Maximal chains in $^{\omega}\omega$ and ultrapowers of the integers

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Summary. Various questions posed by P. Nyikos concerning ultrafilters on ω and chains in the partial order $(\omega, <^{*})$ are answered. The main tool is the oracle chain condition and variations of it.

1 Introduction

In [5] various axioms related to maximal chains in ultrapowers of the integers were classified and studied. The purpose of the present paper is to answer several of the questions posed in that paper and to pose some new ones.

The notation and terminology of this paper will adhere as much as possible to accepted standards but some of the main points are listed here. The relation $a \subset^{*} b$ means that $|a \setminus b| < \aleph_{0}$ while $f \leq^{*} g$ means that f and g belong to $^{\omega}\omega$ – or, perhaps, $^{A}\omega$ where $A \subseteq \omega$ is infinite – and $f(n) \leq g(n)$ for all but finitely many integers n. If $f(n) < g(n)$ for all but finitely many integers n then this will be denoted by $f <^{*} g$. By a *chain* in $^{\omega}\omega$ will be meant a subset of $^{\omega}\omega$ which is well ordered by $<^{*}$ and consists of nondecreasing functions. In the next section the effects of modifying this definition of a chain will be discussed. A subset $\mathscr{S} \subseteq {}^{\omega}\omega$ will be said to be unbounded if for every $f \in {}^{\omega}\omega$ there is $g \in \mathscr{S}$ such that $g \not\leq^{*} f$. The least cardinality of an unbounded subset of $^{\omega}\omega$ is denoted by $\mathfrak{b}$ while the least cardinality of a cofinal subset of $^{\omega}\omega$ is denoted by $\mathfrak{d}$. The term *ultrafilter* will be reserved for ultrafilters on ω which contain no finite sets. A P-point is an ultrafilter on ω, $\mathscr{U}$, such that for every $\mathscr{A} \in [\mathscr{U}]^{\aleph_{0}}$ there is $B \in \mathscr{U}$ such that $B \subset^{*} A$ for every $A \in \mathscr{A}$. If $\mathscr{U}$ is a filter then $\mathscr{U}^{*}$ will denote the dual ideal to $\mathscr{U}$.

If $\mathcal{U}$ is an ultrafilter then *the integers modulo $\mathcal{U}$* will refer to the ultrapower of the integers with respect to $\mathcal{U}$ and will be denoted by $^{\omega}\omega/\mathcal{U}$. If $\mathcal{U}$ is an ultrafilter then $\mathcal{E} \subseteq {}^{\omega}\omega$ will be said to be unbounded modulo $\mathcal{U}$ if, letting $[f]_{\mathcal{U}}$ represent the equivalence class of $f \in {}^{\omega}\omega$ in $^{\omega}\omega/\mathcal{U}$, the set $\{[f]_{\mathcal{U}} : f \in \mathcal{E}\}$ is unbounded in the linear order $^{\omega}\omega/\mathcal{U}$. The least ordinal which can be embedded cofinally in a linear ordering L is denoted by $\mathrm{cof}(L)$ – $\mathrm{cof}(^{\omega}\omega/\mathcal{U})$ will be an important invariant of $\mathcal{U}$ in the following discussion.

For reference, here are Nyikos' axioms (throughout $\mathcal{E}$ refers to a maximal chain of nondescending functions in $^{\omega}\omega$ and $\mathcal{U}$ refers to an ultrafilter)

- Axiom 1 $(\forall \mathcal{U})\,(\forall \mathcal{E})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 2 $(\exists \mathcal{E})\,(\forall \mathcal{U})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 3 $(\exists \mathcal{U})\,(\forall \mathcal{E})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 4 $(\forall \mathcal{U})\,(\exists \mathcal{E})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 5 $(\forall \mathcal{E})\,(\exists \mathcal{U})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 5.5 $(\exists \mathcal{U})\,(\mathrm{cof}(^{\omega}\omega/\mathcal{U}) = \mathfrak{b})$
- Axiom 6 $(\exists \mathcal{E})\,(\exists \mathcal{U})\,(\mathcal{E}$ is unbounded modulo $\mathcal{U})$
- Axiom 6.5 $(\exists \mathcal{E})\,(\exists \mathcal{U})\,(\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(^{\omega}\omega/\mathcal{U}))$

Various implications and non-implications between these axioms are established in [5]. As well, it is observed that Axiom 2 is equivalent to the equality $\mathfrak{b} = \mathfrak{d}$.

2 Non-monotone functions

The definition of chains as $<^{*}$-increasing sequences of nondecreasing functions in the axioms which appeared in [5] may appear to be somewhat arbitrary and one may wonder what results if chains are defined differently. For the record, therefore, the following definitions are offered.

Definition 2.1. If $\mathcal{A}$ is a subset of $^{\omega}\omega$ then by a $(<^{*}, \mathcal{A})$-*chain* will be meant a subset of $\mathcal{A}$ which is well ordered by $<^{*}$. By a $(\leq^{*}, \mathcal{A})$-*chain* will be meant a subset of $\mathcal{A}$ which is well ordered by $\leq^{*}$. For the pruposes of this definition the most important subsets of $^{\omega}\omega$ are: the nondecreasing functions, which will be denoted by $\mathcal{N}$, the strictly increasing functions, which will be denoted by $\mathcal{S}$, and $^{\omega}\omega$.

If $x \in \{<^{*}, \leq^{*}\}$ then Axiom $N(x, \mathcal{A})$ will denote the Axiom N with $\mathcal{E}$ being a variable ranging over $(x, \mathcal{A})$-chains – so Axiom N is the same as Axiom $N(<^{*}, \mathcal{N})$.

Fortunately, many of these axioms turn out to be equivalent and others are simply false. The following simple observation of Nyikos can be used to see this.

Lemma 2.1. *There is a mapping $\Psi : {}^{\omega}\omega \to {}^{\omega}\omega$ such that*
- $\Psi(f)$ *is strictly increasing for every $f \in {}^{\omega}\omega$*
- $f \leq \Psi(f)$ *for every $f \in {}^{\omega}\omega$*
- *if $f \leq g$ and $f \neq^{*} g$ then $\Psi(f) <^{*} \Psi(g)$*
- *if f is nondecreasing and $f(n) < g(n)$ then $\Psi(f)(n) < g(n)(n+1) + n$*

Proof. Define $\Psi(f)(n) = \left(\sum_{i=0}^{n} f(i) \right) + n$. It is easy to check that $f \leq^{*} \Psi(f)$ and that $\Psi(f)$ is strictly increasing. If $f \leq g$ and $f \neq^{*} g$ then there is some $m \in \omega$ such that $f(i) \leq g(i)$ for all $i \geq m$. Since there are infinitely many $k \in \omega$ such that $f(i) \neq g(i)$ is follows that there is some $M > m$ such that $\sum_{i=0}^{M} g(i) > \sum_{i=0}^{m}(f(i) - g(i))$. Hence $\Psi(f)(j) < \Psi(g)(j)$ for all $j \geq M$.

Finally observe that if $f(n) < g(n)$ and f is nondecreasing then $f(i) < g(n)$ for each $i \leqq n$. Hence $\Psi(f)(n) \leqq f(n)(n+1) + n < g(n)(n+1) + n$. $\square$

A consequence of Lemma 2.1 is that given any $(\leqq^*, {}^\omega\omega)$-chain $\{f_\xi : \xi \in \lambda\}$ there is a $(<^*, \mathscr{S})$-chain $\{f'_\xi : \xi \in \lambda\}$ such that $f_\xi \leqq f'_\xi$ for each $\xi \in \lambda$. Consequently, Axiom $N(<^*, \mathscr{S})$, Axiom $N(<^*, \mathscr{N})$, Axiom $N(<^*, {}^\omega\omega)$, Axiom $N(\leqq^*, {}^\omega\omega)$, Axiom $N(\leqq^*, \mathscr{S})$ and Axiom $N(\leqq^*, \mathscr{N})$ are all equivalent for $N \in \{2, 4, 6, 6.5\}$. Therefore, from now, if $N \in \{2, 4, 6, 6.5\}$, then Axiom N will be used to denote any and all of the Axioms $N(x, \mathscr{A})$ where $x \in \{<^*, \leqq^*\}$, $\mathscr{A} \in \{\mathscr{N}, \mathscr{S}, {}^\omega\omega\}$.

