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Dilatant double shearing theory applied to granular chute flow

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Summary. Although the steady flow of a granular material down a plane inclined slope has been exhaustively examined from both theoretical and experimental points of view, there is still no general agreement concerning the basic flow properties such as density and velocity profiles. The majority of studies assume that the velocity component of the material perpendicular to the inclined plane is sufficiently small to assume that it is everywhere zero. However, recent dynamical modelling of granular chute flow indicates that this component of velocity, although small, is actually non-zero. In this paper, we examine a dilatant double shearing theory for chute flow assuming that the perpendicular component of velocity is non-zero. An explicit analytical form for the perpendicular velocity profile is deduced which gives rise to an integral expression for the chute stream velocity. Assuming a linear decreasing density profile, numerical integration for the chute stream velocity predicts a non-linear profile which is concave in shape and which is in agreement with recent results from computer simulation and existing experimental data in the literature.

1 Introduction

Granular media is the general term referring to systems involving solid particles such as soil, sand, powder, minerals, grains, beads or rocks which are immersed in a fluid environment which might be a vacuum, such as particles in outer space, air or gas such as in bulk material handling and fluidized beds or a liquid such as particles in suspension and sedimentation. Such combinations all constitute important engineering systems [1]–[3] and because of their complexity, the study of granular media presents a fundamental challenge for basic science [4], [5].

A good deal of research has gone into describing the flow of granular materials and considerable progress has been made between theory, experiment and computer simulation [1]–[3]. From a theoretical perspective, there are essentially two different approaches for the description of granular flows. As a discrete many particle system, one approach is to consider the individual particles while the other is to view granular media as a macroscopic system, that is as a continuum. Most successful theories correspond to these two different approaches, namely kinetic theories [6]–[9] based on gas dynamics, and double shearing theories [1], [10] based on continuum soil mechanics, and both approaches have been extensively applied to many practical granular flows. For the kinetic theories, see for example, the review articles of Savage [2] and Campbell [3] and for the double shearing theories, see for example, the work of Spencer [1], [11] and the recent work of Hill and Wu [12]–[15]. It is generally believed that kinetic theories best describe rapid granular flows while double shearing theories apply for initial failure or quasi-static granular flows.

