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On a Trace Functional for Formal Pseudo-Differential Operators and the Symplectic Structure of the Korteweg-Devries Type Equations*

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1. Introduction

This paper developed out of an attempt to understand the results of Gel'fand-Dikii [1] and P. van Moerbeke (unpublished version of [2]), in a unified way. The Korteweg-deVries equation and Toda systems are completely integrable Hamiltonian systems whose equations of motion are expressible in terms of the Lax isospectral equation. Gel'fand-Dikii and P. van Moerbeke recently generalized these two types of systems respectively, and in a strikingly analogous fashion from the computational viewpoint. Indeed, we shall show that the analogy lies much deeper, and in the realm of Lie algebra.

We make the crucial observation that in both cases the relevant symplectic structure is the orbit symplectic structure of Kostant-Kirillov [12, 18].[1] In addition, the splitting of a Lie algebra into a vector space direct sum of Lie algebras is responsible for the complete integrability of the above systems and the Lax isospectral equations associated with such systems. The last statement is seen from a theorem due independently to B. Kostant and B. Symes which will be briefly mentioned, and thus the above mentioned analogy will be made precise. At this point it is good to note that for the Calogero-Moser type integrable systems, the real compact decomposition of a complex semi-simple Lie algebra plays the crucial role [14]. Thus integrable systems are seen to be related to Lie algebra decompositions. We also observe that the orbit symplectic structure plays a crucial role in the n-dimensional Euler spinning top problem of V. Arnold [11], as was observed by L. Dikii [15]. In addition, to the best of my knowledge, the first Lax-isospectral equation associated with a mechanical systems also appears in [11] in the context of the top. We also mention that quotient symplectic structures though Hamiltonian group actions [12], of which

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[1] For the Toda system, this fact was also discovered by B. Kostant. Personal communication from B. Kostant

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the Kostant-Kirillov construction is but an example, have been seen to play a role in the Moser-Calogero systems, see [13, 14].

We shall define the generalized Toda systems in Sect. 2, of which the simplest has the following form:

$$\dot{b}_i = 2(a_{i-1}^2 - a_i^2), \quad \dot{a}_i = a_i(b_i - b_{i+1}), \quad i = 1, \ldots, n, \quad a_0 = a_n = b_{n+1} = 0. \quad (1.1)$$

For this system, the relevant group G is the group of lower triangular matrices with nonzero diagonal elements, envisioned as contained in $SL(n, R)$. We identify the dual algebra of G, $\mathscr{L}^*$, with the upper triangular matrices though the trace form. The orbit Hamiltonian phase space θ_A of interest is of the form

$$\theta_A = \{[U^{-1} A U]_+ \mid U \in G\}, \quad A \in \mathscr{L}^*\},$$

where $[B]_+$ denotes the matrix formed from B by setting its lower triangular entries equal to zero, and A is subject to certain conditions.

For the case of the generalized Korteweg-deVries equation, to be defined in Sect. 3, the relevant group G is the formal pseudo-differential symbols of negative type translated by the identity element 1, whose dual $\mathscr{L}^*$ we may identify, through a trace form to be specified, with the differential symbols of nonnegative type, which are identified with formal differential operators. Thus the large algebra in which everything takes place is the formal pseudo-differential operators. The Hamiltonian orbit space of interest is of the same form as θ_A above, where the operation $[\ ]_+$ denotes the natural projection of a pseudo-differential symbol onto the nonnegative symbols. For the case of the Korteweg-deVries equation

$$q_t = 6 q q_x - 2 q_{xxx}, \quad (1.2)$$

one takes as an orbit θ_A, $A = -D^2 + q$, which, roughly speaking, is specified by the condition $\int q\, dx = \text{constant}$. We also point out that although the formula "in coordinates" for the correct symplectic structure of the generalized Kortweg-deVries equations appears in [1], it comes about through strictly computational procedures, and it is our purpose to place the formula in its proper geometric setting. This also necessitates redoing parts of [1], so as to give complete and consistent proofs.

We amplify the previous paragraph. The algebra of pseudo-differential operators[2] has a subalgebra consisting of those symbols which have an asymptotic description at ∞. One may think of taking this description as a set of normal coordinates for the subalgebra, since one identifies symbols modulo their behavior on compact sets. This suggests that one should abstract the algebraic content of the above situation, and work in a formal setting. We work with formal Laurent series in a variable ξ over some formal differential ring, with the multiplication rule inherited from pseudo-differential operator theory. We shall intuitively think of the formal Laurent series as being in a real variable ξ

[2] I am indebted to Prof. S. Sternberg for pointing out to me that in [20], Guillemin, Quillen, and Sternberg introduced an algebra of formal pseudo-differential operators which is intimately related to the algebra introduced here

embedded in a complex neighborhood of ∞. We then define the trace functional (which is of course commutative) as the equivalence class of the coefficient of the ξ^{-1} term, where we identify two elements if their difference is a total derivative of a ring element. The trace functional makes possible the identification of a cotangent bundle, and from then on, the Korteweg-deVries case proceeds in perfect analogy with the Toda system case, a point which we wish to stress. The method also works automatically for a) the self-adjoint case, which merely involves a slightly different coordinization of the formal Laurent series than in the previous case, or b) for the case where the relevant ring is composed of square matrices (see [19]) whose entries are themselves differential ring elements. In the latter case, we must take for the trace functional the equivalence class of the matrix trace of the coefficient of the ξ^{-1} term.

We point out that the decomposition of a matrix along its diagonals has a place in the Fourier theory of say Z, or Z_n, the integers and the integers modulo n, respectively. This decomposition is analogous to the Fourier decomposition of pseudo-differential operators, and strengthens the formal analogy between the Toda and Korteweg-de Vries type systems, leading to Kac-Moody algebras.

We conclude in Sect. 4 with a generalized set of so-called Lenard relations, which provides an alternate method of constructing all the relevant quantities in two examples; in addition, in the case of self-adjoint operators, the relations contain formal spectral information concerning the operators studied. These relations are essentially a consequence of the law of exponents for operators. We conjecture that they yield a second symplectic structure. We note that Bill Symes has discussed recursion relations in [8], and also that the author has discussed recursion relations in [5].

I wish to thank Bill Symes for many stimulating discussions. I am also indebted to J. Moser who provided the stimulus and encouragement for this line of work, along with the clarification of some points. In addition, my thanks extend to C. Conley and J. Robbins for helpful suggestions. I also wish to thank Conley for encouraging me to present these results at his seminar at the University of Wisconsin, Madison, even when they were still in a crude state.

2. The Generalized Toda Systems

As motivation, and of interest in its own right, we first discuss the Toda system and its generalizations, which display the relevant structures in both examples.

The Toda system (see [21]) is most easily introduced by studying the Hamiltonian system of ordinary differential equations, $\dot{x}_i = \frac{\partial H}{\partial y_i}$, $\dot{y}_i = -\frac{\partial H}{\partial x_i}$, $i = 1, 2, \ldots, n$, with Hamiltonian

$$H = H_2(x, y) = \tfrac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}}, \qquad x_0 = x_{n+1} = 0,$$

yielding

$$\dot{x}_i = y_i, \qquad \dot{y}_i = e^{x_{i-1} - x_i} - e^{x_i - x_{i+1}}, \qquad i = 1, \ldots, n.$$

H. Flaschka made the important discovery [22] that if one introduces the change of coordinates

$$a_i = \tfrac{1}{2} e^{\frac{1}{2}(x_i - x_{i+1})}, \quad i=1,\dots,n-1, \qquad b_i = \tfrac{1}{2} y_i, \quad i=1,\dots,n,$$

then the above differential equations go over into (1.1) and moreover can be written as matrix differential equations in the following Lax isospectral form

$$\dot{L} = [P, L].$$

Here $[P, L] = P \cdot L - L \cdot P$, L, P being the matrices

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & b_n \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & -a_{n-1} & 0 \end{bmatrix}.$$

In addition, if one transforms the standard Poisson bracket $\{f, g\}$ in (x, y) coordinates into (a, b) coordinates, imposing $\Sigma b_i = \text{constant}$, one finds,

$$\{f, g\} = \dot{\Sigma}(a_{i-1} f_{a_{i-1}} - a_i f_{a_i}) g_{b_i} + \dot{\Sigma}(g_{b_i} - g_{b_{i+1}}) a_i f_{a_i},$$

where the $\cdot$ means to omit terms with undefined elements, i.e. terms involving a_0, a_n, b_{n+1}. One also finds [21],

$$\{\operatorname{tr} L^j, \operatorname{tr} L^k\} = 0 \qquad \text{for } j, k = 2, 3, \dots, n,$$

i.e., the Toda system, which has the $2n-2$ degrees of freedom, $b_1, \dots, b_{n-1}$, $a_1, \dots, a_{n-1}$, has $n-1$ linearly independent integrals $H_j = \frac{1}{j} \operatorname{tr} L^j$, $j = 2, \dots, n$ in involution, and hence is completely integrable. The flow associated with H_j is also described by an isospectral deformation equation.

If instead of restricting $L = \{L_{ij}\}$ to be a symmetric matrix such that $L_{ij} = 0$ if $|i-j| > 1$, we allow it to be an 'm-ogonal' symmetric matrix, i.e., a matrix such that $L_{ij}=L_{ji}$, $L_{ij}=0$ if $|i-j|>m$, we can get new isospectral systems. Simply define P to be a skew-symmetric matrix whose strictly upper triangular part agrees with the 'm-ogonal' matrix L, and then the differential equation $\dot{L}=[P,L]$ defines a generalized Toda system of P. van Moerbeke, including (1.1) as a special case. It is these systems which we study as regards their Poisson structure, isospectral properties, and the involutive character of their 'spectral' integrals L^j, $j=2,3,\dots,n$.

To this end, we introduce the group G of lower triangular matrices with nonzero diagonal entries. Its Lie algebra $\mathscr{L}$ is just the lower triangular matrices. Using the pairing

$$\langle E,F\rangle=\mathrm{tr}(EF),\qquad \mathrm{tr}(A_{ij})=\Sigma A_{ii}, \tag{2.1}$$

we may identify the dual of $\mathscr{L}$, $\mathscr{L}^*$, with the upper triangular matrices. Since $g\in G$ acts on $\mathscr{L}$ via conjugation, it naturally acts on $l^*\in\mathscr{L}^*$ through duality.

$$g\colon l^*\to[g^{-1}l^*g]_+, \tag{2.2}$$

where $[\ ]_+$ denotes the projection operation of setting all terms below the diagonal equal to zero. This is the co-adjoint representation of G. We denote an orbit of this action through $l^*\in\mathscr{L}^*$ by $\theta=\theta_{l^*}=\{[g^{-1}l^*g]_+|g\in G\}$. From (2.2), the tangent space of θ_{l^*} at l^*, $T_{l^*}\theta_{l^*}$ is described by

$$T_{l^*}\theta_{l^*}=\{[l^*,l]_+|l\in\mathscr{L}\}, \tag{2.3}$$

and the natural symplectic 2-form ω of Kostant-Kirillov [12], associated with the orbit space θ_{l^*} is

$$\omega([l^*,l_1]_+,[l^*,l_2]_+)(l_*)=\langle l^*,[l_1,l_2]\rangle=\langle[l^*,l_1]_+,l_2\rangle. \tag{2.4}$$

It is standard that ω is well defined by this relation. Let $A=[A_{ij}]$ be the running variable on $\mathscr{L}^*$, and suppose $H=H(A)=H([A_{ij}])$ is a function on $\mathscr{L}^*$. We may uniquely identify the gradient at A of the function $H=H(A)$ with respect to the pairing, $\nabla H_{|A}$, as an element of $\mathscr{L}$. And so we have

$$X(H)_{|A}=\langle X,\nabla H\rangle_{|A}, \tag{2.5}$$

with X a vector field on $\mathscr{L}^*$, and hence identifiable as an element of $C^\infty(\mathscr{L}^*,\mathscr{L}^*)$. In order to compute X_H, the Hamiltonian vector field associated with $H=H[A]$ through the symplectic structure ω, we use the general definition of X_H, $\omega(X_H,Y)\equiv Y(H)$. We apply it to the formula

$$\omega([A,-\nabla H]_+,[A,l]_+)=\langle[A,l]_+,\nabla H\rangle=[A,l]_+(H),$$

where $Y=[A,l(A)]_+$ in the above definition, which yields

$$X_H=[\nabla H,A]_+. \tag{2.6}$$

We note that the $H\in\{H|[A,\nabla H]_+=0$, for all $A\in\theta_B\}$ form an algebra, namely the algebra which characterize the invariants, or constants of the orbit θ

$=\theta_B$. The description of an orbit, θ_B, consists in finding these constants. We also note from (2.2, 4, 6), that the Poisson bracket $\{\ ,\ \}$ based upon ω is

$$\{H^{(1)}, H^{(2)}\}(A) \equiv \omega(X_{H^{(1)}}, X_{H^{(2)}})(A) = \langle A, [\nabla H^{(1)}, \nabla H^{(2)}]\rangle. \tag{2.7}$$

We now spezialize these considerations, and for that we introduce some notation. Let the shift operators ξ, ξ^{-1}, acting on R^n be defined by

$$(\xi v)_i = v_{i+1}, \quad (\xi^{-1} v)_i = v_{i-1}, \quad v = (v_1, \ldots, v_n)^T \in R^n, \tag{2.8}$$

where we define $v_i \equiv 0$ if $i \notin \{1. 2, \ldots, n\}$. We define $\xi^{\pm j} = (\xi^{\pm})^j$, $j = 0, 1, 2, \ldots$.