Another consequence of Lemma 2.1 is that given any $(\leqq^*, \mathscr{N})$-chain $\{f_\xi : \xi \in \lambda\}$ there is a $(<^*, \mathscr{S})$-chain $\{f'_\xi : \xi \in \lambda\}$ such that $f_\xi \in f'_\xi$ for each $\xi \in \lambda$ and, for each ultrafilter $\mathscr{U}$ and $g \in {}^\omega\omega$, the function g is an upper bound for $\{f_\xi : \xi \in \lambda\}$ modulo $\mathscr{U}$ if and only if $g \cdot (n+1) + n$ is an upper bound for $\{f'_\xi : \xi \in \lambda\}$ modulo $\mathscr{U}$. Consequently, Axiom $N(<^*, \mathscr{S})$, Axiom $N(<^*, \mathscr{N})$, Axiom $N(\leqq^*, \mathscr{S})$ and Axiom $N(\leqq^*, \mathscr{N})$ are all equivalent for $N \in \{1, 3, 5\}$. Moreover, Axiom $3(<^*, {}^\omega\omega)$ and Axiom $3(\leqq^*, {}^\omega\omega)$ are obviously both false because if $\mathscr{U}$ is any ultrafilter then it is possible to choose $X \subseteq \omega$ such that $X \notin \mathscr{U}$ and then find, using Lemma 2.1, a $(<^*, {}^\omega\omega)$-chain $\mathscr{C}$ such that $f(n) = 0$ for $f \in \mathscr{C}$ and $n \in X$. It follows from this that Axioms $1(<^*, {}^\omega\omega)$ and $1(\leqq^*, {}^\omega\omega)$ are also false. Therefore, from now on Axiom N can be used to denote any and all of the Axioms $N(x, \mathscr{A})$ where $x \in \{<^*, \leqq^*\}$, $\mathscr{A} \in \{\mathscr{N}, \mathscr{S}\}$ and $N \in \{1, 3\}$. Also the notation Axiom $5(\mathscr{N})$ can be used to denote any and all of Axiom $5(<^*, \mathscr{S})$, Axiom $5(<^*, \mathscr{N})$, Axiom $5(\leqq^*, \mathscr{S})$ and Axiom $5(\leqq^*, \mathscr{N})$.

It is worth noting that Axiom $5(\mathscr{N})$ is not equivalent to Axiom $5(<^*, {}^\omega\omega)$ or Axiom $5(\leqq^*, {}^\omega\omega)$. The reason for this is that it will been shown, in Theorem 2.4, that Axiom 2 implies Axiom $5(\mathscr{N})$; but the same is not true of Axiom $5(<^*, {}^\omega\omega)$ since the next result shows that Axiom $5(<^*, {}^\omega\omega)$ fails assuming $2^{\aleph_0} = \aleph_1$. It is obvious that Axiom 2 holds if $2^{\aleph_0} = \aleph_1$. The following definition will be used to establish this and appears to be central in the context of non-monotone functions.

Definition 2.2. If $\mathscr{A} \subseteq {}^\omega\omega$ then define $\mathscr{I}_b(\mathscr{A})$ to be the set of all $X \subseteq \omega$ such that $\{f \restriction X : f \in \mathscr{A}\}$ is bounded.

Notice that $\mathscr{I}_b(\mathscr{A})$ is an ideal and that $\mathscr{I}_b(\mathscr{A})$ is proper if and only if $\mathscr{A}$ is an unbounded subset of $^\omega\omega$. It is also worth observing that if $\mathscr{C} \subseteq {}^\omega\omega$ and $\mathscr{U}$ is an ultrafilter and $\mathscr{C}$ is unbounded modulo $\mathscr{U}$ then $\mathscr{U} \cap \mathscr{I}_b(\mathscr{C}) = \emptyset$.

Lemma 2.2. *If there is a sequence $\{X_\xi : \xi \in \omega_1\}$ of subsets of ω such that*
— $X_\xi \subseteq^* X_\eta$ if $\xi \in \eta$
— $X_\eta \backslash X_\xi$ is infinite if $\xi \in \eta$
— there exists a family $\{g_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$ such that for every $f \in {}^\omega\omega$ there is $\xi \in \omega_1$ such that $g_\xi \restriction X_{\xi+1} \backslash X_\xi \not\leqq^* f \restriction X_{\xi+1} \backslash X_\xi$
then there is an unbounded $(<^*, {}^\omega\omega)$-chain, $\mathscr{C}$, such that $\mathscr{I}_b(\mathscr{C})$ contains $\{\{n \in \omega : f(n) \geqq n\} : f \in \mathscr{C}\}$.

Proof. Let $\{X_\xi : \xi \in \omega_1\}$ and $\{g_\xi : \xi \in \omega_1\}$ satisfy the hypothesis of the lemma and, without loss of generality, assume that $g_\xi(n) \geqq n$ for all ξ and n. Let $\{h_\xi : \xi \in \omega_1\}$ be a $<^*$-increasing sequence of functions such that $h_\xi(n) < n$ for all $n \in \omega$. A standard induction argument can now be used to construct $\{f_\xi : \xi \in \omega_1\}$ such that

- $f_\xi \upharpoonright \omega \backslash X_{\xi+1} = h_\xi \upharpoonright \omega \backslash X_{\xi+1}$
- if $\xi \in \eta$ then $f_\xi \leq^* f_\eta$
- if $\xi \in \eta$ then $f_\xi \upharpoonright X_\xi =^* f_\eta \upharpoonright X_\xi$
- $f_\xi \upharpoonright X_{\xi+1} \backslash X_\xi = g_\xi \upharpoonright X_{\xi+1} \backslash X_\xi$

and this clearly suffices. □

Notice that ω_1 is crucial to the proof of Lemma 2.2 and can not be replaced by a larger cardinal. The reason is that the inductive construction relies on the fact that if $\{f_n : n \in \omega\}$ is a family of partial functions from ω to ω such that $f_n =^* f_{n+1} \upharpoonright \mathrm{dom}(f_n)$ then, there is a single function f such that $f_n \subseteq^* f$ for each $n \in \omega$. A Hausdorff gap type of construction shows that this is not possible if ω_1 is replaced by some larger cardinal. It is for the same reason that ω_1 appears in the next corollary.

Corollary 2.1. *If $\mathfrak{d} = \aleph_1$ then there is an unbounded $(<^*, {}^\omega\omega)$-chain, $\mathscr{C}$, such that $\mathscr{I}_b(\mathscr{C})$ contains $\{\{n \in \omega : f(n) \geq n\} : f \in \mathscr{C}\}$.*

Proof. Let $\{X_\xi : \xi \in \omega_1\}$ be any sequence of subsets of ω such that $X_\xi \subseteq^* X_\eta$. $X_\eta \backslash X_\xi$ is infinite if $\xi \in \eta$. Then $\{g_\xi : \xi \in \omega_1\}$ can be any dominating family. □

The next result shows that if $\mathfrak{d} = \aleph_1$ then Axiom $5(<^*, {}^\omega\omega)$ fails.

Theorem 2.1. *If $\mathfrak{d} = \aleph_1$ then there is an unbounded $(<^*, {}^\omega\omega)$-chain which is bounded modulo any ultrafilter.*

Proof. Use Corollary 2.1 to find an unbounded chain $\mathscr{C} \subseteq {}^\omega\omega$ such that $\mathscr{I}_b(\mathscr{C})$ contains $\{\{n \in \omega : f(n) \geq n\} : f \in \mathscr{C}\}$. If $\mathscr{V}$ is any ultrafilter such that $\mathscr{C}$ is unbounded modulo $\mathscr{V}$ then it must be that $\mathscr{V} \cap \{\{n \in \omega : f(n) \geq n\} : f \in \mathscr{C}\} = \emptyset$. Then, it is clear that the identity function is an upper bound for $\mathscr{C}$. □

It seems that Axiom $5(<^* \; \mathscr{N}, {}^\omega\omega)$ is very strong and Axiom $5(\leq^*, {}^\omega\omega)$ is potentially even stronger. nevertheless, Axiom $5(\leq^*, {}^\omega\omega)$ is consistent and does not imply Axiom 1. This is implied by the next sequence of results. The question of which of the axioms are implied by Axiom $5(\leq^*, {}^\omega\omega)$ is mostly open however.

The Open Colouring Axiom was first considered by Abraham et al. in [1] and later strengthened by Todorcevic [8].

Definition 2.3. The Open Colouring Axiom states that if $X \subseteq \mathbb{R}$ and $\mathscr{V} \subset [X]^2$ is an open set [1] then either there is $Y \in [X]^{\aleph_1}$ such that $[Y]^2 \subset \mathscr{V}$ or there exists a partition of $X = \bigcup_{n \in \omega} X_n$ such that $[X_n]^2 \cap \mathscr{V} = \emptyset$ for each $n \in \omega$. $\mathbb{R}$ can be replaced by any second countable space in the statement of the Open Colouring Axiom.

Theorem 2.2. *If the Open Colouring Axiom holds and $\{(h_\alpha, g_\alpha) : \alpha \in \lambda\}$ satisfies*
- *λ is a regular cardinal greater than ω_1*
- *$\mathrm{dom}(h_\xi) = \mathrm{dom}(g_\xi) = X_\xi$ for $\xi \in \lambda$*
- *if $\xi \in \eta$ then $X_\xi \subseteq^* X_\eta$*
- *if $n \in X_\xi$ then $g_\xi(n) \leq h_\xi(n)$*
- *if $\xi \in \eta$ then $g_\xi \leq^* g_\eta \upharpoonright X_\xi \leq^* h_\xi$*

then there exists a function $f : \omega \to \omega$ such that $g_\xi \leq^* f \upharpoonright X_\xi$ for all $\xi \in \lambda$.