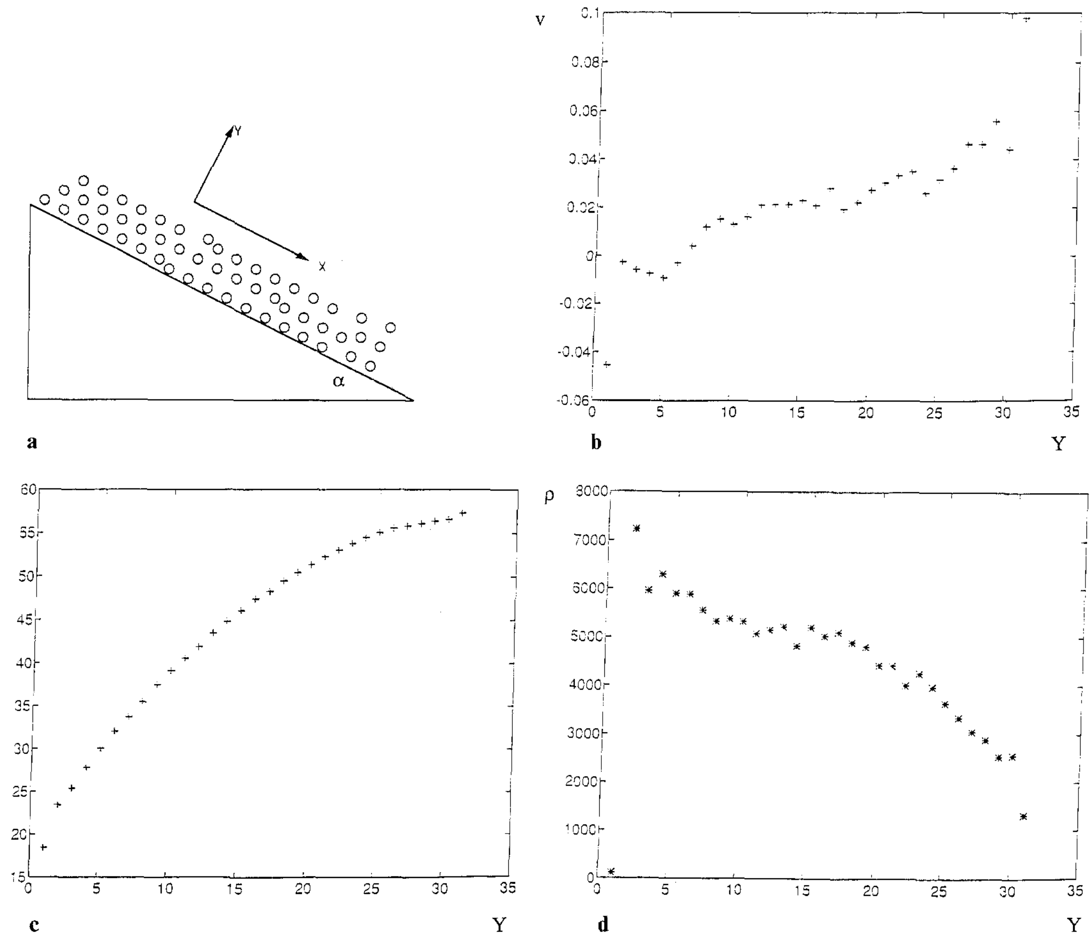

Fig. 1. Typical computer simulation results for $\alpha = 30^\circ$ and $\tan\delta = 0.4$: **a** schematical diagram for granular chute flow, **b** perpendicular velocity profile $v(y)$, **c** chute stream velocity profile $u(y)$, **d** density profile $\varrho(y)$

Granular chute flow (indicated in Fig. 1 a) is perhaps one of the most studied systems theoretically [16]–[22], experimentally [23]–[30], and by computer simulation [31]–[34]. Despite these extensive studies, there is no general agreement regarding the basic flow properties such as density and velocity profiles. For example, it is still a matter of speculation as to whether there exists a low density zone at the base boundary with a maximum at the middle of the flow depth or whether the density profile is simply a monotonically decreasing function. Similarly for the chute stream velocity, the question arises as to whether the profile is linear or non-linear and which is convex or concave in shape [35], [36]. In our recent computer simulation work on chute flow [34], we find that there is no low density zone at the base boundary, and a typical density profile $\varrho(y)$ is shown in Fig. 1 d where y is the height above the chute base. A typical chute stream velocity profile $u(y)$ is non-linear and is concave in shape as shown in Fig. 1 c.

The chute flow of a dry granular material is generally regarded as a rapid flow for which the kinetic theory should be the most appropriate. However, recent work of Anderson and Jackson [21] argues that this is not necessarily the case and claims that a model based on a simple combination of kinetic and frictional theories [37], [19] gives better results. The question therefore arises as to whether or not double shearing theory can correctly describe the main qualitative features for chute flow of a dry granular material.

In the majority of the analysis of shear flows, the perpendicular component of velocity is assumed to be zero or at least sufficiently small to be ignored [16]. This appears to be a reasonable assumption since the perpendicular velocity component is in general much smaller than the chute stream velocity component [34], [27] (see Figs. 1 b and 1 c). In our recent work on dynamical modelling of both chute and Couette flows [34], [38], it has been observed that the perpendicular velocity component in chute flow is much more likely to be non-zero than that for simple Couette flow (Fig. 1 b). This is because physically in chute flow, in order to maintain an averaged steady flow, there should be some averaged perpendicular velocity which balances the downwards velocity caused by gravitation. This perpendicular velocity component depends on the averaged particle fraction (or density) and therefore the averaged interparticle distance. The larger the interparticle distance, the larger the perpendicular velocity that is required. If the averaged particle fraction is constant across the flow field, then the perpendicular velocity will also be a constant profile. It is expected that a zero perpendicular velocity occurs only under the condition of zero inter-particle distance. In a recent paper, Jenkins and Hanes [39] also discuss similar questions.

The aim of this present paper is to investigate whether or not the dilatant double shearing theory applied to chute flow of granular materials and assuming the perpendicular velocity component is non-zero, predicts results which are qualitatively in agreement with our recent computer simulation results. We find by taking the perpendicular velocity component into account that dilatant shearing theory actually provides an excellent description of chute flow. We emphasise that the present analysis leads to a complicated integral equation for the determination of the density profile, which strictly speaking should be solved by an integral iteration scheme. However, the details are complicated and therefore we simply examine the predictions of our analysis assuming either constant or linearly decreasing density profiles. In particular, assuming a linear decreasing density profile, the predicted chute stream velocity profile is in excellent agreement with our computer simulation results and some data in the literature but despite this successful agreement our approach involves assuming a density profile which does not coincide with that presicted by our theory. In the following section, we briefly state the governing equations for plane granular flow assuming dilatant double shearing theory. In Section 3 we derive analytical solutions for the chute flow variables, and in Section 4 we compare our solutions with our recent dynamical simulation results.