In addition, we associate with $a \in R^n$, the multiplication operator $a\cdot$, namely

$$(a \cdot v)_j = a_j v_j, \quad j = 1, 2, \ldots, n. \tag{2.9}$$

We now define the linear subspace $\mathscr{A}_{k,j}$ of operators A by

$$\begin{aligned} \mathscr{A}_{k,j} = \{A \mid A = &\sum_{k \leqq i \leqq j} a^{(i)} \cdot \xi^i, \\ &a^{(i)} = (a_0^{(i)}, a_1^{(i)}, \ldots, a_{(n-1-i)}^{(i)}, 0, 0, \ldots, 0)^T, a^{(i)} \in R^n\}. \end{aligned} \tag{2.10}$$

Note $a_i^{(j)} = A_{i,i+j}$, the $(i, i+j)$ entry in the matrix A. For example, if $A = \sum_{i \geqq 0} a^{(i)} \cdot \xi^i$, $B = \sum_{i \geqq 0} \xi^{-i}(b^{(i)} \cdot\ )$ then $\langle A, B\rangle = \sum_{i \geqq 0} (a_i, b_i)$, with $(\ ,\ )$ the scalar dot product in R_n, $\langle\ ,\ \rangle$, defined by (2.1). In the future, we shall omit the dot in $a \cdot \xi$ when there is no possibility of misunderstanding. Clearly $\mathscr{L} = \mathscr{A}_{-n,0}$, $G \subset \mathscr{A}_{-n,0}$, and $\mathscr{L}^* = \mathscr{A}_{0,n}$. Along with the grading inherent in the specification of the $\mathscr{A}_{k,j}$'s, we have the projections $P_{k,j}$ into the $\mathscr{A}_{k,j}$ defined by $P_{k,j}(\sum a^{(i)} \xi^i) = \sum_{k \leqq i \leqq j} a^{(i)} \xi^i$.

We shall restrict $A \in \mathscr{L}^*$, so that $A \in \mathscr{A}_{0,m}$, $0 < m \leqq n$, and we observe from (2.2) that $B \in \mathscr{A}_{0,m}$ implies $\theta_B \subset \mathscr{A}_{0,m}$. In addition $[\nabla H, B] \in \mathscr{A}_{-n,m}$. This we shall indicate by replacing the subscript $+$ in (2.6) with a superscript m, i.e., we shall write

$$X_H = [\nabla H, A]^m. \tag{2.11}$$

Here we define $[\ ]^m = P_{0,m}[\ ]$. Since from (2.11), $[\nabla H, A]^m$ only depends on the part of ∇H contained in $\mathscr{A}_{-m,0}$, we may as well restrict H to depend only on the $A_{i,j}$'s such that $0 \leqq j - i \leqq m$. Note that clearly $\mathscr{A}_{0,m}$ is not an orbit, for if $D \subset \theta_B$, $\operatorname{tr} B = \operatorname{tr} D$. This is a consequence of $H = \operatorname{tr} A$, $\nabla H = I$, implies $[\nabla H, A]^m = 0$ in (2.11), and so $\operatorname{tr} A$ is an orbit invariant. In general there must be more invariants and we refer the reader to [18], which discusses this problem for the special case of orbits of maximum possible dimensionality.

We compute (2.11) in coordinates, using the notations (2.8)–(2.10). We have

$$A = \sum_{k=0}^{m} a^{(k)} \xi^k, \ \nabla H = \sum_{j=0}^{m} \xi^{-j} \frac{\partial H}{\partial a^{(j)}}, \quad \text{where } \left(\frac{\partial H}{\partial a^{(j)}}\right)_s = \frac{\partial H}{\partial A_{s,s+j}}, \tag{2.12}$$

hence by (2.11),

$$X_H = [\nabla H, A]^m = \sum_{\nu=0}^{m} \sum_{j\geq 0} \left\{ \left[\xi^{-j} \left(\frac{H}{\partial a^{(j)}} \cdot a^{(\nu+j)} \right) \right] - \left[a^{(\nu+j)} \cdot \left(\xi^{\nu} \frac{\partial H}{\partial a^{(j)}} \right) \right] \right\} \xi^{\nu},$$

and so Hamiltonian's equations are, setting $\dot{A} = X_H$, and using (2.12),

$$\dot{a}_t^{(\nu)} = \dot{A}_{t,t+\nu} = \sum_{j\geq 0} \left\{ \frac{\partial H}{\partial A_{t-j,t}} A_{t-j,\nu+t} - A_{t,t+\nu+j} \frac{\partial H}{\partial A_{t+\nu,t+\nu+j}} \right\},$$

$$0 \leqq \nu \leqq m, \qquad 0 \leqq t \leqq n-1-\nu.$$

For example, if we set $m=1$, $A_{i,i}=b_i$, $A_{j,j+1}=a_j$, then the above expression yields for (2.11),

$$\dot{b}_i = (a_{i-1} H_{a_{i-1}} - a_i H_{a_i}), \qquad \dot{a}_i = a_i (H_{b_i} - H_{b_{i+1}}),$$

and since $\{F, H\} = X_H F$, we have

$$\{F, H\} = \Sigma (a_{i-1} H_{a_{i-1}} - a_i H_{a_i}) F_{b_i} + \Sigma a_i (H_{b_i} - H_{b_{i+1}}) F_{a_i},$$

which is nothing but the Poisson bracket of the Toda system discussed at the beginning of this section. In this case it is only necessary to impose the condition $\Sigma b_i = \text{constant}$ to specify an orbit, assuming none of the a_i's are zero, as the property of a_i being zero is orbit invariant.

We now make an important observation due to P. van Moerbeke in an unpublished version of [2] (for the periodic case, see Remark 2). We write every matrix $M = M^+ + M^0 + M^-$, with M^+ the strictly upper triangular part, M^0 the diagonal part, etc.

Theorem 1 (P. van Moerbeke). *Let L be the real symmetric matrix* $A + (A^+)^T$, $A \in \mathscr{A}_{0,m}$. *Then if* $H = H_f(A) = \operatorname{tr} f(L)$, *the Hamiltonian equation,* $\dot{A} = [\nabla H, A]^m$, *implies that L satisfies the Lax isospectral equation*

$$\dot{L} = [P, L], \quad \text{with } P = P_f = f'(L)^+ - f'(L)^-. \tag{2.13}$$

Moreover this implies that $\{H_f, H_g\} = 0$ *for all polynomial f, g, i.e., the* H_f *are in involution with respect to* $\{\ ,\ \}$.

Proof. We give a 'functional' version of P. van Moerbeke's proof in the unpublished version of [2]. In the course of the proof we shall show the necessary fact $[P, L] \in \mathscr{A}_{-m,m}$.

We compute, using the notation of (2.8)–(2.10),

$$\frac{\partial H}{\partial a^{(i)}} = \frac{\partial (\operatorname{tr} f(L))}{\partial a^{(i)}} = \left\langle f'(L), \frac{\partial L}{\partial a^{(i)}} \right\rangle = \begin{cases} \langle f'(L), \xi^i + \xi^{-i} \rangle, & 1 \leqq i \leqq m, \\ \langle f'(L), \xi^0 \rangle & i = 0 \end{cases}.$$

Hence if $f'(L)=\sum_{i=0}^{n}(f'(L))^{(i)}\xi^i+\sum_{i=1}^{n}\xi^{-i}\cdot(f'(L))^{(i)}$,

$$\frac{\partial H}{\partial a^{(i)}}=2(f'(L))^{(i)}-\delta_{i,0}(f'(L))^{(i)}, \qquad i=0,1,\ldots,m,$$

and thus

$$\nabla H=[2f'(L)^{-}+f'(L)^{0}]_{-m}, \tag{2.14}$$

where $[\;]_{-m}=P_{-m,0}[\;]$.

Now $0=[f'(L),L]=[f'(L)^{+}+f'(L)^{0}+f'(L)^{-},L]$, hence for $P=f'(L)^{+}-f'(L)^{-}$

$$[P,L]=[f'(L)^{+}-f'(L)^{-},L]=[L,f'(L)^{0}+2f'(L)^{-}].$$

Substituting $L=A+L^{-}$ we find

$$[L,f'(L)^{0}+2f'(L)^{-}]=[A+L^{-},f'(L)^{0}+2f'(L)^{-}],$$

and so the former equation implies $[P,L]_{+}=[A,f'(L)^{0}+2f'(L)^{-}]_{+}=[A,f'(L)^{0}+[2f'(L)]_{-m}]^{m}$. As a consequence of (2.14) the above yields $[P,L]_{+}=[A,\nabla H]^{m}$. By the skew-symmetry of P, we must have $[P,L]\in\mathscr{A}_{-m,m}$. We thus conclude

$$\dot{A}=[A,\nabla H]^{m} \quad \text{implies } \dot{L}=[P,L], \quad H=H_f.$$

From $\dot{L}=[P,L]$, $P=P_f$ when $H=H_f$, we have

$$\frac{d}{dt}H_g=\frac{d}{dt}(\operatorname{tr} g(L))=\langle g'(L)\cdot\dot{L}\rangle=\langle g'(L),[P,L]\rangle=\langle P,[L,g'(L)]\rangle=0,$$

as a consequence of $[L,g'(L)]=0$. Since $\frac{d}{dt}H_g=\{H_g,H_f\}$, we have proven the statement concerning the involution of the H_f's. This concludes the proof of Theorem 1. We refer the reader to [9, 10] for theorems analogous to Theorem 1, concerning other finite systems.

There is more information, namely recursion relations, or so-called Lenard relations to be gleaned from the companion identity to the fundamental relation $[f'(L),L]=0$, namely the stronger statement, $L^{j+1}=L^{j}\cdot L^{1}$, which we shall discuss in Sect. 4, but we go in the next section to the differential operator case. In preparation for the next section we mention immediate generalizations of the above discussion.

Remark 1. B. Kostant and B. Symes have independently generalized Theorem 1. We give an easily provable form of their generalization with a view towards the Kortweg-deVries type systems to be discussed in the next section.

Theorem (Kostant, Symes). *Let L be a Lie algebra which has the following vector space direct sum decomposition*

$$L=K+N,$$

with K, N Lie algebras. Let $\langle \cdot, \cdot \rangle$ *be a nondegenate pairing between L, and a vector space we shall identify with* L^*. *Therefore* L^* *has the direct sum decomposition*

$$L^* = K^\perp + N^\perp,$$

with the $\perp$ *being with respect to* $\langle \cdot, \cdot \rangle$. *We may thus naturally associate* $K^\perp \approx N^*$, $N^\perp \approx K^*$, *and therefore* $K^\perp$ *inherits the Kostant-Kirillov orbit symplectic structure when identified with* N^*. *Under this symplectic structure, the* Ad* *invariant functions on* L^*, *when restricted to* $K^\perp \approx N^*$, *form an involutive system, to be interpreted as a system of commuting integrals. Moreover if f is such a function, g* $= f|_{K^\perp}$ has the associated Hamiltonian vector field

$$X_g(s) = \pi_{K^\perp}(\mathrm{ad}^*_{\nabla_1 g} s), \qquad s \in K^\perp,$$

with $\pi_{K^\perp}$ *projection onto* $K^\perp$ along $N^\perp$, and $\nabla_1 g(s) \in N$ *is the gradient of g viewed as a function of* $K^\perp \approx N^*$. *In addition the equation* $\dot{s} = X_g(s)$ *has the following Lax form,*

$$\dot{s} = \mathrm{ad}^*_b s, \qquad b = -\pi_K \nabla f(s) \in K, \tag{*}$$

where π_K *is the projection onto K along N. In the event that* $\langle\ ,\ \rangle$ *is a symmetric pairing between L and itself, then* (*) *becomes*

$$\dot{s} = [s, b],$$

with [,] *the Lie bracket of L. In the case that* $s \in M \subset K^\perp$, *with M thought of as a submanifold of* N^*, *invariant under the co-adjoint action, then automatically* $X_g(s) \in T_s M$.