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[1] Here $[X]^2$ can be thought of as the set of points in X^2 above the diagonal

Proof. To begin, identify λ with the subspace of the reals $\{(h_\xi, g_\xi) : \xi \in \lambda\}$ – the reals are being considered as $(^\omega\omega)^2$ or, in other words, the irrationals. Define

$$V = \{\{\alpha, \beta\} \in [\lambda]^2 : \alpha \in \beta \text{ and } (\exists n)(g_\beta(n) > h_\alpha(n))\}$$

and observe that V is open. From the Open Colouring Axiom it follows that there are only two possibilities.

The first is that there is a partition $\lambda = \bigcup_{n \in \omega} X_n$ such that $[X]^2 \cap V = \emptyset$ for each $n \in \omega$. In this case there must be some $n \in \omega$ such that X_n is cofinal in λ. Choose $\xi \in \lambda$ such that $\bigcup_{\eta \in \xi} X_\eta = \bigcup_{\eta \in \lambda} X_\eta$ and let $f(n) = \min\{h_\eta(n) : \eta \in \xi\}$. Now, if $\beta > \xi$ and $n \in X_\beta$ then there is some $\eta \in \xi$ such that $n \in X_\eta$ and hence $n \in \mathrm{dom}(f)$. Moreover, if $\eta \in \xi$ and $n \in X_\eta$ then $g_\beta(n) \leq h_\eta(n)$ and so $g_\beta(n) \leq f(n)$. So f is the desired function.

The second possibility is that there is $X \in [\lambda]^{\aleph_1}$ such that $[X]^2 \subset V$. Since $\lambda \geq \omega_2$ it is possible to choose some $\xi \in \lambda$ such that $X \subseteq \xi$. It is then possible to choose $M \in \omega$, $g : M \to \omega$ and $Y \in [X]^{\aleph_1}$ such that
- if $\eta \in Y$ then $X_\eta \setminus M \subseteq X_\xi$
- if $\eta \in Y$ and $n \in X_\eta \setminus M$ then $g_\eta(n) \leq g_\xi(n) \leq h_\eta(n)$
- $g_\eta \restriction M = g$

Then if $\mu \in \eta$ and $\{\mu, \eta\} \in [Y]^2$ and $n \in X_\mu \cap X_\eta$ it must be that either $n \geq M$ or $n < M$. In the first case it follows that $n \in X_\xi$ and so $g_\eta(n) \leq g_\xi(n) \leq h_\mu(n)$. In the second case it may be concluded that $g_\eta(n) = g(n) = g_\mu(n) \leq h_\mu(n)$. It follows that $\{\mu, \eta\} \notin V$ which is a contradiction. $\square$

Theorem 2.3. *The conjunction of Axiom 2 and the Open Colouring Axiom implies Axiom 5($\leq^*, ^\omega\omega$).*

Proof. To begin, recall that it was shown in [5] that Axiom 2 implies that $\mathfrak{b} = \mathfrak{d}$. Hence it is possible to choose a $(\leq^*, ^\omega\omega)$-chain $\{d_\xi : \xi \in \mathfrak{d}\}$ which is also a dominating family in $^\omega\omega$. Also, if $\mathscr{C}$ is any $(\leq^*, ^\omega\omega)$-chain then $\mathscr{C}$ is of the form $\{g_\xi : \xi \in \mathfrak{d}\}$. Define $E(\eta, \xi) = \{n \in \omega : g_\xi(n) \geq d_\eta(n)\}$.

Next, let $\{\mathfrak{M}_\xi : \xi \in \mathfrak{d}\}$ be a sequence of elementary submodels of $(H(\mathfrak{c}^+), \in)$ such that
- $|\mathfrak{M}_\xi| < \mathfrak{d}$ for each $\xi \in \mathfrak{d}$
- $\{g_\xi : \xi \in \mathfrak{d}\} \in \mathfrak{M}_\eta$ and $\{d_\xi : \xi \in \mathfrak{d}\} \in \mathfrak{M}_\eta$ for each $\eta \in \mathfrak{d}$
- $\eta \in \mathfrak{M}_\eta$
- $\mathfrak{M}_\eta \in \mathfrak{M}_\zeta$ for each $\eta \in \zeta \in \mathfrak{d}$
and let $\mu(\xi) = \mathfrak{M}_\xi \cap \mathfrak{d}$. Define $\mathscr{F}$ to be the filter generated by

$$\{E(\mu(\xi), \mu(\xi + 1)) : \xi \in \mathfrak{d} \text{ and } \xi \text{ is odd}\}$$

and observe that if $\mathscr{F}$ is a proper filter then $\mathscr{C}$ will be cofinal in $^\omega\omega$ modulo $\mathscr{U}$ for any ultrafilter extending $\mathscr{F}$.

Hence it suffices to show that $\mathscr{F}$ is proper. To this end let $\mathscr{F}_\varrho$ be the filter generated by

$$\{E(\mu(\xi), \mu(\xi + 1)) : \xi \in \varrho \text{ and } \xi \text{ is odd}\}$$

and prove by induction that each $\mathscr{F}_\varrho$ is proper. Moreover, it will be shown by induction that $\mathscr{F}_\varrho \cap \mathscr{F}_{\mathfrak{b}}(\mathscr{C}) = \emptyset$. If $\varrho = 0$, ϱ is odd or ϱ is a limit then there is nothing to do

so suppose that $\varrho = \varrho' + 1$, where ϱ' is odd, and that $\mathscr{F}_{\varrho'}$ is a proper filter such that $\mathscr{F}_{\varrho'} \cap \mathscr{T}_b(\mathscr{C}) = \emptyset$.

Notice that $\mathscr{F}_{\varrho'} \in \mathfrak{M}_{\varrho'}$ because ϱ' is odd. Therefore it suffices to show that for each $B \in \mathscr{T}_b(\mathscr{C})^+$ there is some $\theta \in \mathfrak{d}$ such that $E(\mu(\varrho'), \theta) \cap B \in \mathscr{T}_b(\mathscr{C})^+$ – the reason being that the elementarity of $\mathfrak{M}_{\varrho'+1}$ will guarantee that $E(\mu(\varrho'), \mu(\varrho' + 1)) \cap B \in \mathscr{T}_b(\mathscr{C})^+$ for each $B \in \mathscr{T}_b(\mathscr{C})^+$. Elementarity also assures that it may as well be assumed that $B \in \mathfrak{M}_{\varrho'}$. But if there is some $B \in \mathscr{T}_b(\mathscr{C})^+$ such that $E(\mu(\varrho'), \theta) \cap B \in \mathscr{T}_b(\mathscr{C})$ for each $\theta \in \mathfrak{c}$ then it is possible to find h_θ such that
- $\mathrm{dom}(h_\theta) = B \cap E(\mu(\varrho'), \theta)$ for each $\theta \in \mathfrak{d}$
- $h_\theta(n) \geq g_\theta(n)$ for every $n \in B \cap E(\mu(\varrho'), \theta)$ and for each $\theta \in \mathfrak{d}$
- $g_\xi \restriction \mathrm{dom}(h_\theta) \leq^* h_\theta$ for each $\theta \in \xi \in \mathfrak{c}$

It follows that $\{(h_\theta, g_\theta \restriction B \cap E(\mu(\varrho'), \theta)) : \theta \in \mathfrak{d}\}$ satisfies the hypothesis of Lemma 2.2. Since the Open Colouring Axiom is being assumed, there is a function $f \in {}^\omega\omega$ such that $g_\theta \leq^* f \restriction B \cap E(\mu(\varrho'), \theta)$ for each $\theta \in \mathfrak{d}$. It follows that for each $\theta \in \mathfrak{d}$ there are only finitely many $n \in B$ such that $g_\theta(n) > \max\{d_{\varrho'}(n), f(n)\}$ contradicting that $B \notin \mathscr{T}_b(\mathscr{C})$. $\square$

Notice that it is shown in [8] that the Proper Forcing Axiom implies the hypothesis of Theorem 2.3. Moreover it is a Corollary that Axiom $5(\leq^*, {}^\omega\omega)$ does not imply Axiom 1 because it is easy to check that Martins' Axiom – and hence the Proper Forcing Axiom – implies that Axiom 1 fails. In particular, it is possible to inductively define a $(<^*, \mathscr{S})$-chain no member of which dominates the exponential [2] function.

It has already been mentioned thtat the next lemma can be used to show that Axiom $5(\mathscr{N})$ is not equivalent to Axiom $5(\leq^*, {}^\omega\omega)$ or Axiom $5(<^*, {}^\omega\omega)$. It will also be used in the proof of Theorem 3.2 but also has some interest on its own since it provides a sufficient condition for Axiom $5(\mathscr{N})$ to hold. Thus, it will be used to show that Axiom 2 implies Axiom $5(\mathscr{N})$.

