_0 to express Z_3 in terms of the physical charge e_R and physical mass M in *any* dimension D. A short REDUCE program yields

$$\frac{e_R^2}{e_0^2}=Z_3=1-\frac{4}{3}\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)+B(D)\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)^2+O(e_R^6) \quad (44)$$

where e_R is the D-dimensional physical coupling constant, measured at zero momentum, and

$$B(D)=-\frac{2(D-4)}{D(D-3)(D-5)}\{2+(D-4)(D^2-8D+9)\}. \quad (45)$$

The simple rationality of (45) belies its power. It determines not only how the two-loop coupling of any off-shell scheme runs, but also the boundary condition for the integral solution to the renormalization-group equation for the running coupling. The former information is encoded by the leading behaviour as $D\to 4$: $B'(4)=1$; the latter by the next-to-leading behaviour: $B''(4)=-15/2$. Merely by manipulating gamma matrices and gamma functions in D-dimensions at zero momentum, we obtain these two crucial numbers, which require the running coupling $\bar{\alpha}(\mu)$ of the MS scheme to satisfy

$$\frac{\pi}{\bar{\alpha}(\mu)}=\frac{\pi}{\alpha}+\frac{2}{3}\log\frac{M}{\mu}+\frac{\alpha}{\pi}\left(\frac{1}{2}\log\frac{M}{\mu}-\frac{15}{16}\right)+O(\alpha^2)+O(\omega) \quad (46)$$

where $\alpha\equiv\lim_{D\to 4} e_R^2/4\pi$ is the fine structure constant, as *measured* in 4 dimensions. To obtain (46), one has merely to equate the D-dimensional MS *ansatz*

$$\frac{e_0^2}{4\pi}=\left(\frac{\mu^2 e^{\gamma}}{4\pi}\right)^{\omega}\bar{\alpha}(\mu)\left(1+\frac{\bar{\alpha}(\mu)}{\pi}\frac{Z_{11}}{\omega}+\frac{\bar{\alpha}^2(\mu)}{\pi^2}\left(\frac{Z_{22}}{\omega^2}+\frac{Z_{21}}{\omega}\right)+O(\bar{\alpha}^3)\right) \quad (47)$$

to $Z_3^{-1}e_R^2/4\pi$ and require that $\bar{\alpha}(\mu)$ be finite in 4 dimensions. This physical constraint on the MS scheme yields the subtraction constants of (47) and the *solution* (46) to the renormalization-group equation [31]

$$\beta(\bar{\alpha}(\mu))\equiv 2\omega+\frac{\mathrm{d}\log\bar{\alpha}(\mu)}{\mathrm{d}\log\mu}=\frac{2}{3}\frac{\bar{\alpha}(\mu)}{\pi}+\frac{1}{2}\frac{\bar{\alpha}^2(\mu)}{\pi^2}+O(\bar{\alpha}^3). \quad (48)$$

Thus the *on-shell* Z_3 contains, in its finite part, more information than can be obtained by ultraviolet subtraction: it tells the QED MS coupling where to run *to*, in order to agree with on-shell data, rather than leaving it with an integration constant like the astronomic value of $\Lambda_{\mathrm{QED}}\sim M\exp(3\pi/2\alpha)$ [31, 33]. This finite information is as easy to obtain from (44, 45) as is the beta function.

Lest it be thought that this virtue of on-shell renormalization is peculiar to the infrared freedom of QED, we remark that an analogous situation arose concerning the relationship between the pole and MS masses of heavy quarks in QCD [7]. There one was in the ironic situation of knowing the three-loop anomalous dimension γ_m [11], but being unable to use it to relate constituent and current quark masses, for lack of the finite two-loop part of Z_m. This state of affairs was remedied in [7], where it was shown that the finite on-shell two-loop term dominates the next-to-leading corrections.