In the case of the generalized Toda systems, one takes L semi-simple, $\langle\ ,\ \rangle$ the Killing form, and one uses the Isawa decomposition. A filtration of co-adjoint invariant manifolds yield generalizations of the m-ogonal symmetric matrices, and are produced by the weighting of algebra elements into levels inherent in picking a root basis. The weighting also generates the Isawa decomposition. The generalizations of one-ogonal symmetric matrices automatically give rise to completely integrable systems. This fact was previously discovered by Bogoyavlensky [25]. In the case of the Kortweg-deVries systems, the algebra decomposition shall be particularly transparent.

Remark 2. We note that Theorem 1 may also be generalized to the case where the matrix A is thought of as being n-periodic, which is the case considered by P. van Moerbeke. In that case we may either think of

$$A = \sum_{0 \leq i \leq m} a^{(i)} \xi^i + \sum_{-m \leq i \leq -1} \xi^i \cdot a^{(i)},$$

$a^i = (\ldots, a^{(i)}_{-1}, a^{(i)}_0, \ldots)$, where $a^{(i)}_k = a^{(i)}_{k+n}$, or we may represent A by an $n \times n$ matrix in an obvious fashion. In either case, the above representation of A is nothing but the Fourier representation of the operator A. This is most easily seen by noting the operators ξ^{in} have "characters" as their eigenfunctions. It is this decomposition of A which will motivate the choice of an appropriate Lie algebra for the Kortweg-deVries type systems, and thus enable us to apply the considerations of Remark 1. This will be reported on elsewhere in a joint work with P. van Moerbeke and T. Ratu, along with a discussion of the spinning top [15] in this framework, which uses the Kac-Moody algebras.

3. The Generalized Korteweg-deVries Equations

The Korteweg-deVries equation for $q \in C_0^\infty(R)$,

$$q_t = 6\, q\, q_x - 2\, q_{xxx}, \tag{3.1}$$

where q describes the amplitude of water waves in a narrow channel of finite depth, has been described from many points of view by a hugh collection of investigators in recent years. Gardner [23] observed that it could be written in the following Hamiltonian form

$$\frac{d}{dt} q = J \frac{DH}{Dq},$$

where $J = \frac{d}{dx}$, $H = H_2(q) = \int_R (q^3 + \frac{1}{2} q_x^2)\, dx$, $\frac{DH}{Dq}$ being the directional derivative of H with respect to q. The system is called Hamiltonian precisely because J defines a Poisson bracket via

$$\{H(q), F(q)\} = \int_R \left(\frac{DH}{Dq} J \left(\frac{DF}{Dq} \right) \right) dx.$$

The system (3.1) was observed by Gardner et al. [23] to have a denumerable sequence of conserved integrals of motions with polynomial densities P_j, $H_j = \int_R P_j(q, q_x, \ldots, (\partial_x^{k(j)} q))\, dx$, $j = 2, 3, \ldots$ which moreover are in involution, i.e., $\{H_j, H_k\} = 0$, $j = 2, 3, \ldots$. Actually there is a previous integral $H_1 = \int q\, dx$, and since $J \frac{DH_1}{Dq} = 0$, we think of initially specifying our Hamiltonian phase space by constraining q such that $\int q\, dx = \text{constant}$. The existence of these integrals motivated Lax [24] to describe the Korteweg-deVries equation in the form

$$\frac{dL}{dt} = [B, L]$$

$$L = -\partial_x^2 + q(x, t), \qquad B = -4\, \partial_x^3 + 2(q\, \partial_x + \partial_x\, q\cdot),$$

and the differential equations $\frac{d}{dt} q = J \frac{DH_j}{Dq}$ can also be described in this form with $B = B_j$. In [1] Gelfand-Dikii showed that if

$$L = (-i\, \partial_x)^n + \sum_{j=0}^{n-2} q_j (-i\, \partial_x)^j,$$

then the equation $\frac{dL}{dt}=[B, L]$, for appropriate choice of B, is a Hamiltonian system with an infinite sequence of involutive polynomial integrals, in complete analogy with the case $L=-\partial_x^2+q(x,t)$. The B's can be described in a natural fashion. These are the generalized Korteweg-deVries equations.

Since we wish to apply the considerations of Sect. 2 to the generalized Korteweg-deVries equation, we need to define the appropriate formal Lie group G. To this end we introduce a commutative ring R with identity over the complex numbers, equipped with a derivation D. One defines the 'indefinite integrals', $I=R/DR$, i.e. R modulo DR, and we shall use $\doteq$ for the equality sign in I. If $\pi\colon R\to R/DR=I$ denotes the projection associated with I, we shall write $\pi(\varphi)$ as $\bar{\varphi}$, the I equivalence class of φ. We let $R[b_0,b_1,\dots,b_n]\subset R$ denote the differential ring of polynomials in $b_0, b_1, \dots, b_n$ and their derivatives, similarly for $I[b_0, b_1, \dots, b_n]$, $\pi[b_0, b_1, \dots, b_n]$. Providing the operations occurring take place in this smaller ring, this notation shall allow for a more detailed discription of events. We shall remind the reader of this point if it is pertinent.

We now define the ring of formal pseudo-differential operators to be the formal Laurent series in the variable ξ over the ring R:

$$\Phi=\{\phi=\sum_{-\infty<i\leqq N<\infty} a_i\,\xi^i \,|\, a_i\in R\},$$

with the rule of multiplication given by

$$\phi_1\circ\phi_2=\sum_{\nu\geqq 0}(\partial\xi)^\nu\phi_1\cdot(-iD)^\nu\phi_2. \tag{3.2}$$

The $\frac{\partial}{\partial\xi}$ indicates formal series differentiation, and the $\cdot$ indicates formal series multiplication. Also note that since the coefficients of the formal series are R elements, D extends automatically to Φ. In analogy to Sect. 2, we define $\mathscr{A}_{ij}=\left\{\sum_{s=i}^{j} a_s\,\xi^s \,|\, a_s\in R\right\}$, and in the same fashion define the projections $P_{i,j}$ onto the $\mathscr{A}_{i,j}$'s.

For motivation, it is useful to think of ξ as a real variable contained in a complex neighbourhood of ∞, as follows from the brief discussion in the introduction. We give a quick explanation of (3.2).

If R elements were just C^∞ functions of x, and $D=\frac{d}{dx}$, then the differential operator $P(-iD)$, with P a polynomial, acts on $e^{ix\xi}$ via $P(-iD)(e^{ix\xi})=P(\xi)\cdot e^{ix\xi}$; and clearly $[P_1(-iD)(P_2(-iD)(e^{ix\xi}))]=(P_1(\xi)\circ P_2(\xi))\cdot e^{ix\xi}$, with $\circ$ defined in (3.2) Thus Φ models the image of the algebraic isomorphism implicit in the above comments, $P(-iD)\mapsto P(\xi)$, and extends the image of the isomorphism. This also explains why $\circ$ is associative.

The care taken in the definition of Φ was in order to define the trace functional $\langle\ \rangle\colon \Phi\mapsto I$, for $\phi=\sum a_i\,\xi^i$, by means of

$$\operatorname{tr}\phi\equiv\langle\phi\rangle\doteq\bar{a}_{-1}\in I, \tag{3.3}$$

where $\bar{a}_{-1}$ denotes the I equivalence class of a_{-1}. We now state the fundamental result concerning $\langle\ \rangle$.

Theorem 2. *If* $[\phi_1, \phi_2] = \phi_1 \circ \phi_2 - \phi_2 \circ \phi_1$, *we have*

$$\langle [\phi_1, \phi_2] \rangle \doteq 0. \tag{3.4}$$

Proof. It's here that the duality between x, ξ, in Fourier theory plays a crucial role, as evidenced by its appearance in the rule of multiplication (3.2). It's immmediate from (3.2) that

$$[\phi_1, \phi_2] = \frac{\partial}{\partial \xi} \alpha + D\beta, \qquad \alpha, \beta \in \Phi,$$

and so projecting into $I = R/DR$ we have

$$\overline{[\phi_1, \phi_2]} \doteq \frac{\partial \bar{\alpha}}{\partial \xi}.$$

Clearly $\overline{[\phi_1, \phi_2]}$ can have no ξ^{-1} term, which, by (3.3), concludes the proof of Theorem 2.

We also remark that one can prove Theorem 2 by direct computation, which is useful for Sect. 4, but then one fails to see why our choice for $\langle\ \rangle$ is really unique. Note that if $\phi_1 = \sum a_i \xi^i$, $a_i \in R[b_0, \ldots, b_n]$, etc. for ϕ_2, the equality sign, $\doteq$, in (3.4) holds in $I[b_0, \ldots, b_n]$, and thus the residue coefficient of $[\phi_1, \phi_2]$ is an exact derivative in $b_0, \ldots, b_n$ elements.

As a consequence of (3.4) we make the important definition, analogous to (2.1) of $\langle\ ,\ \rangle$,

$$\langle \phi_1, \phi_2 \rangle \equiv \langle \phi_1 \circ \phi_2 \rangle. \tag{3.5}$$

By Theorem 2, $\langle\ ,\ \rangle$ is symmetric in its arguments, which is the crucial point in order that the operation $[\phi, \cdot\,]$ be skew-symmetric with respect to $\langle\ ,\ \rangle$. This played an important role in the computations of Sect. 2 (see Remark 1).

We now single out a 'submanifold' of Φ:

$$\Phi^{(n)} = \{\phi = \xi^n + \sum_{0 \leqq i \leqq n-2} a_i \xi^i \mid a_i \in R\}, \tag{3.6}$$

which we shall think of as being parametrized by $a_0, a_1, \ldots, a_{n-2}$. In order to define the resolvent of $\Phi^{(n)}$ elements we define Φ_λ as the ring over, $R_n = R_n[a_0, \ldots, a_{n-2}]$, of formal joint power series in the variables $(\lambda - \xi^n)^{-1}$, ξ. We then define the resolvent operator R_n,

$$R_n\colon \Phi^{(n)} \to \Phi_\lambda, \qquad R_n(\phi) \equiv \phi_\lambda,$$

by requiring

$$\phi_\lambda \circ (\lambda - \phi) = (\lambda - \phi) \circ \phi_\lambda = 1. \tag{3.7}$$

We compute ϕ_λ in the following way, writing $\phi=\xi^n+\hat{\phi}$, $\phi_\lambda\circ(\lambda-\phi)=\phi_\lambda\circ\{(\lambda-\xi^n)-\hat{\phi}\}=\phi_\lambda\circ(\lambda-\xi^n)-\phi_\lambda\circ\hat{\phi}=1$, hence

$$\phi_\lambda-\frac{\phi_\lambda\circ\hat{\phi}}{\lambda-\xi^n}=\frac{1}{\lambda-\xi^n}.$$

Thus ϕ_λ satisfies an equation of the form $(I-T)x=y$, with $x=\phi_\lambda$, $y=(\lambda-\xi^n)^{-1}$, $T(x)=\frac{x\circ\hat{\phi}}{(\lambda-\xi^n)}$, and so $\phi_\lambda=\sum_{j\geqq 0}T^j(\lambda-\xi^n)^{-1}$. From (3.2) it's easy to see that

$$\phi_\lambda=\sum\phi_{m,l}(-1)^{\frac{m}{n}}\xi^m(\xi^n-\lambda)^{-1-\frac{m+l}{n}},\quad m\geqq 0,\quad l\geqq 0,\quad \frac{m+l}{n}\text{ an integer,}\tag{3.8}$$

with $\phi_{m,l}\in R_n=R_n[a_0,a_1,\ldots,a_{n-2}]$, and so $\phi_\lambda\in\Phi_\lambda$. Using (3.8), we mimic the construction of R. Seeley [7] in a formal way, namely we define for 'nice' f,

$$f(\phi)\equiv[f(\lambda)\phi_\lambda]_{(\lambda,\xi^n)}.\tag{3.9}$$

If $h=h(\lambda)$, by $[h]_{(\lambda,\xi^n)}$ we mean the formal residue term of h at ξ^n, i.e., the coefficient of the $(\lambda-\xi^n)^{-1}$ term of the Laurent series in $(\lambda-\xi^n)$, computed by formally expanding $h=h(\lambda)$ about ξ^n. Since (3.9) necessitates expanding $f(\lambda)$ about $\lambda=\xi^n$, this places restrictions on the choice of f. In practice (3.8) is used to compute (3.9), and also it may be necessary to extend $\Phi\mapsto\hat{\Phi}$ to include $f(\Phi)$. As a simple example, by (3.8), (3.9), one easily computes fractional powers of $\phi\in\Phi^{(n)}$;

$$\phi^s=\sum\phi_{m,l}\Gamma^s_{m,l}\xi^m(\xi^n)^r=\sum_{l\geqq 0}\Delta_{l,s\cdot n}(\xi^n)^s\xi^{-l}\in\Phi,$$

$$r=s-\frac{l+m}{n},\quad \Gamma^s_{m,l}=\binom{s}{t},\quad t=\frac{m+l}{n},\quad \Delta_{l,s\cdot n}=\sum_m\phi_{m,l}\Gamma^s_{m,l}.$$