Lemma 2.3. *If $\mathfrak{d}$ is regular and $\{c_\xi : \xi \in \mathfrak{d}\} \subseteq {}^\omega\omega$ is $\leq^*$ increasing and, moreover, $\{c_\xi \restriction A : \xi \in \mathfrak{d}\}$ is unbounded in ${}^A\omega$ for each $A \in [\omega]^{\aleph_0}$ then there is an ultrafilter $\mathscr{U}$ such that $\{c_\xi : \xi \in \mathfrak{d}\}$ is cofinal in ${}^\omega\omega/\mathscr{U}$.*

Proof. Let $\mathscr{D} \subseteq {}^\omega\omega$ be a cofinal family in ${}^\omega\omega$ of cardinality $\mathfrak{d}$. Let $\{\mathfrak{M}_\xi : \xi \in \mathfrak{d}\}$ be an increasing sequence of elementary submodels of
$$(H(\mathfrak{c}^+), \{c_\xi : \xi \in \mathfrak{d}\}, \mathscr{D}, \in)$$
such that $\mathfrak{M}_\xi \cap \mathfrak{d} = \alpha(\xi) \in \mathfrak{c}$ for each $\xi \in \mathfrak{c}$ and $\bigcup_{\xi \in \mathfrak{d}} \mathfrak{M}_\xi \supseteq \mathscr{D}$ – this is possible because $\mathfrak{d}$ is regular. Let
$$\mathscr{U} = \{\{n \in \omega : f(n) \leq c_{\alpha(\xi)}(n)\} : \xi \in \mathfrak{d} \text{ and } f \in \mathfrak{M}_\xi\}$$
and note that it suffices to show that this is a base for a filter.

That $\mathscr{U}$ has the finite intersection property can be established by induction. Let $B(\xi, f) = \{n \in \omega : f(n) \leq c_{\alpha(\xi)}(n)\}$ for $\xi \in \mathfrak{c}$ and $f \in \mathfrak{M}_\xi$ and suppose that $|\cap \mathscr{A}| = \aleph_0$ for each $\mathscr{A} \in [\mathscr{U}]^m$ – the case $m = 1$ is an easy consequence of elementarity. Now let
$$\{B(\xi_0, f_0), B(\xi_1, f_1), \ldots, B(\xi_m, f_m)\} \in [\mathscr{U}]^{m+1}$$

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[2] The exponential function is not crucial here but some quickly growing function must be used. For example, although the identity function is strictly increasing it can not be used because it is the minimal strictly increasing function

and suppose that $\xi_i \leqq \xi_{i+1}$ for each i. If $\xi_{m-1} = \xi_m$ then $\{f_{m-1}, f_m\} \subseteq \mathfrak{M}_{\xi_m}$ and so the elementarity of $\mathfrak{M}_{\xi_m}$ ensures that there is some $g \in \mathfrak{M}_{\xi_m}$ such that $f_{m-j} \leqq^* g$ for each $j \in 2$. Hence $B(\xi_0, f_0) \cap B(\xi_1, f_1) \ldots B(\xi_m, f_m)$ contains

$$B(\xi_0, f_0) \cap B(\xi_1, f_1) \cap \ldots \cap B(\xi_{m-2}, f_{m-2}) \cap B(\xi_m, g)$$

and this set is infinite by the induction hypothesis.

On the other hand, if $\xi_{m-1} \in \xi_m$ then

$$B = B(\xi_0, f_0) \cap B(\xi_1, f_1) \cap \ldots \cap B(\xi_{m-1}, f_{m-1})$$

is infinite by the induction hypothesis and, moreover, B belongs to $\mathfrak{M}_{\xi_m}$ because all the parameters defining it do. Since $\{c_\xi \restriction B : \xi \in \mathfrak{d}\}$ is unbounded in $^B\omega$ it follows that there must be some $\mu \in \mathfrak{M}_{\xi_m}$ such that $f_m \restriction B \nleqq^* c_\mu \restriction B$ and so $f_m \restriction B \nleqq^* c_{\alpha(\xi)} \restriction B$. Since $B(\xi_0, f_0) \cap B(\xi_1, f_1) \cap \ldots \cap B(\xi_m, f_m) = B \cap B(\xi_m, f_m)$ this is enough. $\square$

Theorem 2.4. *Axiom 2 implies Axiom 5($\mathcal{N}$).*

Proof. In [5] it is shown that Axiom 2 is equivalent to the equality $\mathfrak{b} = \mathfrak{d}$. Since $\mathfrak{b}$ is regular it follows that $\mathfrak{d}$ is regular. Moreover, if $\mathscr{C} \subseteq {}^\omega\omega$ is an unbounded ($\leqq^*, \mathcal{N}$)-chain then $\mathrm{cof}(\mathscr{C}) = \mathfrak{d}$. Since $\mathscr{C}$ consists of nondecreasing functions it is clear that $\{c \restriction A : c \in \mathscr{C}\}$ is unbounded for each infinite set A. Hence, by Lemma 2.3, it follows that there is an ultrafilter $\mathscr{U}$ such that $\mathscr{C}$ is unbounded modulo $\mathscr{U}$. $\square$

3 Oracle chain conditions and locally Cohen partial orders

It will be shown that there is a model of set theory where Axiom 6.5 fails. This answers the first two questions in Problem 5 of [5]. C. Laflamme has remarked that in some models of NCF (see [2] for any overview of this area) Axiom 6.5 fails as well because it is possible to provide a classification of chains in these models. The restriction to chains does not play an important role in this theorem and, in fact, the theorem is slightly stronger than required – at least formally – because of this.

Theorem 3.1. *There is a model where* $\mathrm{cof}(^\omega\omega/\mathscr{U}) = \omega_2$ *for every ultrafilter* $\mathscr{U}$ *but every unbounded subset of* $^\omega\omega$ *has an unbounded subset of size* $\aleph_1$.

Proof. The plan of the proof is to start with a model V in which $\Diamond^*_{\omega_1}$ and $\Diamond_{\omega_2}(\omega_1)$ – in other words, the trapping of subsets of ω_2 occurs at ordinals of cofinality ω_1 in ω_2 – both hold. In this model a finite support iteration $\{(\mathbb{P}_\xi, \mathbb{Q}_\xi) : \xi \in \omega_2\}$ will be constructed along with a sequence of oracles [6] $\{\mathfrak{M}_\xi : \xi \in \omega_2\}$ – more precisely, $\mathfrak{M}_\xi$ is a $\mathbb{P}_\xi$-name for an oracle. The oracles will be chosen so that if $\{g_\eta : \eta \in \omega_1\}$ is a $\mathbb{P}_\xi$-name, guessed by the $\Diamond_{\omega_2}(\omega_1)$ sequence, for an unbounded subset of $^\omega\omega$ then $\mathfrak{M}_\xi$ is chosen so that if $\mathbb{Q}$ is any partial order satisfying the $\mathfrak{M}_\xi$-chain condition then forcing with $\mathbb{Q}$ does not destroy the unboundedness of $\{g_\eta : \eta \in \omega_1\}$. Provided that $\mathbb{P}_{\omega_2}/\mathbb{P}_\xi$ satisfies the $\mathfrak{M}_\xi$-chain condition, it will follow that every unbounded subset of $^\omega\omega$ has cofinality ω_1 because every unbounded subset is reflected at some initial stage by the $\Diamond_{\omega_2}(\omega_1)$ sequence. The rest of the result will follow once it is shown how to construct $\mathbb{Q}_\xi$ satisfying the $\mathfrak{M}_\xi$-chain condition and adding an upper bound to any given sequence from some ultrapower of the integers.

The construction of $\{(\mathbb{P}_\xi, \mathbb{Q}_\xi) : \xi \in \omega_2\}$ and $\{\mathfrak{M}_\xi : \xi \in \omega_2\}$ is, of course, done by induction. If ξ is a limit then $\mathbb{P}_\xi$ is simply the direct limit of $\{\mathbb{P}_\mu : \mu \in \xi\}$. The construction of $\mathfrak{M}_\xi$ and $\mathbb{Q}_\xi$ does not depend on whether or not ξ is a limit.

Given $\mathbb{P}_\xi$, use the results of p. 124 to 127 of [6] to find a $\mathbb{P}_\xi$-name for a single oracle $\mathfrak{N}_\xi$ such that if $\mathbb{Q}$ satisfies the $\mathfrak{N}$-chain condition then it satisfies the $\mathfrak{M}_\mu$ chain condition for each $\mu \in \xi$. Let C_ξ be the set guessed by the $\Diamond_{\omega_2}(\omega_1)$ sequence at ξ. If C_ξ is not a $\mathbb{P}_\xi$-name for an unbounded subset of $^\omega\omega$ then let $\mathfrak{M}_\xi = \mathfrak{N}_\xi$. Otherwise, use Lemma 2.1 on p. 122 of [6] to find an oracle $\mathfrak{M}$ such that if $\mathbb{Q}$ satisfies the $\mathfrak{M}$-chain condition then the subset $C_\xi \subseteq {}^\omega\omega$ remains unbounded after forcing with $\mathbb{Q}$. The use of Lemma 2.1 requires checking that if $C \subseteq {}^\omega\omega$ is an unbounded chain then adding a Cohen real will not destroy its unboundedness. This is a result of the folklore which can be found in [7]. Then use the results of p. 124 to 127 of [6] to find a single oracle $\mathfrak{M}_\xi$ such that any $\mathbb{Q}$ which satisfies the $\mathfrak{M}_\xi$-chain condition will also satisfy the $\mathfrak{M}$-chain condition and the $\mathfrak{N}_\xi$-chain condition.