2 Governing equations

The basic governing equations for plane deformations of a granular material, assuming double shearing theory for the determination of velocities, are fully presented by Spencer [1], and compact accounts can be found in Spencer [40] and Spencer and Bradley [11]. Here we use an extension of this theory, in which double shearing is accompanied by an expansion in the normal directions to the shear plane. This theory is referred to as dilatant double shearing and is due originally to Mehrabadi and Cowin [10], [41], [42], and the basic equations can also be found in Harris [43], [44].

With reference to rectangular Cartesian coordinates (x, y, z), as indicated in Fig. 1 a, and assuming that all stress and velocity components are independent of z and that σ_{zz} is the intermediate principal stress, the basic equations for the tensor $\boldsymbol{\sigma}$ are as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -\varrho g \sin \alpha, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \varrho g \cos \alpha, \tag{1}$$

which together with the Coulomb-Mohr yield condition

$$(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 = [2c\cos\delta - (\sigma_{xx} + \sigma_{yy})\sin\delta]^2, \tag{2}$$

constitute three equations for the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$. In these equations ϱ denotes the density, α is the slope angle, δ is the angle of internal friction (assumed constant) and c is the cohesion of the material. In conjunction with these equations there are additional relations

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi, \tag{3}$$

where p and q are defined by

$$p = -\frac{1}{2}(\sigma_1 + \sigma_2) = -\frac{1}{2}(\sigma_{xx} + \sigma_{yy}), \tag{4}$$

$$q = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\}^{1/2}, \tag{5}$$

where σ_1 and σ_2 denote the maximum and minimum principal stresses respectively ($\sigma_2 \leqq \sigma_{zz} \leqq \sigma_1$), and ψ is the angle the principal stress σ_1 makes with the x-axis. In this terminology we note that the Coulomb-Mohr yield condition (2) becomes

$$q = p\sin\delta + c\cos\delta, \tag{6}$$

and for known stress components, the angle ψ can be determined from either of the following equations:

$$\tan 2\psi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \quad \cos 2\psi = \frac{\sigma_{xx} - \sigma_{yy}}{2c\cos\delta - (\sigma_{xx} + \sigma_{yy})\sin\delta}. \tag{7}$$

Assuming that the stress components are known and therefore the angle ψ, the velocities (u, v) in the (x, y) directions, assuming dilatant double shearing, are obtained as follows:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\cos 2\psi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin 2\psi = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\cos(\delta - \gamma)}{\sin\gamma}, \tag{8}$$

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\sin 2\psi - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos 2\psi = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + 2\Omega\right)\frac{\sin(\delta - \gamma)}{\cos\gamma}, \tag{9}$$

where Ω denotes the material derivative of ψ with respect to time, that is

$$\Omega = \frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y}, \tag{10}$$

and γ is a constant referred to as the angle of dilatancy. If the velocity components are known, the density $\varrho(x, y, t)$ is determined from the continuity equation

$$\frac{\partial\varrho}{\partial t} + u\frac{\partial\varrho}{\partial x} + v\frac{\partial\varrho}{\partial y} + \varrho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{11}$$

We comment here that for a highly dilatant material the assumption that the material parameters γ, δ and c are constant is physically unrealistic, but acceptable within the context of demonstrating the qualitative behaviour of the theory. We also note that Morrison and Richmond [47] extend Spencer's incompressible double shearing theory to include the inertia terms in Eq. (1), and approximate solutions to hopper and chute problems are presented. A similar extension could be obtained for the dilatant double shearing theory used here, although the solution details would be more complicated. In the following section we present new solutions to the above equations for the problem of steady granular flow down a plane inclined chute.