For the case $s=\frac{N}{n}$, $N\in Z$, but not a multiple of n, we have

$$\phi^{\frac{N}{n}}=\sigma^N\cdot\sum_{l\geqq 0}\Delta_{l,N}\xi^{N-l},\quad\text{and so }\langle\phi^{\frac{N}{n}}\rangle=\bar{\Delta}_{N+1,N}\cdot\sigma^N.\tag{3.10}$$

Here, in order to recover the freedom one enjoys in picking a branch of a root, one formally adjoins to R an element σ obeying the rule $\sigma^2=1$. One may think of $\sigma=\begin{Bmatrix}\text{sign}(\xi), & n\text{-even}\\ 1 & n\text{-odd}\end{Bmatrix}$ for the purpose of motivation. In addition we now have $\phi^{\frac{N}{n}}\in\hat{\Phi}$, $\hat{\Phi}$ an extension of Φ, $\Phi=\Phi\oplus\sigma\cdot\Phi$. Upon interpreting $\hat{\Phi}$ elements as Laurent series about ∞, the interpretation of σ becomes clear. Implicitly we have and shall be working with a modified R, i.e., $R\to R\oplus\sigma\cdot R$, hence $I\to I\oplus\sigma\cdot I$, but we shall not change notation. That the trace, $\langle\cdot\rangle$, has all of its usual properties follows from $\frac{\partial}{\partial\xi}\sigma=0$. In general for r a complex number

$\phi\in\Phi^{(n)}$, $\phi^r\in\sum_{\tau,s}\oplus(\sigma_\tau\circ\Phi\circ\xi^s)$, where the double sum in τ,s extends over the complex numbers, and the adjoined elements σ_τ obey the rule $\sigma_{\tau_1}\cdot\sigma_{\tau_2}=\sigma_{\tau_1+\tau_2}$. And so if r is not a real number, or real but either irrational, an integer, or less than $-\frac{1}{n}$, ϕ^r has no residue term and so $\langle\phi^r\rangle=0$.

We can also compute $\phi^{\frac{1}{n}}$, and hence $\phi^{\frac{N}{n}}=(\phi^{\frac{1}{n}})^N$, by the following, more natural procedure. Just look for $\phi^{\frac{1}{n}}$ to have the form $\phi^{\frac{1}{n}}=\xi+\sum_{i\geq 0} b_i\xi^{-i}$. Then define $\xi=\delta_1$, $\xi+\sum_{i=0}^{i=k} b_i\xi^{-i}=\delta_{k+2}$, $k=0,1,\dots$, and one computes the b_i inductively (and uniquely) by requiring $\delta_k^n=\phi+\mu_{n-k}$, $\mu_{n-k}\in\mathscr{A}_{-\infty,n-k}$. For assuming we have computed δ_k, and thus b_{k-2}, to compute b_{k-1}, we observe, using induction,

$$(\delta_k+b_{k-1}\xi^{-(k-1)})^n=(\phi+\mu_{n-k})+n\,b_{k-1}\xi^{n-k}+\nu,\quad \nu\in\mathscr{A}_{-\infty,n-k-1}. \tag{3.11}$$

Hence if $\mu_{n-k}=c_k\xi^{n-k}+\beta,\beta\in\mathscr{A}_{-\infty,n-k-1}$, just define $b_{k-1}=-(n^{-1})\cdot c_k$, and so we have computed δ_{k+1}. Let $\phi^{\frac{1}{n}}\mapsto\sigma\phi^{\frac{1}{n}}$ to obtain the 'branched' solution. Similarly one computes ϕ^{-1} by looking for a solution of the form $\xi^{-n}(1+\sum_{k\geq 0} d_k\xi^{-k})$, letting $\bar{\delta}_k=\xi^{-n}\left(1+\sum_{s=0}^{k} d_s\xi^{-s}\right)$, and requiring $\phi\bar{\delta}_k-1\in\mathscr{A}_{-\infty,-(k+2)}$. If we allowed division in our ring we could compute $\phi^{\frac{1}{n}}$ for $\phi=b^n\xi^n+\sum_{i=0}^{n-1} a_i\xi^i, b$ invertible; for in the above we would now have, (see (3.11)), $\phi^{\frac{1}{n}}=b\xi+\sum_{i\geq 0} b_i\xi^{-i}$, $b_{k-1}=-(nb)^{-1}\cdot c_k$, etc. for ϕ^{-1}. The reason we go through the artifice of the resolvent is to be found in the proof of Lemma 1, where we need a tool to compute variations in $\phi^{\frac{N}{n}}$.

We shall work with the formal Lie group $G=1+\mathscr{A}_{-\infty,-1}$, with formal Lie algebra $\mathscr{L}=\mathscr{A}_{-\infty,-1}$. Alternatively, we shall choose to represent $\mathscr{L}$ in the following form

$$\mathscr{L}=\{\sum_{j\geq 0}(\xi-iD)^{-j-1}a_j\,|\,a_j\in R\}, \tag{3.12}$$

where $(\xi-iD)^{-j-1}b\equiv\sum_{v\geq 0}\xi^{-j-1-v}\binom{j+v}{v}(iD)^v b$, $j\geq 0$. It is clear that the two definitions of $\mathscr{L}$ are the same: in fact,

$$\begin{aligned}\sum_{k\geq 0}a_k\xi^{-k-1}&=\sum_{j\geq 0}(\xi-iD)^{-j-1}b_j=\sum_{\substack{v\geq 0\\ j\geq 0}}\binom{j+v}{v}(iD)^v b_j\xi^{-1-j-v}\\ &=\sum_{k\geq 0}\left(\sum_{v=0}^{k}\binom{k}{v}(iD)^v b_{k-v}\right)\xi^{-k-1}.\end{aligned} \tag{3.13}$$

We must have

$$a_k = \sum_{v=0}^{k} \binom{k}{v} (iD)^v b_{k-v}, \quad \text{and} \quad b_k = \sum_{v=0}^{k} \binom{k}{v} (-iD)^v a_{k-v}, \tag{3.14}$$

the second relation in (3.14) being an easy consequence of the first. We shall denote by $\tilde{\mathscr{A}}_{k,j} = \left\{ \sum_{s=k}^{j} (\xi - iD)^s \cdot b_s | b_s \in R \right\}$, and by $\tilde{P}_{k,j}$ the projections onto $\tilde{\mathscr{A}}_{k,j}$. We define the dual of $\mathscr{L}$, $\mathscr{L}^* \hookrightarrow \mathrm{Hom}(\mathscr{L}, I)$ (over the complex numbers), to be the differential operators, i.e.,

$$\mathscr{L}^* = \left\{ \sum_{\infty > n \geqq i \geqq 0} a_i \xi^i | a_i \in R \right\} = \mathscr{A}_{0,\infty}. \tag{3.15}$$

For if $A = \sum_{i \geqq 0} a_i \xi^i \in \mathscr{L}^*$, $B = \sum_{j \geqq 0} (\xi - iD)^{-j-1} b_j \in \mathscr{L}$, then

$$\langle A, B \rangle \doteq \langle A \circ B \rangle \doteq \sum_{i \geqq 0} \overline{a_i b_i}, \tag{3.16}$$

as easily follows from (3.14), (3.2). Note this is in complete analogy with the formula following (2.10). In short, $[(\xi - iD)^{-j-1}]$ is due to ξ^j, i.e.,

$$(\xi^j)^* = [(\xi - iD)^{-j-1}], \qquad [(\xi - iD)^{-j-1}]^* = \xi^j, \qquad j \geqq 0, \tag{3.17}$$

where * denotes duality under $\langle \ , \ \rangle$. In words, 'total integration' $j+1$ times is dual to 'differentiation' j times. We now are in a position to state the main theorem of this paper.

Theorem 3. *Let θ_B be the orbit through B of the co-adjoint action of G on $\mathscr{L}^*$, i.e., $\theta_B = \{[g^{-1} B g]_+ | g \in G\}$, where + denotes projection into $\mathscr{A}_{0,\infty}$. If $B \in \mathscr{A}_{0,m}$, then so is $A \in \theta_B$, and we may write $A = \sum_{i=0}^{m} a_i \xi^i$. Then the (formal) Kostant-Kirillov symplectic form ω on θ_B at the point A is given by*

$$\omega([A, l_1]_+, [A, l_2]_+) \doteq \langle A, [l_1, l_2] \rangle \doteq \langle [A, l_1]_+, l_2 \rangle,$$

with $l_1, l_2 \in \mathscr{L}$, hence $[A, l_1]_+ \in T_A \theta_B$. Let $H \doteq H(A) \doteq \bar{P}(a)$, $P(a)$ a polynomial in a_i, $i = 0, 1, \ldots, m$, and its derivatives. Then the Hamiltonian vector-field X_H induced by ω is given by

$$X_H = [\nabla H, A]^{m-2} \tag{3.18}$$

with $[\ ]^n = P_{0,n}[\ ]$, and

$$\nabla H = \sum_{j=0}^{m} (\xi - iD)^{-j-1} \frac{DH}{Da_j}. \tag{3.19}$$

In the above $\frac{DH}{Da_j}$ is the formal variational derivative of $H \doteq H(a)$ with respect to a_j. In addition, the (formal) Poisson bracket, based on ω, $\{ \ , \ \}$, is given by

$$\{H, F\} \doteq \langle A, [\nabla H, \nabla F]\rangle. \tag{3.20}$$

Finally, this Poisson structure agrees with the structure of Gel'fand-Dikii [1] *if we take* $a_m = 1$, $a_{m-1} = 0$. *We can do this as* a_m, a_{m-1} *are orbit invariants. In general we can think of the Hamiltonian structure as being parametrized by the invariants* a_m, a_{m-1}. *At this point we may restrict our differential ring* R *to* $R_m = R[a_0, a_1, \ldots, a_m]$, *and correspondingly* $I_m = I[a_0, \ldots, a_m]$.

Before we proceed to the proof, which entails making the above statements rigorous, we must clarify some concepts. The setting for these concepts is the formal variational calculus of Gel'fand-Dikii, and we give our own version of some necessary aspects of it, referring the read to [1, 4] for amplification and further references. Given the group $G = 1 + \mathscr{A}_{-\infty, -1}$, we define curves on G as polynomial maps from the real numbers to G, i.e., functions of the form $g(t) = 1 + \sum_{i=0}^{N} t^i l_i$, $l_i \in \mathscr{A}_{-\infty, -1}$, and we define the derivation $\frac{d}{dt}\Big|_b$ by $\frac{dg(t)}{dt}\Big|_b = \sum_{i=0}^{N} i b^{i-1} l_i$. We shall always employ such a definition in differentiating polynomials. Then $TG|_{g_0}$ equals the linear span of elements of the form $\frac{dg(0)}{dt}$, with $g(t)$ a curve and $g(0) = g_0$. Given the conjugation action, $g\colon g_1 \to g g_1 g^{-1}$, then $\mathrm{Ad}\, g\colon \mathscr{L} \to \mathscr{L}$ is (well) defined by $\mathrm{Ad}\, g(l) = \frac{d}{dt}(g g_1(t) g^{-1})|_{t=0}$, with $g(t)$ a curve, $g_1(0) = 1$, $\frac{d}{dt} g_1(0) = l$, and so $\mathrm{Ad}\, g(l) = g l g^{-1}$. Through the inner product $\langle\ ,\ \rangle$, we have defined $\mathscr{A}_{0,\infty} = \mathscr{L}^* \hookrightarrow \mathrm{Hom}(\mathscr{L}, I)$, and so it makes sense to compute the co-adjoint action of G in $\mathscr{L}^*$. Let $l^* \in \mathscr{L}^*$, and we compute $\mathrm{Ad}^* g(l^*)$ as follows:

$$\begin{aligned}\mathrm{Ad}^* g(l^*) &= \langle l^*, \mathrm{Ad}\, g(l)\rangle = \langle l^*, g l g^{-1}\rangle = \langle g^{-1} l^* g, l\rangle \\ &= \langle [g^{-1} l^* g]_+, l\rangle, \quad \text{and so } \mathrm{Ad}^* g(l^*) = [g^{-1} l^* g]_+.\end{aligned}$$