Suppose that the $\Diamond_{\omega_2}(\omega_1)$ sequence has also trapped a filter $\mathscr{U}_\xi$ – which is an ultrafilter in the intermediate generic extension by $\mathbb{P}_\xi$ – and an increasing sequence $\{f_\mu^\xi : \mu \in \omega_1\}$ in the reduced power of the integers modulo $\mathscr{U}_\xi$. (So it is being assumed that, by some coding, the $\Diamond_{\omega_2}(\omega_1)$ sequence traps triples of sets – the first component of the triple at ξ is a candidate for C_ξ in the construction of $\mathfrak{M}_\xi$ while the second and third components are candidates for the ultrafilter $\mathscr{U}_\xi$ and the sequence $\{f_\mu^\xi : \mu \in \omega_1\}$.) The only thing left to do is to construct $\mathbb{Q}_\xi$ satisfying the $\mathfrak{M}_\xi$-chain condition and adding an upper bound for $\{f_\mu^\xi : \mu \in \omega_1\}$ in the reduced power modulo $\mathscr{U}_\xi$.

Let $\mathfrak{M}_\xi = \{M_\xi^\mu : \mu \in \omega_1\}$. The partial order $\mathbb{Q}_\xi$ is constructed by induction on ω_1 in $V^{\mathbb{P}_\xi}$ – it will be similar to the forcing which adds a dominating real but with extra side conditions. In particular, a sequence of partial functions $\{S_\mu : \mu \in \omega_1\} \subseteq {}^\omega\omega$ is constructed by induction on ω_1 and $\mathbb{Q}_\xi^\mu$ is defined to be the set of all pairs (F, Γ) such that $F : k \to \omega$ is a finite partial function and $\Gamma \in [\mu]^{<\aleph_0}$. The ordering on $\mathbb{Q}_\xi^\mu$ is defined by $(F, \Gamma) \leqq (F', \Gamma')$ provided that $\Gamma \subseteq \Gamma'$, $F \subseteq F'$ and $F'(j) \geqq S_\gamma(j)$ for $\gamma \in \Gamma$ and $j \in (\mathrm{dom}(S_\gamma) \backslash \mathrm{dom}(F))$. Moreover, the functions S_μ will be constructed so that $\mathrm{dom}(S_\mu) \in \mathscr{U}_\xi$ and $S_\mu(j) \geqq f_\mu^\xi(j)$ for each $j \in \mathrm{dom}(S_\mu)$. It is easy to see that $\{(F, \Gamma) : \mu \in \Gamma\}$ is dense in $\mathbb{Q}_\xi^\eta$ for every $\mu \in \eta$ and so if G is $\mathbb{Q}_\xi = \mathbb{Q}_\xi^{\omega_1}$ generic over $V^{\mathbb{P}_\xi}$ then $\cup \{F : (\exists \Gamma)((F, \Gamma) \in G)\}$ is an upper bound for $\{f_\mu^\xi : \mu \in \omega_1\}$ in the reduced power with respect to $\mathscr{U}_\xi$. It therefore suffices to construct $\{S_\mu : \mu \in \omega_1\}$ so that for every $\mu \in \omega_1$, every dense open subset of $\mathbb{Q}_\xi^\mu$ which belongs to M_ξ^μ remains predense in $\mathbb{Q}_\xi^{\mu+1}$. This, of course, will ensure that $\mathbb{Q}_\xi = \mathbb{Q}_\xi^{\omega_1}$ satisfies the $\mathfrak{M}_\xi$-chain condition.

Suppose that $\{S_\mu : \mu \in \eta\}$ have been constructed. Let $\mathscr{A}$ be the set of all of the dense open subsets of $\mathbb{Q}_\xi^\eta$ which belong to M_η – this includes all those dense open subsets of $\mathbb{Q}_\xi^\eta$ which belong to M_ζ some $\zeta \in \eta$. Choose $h \in {}^\omega\omega$ to be some function which dominates all members of M_η; in other words, if $g \in {}^\omega\omega \cap M_\eta$ then $g \leqq^* h$. Let $\{(A_i, (F_i, \Gamma_i)) : i \in \omega\}$ enumerate $\mathscr{A} \times \mathbb{Q}_\xi^\eta$. Now choose, by induction on ω, integers $\{K_i : i \in \omega\}$ such that $K_i < K_{i+1}$ and $K_0 = 0$. Given K_i, define $\bar{F}_j^i \supset F_j$ for each $j \leqq i$ such that if $\mathrm{dom}(F_j) \subseteq K_i$ then $\mathrm{dom}(\bar{F}_j^i) = K_i$, $(F_j, \Gamma_j) \leqq (\bar{F}_j^i, \Gamma_j)$ and $\bar{F}_j^i(k) \geqq h(k)$ if $k \in \mathrm{dom}(\bar{F}_j^i \backslash F_j)$. Now choose $(F_j^i, \Gamma_j^i) \in A_j$ such that $(F_j^i, \Gamma_j^i) \geqq (\bar{F}_j^i, \Gamma)$. Let K_{i+1} be such that $\mathrm{dom}(F_j^i) \subset K_{i+1}$ for each $j \leqq i$.

Let $X_m = \bigcup_{i \in \omega} [K_{2i+m}, K_{2i+m+1})$ for $m \in 2$ and note that there exists $m' \in 2$ such that $X_{m'} \in \mathscr{U}_\xi$. Let $S_\mu = h \upharpoonright X_{m'}$; the reason being that $\mathscr{U}_\xi$ is an ultrafilter in

$V^{\mathbb{P}}\xi$ and $X_0 \in V^{\mathbb{P}}\xi$. To see that this definition of S_μ ensures that every dense open subset of $\mathbb{Q}^\mu_\xi$ which belongs to M^μ_ξ remains predense in $\mathbb{Q}^{\mu+1}_\xi$ let $(F, \Gamma) \in \mathbb{Q}^{\mu+1}_\xi$ and let $D \in M_\mu$ be dense open in $\mathbb{Q}^\mu_\xi$. It follows that $(F, \Gamma \backslash \{\mu\}) \in \mathbb{Q}^\mu_\xi$. To simplify notation assume that $m = 1$. Choose j such that $\mathrm{dom}(F) \subseteq K_{2j}$ and such that $(D, (F, \Gamma \backslash \{\mu\})) = (A_k, (F_k, \Gamma_k))$ for some $k \leq 2j$. Since $\mathrm{dom}(F^{2j}_k) \subseteq K_{2j}$ it follows that $\bar{F}^{2j}_k(n) \geq h(n)$ if $n \in \mathrm{dom}(\bar{F}^{2j}_k \backslash F_k)$. Moreover $[K_{2j}, K_{2j+1}) \cap \mathrm{dom}\, S_\mu = \emptyset$. Hence $(F^{2j}_k, \Gamma^{2j}_k)$ is compatible with (F, Γ). $\square$

The methods of the previous theorem can also be used to show that it is consistent that Axiom 6 holds but Axiom 5.5 fails. In establishing this it will be helpful to introduce the following definition.

Definition 3.1. A partial order $(\mathbb{P}, \leq)$ will called *locally Cohen* if for every $X \in [\mathbb{P}]^{\aleph_0}$ there is $Y \in [\mathbb{P}]^{\aleph_0}$ such that $X \subseteq Y$ and Y is completely embedded in $\mathbb{P}$ – in other words, if $A \subseteq Y$ is a maximal antichain in the partial order $(Y, \leq \cap Y \times Y)$ then it is also maximal in $(\mathbb{P}, \leq)$.

The notion of locally Cohen partial orders has already been isolated and investigated by Just in [4] who refers to locally Cohen partial orders as *harmless*. The motivation of Just was that any locally Cohen forcing satisfies the oracle chain condition for every oracle.

Let $\mathbb{S}(\lambda)$ be the canonical partial order for adding a scale of length λ in $^\omega\omega$ with finite conditions. To be precise, a condition p belongs to $\mathbb{S}(\lambda)$ if and only if $p: \Gamma_p \times n_p \to \omega$ is a function and $\Gamma_p \in [\lambda]^{<\aleph_0}$ and $n_p \in \omega$. The ordering on $\mathbb{S}(\lambda)$ is $\leq$ defined by $p \leq q$ if and only if:
- $p \subseteq q$
- if $\{\xi, \eta\} \subseteq \Gamma_p$ and $\xi \in \eta$ then $q(\eta, m) \geq q(\xi, m)$ for every $m \in n_q \backslash n_p$

It should be noted that $\mathbb{S}(\lambda)$ is also the finite support iteration of length λ of the partial orders $\{\mathbb{D}(\xi): \xi \in \lambda\}$ where $\mathbb{D}(\xi)$ is the finite condition forcing for adding a nondecreasing function – which wil be denoted by c_ξ – which dominates all the reals $\{c_\eta: \eta \in \xi\}$.