These two examples show the utility of obtaining on-shell renormalization constants in D-dimensions, in order to extract physically relevant finite parts, as well as anomalous dimensions. We therefore give a complete description of the on-shell two-loop renormalization of QED in any dimension by complementing (44) with the corresponding expansions of Z_m and Z_2 in terms of the physical charge and mass:

$$Z_m=1-\frac{D-1}{D-3}\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)+\sum_{j=1}^{3}M_j(D)\frac{R_j}{R_1}\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)^2+O(e_R^6) \quad (49)$$

$$Z_2=1-\frac{D-1}{D-3}\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)+\sum_{j=1}^{3}F_j(D)\frac{R_j}{R_1}\left(\frac{e_R^2\Gamma(\omega)}{(4\pi)^{D/2}M^{2\omega}}\right)^2+O(e_R^6) \quad (50)$$

where the rational functions multiplying the integral structures (14) are obtained from the coefficients of Tables 1 and 2 as follows:

$$M_j(D)=-2M_{1j}+M_{2j}+2M_{4j}-\frac{4(D-1)}{3(D-3)}\delta_{j1} \quad (51)$$

$$F_j(D)=-2F_{1j}+F_{2j}+2F_{4j}+\frac{D(D-1)}{3(D-3)}\delta_{j1}+(1+D/4)M_j(D) \quad (52)$$

by setting $C_A=N_L=0$ and $C_F=T_F=1$ in (17) and using (44) to transform to the physical charge. The explicit forms of these coefficients involve polynomials in D of orders up to 10. Their Laurent expansions are used in the next section.

We remark that the rationality of D-dimensional calculation extends beyond the calculation of renormalization constants. It is clear that the two-loop anomalous magnetic moment calculation involves only zero-momentum insertions in Fig. 1a, b, d, e, after differentiating with respect to an infinitesimal external photon momentum. Thus $g-2$ to two loops, in D-dimensions, can likewise be reduced to the same three integral structures, by systematic computer algebra, quite free of anything remotely resembling integration over Feynman or Schwinger parameters.

Nor does the avoidance of integration end here, since one of us has found [19] that the sole recalcitrant integral, $I(\omega)$, may be reduced, in any dimension, to Γ functions and a single Saalschützian ${}_3F_2$ series, whose power expansion in ω can be found up to the level required for four-loop calculations by a combination of finite group theory and known special cases of related series, mainly culled from Hardy's lucid exegesis [32] of Chapter XII of Ramanujan's notebook. This expansion involves $\{\mathrm{Li}_p(1), \mathrm{Li}_p(\tfrac{1}{2})|p\leq 5\}$, yet no Spence *integral* is ever encountered; computer algebra suffices.

We defer consideration of $g-2$ and higher-order terms in $I(\omega)$ to subsequent papers, here making the general point that, by mere book-keeping in D dimensions, much may be calculated which previously appeared to entail very difficult integrations in four dimensions, and exemplifying this by our rational results (45, 51, 52), which give the two-loop renormalization constants (44, 49, 50).

5.2 *Laurent expansion for one-lepton QED*

Before giving the $\omega \to 0$ behaviour of (44, 49, 50), there is an important observation to make regarding the D-dimensional physical charge e_{R}, lest our subsequent formulae be misunderstood.

In D dimensions, the on-shell charge, e_{R}, necessarily has mass dimension $(4-D)/2 \equiv \omega$. This is an ineluctable consequence of having a dimensionless action [31]. It follows that the L-loop term of the expansion of a dimensionless quantity (such as $g-2$ or a renormalization constant) will involve e_{R}^{2L} divided by some physical mass or momentum scale (such as M) to the power $2L\omega$, as is the case in (44, 49, 50). There will also be the inevitable factor of $(4\pi/\mathrm{e}^{\gamma})^{L\omega}$ which results from the surface of the unit sphere in D-dimensions, $2\pi^{D/2}/\Gamma(D/2)$, divided by the $(2\pi)^D$ factor of Fourier transformation. It is therefore very *convenient*, though not logically necessary, to introduce the shorthand notation

$$\alpha_M \equiv \frac{e_{\mathrm{R}}^2}{4\pi}\left(\frac{4\pi}{M^2 \mathrm{e}^{\gamma}}\right)^{\omega} \text{ (\textit{not} a running coupling).} \tag{53}$$