Here we have used the symmetry of $\langle\ ,\ \rangle$, and the injective character of $\mathscr{L}^* \hookrightarrow \mathrm{Hom}(\mathscr{L}, I)$. We have also shown the necessary fact that $\mathscr{L}^*$ is invariant under the co-adjoint action. So an orbit through B of the co-adjoint action on $\mathscr{L}^*$ is of the form

$$\theta_B = \{[g^{-1} B g]_+ \mid g \in G\},$$

and since

$$\frac{d}{dt}[g(t)^{-1} B g(t)]_+|_{t=0} = [B, l]_+$$

if g is a curve such that $g(0) = 1$, $\frac{dg}{dt}(0) = l$, we find using our previous notions

$$T_B \theta_B = \{[B, l]_+ \mid l \in \mathscr{L}\}.$$

It is also necessary to enlarge the differential ring (R, D) through the use of its integrals $I = R/DR$. First note that I has a natural additive structure, and we by edict give it a free abelian multiplicative structure, denoting this by $I \mapsto \tilde{I}$. In addition, we adjoin to $\tilde{I}$, without changing notation, a multiplicative identity element. Thus $\tilde{I}$ equals finite sums of finite products of I elements and the identity element. We now define the enlarged ring of formal products and sums

$$\hat{R}=\left\{\sum_{i=0}^{N<\infty} c_i\cdot r_i \equiv \sum_{i=0}^{N<\infty} r_i c_i \,|\, c_i\in\tilde{I}, r_i\in R\right\},$$

with $\hat{R}$ getting its ring and module structure by declaring it to be a two sided Abelian $\tilde{I}$ and R module. Of course $\hat{R}$ is a differential ring, derived from R by considering the $\tilde{I}$ elements as constants with respect to D. Specifically define $\hat{D}$: $\hat{R}\mapsto\hat{R}$ as follows: $\hat{D}(c\cdot r)=c\cdot Dr$, with $c\in\tilde{I}$, $r\in R$, and we just extend $\hat{D}$ to $\hat{R}$ by linearity, yielding the differential ring $(\hat{R},\hat{D})$. We have the enlarged integral class $\hat{I}=\hat{R}/\hat{D}\hat{R}$, which may be identified with the *ring* $\tilde{I}\setminus\{\text{identity element}\}$. Similarly we have the projection $\hat{\pi}$: $\hat{R}\mapsto\hat{I}$, and $\hat{\Phi}$, which is the formal pseudo-differential operators over the ring $\hat{R}$. We may apply this construction to $R[a_0,a_1,\dots,a_m]=R_m$, etc.

Functions on θ_B, R_B and I_B respectively shall just be the restrictions of R_m, $I_m=I_m[a_0,a_1,\dots,a_m]$ respectively to $a_0,\dots,a_m$ such that $A=\sum_{i=0}^{m} a_i\xi^i\in\theta_B$, and similarly for $\hat{R}_B$, $\hat{I}_B$. Hence they shall just be composed of the 'coordinates' on θ_B.

We define the vector fields on θ_B:

$$\chi(B)\equiv\left\{\left[A,\sum_{j=1}^{m-1}(\xi-iD)^{-j-1}\sigma_j\right] \,|\, A\in\theta_B, \sigma_j\in\hat{R}_B\right\}.$$

The vector fields $X(B)$ act on R_B, I_B, for if $X\in\chi(B)$, $f\in R_B$ or I_B, $A\in\theta_B$, define $X(f)|_A=\frac{d}{dt}f(c(t))|_{t=0}$, with $c(t)$ a curve such that $c(0)=A$, $\frac{dc}{dt}(0)=X|_A$. As before, we use the usual definition for differentiating a polynomial. Since $\chi(B)$ acts on I_B, it acts on $\tilde{I}_B$ through the product rule, and hence also on $\hat{R}_B$ through the same rule. This definition is consistent with defining $\chi(B)$'s action on $\hat{R}_B$ directly through differentiating curves. That all these operations are well-defined is obvious. As usual one defines for $X, Y\in\chi(B)$, $[[X,Y]]=X(Y(f))-Y(X(f))$, $f\in R_B$, $\hat{I}_B$, $\hat{R}_B$, etc. Then one verifies in the standard fashion that $[[fX,Y]]=f[[X,Y]]-(Yf)X$. As a consequence of the above definition of $X(f)$, the product rule of differentiation for $\frac{d}{dt}$, and the Jacobi identity for [,], one computes $[[[A,l_1(A)]_+,[A,l_2(A)]_+]]=[A,[l_1,l_2]_+ +(X\,l_2-Y\,l_1)]_+\in\chi(B)$, where $X=[A,l_1]_+$, $Y=[A,l_2]_+$, $l_1=\sum_{j=1}^{m-1}(\xi-iD)^{-j-1}\sigma_j$, etc. for l_2, and $Y(l_1)=\sum_{j=1}^{m-1}(\xi-iD)^{-j}(Y(\sigma_j))$, etc. for $X(l_2)$.

Now we defines covariant k-tensors on θ_B as multilinear maps on $[\chi(B)]^k$ into $\hat{I}_B$, (remember that $\hat{I}_B$ is a ring), multilinear over $\hat{I}_B$, and similarly for differential forms. Then one defines the exterior derivative d using the intrinsic definition:

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})$$
$$+ \sum_{1 \leqq i < j \leqq n} (-1)^{i+j} \omega([[X_i, X_j]], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}).$$

This definition immediately implies that d preserves differential forms, and $d^2 = 0$.

Of course, one may apply these considerations to a more general context, for instance to the 'manifold' $\mathscr{L}^*$. If we parametrize $\mathscr{L}^*$ by $a_0, a_1, \ldots$, i.e. by describing $A^* \in \mathscr{L}^*$ by $A = \sum_{i=0}^{N} a_i \xi^i$, $N < \infty$, then if $H(A) \in \hat{I}_+$, $I_+ = R_+/DR_+$, $R_+ = R_+(a_0, a_1, \ldots)$, the differential ring generated by $a_0, a_1, \ldots$, then the above discussion yields

$$dH(X) \doteq X(H), \qquad X \in X(\mathscr{L}^*),$$

with $X(\mathscr{L}^*)$ being vector fields on $\mathscr{L}^*$ of the form $\sum_{i=0}^{N} X_i \xi^i$, $N < \infty$, $X_i \in \hat{R}_+$. One then defines the gradient of H with respect to $\langle\ ,\ \rangle$, ∇H, (uniquely) by requiring

$$dH(X) \doteq \langle \nabla H, X \rangle, \quad \nabla H \in \hat{\mathscr{L}}. \quad \text{(Note if } H \in I_+,\ \nabla H \in \mathscr{L}.)$$

This implies by (3.16) that

$$\nabla H = \sum_{j \geqq 0} (\xi - iD)^{-(j+1)} \frac{DH}{Da_j}, \qquad \frac{D}{Da_j} = \sum_{k \geqq 0} (-D)^k \frac{\partial}{\partial(D^k a_j)}.$$

Note $\frac{D}{Da_j}$ is a derivation over $\hat{I}_+$.

See [4] for amplication concerning related matters. In the above case the identity $d^2 H = 0$ just says

$$d^2 H(Z, Y) \doteq d(dH)(Y, Z) \doteq \langle Z, \partial(\nabla H(Y)) \rangle - \langle Y, \partial(\nabla H(Z)) \rangle = 0.$$

(since the $[Y, Z]$ term cancels out), where $\partial = \frac{\partial}{\partial A}$ is just the directional derivative, i.e.

$$\partial(\nabla H)|_A = \frac{d\nabla H_A}{dt}(c(t))|_{t=0},$$

$c(t)$ a curve such that

$$c(0) = A, \qquad \left.\frac{dc(t)}{dt}\right|_{t=0} = Y\Big|_A.$$

Thus $\partial(\nabla H)$ is a symmetric operator with respect to $\langle\ ,\ \rangle$. In coordinates we have the operator identy $\frac{\delta}{\delta a_i}\frac{DH}{Da_j}(\cdot) = \frac{\delta}{\delta a_j}\frac{DH}{Da_i}(\cdot)$, all i, j. Note in particular if $H \in I_B$

$$\nabla H = \sum_{j=0}^{m-2} (\xi - iD)^{-j-1} \frac{DH}{Da_j}, \qquad A \in O_B,$$

as a_m, a_{m-1} are orbit invariants, which yields formula (3.19). Now we are prepared to give proof of Theorem 3.

Proof of Theorem 3. We now return to θ_B, and note that the form ω given in the statement of the theorem is automatically multilinear over $\hat{I}_B$, hence a differential form. One checks by a straightforward computation, like the one which went into the computation of $[[X, Y]]$, that $d\omega = 0$. This involves using the formula for ω, $d\omega$, $[[X, Y]]$, the product rule of differentiation, and the Jacobi identity for $[\ ,\ ]$ twice. From the definition

$$\omega(X_H, Y)|_A \equiv dH(Y)|_A = \langle \nabla H, Y \rangle|_A, \qquad A \in \theta_B, \quad H \in I_B,$$

and the definition of ω, one immediately verifies $X_H = [\nabla H, A]^{m-2}$. Note that $X_H \in \chi_B$, as it must be. To show that $\{H, F\} \equiv \omega(X_H, X_F) = \langle A, [\nabla H, \nabla F] \rangle$, $H, F \in I_B$, is a Poisson bracket as stated in the theorem, it is only necessary to verify the Jacobi identity, but that is an immediate consequence of $d\omega = 0$, $d^2 H = 0$. At this point we mention that, as in Sect. 2, one can give an easy direct proof of the Jacobi identity for $\{\ ,\ \}$ on $\mathcal{L}^*$, which uses only the symmetry of $\langle\ ,\ \rangle$, the Jacobi identity for $[\ ,\ ]$, and that $\delta(\nabla H)$ is a symmetric operator with respect to $\langle\ ,\ \rangle$, [26]. However, we preferred to prove the identity using $d\omega = 0$, the precise statement of which we felt was worth going through some extra definitions. The identity $d\omega = 0$ highlights the importance of the orbit, θ_B.

Finally we prove that $\{\ ,\ \}$ is indeed the bracket of Gel'fand-Dikii. We first note that a_m, a_{m-1} are orbit invariants, and hence may be set equal to $1, 0$, respectively. In general, however, the bracket $\{\ ,\ \}$ will depend on a_m, a_{m-1}, as the following computation will make clear. We shall compute (3.18) explicitely, using (3.2), and thus finish the proof of Theorem 3. First observe, setting $\frac{DH}{Da_j} = H_{a_j}$, that

$$\nabla H = \sum_{j \geq 0}^{m-2} (\xi - iD)^{-j-1} H_{a_j} = \sum_{\substack{j \geq 0 \\ s \geq 0}}^{j=m-2} \xi^{-j-1-s} \binom{j+s}{s} (iD)^s H_{a_j},$$

hence

$$X_H = [\nabla H, A]^{m-2} = \left\{ \begin{matrix} \left(\Sigma\, \xi^{-j-1-s} \binom{j+s}{s} ((iD)^s H_{a_j}) \right) \circ (\Sigma\, a_k \xi^k) \\ -(\Sigma\, a_k \xi^k) \circ (\Sigma\, \xi^{-j-1-s} \binom{j+s}{s} ((iD)^s H_{a_j}) \end{matrix} \right\}^{m-2}$$

$$= \sum_{r=0}^{m=2} \left[\begin{matrix} \Sigma \binom{j+s}{s} \binom{j+s+v}{v} ((iD)^s H_{a_j})((iD)^v a_k) \\ -\Sigma \binom{j+s}{j} \binom{k}{v} (-1)^v a_k ((iD)^{v+s} H_{a_j}) \end{matrix} \right] \cdot \xi^{-j-1-s-v+k}$$

(with $-j-1-s-v+k=r\geqq 0$, $v+s=\mu$) and using standard binomial identies we have,

$$=\sum_{r=0}^{m-2}\left[\begin{array}{l}\Sigma\binom{j+\mu}{\mu}\binom{\mu}{s}((iD)^s H_{a_j})\cdot((iD)^{\mu-s}a_{r+j+\mu+1})\\ -\Sigma\binom{r+j+\mu+1}{v}\binom{j+s}{s}(-1)^v a_{r+j+\mu+1}((iD)^\mu H_{a_j})\end{array}\right]\xi^r$$

$$=\sum_{r=0}^{m-2}\left[\begin{array}{l}\Sigma\binom{j+\mu}{\mu}((iD)(H_{a_j}\cdot a_{r+j+\mu+1}))\\ -\Sigma\binom{r+\mu}{\mu}a_{r+j+\mu+1}((-iD)^\mu H_{a_j})\end{array}\right]\cdot\xi^r=\sum_{r=0}^{m}X_H(a_r)\xi^r. \tag{3.21}$$

The sum in the brackets is understood to extend over all nonnegative integers with the proviso that only terms with a_j, $j=0, 1, \ldots, m$, can appear. This formula agrees with the formula of Gel'fand-Dikii [1] if we set $a_m=1$, $a_{m-1}=0$, i.e., in the setting of $\Phi^{(m)}$, (3.6), and so Theorem 3 is proven.