Lemma 3.1. *For any ordinal λ the partial order $\mathbb{S}(\lambda)$ is locally Cohen.*

Proof. Given $X \in [\mathbb{P}]^{\aleph_0}$ let $Y \in [\lambda]^{\aleph_0}$ be any set such that $\mathbb{S}(Y) \supseteq X - \mathbb{S}(Y)$ can be defined for any set of ordinals in the same way that $\mathbb{S}(\lambda)$ is defined for an ordinal λ. To see that $\mathbb{S}(Y)$ is completely embedded in $\mathbb{S}(\lambda)$ let $A \subseteq \mathbb{S}(Y)$ be a maximal antichain in $\mathbb{S}(Y)$. If $p \in \mathbb{S}(\lambda)$ then let $\Gamma' = \Gamma \cap Y$ and $p' = p \upharpoonright \Gamma' \times n_p$. Since $p' \in \mathbb{S}(Y)$ there must be some $q \in A$ such that $q' = p' \cup q \in \mathbb{S}(Y)$ and $p \leq q'$. Define $q'': (\Gamma_q \cup \Gamma_p) \times n_q \to \omega$ by

$$q''(\xi, j) = \begin{cases} q'(\xi, j) & \text{if } \xi \in Y \\ \max\{q'(\eta, j): \eta \in Y \cap \xi\} & \text{if } j \notin n_p \text{ and } \xi \notin Y \\ p(\xi, j) & \text{if } \xi \notin Y \text{ and } j \in n_p \end{cases}$$

It is easy to check that $p \leq q''$. $\square$

Theorem 3.2. *There is a model of set theory where*
- $2^{\aleph_0} = \aleph_2$
- $\mathfrak{b} = \aleph_1$
- *there is a an unbounded $(\leq^*, {}^\omega\omega)$-chain of length ω_2*
- *the cofinality of any ultrapower of the integers is ω_2*

Proof. As in the proof of Theorem 3.1, let V be a model of set theory where $\Diamond^*_{\omega_1}$ and $\Diamond_{\omega_2}(\omega_1)$ are both satisfied. Let H be $\mathbb{S}(\omega_2)$ generic over V and define $H_\xi = H \cap \mathbb{S}(\xi)$. Let $c_\xi(n) = (\cup H)(\xi, n)$ and observe that $\{c_\xi : \xi \in \omega_2\} \subseteq {}^\omega\omega$ is increasing with respect to $\leq^*$ and $\{c_\xi : \xi \in \omega_2\}$ is not bounded. If an oracle chain condition forcing extension of $V[H]$ can be found which preserves the unboundedness of $\{c_\xi : \xi \in \omega_2\}$ and in which the cofinality of any ultrapower of the integers is ω_2 then the result will follow because $\mathfrak{b} = \aleph_1$ is easily preserved by the oracle chain condition.

To do this, construct $\{(\mathbb{P}_\xi, \mathbb{Q}_\xi) : \xi \in \omega_2\}$ and $\{\mathfrak{M}_\xi : \xi \in \omega_2\}$ exactly as in the proof of Theorem 3.1 except that $\mathfrak{M}_\xi$ is chosen to be an oracle in the model $V[G, H_\xi]$ where G is generic over $\mathbb{P}_\xi$. There is no problem in doing this because $\mathbb{S}(\xi)$ is locally Cohen and hence satisfies the $\mathfrak{M}_\mu$ chain condition for each $\mu \in \xi$ – indeed, $\mathbb{S}(\xi)$ satisfies any oracle chain condition. It is therefore easy to use Claim 3.3 on p. 127 of [6] to obtain $\mathfrak{M}_\xi$ exactly as in the proof of Theorem 3.1.

Let G be $\mathbb{P}_{\omega_2}$ generic over $V[H]$. Exactly as in the proof of Theorem 3.1, it can be shown that the cofinality of any ultrapower of the integers is ω_2 while $\mathfrak{b} = \aleph_1$ in $V[G, H]$. On the other hand, the fact that $\{c_\xi : \xi \in \omega_2\}$ is unbounded follows from genericity and the fact that $\mathbb{S}(\xi + 1) = \mathbb{S}(\xi) * \mathbb{D}(\{x_\zeta : \zeta \in \xi\})$ – so c_ξ is not dominated by any function from $V[G_\xi, H_\xi]$ where G_ξ is the restriction of G to $\mathbb{P}_\xi$. □

Corollary 3.1. *Axiom 6 does not imply Axiom 5.5.*

To see that the model constructed in Theorem 3.2 is a model of Axiom 6 but not of Axiom 5.5 observe first that Axiom 5.5 fails because $\mathfrak{b} = \aleph_1$ while the cofinality of any ultrapower of the integers is ω_2. On the other hand, there is a $(\leq^*, {}^\omega\omega)$-chain of length $\omega_2 \geq \mathfrak{d}$. Using Lemma 2.1 it is possible to construct from this a $(<^*, \mathscr{S})$-chain, $\mathscr{C}'$, of nondecreasing functions. From Lemma 2.3 it follows that there is an ultrafilter $\mathscr{U}$ on ω such that $\mathscr{C}'$ is cofinal in the ultrapower of the integers modulo $\mathscr{U}$. This is the statement of Axiom 6. □

The partial order $\mathbb{S}(\lambda)$ can be modified to yield a model where Axiom 4 holds yet Axiom 5.5 fails. Recall that Lemma 2.1 implies that to do this it is only necessary to find a model of Axiom 4($\leq$, ${}^\omega\omega$) and the failure of Axiom 5.5.

Theorem 3.3. *If set theory is consistent then there is a model of set theory where* $\mathfrak{b} = \aleph_1$ *yet for every ultrafilter $\mathscr{U}$ there is a $(\leq^*, {}^\omega\omega)$-chain of length ω_2 which is cofinal in ${}^\omega\omega / \mathscr{U}$.*

Proof. It will be shown that, assuming $\Diamond_{\omega_2}(\omega_1)$, there is a locally Cohen partial order $\mathbb{P}$ such that if G is $\mathbb{P}$ generic then $\mathrm{cof}({}^\omega\omega / \mathscr{U}) = \omega_2$ for every ultrafilter $\mathscr{U}$ in $V[G]$. The fact that $\mathbb{P}$ is locally Cohen will guarantee that $\mathfrak{b} = \aleph_1$ in $V[G]$.

To construct $\mathbb{P}$ some preliminary bookkeeping is required. Let $\{D_\xi : \xi \in \omega_2\}$ be a $\Diamond_{\omega_2}(\omega_1)$ sequence and let $\{g_\xi : \xi \in \omega_2\}$ enumerate names for elements of ${}^\omega\omega$ which arise from countable chain condition forcing partial orders on ω_2. Also, if $\mathbb{Q}$ is any partial order of size $\aleph_2$ and satisfying the countable chain condition then any subset of the reals in a $\mathbb{Q}$ generic extension has a name of size $\aleph_2$. Consequently, it is possible to use subsets of ω_2 to code such names for sets of reals. If $\mathscr{X}$ is some name – in a suitable partial order – for a subset of $[\omega]^{\aleph_0}$ then $c(\mathscr{X})$ will denote the subset of ω_2 which codes it while if $X \subseteq \omega_2$ then $d(X)$ will denote the name it codes. The details of the coding will not be important. Define a partial order $\prec$ on ω_2 by $\xi \prec \eta$ if and only if $c(D_\xi) = c(D_\eta \cap \xi)$.

Now construct $\mathbb{P}$ as a finite support iteration of $\{\mathbb{P}_\xi : \xi \in \omega_2\}$ such that $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \mathbb{C}_\xi \times \mathbb{D}_\xi$ where $\mathbb{C}_\xi$ adds a Cohen real, $A_\xi : \omega \to 2$ and $\mathbb{D}_\xi$ is some partial order

which has yet to be defined. At the same time, construct a partial function $\Theta:\omega_2 \to \omega_2$ so that if μ is in the domain of Θ then $1 \Vdash_{\mathbb{P}_\mu}$ "$d(D_\mu)$ is an ultrafilter" and $\Theta(\mu)$ is the minimum ordinal such that $\Theta(\mu) \notin \{\Theta(\gamma):\gamma \prec \mu\}$ and such that $g_{\Theta(\mu)}$ is a $\mathbb{P}_\mu$ name.

Given $\mathbb{P}_\eta$, define D_η by $p \in D_\eta$ if and only if
- $p = (f_p, \Gamma_p)$
- $f_p \in {}^\omega\omega$
- $\Gamma_p \in [\eta]^{<\aleph_0}$
- if $\gamma \in \Gamma_p$ then $\gamma \prec \eta$

and $p \leq q$ is defined to hold if and only if
- $f_p \subseteq f_q$
- $\Gamma_p \subseteq \Gamma_q$
- if $\gamma \in \Gamma_p$, $A_\gamma^{-1}\{k\} \in d(D_\eta)$, $A_\gamma(n) = k$ and $n \in \text{dom}(f_q \backslash f_p)$ then $f_q(n) \geq \max\{g_{\Theta(\gamma)}(n), F_\gamma(n)\}$ where, for any $\xi \in \omega_2$, F_ξ is the generic function added by the partial order D_ξ – to be precise, $F_\xi = \cup\{f_p:p \in G\}$ where G is D_ξ generic.

