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Quadrature Formulas for Oscillatory Integral Transforms*

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Summary. Quadrature formulas are obtained for the Fourier and Bessel transforms which correspond to the well-known Gauss-Laguerre formula for the Laplace transform. These formulas provide effective asymptotic approximations, complete with error bounds. Comparison is also made between the quadrature formulas and the asymptotic expansions of these transforms.

Subject Classifications AMS: 65D30, 65R10; CR: 5.16.

1. Introduction

If $w(t)$ is a positive function on $(0, \infty)$ which is rapidly decreasing at infinity, and if $f(t)$ is sufficiently smooth in $(0, \infty)$, then it is well-known that integrals of the form

$$\int_0^\infty f(t)\,w(t)\,dt \tag{1.1}$$

can be numerically evaluated by Gaussian quadrature rules. However, if $w(t)$ is an oscillatory function, such as e^{it} or the Bessel function $J_\nu(t)$, and if $f(t)$ decreases slowly at infinitely, then the problem of numerical computation of these integrals becomes considerably more difficult. Although there are several different methods of treating this problem, (see, e.g., [2, § 3.9], [11, § 5], [14, § 6] and [17]), there do not seem to be any simple quadrature rules for oscillatory integrals over $(0, \infty)$ which correspond to the Gauss-Laguerre quadrature formula for the case when $w(t)$ in (1.1) is e^{-t}. Exceptions are the n-point formulas with weight function $w(t)=\left(1+\begin{matrix}\cos\\ \sin\end{matrix}\, t\right)(1+t)^{-(2n-1+s)}$, n

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$=1(1)10$, $s=1.05(0.05)4$, given in [7]. The purpose of this note is to derive quadrature formulas for the Fourier and the Bessel transforms

$$F(x)=\int_0^\infty t^\mu f(t)\,e^{ixt}\,dt, \tag{1.2}$$

$$H_i(x)=\int_0^\infty t^\mu f(t)\,H_\nu^{(i)}(xt)\,dt, \qquad i=1,2, \tag{1.3}$$

where x is a real parameter and $H_\nu^{(i)}(t)$, $i=1,2$, are the Hankel functions. In (1.2) we require $\mu>-1$, and in (1.3) we require $\mu\pm\nu>-1$. These quadrature formulas in fact also provide asymptotic approximations for the integrals in (1.2) and (1.3), as $x\to\infty$, complete with error bounds. A comparison is then made between these quadrature formulas and the asymptotic expansions of these integrals. Our analysis is based on the use of analytic continuation and an integral analogue of Abel's limit theorem for power series. The basic assumption for our argument is that the function $f(t)$ is holomorphic in the half plane $Re\,t>0$.

2. The Fourier Transforms

We first recall the Gauss-Laguerre formula [2, p. 174]

$$\int_0^\infty t^\mu f(t)\,e^{-t}\,dt=\sum_{k=1}^n w_k\,f(t_k)+E_n(f), \tag{2.1}$$

where

$$E_n(f)=\frac{n!\,\Gamma(n+\mu+1)}{(2n)!}\,f^{(2n)}(\xi), \qquad 0<\xi<\infty. \tag{2.2}$$

The abscissas t_k are the zeros of the Laguerre polynomial

$$L_n^{(\mu)}(t)=e^t\,t^{-\mu}\,\frac{d^n}{dt^n}(e^{-t}\,t^{\mu+n}), \tag{2.3}$$

and the weights w_k are given by

$$w_k=\frac{n!\,\Gamma(n+\mu+1)\,t_k}{[L_{n+1}^{(\mu)}(t_k)]^2}. \tag{2.4}$$

Some tables of t_k and w_k can be found in [9] and [1, p. 923]. If z is real and positive, then it is easy to see that (2.1) can be written in the more general form

$$\int_0^\infty t^\mu f(t)\,e^{-zt}\,dt=z^{-\mu-1}\sum_{k=1}^n w_k\,f(t_k/z)+E_n(f;z), \tag{2.5}$$

where

$$E_n(f;z)=\frac{n!\,\Gamma(n+\mu+1)}{(2n)!\,z^{2n+\mu+1}}\,f^{(2n)}(\xi/z), \qquad 0<\xi<\infty. \tag{2.6}$$

We wish to show that the result (2.5)-(2.6) in fact holds when z is purely imaginary, provided that $f(t)$ is a holomorphic function in the half-plane $Re\, t>0$. To prove this result, we need the following integral analogue of the Abel limit theorem for power series. For a proof of this result, we refer to [15, p. 26].