Remark 3. In fact in [1], Gel'fand-Dikii arrived at this formula from computational considerations which will become apparent in Theorem 4, and then they proceeded to verify by a difficult computation that (3.21) in fact yielded a Poisson bracket via

$$\{G, H\}\equiv\left.\frac{dG}{dt}\right|_H, \quad \text{with } H \text{ the Hamiltonian in (3.21).}$$

Every vector field, when viewed as acting on I_B, can be uniquely written $X(F)\doteq\Sigma\gamma_i\frac{DF}{Da_i}$, $F\in I_B$. So (3.21) just says that every Hamiltonian vector X_H when acting on I_B is of the form

$$X_H=\sum_{i=0}^{m-2}\left[(\mathscr{I})\cdot\frac{DH}{Da}\right]_i\cdot\frac{D}{Da_i},$$

where $\frac{D}{Da}=\left(\frac{D}{Da_0}, \ldots, \frac{D}{Da_{m-2}}\right)^T$, $\mathscr{I}=\Delta-\Delta^*$.

In the above Δ is the $(m-1)\times(m-1)$ matrix differential operator with components

$$\Delta_{rs}=\sum_\gamma\binom{\gamma+r}{r}a_{\gamma+1+r+s}(-iD)^\gamma, \quad 0\leqq r\leqq m-2, \quad 0\leqq s\leqq m-2,$$

Δ^* is the dual of Δ with respect to the natural inner product on $R_m^{(m-1)}$,

$$v, w\in R_m^{(m-1)}, \quad (v, w)\doteq\sum_{i=0}^{m-2}\overline{v_i w_i}\in I, \quad (\text{— is } \textit{not} \text{ complex conjugation})$$

and the remark after (3.21) concerning summation applies above. For $m=2$, $A=\xi^2+a_0$, $X_H=-2iD\cdot\frac{DH}{Da_0}$, which yields the well-known Poisson structure of

Gardner for the Korteweg-deVries equation. This was mentioned at the beginning of the section. Equivalently the Poisson bracket is given by

$$\{G, H\} \doteq X_H G \doteq \left(\Delta\left(\frac{DH}{Da}\right), \frac{DG}{Da}\right) - \left(\frac{DH}{Da}, \Delta\left(\frac{DG}{Da}\right)\right).$$

Remark 4. It is interesting to study the orbit of $\xi^m + \sum_{i=0}^{m-2} a_i \xi^i$ under the group G. We claim that $\bar{a}_{m-2}, \bar{a}_{m-3}$ are always orbit invariants. A typical element of G is of the form $1-x$, $x \in \mathscr{A}_{-\infty,-1}$, i.e., $x = \sum_{j \geqq 1} x_j \xi^{-j}$, and so we compute (using (3.2)) that under the action of G

$$\begin{aligned}\delta A &= ((1-x)A(1+x+x^2+\dots)]_+ - A = [[A, x]_+ \cdot \{1+x+x^2+\dots\}]_+ \\ &= (c_1 D x_1)\xi^{m-2} + (c_2 x_1 D x_1 + c_3 D^2 x_1 + c_4 D x_2)\xi^{m-3} + \eta, \qquad \eta \in \mathscr{A}_{0,m-4},\end{aligned}$$

with c_1, c_2, c_3, c_4, constants depending on m. The above shows that δa_{m-2}, $\delta a_{m-3} \in DR$, and so the assertion is proven. In fact more is true, namely $\langle A^{\frac{s}{m}} \rangle$, $1 \leqq s \leqq m-1$ are also orbit invariants, which in particular implies the above assertion. This will be proven in Corollary 1 at the end of this section. In the case of the Korteweg-deVries equation, an orbit of $-D^2+q$ is specified by $\bar{q}$ $=$constant, and so the apparent one-dimensional degeneracy of the Gardner structure comes from the representation of an orbit by q. This phenomena is well known in the Toda system, where as previously discussed in Sect. 2, one has the similar orbit relation, trace $A=$constant.

Remark 5. Suppose we had instead defined

$$\Phi^{(m)} = \left\{\xi^m + \sum_{j=0}^{m-2} (a_j \xi^j + (\xi - iD)^j a_j) | a_j \in R\right\}, \qquad m < \infty,$$

i.e. we coordinize $\Phi^{(m)}$ by these new a_j's. Then as our arguments have shown, (3.10) defines a Hamiltonian structure. Of course we have to compute ∇H, as in (3.18), which means we must compute the dual of the $\eta_j = \xi^j + (\xi - iD)^j$, $0 \leqq j \leqq \infty$ in $\mathscr{L}$. The computation of the dual of ξ^j depended on inverting the first relation in (3.14), but in fact, it was only necessary to invert that linear, strictly triangular relation up to degree $m+1$ to compute the Hamiltonian structure from (3.18). This is always a trivial matter, since m is finite, and the relation to be inverted is upper triangular. In the above manner we can compute the Hamiltonian structure of $\Phi^{(m)}$ in the coordinates of the 'formally self-adjoint operators', i.e., with the above coordinization of $\Phi^{(m)}$. Although the results of this section and the next do not depend on the coordinization of $\Phi^{(m)}$, certainly the formulas involving coordinates are coordinate dependent. In the case of the formally self-adjoint operators, we have extra reality conditions if we require the a_j's to be real. For instance, the Poisson bracket and traces, (3.10), must then be real, that is up to an inessential factor. For the rest of the paper, we shall mean either of the two coordinatizations by our expression $\Phi^{(m)}$, unless otherwise stated. For the 'self-adjoint' case, we have for $\mathscr{J}$, the symplectic matrix operator, for $m=3$,

$\mathscr{J}=\begin{bmatrix}0, & D\\ D, & 0\end{bmatrix}$, which applies to the Boussinesq equation, see [5, 6]. For $m=4$, one can show

$$\mathscr{J}=\begin{bmatrix}0 & 0 & D\\ 0 & -1/2\,D & 0\\ D & 0 & k\end{bmatrix}, \qquad k=-1/2\,D^3+a_2\,D+D\,a_2\,\cdot, \qquad \text{see [5].} \tag{3.22}$$

Remark 6. If we had chosen the coefficient ring of Φ to be $ML(n, R)$, the ring of $n\times n$ matrices over R, then we would have to redefine for $\phi\in\Phi$, (see (3.3)). $\operatorname{tr}\phi \doteq$ matrix trace $\bar{a}_{-1}\in I$. This extends definition (3.3). All the constructions and arguments of this section apply to this case. For instance, analogous to formula (3.17), we have that $(E_{ij}\,\xi^k)^*$, where E_{ij} is the matrix with its i, j component unity, all other elements zero, equals $(\xi-i\,D)^{-k-1}\,E_{ji}$. The H_{a_k} is the matrix such that $[H_{a_k}]_{jl}=\dfrac{DH}{D\,a_k^{(lj)}}$, which would have to be substituted in (3.21). Also 1 should be interpreted to mean the $n\times n$ identity matrix. In addition, one may specialize the coefficients of A to lie in some ring of matrices, which amounts to changing the base ring to that ring of matrices. As an example we note that the orbit symplectic structure associated with the operator $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\xi+\begin{bmatrix}0 & q\\ r & 0\end{bmatrix}$ is, up to a constant, the Darboux structure, i.e., the Hamiltonian vector fields $X_H=\dfrac{DH}{Dq}\cdot\dfrac{D}{Dr}-\dfrac{DH}{Dr}\cdot\dfrac{D}{Dq}$.

We now pursue the result analogous to Theorem 1, but first some brief observations. Suppose $H\doteq\langle A^\nu\rangle$, $\nu=\dfrac{N}{n}$, N an integer. We need to compute ∇H. First we write $A=[\lambda^\nu\,\phi_\lambda]_{(\lambda,\,\xi^n)}$, by (3.19), and so $H\doteq\langle[\lambda^\nu\,\phi_\lambda]|_{(\lambda,\,\xi^n)}\rangle$. In addition, by (3.7) we have $\delta\varphi_\lambda=\phi_\lambda\circ\delta A\circ\phi_\lambda$, $\dfrac{d\varphi_\lambda}{d\lambda}=-\varphi_\lambda^2$, with δ indicating an increment, and so

$$\begin{aligned}\delta H &\doteq \langle[\lambda^\nu\,\phi_\lambda\circ\delta A\circ\phi_\lambda]_{(\lambda,\,\xi^n)}\rangle \doteq \langle[\lambda^\nu\,\phi_\lambda^2]|_{(\lambda,\,\xi^n)},\,\delta A)+\langle\partial_\xi\{\cdot\}\rangle\\ &=\langle[\lambda^\nu\,\phi_\lambda^2]_{(\lambda,\,\xi^n)},\,A\rangle=\left\langle\left[-\left(\frac{\partial\phi_\lambda}{\partial\lambda}\right)\lambda^\nu\right]_{(\lambda,\,\xi^n)},\,\delta A\right\rangle\\ &=\left\langle\left[\frac{\partial\lambda^\nu}{\partial\lambda}\cdot\phi_\lambda\right]_{(\lambda,\,\xi^n)}+[\partial_\lambda\{\cdot\}]_{(\lambda,\,\xi^n)},\,\delta A\right\rangle=\langle[\nu\,\lambda^{\nu-1}\,\phi_\lambda]_{(\lambda,\,\xi^n)},\,\delta A\rangle\\ &=\langle\nu\,A^{\nu-1},\,\delta A\rangle.\end{aligned}$$

In the above we have repeatedly used that a perfect derivative can have no residue term. We thus have

$$\nabla\langle A^\nu\rangle=(\nu\,A^{-1})_{-(m-1)}, \qquad A\in\Phi^{(m)}, \tag{3.23}$$

where $(\cdot)_{-n}=\tilde{P}_{-n,\,0}(\cdot)$, and so can conclude:

Lemma 1.

$$\nabla\langle A^{\frac{N+m}{m}}\rangle=\left(\frac{N+m}{m}\right)(A^{\frac{N}{m}})_{-(m-1)}. \tag{3.24}$$

The computational equivalents of Lemma 1 can be found in Gel'fand-Dikii [1], but of course proved and viewed in a different fashion. We can now prove the analog of Theorem 1, namely

Theorem 4 (Gel'fand-Dikii). *If* $H=H^N=\frac{m}{N+m}\langle A^{\frac{N+m}{m}}\rangle$, $A\in\Phi^{(m)}$, *then the Hamiltonian equation,* $\dot{A}=[\nabla H, A]^{m-2}$, *implies the Lax isospectral equation*

$$A=[A,P_N],\qquad P_N=(A^{\frac{N}{n}})_+, \tag{3.25}$$

with $(\ )_+$ *as usual indicating projection onto* $\mathscr{A}_{0,\infty}$. *In addition, the* $\langle A^{\frac{N}{m}}\rangle$ *are in involution with respect to the Poisson bracket.*

Proof. Theorem 4, like Theorem 1, depends on the crucial observation $[A^s, A]=0$ for all s. This is a consequence of the definition of $f(L)$ by residues (3.8) and the well-known functional equation of the resolvent $\frac{\phi_{\lambda_1}-\phi_{\lambda_2}}{\lambda_1-\lambda_2}=\phi_{\lambda_1}\circ\phi_{\lambda_2}$, which implies $A^s\cdot A^t=A^{s+t}$. It is also an immediate consequence of the inductive method of computing A^s, if s is rational. Since $A^{\frac{N}{m}}=(A^{\frac{N}{m}})_+ +(A^{\frac{N}{m}})_-$, with $(\ )_-$ denoting projection into $\mathscr{A}_{-\infty,-1}$, we have $0=[A^{\frac{N}{m}}, A]=[(A^{\frac{N}{m}})_+ +(A^{\frac{N}{m}})_-, A]$, and so $[(A^{\frac{N}{m}})_-, A]=[A,(A^{\frac{N}{m}})_+]$. We thus have

$$[A,(A^{\frac{N}{m}})_+]=[(A^{\frac{N}{m}})_{-(m-1)}, A]+[\theta_{-m}, A], \tag{3.26}$$

where θ_{-m} is contained in $\tilde{\mathscr{A}}_{-\infty,-m}$. We now project (3.26) onto $\mathscr{A}_{0,\infty}$, using $P_{0,\infty}$, observing that

$$[A,(A^{\frac{N}{m}})_+]\in\mathscr{A}_{0,\infty},\qquad [\theta_{-m}, A]\in\mathscr{A}_{-\infty,-1},\quad\text{and}\quad [(A^{\frac{N}{m}})_{-(m-1)}, A]\in\mathscr{A}_{-\infty,m-2}.$$

Thus after projection onto $\mathscr{A}_{0,\omega}$, we arrive at $[A,(A^{\frac{N}{m}})_+]=[(A^{\frac{N}{m}})_{-(m-1)}, A]^{m-2}$, which is a remember of $\mathscr{A}_{0,m-2}$, and so as a consequence of Lemma 1, the first part of Theorem 4 is proven.