3 Solutions for granular chute flow

For steady plane chute flow we would except that all quantities are independent of t and x, assuming that the chute is infinitely long. Accordingly all quantities only depend upon the height y above the chute base and the two stress equations become simply

$$\frac{d\sigma_{xy}}{dy} = -\varrho(y)\, g \sin\alpha, \qquad \frac{d\sigma_{yy}}{dy} = \varrho(y)\, g \cos\alpha, \tag{12}$$

so that we have

$$\sigma_{xy} = -\lambda(y)\sin\alpha, \tag{13}$$

$$\sigma_{yy} = \lambda(y)\cos\alpha, \tag{14}$$

where $\lambda(y)$ is defined by

$$\lambda(y) = g\int_h^y \varrho(s)\, ds, \tag{15}$$

where h denotes the height of the free surface on which the stresses are zero. On introducing the quantity $\mu = \sigma_{xx} - \sigma_{yy}$ we may deduce from Eq. (2) the following quadratic equation:

$$\mu^2 + 4\lambda^2 \sin^2\alpha = [2(c\cos\delta - \lambda\cos\alpha\sin\delta) - \mu\sin\delta]^2, \tag{16}$$

which simplifies to give

$$\left(\frac{\mu}{2}\right)^2 \cos^2\delta + 2\left(\frac{\mu}{2}\right)\sin\delta\,(c\cos\delta - \lambda\cos\alpha\sin\delta) + \lambda^2\sin^2\alpha - (c\cos\delta - \lambda\cos\alpha\sin\delta)^2 = 0. \tag{17}$$

Using the notation

$$a = c\cos\delta - \lambda\cos\alpha\sin\delta, \tag{18}$$

$$b = [(c\cos\delta - \lambda\cos\alpha\sin\delta)^2 - \lambda^2\sin^2\alpha\cos^2\delta]^{1/2}, \tag{19}$$

there are two roots of (17),

$$\frac{\mu}{2} = \frac{-a\sin\delta \pm b}{\cos^2\delta}, \tag{20}$$

which gives rise to two values of the angle ψ, thus

$$\tan 2\psi = \frac{\lambda \sin \alpha \cos^2 \delta}{a \sin \delta \mp b}, \tag{21}$$

both of which appear to be physically reasonable. We observe that from the definition of $\lambda(y)$ it is clear that ψ is zero on the free surface ($y = h$). Also from the above equations we see that for a cohesionless material ($c = 0$) the angle ψ is constant. Accordingly, in the subsequent analysis we assume throughout that c is non-zero.

For the determination of the velocity field and with the assumption

$$u = u(y), \quad v = v(y), \quad \varrho = \varrho(y), \tag{22}$$

Eqs. (8), (9) and (11) simplify to yield

$$\frac{du}{dy} \sin 2\psi - \frac{dv}{dy} \cos 2\psi = \frac{dv}{dy} \frac{\cos(\delta - \gamma)}{\sin \gamma}, \tag{23}$$

$$\frac{du}{dy} \cos 2\psi + \frac{dv}{dy} \sin 2\psi = -\left(\frac{du}{dy} + 2v \frac{d\psi}{dy}\right) \frac{\sin(\delta - \gamma)}{\cos \gamma}, \tag{24}$$

$$v \frac{d\varrho}{dy} + \varrho \frac{dv}{dy} = 0, \tag{25}$$

from which it is a simple matter to deduce

$$\frac{dv}{d\psi} (A + B \cos 2\psi) = -2 \frac{\sin(\delta - \gamma)}{\cos \gamma} v \sin 2\psi, \tag{26}$$

where the constants A and B are defined by

$$A = 1 + \frac{\sin(\delta - \gamma) \cos(\delta - \gamma)}{\sin \gamma \cos \gamma}, \tag{27}$$

$$B = \frac{\cos(\delta - \gamma)}{\sin \gamma} + \frac{\sin(\delta - \gamma)}{\cos \gamma}. \tag{28}$$

On integrating (26) we may readily obtain the explicit expression

$$v(y) = v_0(A + B \cos 2\psi)^m, \tag{29}$$

where v_0 denotes the arbitrary constant of integration and m is a further constant defined by

$$m = \frac{\tan \gamma \tan(\delta - \gamma)}{1 + \tan \gamma \tan(\delta - \gamma)}. \tag{30}$$