The important point is that when one has obtained a result for a finite quantity, such as $g-2$, one may take the limit $\omega \to 0$ and express the answer in terms of the experimentally determined 4-dimensional coupling

$$\alpha \equiv \lim_{D\to 4} \alpha_M = 1/137.036\ldots \text{ (for \textit{all} } M). \tag{54}$$

By this device we are able to present two-loop results uncluttered by factors from the expansion

$$\left(\frac{\alpha_M}{\pi\omega}\right)^2 = \left(\frac{e_{\mathrm{R}}^2}{4\pi^2}\right)^2 \{\omega^{-2} + 2(\log 4\pi - \gamma - \log M^2)\,\omega^{-1} + 2(\log 4\pi - \gamma - \log M^2)^2 + O(\omega)\}$$

which, whilst formally correct, looks dimensionally puzzling at first sight. What it means is that one should use the *same* mass unit to express the values both of M and of e_{R}, for $\omega \neq 0$. Thus one might as well work with units in which $M^2 = 4\pi/\mathrm{e}^{\gamma}$. Only when there is another mass scale in the problem, as in the next Sect., need one concern oneself with logarithms.

In terms of α_M, we find

$$Z_3 = 1 - \{\tfrac{1}{3}\omega^{-1} + \tfrac{1}{6}\zeta(2)\,\omega + O(\omega^2)\}\,\frac{\alpha_M}{\pi} - \{\tfrac{1}{8}\omega^{-1} + \tfrac{15}{16} + O(\omega)\}\,\frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \tag{55}$$

$$Z_2 = 1 - \{\tfrac{3}{4}\omega^{-1} + 1 + (\tfrac{3}{8}\zeta(2) + 2)\,\omega + O(\omega^2)\}\,\frac{\alpha_M}{\pi} + \{\tfrac{9}{32}\omega^{-2} + \tfrac{55}{64}\omega^{-1} + \pi^2 \log 2 - \tfrac{3}{2}\zeta(3) - \tfrac{211}{32}\zeta(2) + \tfrac{7685}{1152} + O(\omega)\} \cdot \frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \tag{56}$$

$$Z_{\mathrm{m}} = 1 - \{\tfrac{3}{4}\omega^{-1} + 1 + (\tfrac{3}{8}\zeta(2) + 2)\,\omega + O(\omega^2)\}\,\frac{\alpha_M}{\pi} + \{\tfrac{5}{32}\omega^{-2} + \tfrac{155}{192}\omega^{-1} + \tfrac{1}{2}\pi^2 \log 2 - \tfrac{3}{4}\zeta(3) - \tfrac{87}{32}\zeta(2) + \tfrac{1169}{384} + O(\omega)\} \cdot \frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \tag{57}$$

where, as ever in on-shell two-loop renormalization, one should retain the one-loop $O(\omega)$ terms, since they may later be multiplied by the one-loop $O(1/\omega)$ terms of another expansion. The numerical values of the finite parts of the coefficients of α_M^2/π^2 in (56) and (57) are 0.86 and 1.09, respectively, indicating considerable cancellations between the four terms in each analytical result.

5.3 *Laurent expansion for multi-lepton QED*

To two loops, the effect of adding more leptons is easy to specify in the case of Z_3: given a set of leptons of masses $\{M_i | i = 1, N_{\mathrm{lep}}\}$, one merely replaces α_M in the one-loop term of (55) by $\sum_i \alpha_{M_i}$, and α_M^2 in the two-loop term by $\sum_i \alpha_{M_i}^2$. There are no cross terms, to two loops. At first sight this might seem odd, since the bare self energy is iterated in $Z_3 = 1/(1 + \Pi(0))$, which does produce cross terms in the expansion in powers of the bare charge. However, these are removed when one performs one-loop charge renormalization. The corresponding effect on (46) is to replace $\log M/\mu$ by $\sum_i \log M_i/\mu$. Thus the effect of the μ and τ leptons on the MS coupling at the electron mass is rather substantial:

$$\frac{\pi}{\bar{\alpha}(M_e)} = \frac{\pi}{\alpha} + \frac{2}{3}\log\frac{M_\mu M_\tau}{M_e^2} + \frac{\alpha}{\pi}\left(\frac{1}{2}\log\frac{M_\mu M_\tau}{M_e^2} - \frac{15}{16}\right) + O(\alpha^2) + O(\omega). \tag{58}$$

Only in one-lepton QED is it a good approximation [31] to take $\bar{\alpha}(M_e) \approx \alpha$.