We now prove the involution statement, namely that

$$\{\langle A^{\frac{N}{m}}\rangle,\langle A^{\frac{K}{m}}\rangle\}=0,\qquad N, K\ \text{integers},$$

where $\{\ ,\ \}$ is the Poisson bracket, (3.20), i.e., we show that the traces of A are in involution. We first make the critical observation

$$\langle\mathscr{A}_{0,\infty},\mathscr{A}_{0,\infty}\rangle=0\qquad\langle\mathscr{A}_{-\infty,-1},\mathscr{A}_{-\infty,-1}\rangle=0. \tag{3.27}$$

By the Hamiltonian formalism, $\{\langle \overset{N}{\overline{A^m}}\rangle, \langle \overset{K}{\overline{A^m}}\rangle\} = \frac{d}{dt|_H}\langle \overset{N}{\overline{A^m}}\rangle$, with $H = \langle \overset{K}{\overline{A^n}}\rangle$, i.e.

$$\{\langle \overset{N}{\overline{A^m}}\rangle, \langle \overset{K}{\overline{A^m}}\rangle\} = \langle \dot{A}, \nabla \langle \overset{N}{\overline{A^m}}\rangle\rangle \text{ (by the definition of gradient)}$$
$$= c\langle [A, P_K], (\overset{N'}{\overline{A^m}})_{-(m-1)}\rangle = c\langle [A, P_K], (\overset{N'}{\overline{A^m}})_-\rangle$$

$\left(\text{since } [A, P_K] \in \mathscr{A}_{0,m-2}, \text{ see (3.16)}, c = \frac{N}{m}, N' = N - m\right)$

$$= c\langle [A, P_K], (\overset{N'}{\overline{A^m}})\rangle \text{ (by (3.27))} = \langle P_K, [\overset{N'}{\overline{A^m}}, A]\rangle = 0.$$

This completes the proof of Theorem 4.

Remark 7. In Remark 1 we discussed an abstract Lie algebra theorem which contained Theorem 1 as a special case. This abstract theorem can be adapted to the setting of our formal Lie algebra over a differential ring. In fact, the proof of the Kostant-Symes Theorem in our setting only requires the machinery developed in the proof of Theorem 3, and otherwise formally proceeds as if one were in the case of an ordinary Lie group.

Using the notation of Remark 1, one takes $\Phi = L = L^*$, $K = K^\perp = \mathscr{A}_{0,\infty}$, $N = N^\perp = \mathscr{A}_{-\infty,-1}$, with $\langle\ ,\ \rangle$ given by (3.3). The ad* invariant manifolds are just the $\Phi^{(m)}$. One uses the method of proof of Lemma 1 to show $\nabla_H\langle A^\nu\rangle = \nu A^{\nu-1}$, and thus the crucial statement $[A, \nu A^{\nu-1}] = 0$, is nothing but the ad* invariance of the function $\langle A^\nu\rangle$. Similarly $\nabla|_K\langle A^\nu\rangle = \nu[A^{\nu-1}]_-$, $\nabla|_N\langle A^\nu\rangle = \nu[A^{\nu-1}]_+$, and so by the abstract theorem $\dot{A} = \pi_K[\nabla_K\langle A^\nu\rangle, A]$ implies $\dot{A} = [A, \nabla_N\langle A^\nu\rangle] = [A, \nu[A^{\nu-1}]_+]$, and moreover the $\langle A^\nu\rangle|_K$ and hence $\langle A^\nu\rangle|_{\Phi^{(m)}}$ from an involutive system of functions. The only point requiring care is that the function $\langle A^\nu\rangle$ is not everywhere defined as it is for the matrix case. Thus using the Kostant-Symes Theorem we have sketched another proof of Theorem 4.

Remark 8. B. Symes has tried to generalize Theorem 4 to the case of more than one space variable without success. In this case, take for R a differential ring with n commuting derivations D_i, $i = 1, \ldots, n$, and $I = R/\Sigma D_i R$, while for Φ take formal series over R in formal roots of polynomials of the formal variables ξ_i, $i = 1, \ldots, n$. Use (3.2) for the rule of multiplication, thinking of ν as a multivector $\nu = (\nu_1, \ldots, \nu_n)$, and use the usual multi-vector notation. If $\phi \in \Phi$, $\phi = \sum_{-\infty < i < N < \infty} a_i h_i(\xi)$, with $h_i(\xi)$ homogeneous of order i in ξ. One defines $\operatorname{tr}\phi \equiv \langle\phi\rangle \doteq \int h_{-n}\, d\Omega(\xi)$, where by $\int f(\xi)\, d\Omega(\xi)$ we mean integrate $f(\xi)$ over the unit sphere $\Sigma \xi_i^2 = 1$ with the usual polar measure, for the moment thinking of $\xi \in R^n$. That $\langle[\phi_1, \phi_2]\rangle = 0$ is seen by using (3.2) and expressing the divergence formula in polar coordinates. One can in fact duplicate all the constructions of this section, but Theorem 4 fails because we have no Lie algebra decomposition $\Phi = L = K + N$. One again can take N to be the formal Lie algebra of smoothing operators, but the operators of homogeneity greater than or equal to zero don't from a Lie algebra, as they do in the case of one variable. This seems to be the crucial obstruction towards extending Theorem 4, at least from the point of view of Lie algebra.

Corollary 1. *If* $A\in\Phi^{(m)}$, $\langle A^{\frac{s}{m}}\rangle$, $s=1,2,\dots,m-1$, *are orbit invariants and consequently* $\bar{a}_{m-2}$, $\bar{a}_{m-3}$ *are orbit invariants.*

Proof. Since $P_N=(A^{\frac{N}{m}})_+$, and $P_N=0$ for N a negative integer, we have $[A, P_N]=0$. And so in Theorem 4, for H^N, $N<0$, $\dot{A}=0$, hence $\dot{A}=[\nabla H^N, A]^{m-2}=0$. But by (3.20), $[\nabla H, A]^{m-2}=0$ is a necessary and sufficient condition for a function H to be an orbit invariant, and so we conclude that $\langle A^{\frac{s}{n}}\rangle$, $s=1,2,\dots$, $m-1$, are orbit invariants.

To see that $\bar{a}_{m-2}$, $\bar{a}_{m-3}$ are orbit invariants, observe that $A=\xi^m+\sum_{i=0}^{m-2} a_i\xi^i$ has weight m if we assign ξ weight 1, and a_i weight $m-i$. From the definition of $A^{\frac{N}{m}}$ (3.9), it is easy to see $A^{\frac{N}{m}}$ has weight N if we further define $D^j a_i$ to have weight $j+(m-i)$, i.e., D to have weight one. Thus if $\langle A^{\frac{N}{m}}\rangle \doteq \bar{E}_N$, equality being in I_m, E_N can be taken to have weight $N+1$, and so we must have

$$\langle A^{\frac{1}{m}}\rangle \doteq c_1\,\bar{a}_{m-2}, \qquad \langle A^{\frac{2}{m}}\rangle \doteq c_2\,\bar{a}_{m-3},$$

c_1, c_2 being (nonzero) constants which can be computed explicitly. One expects that the algebra formed by the invariants of the corollary provide a complete list of 'generic' orbit invariants, i.e. those which can be expressed as I_m elements.

4. Lenard Relations

From the considerations of the previous two sections, we are in a position to compute recursion relations, or so-called Lenard relations for various quantities of interest. The reader may see [5] for instance, where such relations are discussed. The skew-symmetry of such relations can be used to prove involution statements concerning integrals. We first consider the setup in Sect. 3. The crucial tool in these relations is the identity $A^s\cdot A=A^{s+1}$, which plays such an important role in Theorems 1, 4 in the weaker form $[A^s, A^1]=0$.

We decompose Φ in the usual way, namely $B\in\Phi$ implies $B=B_+ + B_-$, with $B_+\in\mathscr{L}^*=\mathscr{A}_{0,\infty}$, $B_-\in\mathscr{L}=\mathscr{A}_{-\infty,-1}$. We thus have, with $A\in\Phi^{(n)}$,

$$(A^{\frac{N}{n}+1})_+=(A^{\frac{N}{n}}\cdot A)_+=(A^{\frac{N}{n}})_+\cdot A+((A^{\frac{N}{n}})_-\cdot A)_+.$$

We write, using Lemma 1,

$$(A^{\frac{N}{n}})_-=(A^{\frac{N}{n}})_{-(n-1)}+\theta_{-n}=\nabla H^N+\theta_{-n},$$

with $\theta_{-n}\in\tilde{\mathscr{A}}_{-\infty,-n}$. The above yields, again using the notation of Theorem 4,

$$P_{N+n}=P_N A+(\nabla H^N\cdot A)_+ +E_0, \tag{4.1}$$

with E_0 equaling the coefficient of the $(\xi-iD)^{-n}$ term of $(A^{\frac{N}{n}})_-$, i.e., the coefficient of the component of $((A^{\frac{N}{n}})_- -\nabla H^N)$ in $\tilde{\mathscr{A}}_{-n,-n}$. In addition by

Theorem 2, the ξ^{-1} coefficient of $[(A^{\frac{N}{n}})_-, A]$ is an exact derivative, identically in the coefficients of $(A^{\frac{N}{n}})_-, A$. By the rule of multiplication (3.2), the coefficient of the ξ^{-1} term of the bracketed expression is of the form (up to a constant multiple),

$$DE_0+\mathcal{Q}((A^{\frac{N}{n}})_{-(n-1)})=DE_0+\mathcal{Q}(\nabla H^N) \quad \text{(by Lemma 1)},$$

but since $[A^{\frac{N}{n}}, A]=0$, hence $[(A^{\frac{N}{n}})_-, A]_-=0$, both sides of the above equation must equal zero. Here $\mathcal{Q}(\nabla H^N)$ just depends on the coefficients of ∇H^N, A, and their derivatives. Thus the term $\mathcal{Q}(\nabla H^N)$ is an exact derivative in the coefficients of ∇H^N, A, i.e., $\mathcal{Q}(H_A^N)=D\mathcal{Q}'(\nabla H^N)$, and in fact $\mathcal{Q}'(\cdot)$ is easily computable. We thus have $E_0 = -\mathcal{Q}'(\nabla H^N)$, from which we deduce, using (4.1) and a little computation involving (3.2),

$$P_{N+n}=P_N A+\psi(\nabla H^N), \qquad \psi(\nabla H)=(\nabla H\cdot A)_+ +\frac{iD^{-1}}{n}(P_{0,0}([\nabla H, A]\xi)). \tag{4.2}$$

We note that the coefficients of $\psi\in\mathcal{A}_{0,\infty}$ are polynomial functions of $\frac{DH}{Da_i}$, $0\leqq i\leqq n-2$, and their derivatives. We view $\psi(\cdot)$ as an A dependent operator $\psi\colon \tilde{\mathcal{A}}_{-(n-1),-1}\to\mathcal{A}_{0,n-1}$.

We now bracket (4.2) with A, yielding

$$[A, P_{N+n}]=[A, P_N]\cdot A+[A, \psi(\nabla H^N)]. \tag{4.3}$$

By Theorem 4, $[A, P_N]=\dot{A}\equiv\mathcal{K}\left(\mathcal{J}\frac{DH^N}{Da}\right)$, with $\mathcal{J}=\Delta-\Delta^*$ the matrix differential operator which determines Hamilton's equation (see Sect. 3), and $\mathcal{K}$ defined by

$$\mathcal{K}\begin{pmatrix} a_0 \\ \vdots \\ a_{n-2}\end{pmatrix}=A-\xi^n.$$

We define the $(n-1)\times(n-1)$ matrix differential operator $\mathcal{M}$ by

$$\mathcal{K}\left(\mathcal{M}\frac{DH}{Da}\right)\equiv\left[\left(\mathcal{K}\left(\mathcal{J}\frac{DH}{Da}\right)\right)\cdot A+[A, \psi(\nabla H)]\right]^{(n-2)}, \tag{4.4}$$

and since $[A, P_{N+n}]=\mathcal{J}\left(\frac{DH^{N+n}}{Da}\right)$, (4.3), (4.4) implies

$$\mathcal{M}\left(\frac{DH^N}{Da}\right)=\mathcal{J}\left(\frac{DH^{N+n}}{Da}\right). \tag{4.5}$$

This is the standard form of Lenard recursion relations, since $\mathcal{M}$, $\mathcal{J}$ are matrix differential operators. We also note that equation (4.2) is equivalent to

$$P_{s+nt}=\sum_{\nu=0}^{t}\psi_{\nu,s}A^{t-\nu}, \quad \psi_{\nu,s}=\psi(\nabla H^{(\nu-1)\cdot n+s}), \quad 0\leqq s\leqq n-1, \quad P_N=0, \quad N<0. \tag{4.6}$$

We observe that (4.4), (4.5), (4.6) are enough to establish $[A, P_N]=\mathcal{K}(\mathcal{J}\,\nabla H^N)$, i.e., Theorem 4, and we shall think of three equations as the Lenard relations.