From (23), (24) and (29) we have

$$\frac{du}{dy} = -2mv_0 B(A + B \cos 2\psi)^{m-1} \left\{\cos 2\psi + \frac{\cos(\delta - \gamma)}{\sin \gamma}\right\}, \tag{31}$$

and therefore upon integration we obtain

$$u(y) = -2mv_0B \int_{\psi_0}^{\psi} (A + B\cos 2\xi)^{m-1} \left\{\cos 2\xi + \frac{\cos(\delta - \gamma)}{\sin\gamma}\right\} d\xi + u_0, \tag{32}$$

where ψ_0 denotes the value of ψ at the chute base ($y = 0$) and u_0 denotes a further arbitrary constant, which is the slip velocity at the base. Finally from (25), we have on integration

$$\varrho(y) = \frac{\varrho_0}{v(y)}, \tag{33}$$

where ϱ_0 is a constant. From Eqs. (15), (21), (29) and (33) we may formulate a single but complicated integral equation for the determination of the density profile $\varrho(y)$, which in principle we could solve iteratively. However, the details are not simple and therefore in the numerical results of the following section we adopt the strategy of simply investigating the various velocity and density profiles predicted if we assume in Eq. (15) either constant or linearly decreasing density profiles.

We note that from the explicit form of Eq. (29) and the fact that ψ is zero on the free surface we can show that there are no non-trivial solutions of this form for $v(y)$ for which v is zero at the free surface $y = h$. That is, from the condition $A + B = 0$ for $\psi = 0$ (or even from the condition $A - B = 0$ for $\psi = \pi/2$) we may conclude that both A and B are identically zero. For example, the condition $A + B = 0$ can be readily factored to yield either $\cos(\delta - \gamma) = -\sin\delta$ or $\sin(\delta - \gamma) = -\cos\delta$, which can be simplified to yield $\gamma = \delta/2 \pm \pi/4$. However, even for these two restricted values of γ we may show that the constants A and B as defined by Eqs. (27) and (28) are both identically zero. This implies that our solution always predicts a non-zero component v at the free surface, which means that the free surface must be regarded as a statistical surface which has small fluctuations from $y = h$.

4 Numerical results

Equations (21), (29), (32) and (33) are the basic equations describing the chute flow field and the parameters involved depend upon the specific material and the physical properties of the chute. For a given material, the coefficient of internal friction $\tan\delta$, the dilatancy angle γ and the cohesion c are known parameters. For a given chute, the slope angle α is known while the parameters v_0 and u_0 depend on the boundary conditions of the chute base. While the perpendicular velocity component v, the angle of the principal stress axis with respect to the x-axis ψ, and the density ϱ-profiles are all explicit forms, the chute velocity profile u must be integrated numerically.

In order to illustrate the profiles for the above solution, we need to prescribe typical parameters. Here, we assume a coefficient of internal friction to be $\tan\delta = 0.4$, which is the same value as that used in our computer simulation work [34]. We suppose that the dilatancy angle lies within the range $0 \leqq \gamma \leqq \delta$ are here the value $\tan\gamma = 0.2$ is arbitrarily assigned. The acceleration due to gravity is taken as $g = 9.8$, and a typical chute slope angle is $\alpha = 30.0°$, which is also used in our computer simulation results shown in Figs. 1 b – 1 d. For a dry powder according to Rietema [45], a typical value for the cohesion constant is $c = 6.1$. For purposes of illustration we simply set the remaining parameter values to unity, namely v_0, u_0, ϱ_0 and h are all taken to be unity.