The changes to the renormalization constants (56, 57) of one lepton, with mass M, due to another lepton, with mass $M_i = rM$, are to add the following corrections

$$\Delta \bar{Z}_2 = \left\{\frac{1}{16\omega} - \frac{1}{4}\log r - \frac{5}{96} + 2\bar{\Delta}(r)\right\}\frac{\alpha_M^2}{\pi^2} \tag{59}$$

$$\Delta \bar{Z}_{\mathrm{m}} = \left\{-\frac{1}{8\omega^2} + \frac{5 + 24\log r}{48\omega} - \frac{1}{2}\log^2 r + \frac{2}{3}\log r + \frac{3}{8}\zeta(2) + \frac{71}{96} - 2\Delta(r)\right\}\frac{\alpha_M^2}{\pi^2} \tag{60}$$

where, given the gross disparity between lepton masses, it is a good approximation to work with the appropriate limiting forms of the dilogarithms (23, 27), given by (24, 28) when $r \gg 1$, or by (25, 29) when $r \ll 1$.

Note that on-shell charge renormalization ensures that the mass-dependent singular term, $\omega^{-1}\log r$, in the bare correction (22), is absent from the renormalized correction (59). Correspondingly, the absence of such a term from (26) entails its appearance in (60). There seems, in general, to be no particular reason why either renormalization constant should be well-behaved as $r \to \infty$, since only relationships between observable quantities satisfy decoupling theorems. The mass singularities of renormalization will cancel those in the truncated bare Green functions, to ensure decoupling of internal heavy-lepton effects from renormalized light-lepton Green functions.

6 Summary and conclusions

In dimensionally regularized QED, Z_2 is gauge-invariant to all orders, by virtue of the Johnson-Zumino [12] identity (6). We are not aware of a non-abelian generalization of this result. Nevertheless, Z_2 is gauge-invariant at the two-loop level in QCD, thanks to intricate cancellations between the diagrams of Fig. 1. We take this as strong evidence of its gauge invariance in general.

The precise form of Z_2 at two loops provides a link between the MS renormalization of a heavy-quark field in QCD and in the effective field theory [13, 14] obtained by letting $M \to \infty$. To convert MS-renormalized truncated Green functions to on-shell S-matrix elements in QCD and EFT one must multiply by the factors $(Z_2^{\mathrm{MS}}(\mu)/Z_2)^{N_{\mathrm{E}}/2}$ and $(\tilde{Z}_2^{\mathrm{MS}}(\mu)/\tilde{Z}_2)^{N_{\mathrm{E}}/2}$, respectively, for processes with N_{E} external heavy quarks. But in dimensionally regularized EFT, with one or more infinite-mass quarks and N_{L} zero-mass quarks, there is *no* on-shell wave-function renormalization, since the on-shell self energy is scale free. Thus ratios of S-matrix elements differ from ratios of renormalized Green functions by powers of the factor $R(\mu) \equiv (Z_2^{\mathrm{MS}}(\mu)/Z_2^{\mathrm{L}})/\tilde{Z}_2^{\mathrm{MS}}(\mu)$, where Z_2^{L} includes the effects of light quarks and gluons in QCD, but excludes the effects of heavy-quark loops, since these are discarded in EFT. The factor $R(\mu)$ must be finite. Its μ dependence is therefore determined by the singular terms in Z_2^{L}, from which we have obtained the gauge-invariant difference (35) between the anomalous dimensions of the heavy-quark field in QCD and EFT. Renormalization-group improvement then gives

$$R(\mu) \approx R(M)\left(\frac{\bar{\alpha}_s(M)}{\bar{\alpha}_s(\mu)}\right)^{4/b_1}\left(\frac{1+E_2\bar{\alpha}_s(M)/\pi}{1+E_2\bar{\alpha}_s(\mu)/\pi}\right)$$

where $b_1 = 11 - \frac{2}{3}N_{\mathrm{L}}$, $E_2 = \frac{175}{162}$ or $\frac{4253}{3750}$ for $N_{\mathrm{L}} = 3$ or 4, and the integration constant is found from the finite part of Z_2^{L} to be

$$R(M) \approx 1 + \tfrac{4}{3}\bar{\alpha}_s(M)/\pi + (19.23 - 1.33\,N_{\mathrm{L}})\,\bar{\alpha}_s^2(M)/\pi^2$$

whose two-loop term is commensurate with three-loop MS renormalization and, like the corresponding term [7] in $Z_m^{\mathrm{MS}}(M)/Z_m$, is numerically large.