Lemma. *If the integral* $\int_0^\infty \varphi(t)\,dt$ *converges as an improper Riemann integral, then*

$$\lim_{\varepsilon\to 0} \int_0^\infty e^{-\varepsilon t}\varphi(t)\,dt = \int_0^\infty \varphi(t)\,dt.$$

Theorem 1. *Let* $f(t)$ *be a holomorphic function in* $Re\, t>0$, *and suppose that* $f^{(2n)}(t)$ *is continuous in* $Re\, t\geqq 0$. *If the Fourier transform* $F(x)$ *in* (1.2) *converges as an improper Riemann integral, then we have*

$$F(x) = \frac{e^{(\mu+1)\pi i/2}}{x^{\mu+1}} \sum_{k=1}^{n} w_k f(i\,t_k/x) + \varepsilon_n(f; x), \tag{2.7}$$

where

$$\varepsilon_n(f; x) = \frac{n!\,\Gamma(n+\mu+1)}{(2n)!\,x^{2n+\mu+1}}\, e^{(2n+\mu+1)\pi i/2} f^{(2n)}(i\,\xi/x), \qquad 0<\xi<\infty. \tag{2.8}$$

Proof. The integral on the left-hand side of (2.5) can be considered as the Laplace transform of $t^\mu f(t)$. Thus, by a well-known result from Laplace transform theory, this integral converges for $Re\, z>0$ [18, p. 37, Corollary 1a] and defines a holomorphic function there [18, p. 57, Theorem 5a]. Since $f(t)$ is holomorphic for $Re\, t>0$, the terms on the right-hand side of (2.5) are all holomorphic for $Re\, z>0$. By analytic continuation, the identity in (2.5) holds for all z in $Re\, z>0$. Now, write $z=\varepsilon-ix$ and let $\varepsilon\to 0$ in (2.5). The right-hand side of (2.5) clearly tends to the right-hand side of (2.7), as desired. The fact that the Laplace integral in (2.5) tends to the Fourier integral in (2.7) follows from the above lemma. This proves (2.7).

Remark 1. It is sometimes advantageous to express the truncation error in (2.1) in the form

$$E_n(f) = n!\,\Gamma(n+\mu+1)\, f[t_1, t_1, t_2, t_2, \ldots, t_n, t_n, \xi_1], \tag{2.9}$$

where $0<\xi_1<\infty$ and $f[t_1, t_1, \ldots, t_n, t_n, \xi_1]$ is the $2n$-th divided difference of $f(t)$, relative to the abscissas $t_1, t_1, \ldots, t_n, t_n$ and ξ_1; see [6, p. 397, Eq. (8.7.12)]. The corresponding error term in (2.8) can hence be written as

$$\varepsilon_n(f; x) = \frac{n!\,\Gamma(n+\mu+1)}{x^{2n+\mu+1}}\, e^{(2n+\mu+1)\pi i/2} \times f\left[i\frac{t_1}{x}, i\frac{t_1}{x}, \ldots, i\frac{t_n}{x}, i\frac{t_n}{x}, i\frac{\xi_1}{x}\right], \tag{2.10}$$

where $0<\xi_1<\infty$.

Remark 2. If $f^{(2n)}(t)$ is bounded on the imaginary axis, say by M_{2n}, then from (2.8) we have

$$|\varepsilon_n(f;x)| \leqq \frac{n!\,\Gamma(n+\mu+1)}{(2n)!\,x^{2n+\mu+1}} M_{2n}. \tag{2.11}$$

A similar estimate holds if the divided difference in (2.10) is bounded. In either case, (2.7) provides an attractive asymptotic approximation for the Fourier integral $F(x)$. The asymptotic nature of Gauss quadrature approximations and their superiority over ordinary asymptotic expansions were first noted by Todd [16], and later again by Gautschi [4] and Stenger [12].