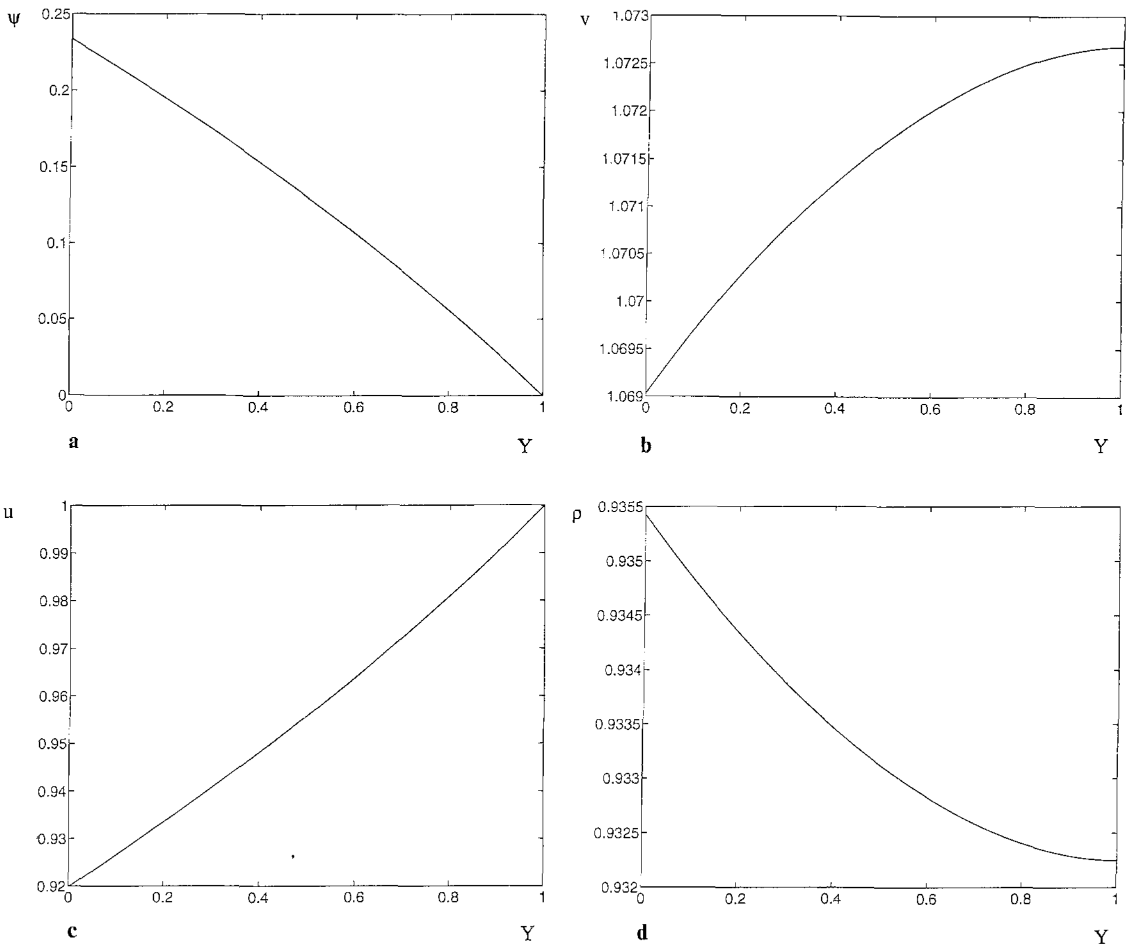

Fig. 2. Typical profiles assuming density constant in equilibrium equations: **a** principal stress angle $\psi(y)$, **b** perpendicular velocity profile $v(y)$, **c** chute stream velocity profile $u(y)$, **d** density profile $\varrho(y)$

In the first instance, the density ϱ in the equilibrium equations (12) is assumed to be a constant $\varrho_0{}^*$ in which case Eq. (15) becomes

$$\lambda(y) = g\varrho_0{}^*(y - h). \tag{34}$$

While both positive and negative values of ψ appear to be physically possible, here we only consider the case $\psi > 0$ and in Eq. (21) the negative sign applies. Figure 2a shows that $\psi(y)$ is a decreasing function of the flow depth y. On the other hand, both $v(y)$ and $u(y)$ are increasing functions of y as shown in Figs. 2b and 2c. We note that the flow stream velocity u is almost a linear function of y. The density profile in Fig. 2d shows a linear decreasing trend although the rate of decrease is small. This appears to be in agreement with our computer simulation prediction shown in Fig. 1d but clearly contradicts our assumption that ϱ is constant.

We next assume a linear decreasing density in the equilibrium equations (12), namely