These results were obtained from the exact D-dimensional rational functions of Tables 1 and 2, found by implementation of the recurrence relations of [7] in a REDUCE [23] program which involves no integration whatsoever. For convenience, the resultant Laurent expansions are given in Table 3, before coupling renormalization, and Table 4, after MS renormalization of the QCD coupling. The EFT anomalous dimension

$$\tilde{\gamma}_{\mathrm{F}} = \frac{(\bar{a}-3)\,C_{\mathrm{F}}\bar{\alpha}_s}{2\pi} + \left\{\left(\frac{\bar{a}^2}{32} + \frac{\bar{a}}{4} - \frac{179}{96}\right)C_{\mathrm{A}} + \frac{2}{3}\,T_{\mathrm{F}}N_{\mathrm{L}}\right\}\frac{C_{\mathrm{F}}\bar{\alpha}_s^2}{\pi^2} + O(\bar{\alpha}_s^3)$$

was obtained from the singular terms of Table 4 and the corresponding QCD result [20]. It has been verified [16] by an analogous implementation of the recurrence relations for the off-shell two-loop integrals of EFT.

This complete avoidance of integration, or infinite summation, is familiar in massless QCD [22] and clearly capable of extension to EFT. What is more surprising is that the two-loop on-shell two- and three-point functions of pure QED fall into the same category of rational simplicity in D dimensions, as exemplified by the complete account of two-loop renormalization given, for all D, by (44, 49, 50) and, for $D \to 4$, by (55–57). The classic two-loop result for $g-2$ may also be viewed as a calculation of the $D \to 4$ limits of the three coefficients of the integral structures (14) to which all on-shell two-loop diagrams of the type of Fig. 1 are systematically reducible. Indeed the value [18]

$$g - 2 = \alpha/\pi + (\zeta(2) - I(0) + \tfrac{197}{72})\,\alpha^2/\pi^2 + O(\alpha^3)$$

clearly demonstrates that $I(0) = \pi^2\log 2 - \frac{3}{2}\zeta(3)$ is central to on-shell two-loop QED. This $D=4$ value of the integral (15) was obtained in [17] by evaluation of trilogarithmic integrals. But even that is unnecessary, since recently it has proved possible [19] to expand $I(\omega)$ through $O(\omega^2)$ by purely algebraic methods. This expansion involves a fifth-order polylogarithm, $\mathrm{Li}_5(\frac{1}{2}) = \sum_{n=1}^{\infty} 2^{-n}n^{-5}$, typical of *four*-loop QED calculations, yet no integration is needed to obtain it.

When one encounters a physically significant mass ratio, such as M_e/M_μ in the calculation of the muon's anomalous magnetic moment, exact two-loop calculation entails the evaluation of dilogarithms, by old-fashioned analytical techniques. We have given the corresponding effects (59, 60) on renormalization constants in terms of the dilogarithms (23, 27), whose limiting forms (24, 28) and (25, 29) are useful in QED.

In conclusion: on-shell renormalization of a theory with a single mass scale enjoys much of the calculational simplicity of deep-euclidean MS renormalization. Its results, however, are more powerful, since they determine both the MS counterterms and the finite parts needed to make contact with physical processes. On-shell renormalization is also satisfyingly gauge-invariant. The physical significance of this is that the gauge dependences of MS renormalization of QCD and EFT cancel. The implications of our results for the two-loop anomalous dimensions of EFT currents [14, 15] linking static and massless quarks are under study [16], as are the prospects of extending our methods for massive Feynman integrals to three loops [19].

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