If it is possible to extend Θ to include η in its domain then do so – there is no ambiguity here because an extension, if it exists, is unique.

Let $\mathbb{P} = \mathbb{P}_{\omega_2}$. It will soon be shown that $\mathbb{P}$ satisfies the countable chain condition. However, first suppose that G is $\mathbb{P}$ generic over V and that $\mathscr{U}$ is the $\mathbb{P}$ name for an ultrafilter in $V[G]$. There is then a stationary set, $S(\mathscr{U})$, such that if $\xi \in S(\mathscr{U})$ then $1 \Vdash_{\mathbb{P}_\xi}$ "$d(c(\mathscr{U}) \cap \xi)$ is an ultrafilter". It will be shown that $\{F_\xi:\xi \in S(\mathscr{U})\}$ is a $(\leq^*, {}^\omega\omega)$-chain which is cofinal in $^\omega\omega/\mathscr{U}$. The fact that it is a $\leq^*$ increasing sequence is an immediate consequence of the definition of D_ξ.

To see that it is cofinal in $^\omega\omega/\mathscr{U}$ let $g \in {}^\omega\omega$. Then, assuming that $\mathbb{P}$ has the countable chain condition, there is some $\theta \in \omega_2$ and $\mu \in \omega_2$ such that g_μ is a $\mathbb{P}_\theta$ name for g. It follows that there is some $\zeta \in S(\mathscr{U})$ such that $\mu = \Theta(\zeta)$. Let $\eta \in S(\mathscr{U})\backslash(\zeta + 1)$ and note that $\zeta \prec \eta$. Hence, the partial order D_η adds a function which dominates g_μ on $A_\zeta^{-1}\{k\}$ for some $k \in 2$ and, moreover, $A_\zeta^{-1}\{k\} \in D_\eta$.

It remains to be shown that $\mathbb{P}$ satisfies the countable chain condition and that $\mathfrak{b} = \aleph_1$ after forcing with $\mathbb{P}$. Both these facts will follow once it has been shown that $\mathbb{P}$ is locally Cohen. To this end, it is worth observing that $\mathbb{P}$ has a dense set of conditions which are somewhat determined – a condition p will be said to be *somewhat determined* if the support of p is $\Sigma^p \in [\omega_2]^{<\aleph_0}$ and there is an integer $n(p)$ such that
- for each $\sigma \in \Sigma^p \cap \text{dom}(\Theta)$ there is $h_0^{p,\sigma}:n(p) \to 2$, $h_1^{p,\sigma}:n(p) \to \omega$ and $\Delta^{p,\sigma} \in [\sigma]^{<\aleph_0}$ such that $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$p(\sigma) = (h_0^{p,\sigma}, h_1^{p,\sigma}, \Delta^{p,\sigma})$"
- $\Delta^{p,\sigma} \subseteq \{\zeta \in \Sigma^p \cap \sigma:\zeta \prec \sigma\}$
- for each $\sigma \in \Sigma^p$ and $\tau \in \Sigma^p$ such that $\sigma \prec \tau$ there is $k(p,\sigma,\tau) \in 2$ such that $p \restriction \tau \Vdash_{\mathbb{P}_\tau}$ "$A_\sigma^{-1}\{k(p,\sigma,\tau)\} \in d(D_\tau)$"
- for each $\sigma \in \Sigma^p$ and $\tau \in \Sigma^p$ such that $\sigma \prec \tau$ there is $M_p(\sigma) \in \omega$ and $G_p(\sigma):M_p(\sigma) \to \omega$ such that $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$g_{\Theta(\sigma)} \restriction M_p(\sigma) = G_p(\sigma)$" and, moreover, $h_0^{p,\sigma}(i) = 1 + k(p,\sigma,\tau) \bmod 2$ provided that $i \in n(p)\backslash M_p(\sigma)$.

The fact that the set of somewhat determined conditions in $\mathbb{P}_\eta$ is dense in $\mathbb{P}_\eta$ will be proved by induction, but an extra induction hypothesis is necessary. What will be shown by induction on η is that, given
- $p \in \mathbb{P}_\eta$
- any finite set W of maximal elements of $\prec \cap(\eta \times \eta)$

– any function $v: W \to 2$

– any function $a: W \to \omega_2$ such that $g_{a(\xi)}$ is a $\mathbb{P}_\xi$ name for each $\xi \in W$

there is a determined condition q – the fact that q is determined is witnessed by $n(q)$ – with the additional properties that for each $\xi \in W$ there is $M(\xi) \in \omega$ and $G(\xi): M(\xi) \to \omega$ such that $q \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "$g_{a(\xi)} \restriction M(\xi) = G(\xi)$" and, moreover,

$h_0^{p,\xi}(i) = v(\xi)$ provided that $i \in n(q) \backslash M(\xi)$.

If $\eta = 0$ this is trivial and if η is a limit ordinal then it follows from the fact that a finite support iteration is being used. Therefore, suppose that the fact has been established for η and that $p = (p \restriction \eta, (h_0, h_1, \Gamma)) \in \mathbb{P}_{\eta+1}$. Suppose also that W, $v: W \to 2$ and $a: W \to \omega_2$ have been given so that W is a finite set of maximal elements of $\prec \cap (\eta + 1)^2$. Notice that $\prec \cap (\eta + 1)^2$ has at most one maximal element, η, which is not maximal in $\prec \cap (\eta \times \eta)$. It is, of course, possible that some maximal element in $\prec \cap (\eta \times \eta)$ is no longer maximal in $\prec \cap (\eta + 1)^2$. If there is such a new non-maximal element, then denote it by θ; if not, then the following argument is a bit easier and so it will be assumed that θ exists. Find $q \geq p \restriction \eta$ and $H_0: I_0 \to 2$ and $H_1: I_1 \to \omega$ as well as $k \in 2$ and $\Delta \in [\eta]^{\aleph_0}$ such that

– $q \Vdash_{\mathbb{P}_\eta}$ "$h_0 = H_0$ and $h_1 = H_1$"

– $q \Vdash_{\mathbb{P}_\eta}$ "$g_{a(\eta)} \restriction I_0 = G$" for some $G: I_0 \to \omega$ (if $\eta \notin W$ this can be ignored)

– $q \Vdash_{\mathbb{P}_\eta}$ "$\Gamma = \Delta$"

– $q \Vdash_{\mathbb{P}_\eta}$ "$A_\theta^{-1}\{k\} \in d(D_\eta)$"

That it is possible to arrange for the first two clauses follows from the fact that $g_{a(\eta)}$ is a $\mathbb{P}_\eta$ name and so any information about it can be obtained without changing h_0 or h_1. To satisfy the last clause, use the fact that $\theta \prec \eta$, which follows because θ is no longer maximal in $\prec \cap (\eta + 1)^2$.

Now define $W' = (W \backslash \{\eta\}) \cup \{\theta\}$ and observe that W' is a set of maximal elements in $\prec \cap (\eta \times \eta)$. Define $v' = v \restriction W' \cup \{(\theta, k + 1 \bmod 2)\}$ and $a' = a \restriction W' \cup \{(\theta, \Theta(\theta))\}$ and observe that both a' and v' are still functions of the right type. Then use the induction hypothesis on η to find $q' \geq q$ which is somewhat determined and such that this is witnessed by $n(q')$ and, such that for each $\xi \in W'$ there is $M(\xi) \in \omega$ and $G(\xi): M(\xi) \to \omega$ such that $q' \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "$g_{a'(\xi)} \restriction M(\xi) = G(\xi)$" and, moreover,

$h_0^{p,\xi}(i) = v'(\xi)$ provided that $i \in n(q) \backslash M(\xi)$. Without loss of generality, $n(q') \geq I_0$ and $n(q') \geq I_1$.

Then let $p' = (q', (h_0', h_1', \Gamma))$ where $h_0': n(q') \to 2$ is the extension of H_0 to $n(q')$ such that $h_0'(i) = v(\eta)$ if $i \in n(q') \backslash I_0$ and h_1' is the extension of H_1 such that

$$h_1'(i) = \max(\{f_\gamma(i): \gamma \prec \eta \text{ and } \gamma \in \Gamma\} \cup \{G(\Theta(\gamma))(i): \gamma \prec \eta \text{ and } \gamma \in \Gamma\}$$

for $i \in n(q') \backslash I_1$. Notice that maximum is taken over actual integers rather than names for integers. The definition also respects the requirements of extension in the partial order $\mathbb{P}_\eta$. Defining $M(\eta) = I_0$ and $G(\eta) = G$ satisfies the extra induction hypothesis.