One computes

$$\mathcal{M}\left(\frac{DH}{Da}\right)=[A(\nabla H(A))_+ -(A(\nabla H))_+ A]^{(n-2)} + \frac{1}{n}\sum_{v=1}^{v=n}\frac{1}{v!}\left(\frac{\partial}{\partial \xi}\right)^v A\cdot(-iD)^{v-1}(P_{0,0}([\nabla H, A]\xi)).$$

We also note that Eq. (4.5) immediately implies Corollary 1.

Remark 9. The self-adjoint case discussed in Remark 5 may be done in precisely the same way as above, but has some novel features. To fix ideas, let $(R, D) = \left(C_0^\infty \text{ (real line)}, \frac{d}{dx}\right)$. For if $A=P(\xi)=\xi^n+\sum_{j=0}^{n-2}(a_j\xi^j+(\xi-iD)^j a_j)$, with a_j real, and if ψ formally satisfies $P(-iD)\psi=\lambda\psi$, then by the self-adjointness of A, we have

$$\langle\psi\bar{\psi}\rangle\cdot\frac{D\lambda}{Da_j}=[((iD)^j\psi)\psi^*+\psi((iD)^j\psi)^*], \quad j=0, 1, \ldots, n-2, \tag{4.7}$$

where * denotes complex conjugation. From Eq. (4.5) one infers the identity

$$\mathcal{M}\left(\frac{D\lambda}{Da}\right)=\lambda\mathcal{J}\left(\frac{D\lambda}{Da}\right). \tag{4.8}$$

Remark 10. One conjectures from computational evidence that $\mathcal{M}$ in fact defines a symplectic structure, like its counterpart $\mathcal{J}$. One also conjectures that in (4.4), the projection operator, $[\ ]^{(n-2)}$, is unnecessary, i.e., that the right side of (4.4) is automatically contained in $\mathcal{A}_{0,n-2}$, even before projection.

Examples. For the self-adjoint case, $A=-D^2+a_0$, we have

$$\mathcal{J}=D, \quad \mathcal{M}=-\tfrac{1}{4}\{D^3-2(a_0D+Da_0\cdot)\}, \quad \Psi(v)=\tfrac{1}{4}(2vD-(Dv)).$$

These formulas are well known. For

$$A=iD^3+i(a_1D+Da_1\cdot)+a_0,$$

we have

$$\mathcal{J}=\begin{bmatrix}0, & D\\ D, & 0\end{bmatrix},$$

$$\mathcal{M}=\begin{bmatrix}M_1, & M_2\\ M_3, & M_4\end{bmatrix},$$

$$M_1=\tfrac{1}{3}[D^3+a_1D+Da_1\cdot)$$

$$M_2=a_0D+\tfrac{2}{3}(Da_0), \qquad M_3=a_0D+\tfrac{1}{3}(Da_0)$$

$$M_4=\tfrac{1}{9}[D^5+5a_1D^3+D^3(5a_1)\cdot+(8a_1^2-3(D^2a_1))D+D(8a_1^2-3(D^2a_1))\cdot],$$

$$\psi\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\tfrac{1}{3}\{2iv_2D^2+(v_1-i(Dv_2))D+\frac{i}{3}(8a_1v_2+(D^2v_2-Dv_1)\}.$$

These formulas for $\mathcal{J}$, $\mathcal{M}$ have independently been found by H. McKean [6].

For

$$A=D^4+(a_2 D^2+D^2 a_2 \cdot)+i(a_1 D+D a_1 \cdot)+a_0,$$

$$J=\begin{bmatrix} J_{11}, & J_{12}, & J_{13} \\ J_{21}, & J_{22}, & J_{23} \\ J_{31}, & J_{32}, & J_{33} \end{bmatrix}, \qquad \mathcal{M}=\begin{bmatrix} M_{11}, & M_{12}, & M_{13} \\ M_{21}, & M_{22}, & M_{23} \\ M_{31}, & M_{32}, & M_{33} \end{bmatrix}$$

where (as in (3.24)),

$$J_{11}=J_{12}=J_{21}=J_{23}=J_{32}=0,$$

$$J_{13}=J_{31}=D, \quad J_{22}=-\tfrac{1}{2}D, \quad J_{33}=-\tfrac{1}{2}D^3+(a_2 D+D a_2 \cdot),$$

and letting $D^2\psi=\psi''$, etc., we have (where * denotes the formal *real* adjoint)

$$M_{11}=\tfrac{5}{8}D^3+\tfrac{1}{4}(a_2 D+D a_2 \cdot), \qquad M_{12}=M_{21}^*=\tfrac{3}{4}a_1 D+\tfrac{1}{2}a_1',$$

$$M_{13}=-A_{31}^*=-\tfrac{1}{4}D^5+\tfrac{1}{4}a_2 D^3+\tfrac{5}{4}a_2' D^2+(\tfrac{7}{4}a_2''+a_0)D+(\tfrac{3}{4}a'''+\tfrac{3}{4}a_0'),$$

$$M_{22}=\tfrac{1}{2}\{\tfrac{1}{4}D^5+\tfrac{1}{2}[a_2 D^3+D^3 a_2]-\tfrac{1}{2}[(a_2''+a_0-a_2^2)D+D(a_2''+a_0-a_2^2)\cdot]\}$$

$$M_{23}=-A_{32}^*=-a_1 D^3-\tfrac{3}{2}a_1' D^2-(a_1''+\tfrac{1}{2}a_2 a_1)D-(\tfrac{1}{2}a_1' a_2+\tfrac{1}{4}a_1'''),$$

$$M_{33}=\tfrac{1}{8}D^7+\tfrac{1}{2}[(a_2^2-a_2''-\tfrac{1}{2}a_0)D^3+D^3(a_2^2-a_2''-\tfrac{1}{2}a_0)\cdot]$$
$$+[(\tfrac{1}{4}a_2''''-a_2'^2+\tfrac{3}{4}a_1^2+a_2 a_0)D+D(\tfrac{1}{4}a_2''''-a_2'^2+\tfrac{3}{4}a_1^2+a_2 a_0)].$$

while

$$\psi(v)=\Phi\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\tfrac{1}{2}v_3 D^3-\tfrac{1}{4}(v_3'+i v_2)D^2+[\tfrac{1}{4}(v_1+i v_2')+\phi_1 v_3]D$$
$$+\tfrac{1}{4}[\tfrac{1}{2}(-3v_1'-i v_2''+v_3''')-2\phi_1 v_3'+3i\phi_2 v_3-i\phi_1 v_2].$$

These formulas are computed from (4.1)–(4.4). In the case $A=\xi^n+\sum_{i=0}^{n-2} a_i \xi^i$, one can use the above formulas to get general formulas for $\mathcal{M}$, $\mathcal{J}$, ψ, but since in the self-adjoint case, one must compute the dual of $(\xi-iD)^j+\xi^j=\eta_j$ by hand (up to order ξ^{-n-1}), one computes $\mathcal{M}$, $\mathcal{J}$, ψ separately for each n.

Remark 11. The relations of this Section also apply to the systems of Sect. 2. We derive relation (4.2) for these systems which imply relations of the form (4.4), (4.5), (4.6).

We shall use the notation of Theorem 1. Observe

$$L^N-(L^N)^+-(L^N)^-+(L^N)^0+2(L^N)^-=P_N+B_N,$$

where

$$P_N=(L^N)^+-(L^N)^-,$$

$$B_N=(L^N)^0+2(L^N)^-=\nabla(H_N)+\theta_m,$$

by (2.14), where $H_N = \left\langle \frac{1}{N+1} A^{N+1} \right\rangle$, and $\theta_m \in \mathscr{A}_{-n,m-1}$. From this we conclude

$$\begin{aligned} P_{N+1} &= L^{N+1} - B_{N+1} = (P_N + B_N) \cdot L - B_{N+1} \\ &= P_N \cdot L + (B_N \cdot L) - B_{N+1} \\ &= [P_N L + \nabla(H_N) \cdot L] + [\theta_m \cdot L - B_{N+1}], \end{aligned}$$

hence, remembering $(P_{N+1})^+ \in \mathscr{A}_{1,n}$, we have shown

$$(P_{N+1})^+ = (P_N L)^+ + [\nabla(H_N) \cdot L]^+ \equiv (P_N L)^+ - \Delta(\nabla(H_N)). \tag{4.9}$$

The above defines the function $\Delta(\cdot)$. Taking the negative transpose of the above equation, we have $P_{N+1}^- = (LP_N)^- + \Delta^T(\nabla(H_N))$, where we have used $P_N^T = -P_N$, $L^T = L$. We have by Theorem 1,

$$(P_N L)_{-n} - (LP_N)_{-n} = [P_N, L]_{-n} = ([A, \nabla H^N]_+)^T, \tag{4.10}$$

where $(\ )_{-m} = P_{-m,0}(\ )$, and so (4.9), (4.10), and the expression for P_{N+1}^+ imply

$$P_{N+1} = P_N \cdot L + \tilde{\psi}(\nabla(H_N)),$$

with $\tilde{\psi}(\nabla H_A) = (\nabla H \cdot A)^+ - ((A \nabla H)^+)^T + \frac{1}{2}([\nabla H, A]_0)$. Thus a relation of the form (4.2) has been established as claimed. Note $\psi(\nabla H)$ is skew-symmetric precisely if $H(A)$ is an orbit invariant. We also note that the conjectures of Remark 10 are relevant in this case. From (4.4), we compute, using the analogous terminology,

$$\tilde{\mathscr{K}}\left(\tilde{\mathscr{M}}\left(\frac{\partial H}{\partial A}\right)\right) = (L\tilde{\psi} + \tilde{\psi}^T L)_+, \qquad \tilde{\psi} = \tilde{\psi}(\nabla H).$$

We give the results for the Toda system, i.e., $A_{i,i} = b_i$, $A_{i,i+1} = a_i$, all other $A_{i,j}$'s equal to zero. Define $E_{ij} = 1$ if $j = i+1$, zero otherwise and similarly for E_{ij}^T. Let $\tilde{\mathscr{M}}$, $\tilde{\mathscr{J}}$ act on the vector $\begin{pmatrix} H_a \\ H_b \end{pmatrix}$, $H = H(a, b)$, $H_a = \left(\frac{\partial H}{\partial a_0}, \ldots, \frac{\partial H}{\partial a_{n-1}}\right)^T$, etc. for H_b. Then we have

$$\tilde{\mathscr{J}} = \begin{bmatrix} O_n, & S \\ -S^T, & O_{n+1} \end{bmatrix},$$

O_n, O_{n+1} the $n \times n$, and $(n+1) \times (n+1)$ zero matrices respectively, and $[S]_{ij} = (-\delta_{ij} + E_{ij})\, a_i$, $i = 0, \ldots, n-1$, $j = 0, \ldots, n$.

$$\tilde{\mathscr{M}} = \begin{bmatrix} \mathscr{M}_1 & \mathscr{M}_3 \\ -\mathscr{M}_3^T & \mathscr{M}_2 \end{bmatrix},$$

where

$$[\tilde{\mathscr{M}}_1]_{ij} = \frac{a_i a_{i+1}}{2} (E_{ij} - E_{ij}^T), \qquad i, j = 0, \ldots, n-1,$$

$$[\tilde{\mathscr{M}}_2]_{ij} = 2 a_i^2 (E_{ij} - E_{ij}^T), \qquad i, j = 0, \ldots, n,$$

$$[\tilde{\mathscr{M}}_3]_{ij} = a_i (b_{i+1} E_{ij} - b_i \delta_{ij}), \quad i = 0, \ldots, n-1, \quad j = 0, \ldots, n.$$

Finally if $w=(u_0, \ldots, u_{n-1}, v_0, \ldots, v_n)^T$, then we have

$$[\tilde{\psi}(w)]_{ij} = \tfrac{1}{2}\delta_{ij}(a_i u_i - a_{i-1} u_{i-1}) - a_i(E_{ij} v_i - E_{ij}^T v_{i+1}), \qquad i, j = 0, 1, \ldots, n.$$

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