$$\varrho(y) = \varrho_0{}^*(h - y) \tag{35}$$

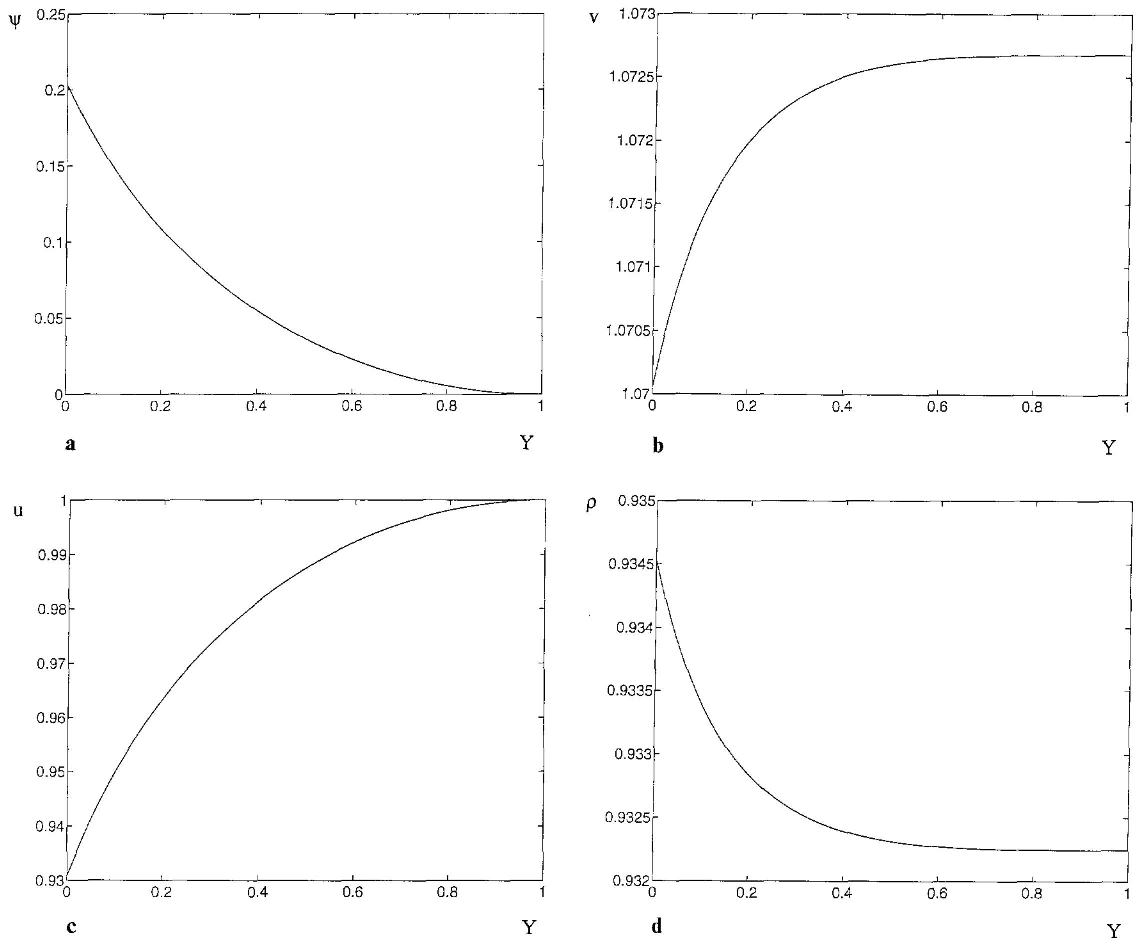

Fig. 3. Typical profiles assuming linearly decreasing density in equilibrium equations: **a** principal stress angle $\psi(y)$, **b** perpendicular velocity profile $v(y)$, **c** chute stream velocity profile $u(y)$, **d** density profile $\varrho(y)$

so that Eq. (15) becomes

$$\lambda(y) = \frac{1}{2} g\varrho_0{}^*(y-h)^2 . \tag{36}$$

and all of the remaining functions are as given above. Again, we only consider the case of $\psi > 0$. Figures 3 a – 3 d show the corresponding profiles of $\psi(y)$, $v(y)$ $u(y)$ and $\varrho(y)$. The most interesting feature is that the $u(y)$ profile which is non-linear and concave in shape is entirely in agreement with our computer simulation result shown in Fig. 1 c and the experimental results given in the literature [46]. The perpendicular velocity component (Fig. 3 b) and the density (Fig. 3 d) profiles are also non-linear and evidently this density profile is again different from that originally assumed. Strictly speaking, the density profile of the flow field should be determined iteratively rather than simply guessing the initial estimate but we do not attempt this approach here.

For our purposes the above two arbitrary cases illustrate the main characteristics of the analytical solution. Figures 4 a – 4 d show the profiles of ψ, $v(y)$, $u(y)$ and $\varrho(y)$ for different slope angles and assuming the linear decreasing density profile (35). These profiles share similar trends