To see that $\mathbb{P}$ is locally Cohen let $X \in [\mathbb{P}]^{\aleph_0}$. Let $\mathfrak{M}$ be a countable elementary submodel of $(H(\omega_3), \mathbb{P}, \{D_\xi: \xi \in \omega_2\}, \Theta, X)$. It suffices to show that $\mathbb{P} \cap \mathfrak{M}$ is completely embedded in $\mathbb{P}$. To see that this is so, let $A \subseteq \mathbb{P} \cap \mathfrak{M}$ be a maximal antichain in $\mathbb{P} \cap \mathfrak{M}$ and let $p \in \mathbb{P}$; without loss of generality p can be assumed to be somewhat determined and, moreover, it may be assumed that this is witnessed by $n(p)$. Let p' be defined so that $\text{dom}(p') = \text{dom}(p) \cap \mathfrak{M}$ and $p'(\xi) = (h_0^{p,\xi}, h_1^{p,\xi}, \Sigma^p \cap \xi \cap \mathfrak{M})$ for $\xi \in \text{dom}(p')$. Note that $p' \in \mathfrak{M} \cap \mathbb{P}$. Hence there is $q' \in A$ and $q \in \mathbb{P} \cap \mathfrak{M}$ such that $q \geq q'$ and $q \geq p'$ – without loss of generality it may be assumed that q is determined and this is witnessed by $n(q)$. It must be shown that p and q are compatible.

As in the proof that $\mathbb{S}(\lambda)$ is locally Cohen, for $\sigma \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ extend $h_1^{p,\sigma}$ to h_1^σ by defining

$$h_1^\sigma(m) = \max(\{h_1^{q,\tau}(m) : \tau \prec \sigma \text{ and } \tau \in \mathrm{dom}(q)\}$$
$$\cup \{G_q(\tau)(m) : \tau \prec \sigma \text{ and } \tau \in \mathrm{dom}(q) \text{ and } m \in M_q(\tau)$$
$$\text{and } A_\tau(m) \neq k(q, \tau, \sigma)\})$$

for $m \in n(q) \setminus n(p)$, recalling that $G_q(\xi)$, $M_q(\xi)$ and $k(q, \xi, \eta)$ are witnesses to the fact that q is somewhat determined. This will certainly assure that if τ and ϱ are in the domain of q and $\tau \prec \sigma \prec \varrho$ then $h_q^{q,\tau}(m) \leq h_q^\sigma(m) \leq h_1^{q,\varrho}(m)$ – the fact that $h_1^\sigma(m) \leq h_1^{q,\varrho}(m)$ follows from the definition of the third coordinates in p'. Also, $g_{\Theta(\tau)}(m) \leq h_1^\sigma(m)$ if $A_\tau(m) \neq k(p, \tau, \sigma)$ will be true if $m \in M(\tau)$ by construction.

Next, if $\sigma \in \mathrm{dom}(p) \setminus \mathfrak{M}$, $\tau \in \mathrm{dom}(p) \cap \mathfrak{M}$ and $\sigma \prec \tau$ then define $h_0^\sigma(m) = 1 + k(p, \sigma, \tau) \bmod 2$. If this can be done then, if $m \geq n(p)$, it is not necessary for $h_1^{q,\tau}(m)$ to be greater than $g_{\Theta(\sigma)}(m)$. If there is no $\tau \in \mathrm{dom}(p) \cap \mathfrak{M}$ such that $\sigma \prec \tau$, do not extend $h_0^{p,\sigma}$ at all. Notice that in this last case it is still possible that there is some $\tau \in \mathrm{dom}(q)$ such that $\sigma \prec \tau$. However, because it is only necessary for $h_1^{q,\tau}(m)$ to be greater than $g_{\Theta(\sigma)}(m)$ in case $\sigma \in \Delta^{q,\tau}$, this will cause no problems because $\Delta^{q,\tau} \subseteq \mathfrak{M}$ if $\tau \in \mathfrak{M}$.

What must be checked, though, is that no conflict arises as a result of this definition of h_0^σ. After all, it is conceivable that $\sigma \prec \tau$ and $\sigma \prec \tau'$ but $k(q, \sigma, \tau) \neq k(q, \sigma, \tau')$. To see that this does not happen, suppose that $\sigma \prec \tau$, $\sigma \prec \tau'$, $k(p, \sigma, \tau) \neq k(p, \sigma, \tau')$ and $\{\tau, \tau'\} \subseteq \mathfrak{M}$. It follows that if

$$\varrho = \sup\{\theta : \theta \prec \tau \text{ and } \theta \prec \tau' \text{ and } \theta \in \mathrm{dom}(\Theta)\}$$

then $\varrho \in \mathfrak{M}$. Hence $\sigma \in \varrho$ and so there is some θ' such that $\theta' \prec \tau$, $\theta' \prec \tau'$ and $\theta \in \mathrm{dom}(\Theta)$ such that $\sigma \in \theta'$. Hence $A_\sigma^{-1}\{0\}$ is measured by the ultrafilter $d(D_{\theta'})$. Since $d(D_{\theta'}) \subseteq d(D_\tau)$ and $d(D_{\theta'}) \subseteq d(D_{\tau'})$ it follows that $k(p, \sigma, \tau) = k(p, \sigma, \tau')$. $\square$

4 Open questions

Table 1 of implications and non-implications summarizes the known results about the axioms discussed in this paper. The key to understanding Table 1 is that
– if there is a "$\Rightarrow$" in the entry in the row headed by Axiom R and the column headed by Axiom C then Axiom R implies Axiom C
– if there is a "$\nRightarrow$" in the entry in the row headed by Axiom R and the column headed by Axiom C then Axiom R is known to be consistent with the negation of Axiom C
– if there is a question mark in the entry in the row headed by Axiom R and the column headed by Axiom C then it is not known whether Axiom R implies Axiom C

Not all the reasons for the assertiona made in Table 1 are contained in this paper. Some will be found in [5] and others must be deduced by modus ponens. Table 2 contains a guide to reasons for the various assertions in Table 1.

A "T" in the row corresponding to Axiom R and the column corresponding to Axiom C in Table 2 indicates that the fact that Axiom R implies Axiom C is a trivial implication – by trivial is meant something which can be deduced by considering the quantifiers in the relevant axioms. An "N" in that entry means that either the

implication or non-implication can be found in [5]. The enumeration of the following list corresponds to the numbered entries in Table 2. So, for example, the second entry of this list refers to Theorem 2.4 because this is the reason there is an "⇒" in Table 1 in the row corresponding to Axiom 2 and the column corresponding to Axiom 5($\mathcal{N}$).

Table 1. Table of implications

Axiom	1	2	3	4	$5 \leq^* {}^\omega\omega$	$5 <^* {}^\omega\omega$	$5\mathcal{N}$	5.5	6	6.5
1	⇒	⇒	⇒	⇒	?	?	⇒	⇒	⇒	⇒
2	⇏	⇒	⇏	⇒	⇏	⇏	⇒	⇒	⇒	⇒
3	⇏	⇏	⇒	⇏	⇏	⇏	⇒	⇒	⇒	⇒
4	⇏	⇏	⇏	⇒	⇏	⇏	⇒	⇏	⇒	⇒
$5 \leq^* {}^\omega\omega$	⇏	?	?	?	⇒	⇒	⇒	⇒	⇒	⇒
$5 <^* {}^\omega\omega$	⇏	?	?	?	?	⇒	⇒	⇒	⇒	⇒
$5\mathcal{N}$	⇏	⇏	⇏	⇏	⇏	⇏	⇒	⇒	⇒	⇒
5.5	⇏	⇏	⇏	⇏	⇏	⇏	?	⇒	⇒	⇒
6	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇒	⇒
6.5	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	?	⇒

Table 2. Table of references

Axiom	1	2	3	4	$5 \leq^* {}^\omega\omega$	$5 <^* {}^\omega\omega$	$5\mathcal{N}$	5.5	6	6.5
1	T	T	T	T	?	?	T	N	T	N
2	N	T	1	T	1	1	2	N	T	N
3	N	N	T	N	9	9	T	N	T	N
4	N	N	N	T	3	3	T	4	T	N
$5 \leq^* {}^\omega\omega$	10	?	?	?	T	T	T	N	T	N
$5 <^* {}^\omega\omega$	10	?	?	?	?	T	T	N	T	N
$5\mathcal{N}$	N	N	N	N	9	9	T	N	T	N
5.5	5	5	5	5	9	9	?	T	N	N
6	5	5	5	5	6	6	8	7	T	N
6.5	5	5	5	5	6	6	5	5	?	T

(1) Theorem 2.1

(2) Theorem 2.3

(3) The fact that Axiom 4 does not imply Axiom 5($<^*, {}^\omega\omega$) follows because it has been shown that Axiom 2 does not imply Axiom 5($<^*, {}^\omega\omega$) in Theorem 2.1 and the fact that Axiom 2 implies Axiom 4 follows from an inspection of the quantifiers involved. Modus ponens yields the rest.

(4) Theorem 3.3

(5) The antecedent of the implication is implied by Axiom 5($\mathcal{N}$) so the non-implication follows from modus ponens because Axiom 5($\mathcal{N}$) does not imply the conclusion.

(6) The antecedent of the implication is implied by Axiom 4 so the non-implication follows from modus ponens.

(7) Corollary 3.1

(8) Axiom 5($\mathcal{N}$) implies Axiom 5.5.

(9) $2^{\aleph_0} = \aleph_1$ is known [3] to imply Axiom 3 and Lemma 2.2, shows that Axiom 5($<^*$, $^\omega\omega$) fails under this assumption. For the rest, use modus ponens.

(10) See the remarks following Theorem 2.3.

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