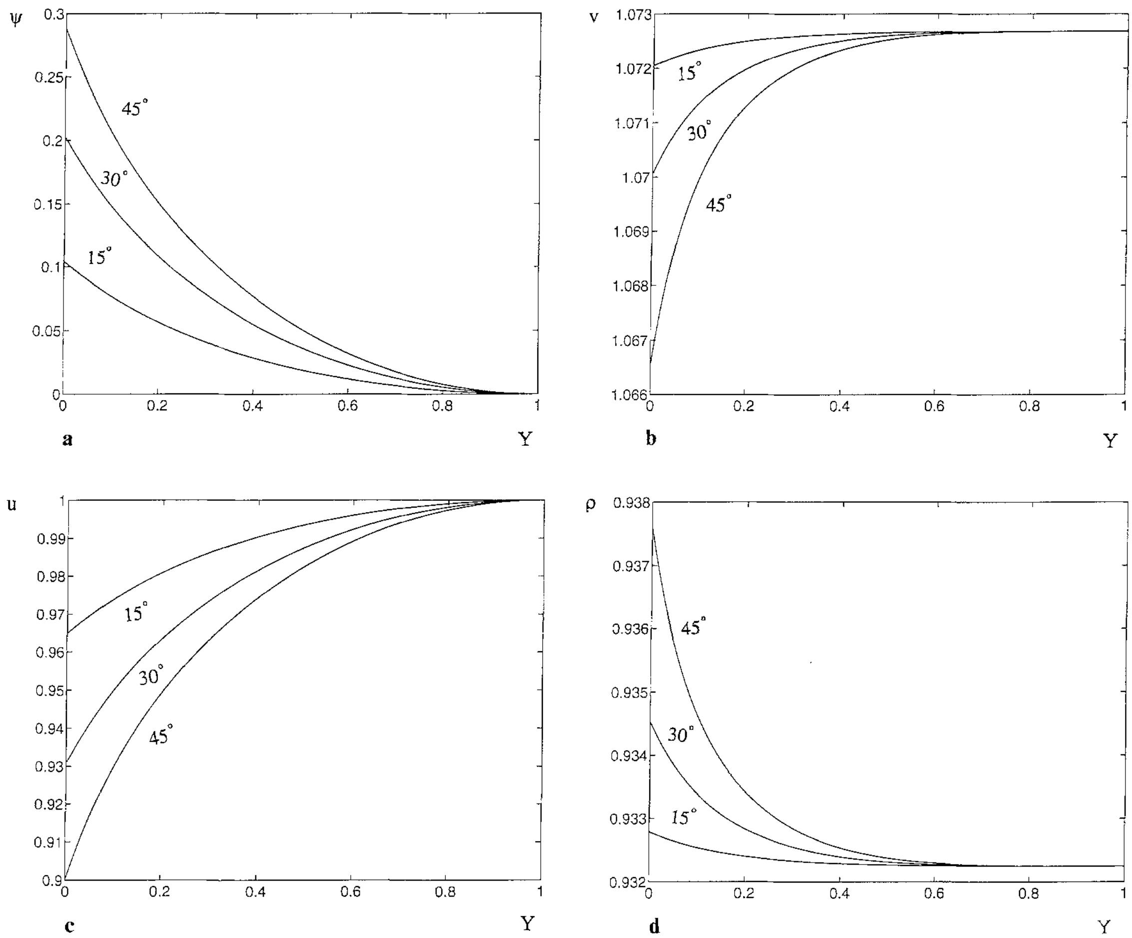

Fig. 4. Typical profiles assuming linearly decreasing density in equilibrium equations for $\alpha = 15°, 30°, 45°$: **a** principal stress angle $\psi(y)$, **b** perpendicular velocity profile $v(y)$, **c** chute stream velocity profile $u(y)$, **d** density profile $\varrho(y)$

such that the larger the slope angle, the more linear the profile becomes. We note that the most linear chute velocity profiles are obtained at small chute angles which is agreement with our computer simulation results [34].

5 Conclusions

For granular flow down a plane inclined chute, the velocity component perpendicular to the chute base is an important flow property which for a realistic analysis should not be assumed negligible. Computer simulation results indicate this, and an explicit analytical form of this perpendicular velocity has been deduced assuming dilatant double shearing and from which an integral expression for the chute stream velocity has been obtained. In general, the density and velocity profiles of the flow field are coupled functions which in principle should be determined iteratively from an integral formulation. By considering a linear density decreasing distribution, a non-linear chute stream velocity profile which is concave in shape is found to be in excellent agreement with recent results obtained from computer simulation and with experimentally obtained data. However, we emphasise that our approach means that the assumed density profile does not generally coincide with that predicted by our theory. We adopt this approach for reasons of simplicity and it does give rise to a chute stream velocity profile which is in agreement with established work. We also emphasise that the solution presented here predicts a small but non-zero perpendicular velocity at the free surface which is consistent with our computer simulation data and which we interpret as meaning that the surface must be regarded as a statistical surface which has small fluctuations from $y = h$.

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