3. Numerical Examples

As a preliminary check on the validity of (2.7), we take $-1<\mu<0$ and $f(t)=1$. Since from (2.1) we have

$$\sum_{k=1}^{n} w_k = \Gamma(\mu+1),$$

it follows from (2.7) that

$$\int_0^\infty t^\mu e^{ixt}\,dt = \exp\left\{\frac{(\mu+1)\pi i}{2}\right\} \frac{\Gamma(\mu+1)}{x^{\mu+1}},$$

which is a well-known identity [8, p. 98].

Example 1. The function

$$E^*(x) = \int_0^\infty \frac{e^{ixt}}{1+t}\,dt \tag{3.1}$$

can be expressed in terms of trigonometric integrals $C_i(x)$ and $S_i(x)$; see [1, Chapt. 5]. From (2.7) we have, with $\mu=0$,

$$E^*(x) \approx \sum_{k=1}^{n} w_k \frac{t_k + ix}{t_k^2 + x^2}. \tag{3.2}$$

The weights w_k and abscissas t_k for $2 \leqq n \leqq 15$ are given in [1, p. 923]. The following table shows the closeness of the approximation when we take $n=10$ in (3.2).

Table 1

x	$I(x)$	Approximation
2	$0.144545 \quad +i0.399021$	$0.144544 \quad +i0.399037$
4	$0.0496782 \ +i0.229193$	$0.0496781 \ +i0.229193$
6	$0.0245215 \ +i0.159306$	$0.0245215 \ +i0.159306$
8	$0.0144597 \ +i0.121624$	$0.0144597 \ +i0.121624$
10	$0.00948854+i0.0981910$	$0.00948854+i0.0981910$

Example 2. Consider the integral

$$S(x)=\int_0^\infty t^{1/2}\frac{\sin xt}{1+t}\,dt. \tag{3.3}$$

With $\mu=\frac{1}{2}$, the quadrature formula (2.7) gives

$$S(x)=\frac{1}{\sqrt{2x}}\sum_{k=1}^{n} w_k\frac{x+t_k}{x^2+t_k^2}+\varepsilon_n(x), \tag{3.4}$$

where

$$|\varepsilon_n(x)|\leqq\frac{n!\,\Gamma(n+\frac{1}{2}+1)}{x^{3/2}(x^2+t_1^2)\dots(x^2+t_n^2)}. \tag{3.5}$$

The weights w_k and abscissas t_k here are given in [10]. Let $s_n(x)$ denote the first term on the right-hand side of (3.4). In Table 2 we tabulate the values of $s_n(x)$ for $n=4, 8, 16$ and $x=2, 3, 4, 5$.

Table 2

n \ x	2	3	4	5
4	0.231422	0.13392163	0.088647098	0.0636329754
8	0.232113	0.13411233	0.088672001	0.0636330084
16	0.232087	0.13410937	0.088672219	0.0636330809

The estimate (3.5) shows that $|\varepsilon_{16}(5)|\leqq 2\times 10^{-10}$. Thus, we have $S(5)=0.0636330809$ accurate to at least eight decimal places. For comparison, we also refer the reader to [11, p. 202] for a different way of computing $S(x)$.

4. The Bessel Transforms

Let $K_\nu(t)$ denote the modified Bessel function of the third kind, and let

$$w(t)=t^\mu K_\nu(t), \qquad \mu\pm\nu>-1, \tag{4.1}$$

be the weight function in (1.1) The moments

$$\rho_n=\int_0^\infty t^{n+\mu}K_\nu(t)\,dt, \qquad n=0,1,2,\dots \tag{4.2}$$

can easily be found to be

$$\rho_n=2^{n+\mu-1}\,\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}\mu+\tfrac{1}{2}+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}\mu+\tfrac{1}{2}-\tfrac{1}{2}\nu). \tag{4.3}$$

A sequence of polynomials, $p_n(t)$, orthonormal with respect to the weight function $w(t)$ in $(0,\infty)$ can then be constructed by using determinants; see [13, §§ 2.1 and 2.2]. The zeros of these polynomials are positive and distinct.

Furthermore, between two consecutive zeros of $p_n(t)$, there is exactly one zero of $p_{n+1}(t)$; see [13, § 3.3].

For fixed n, let $t_1, \ldots, t_n$ denote the zeros of $p_n(t)$, and let A_n be the coefficient of t^n in $p_n(t)$. The general quadrature formula [6, Eqs. (8.4.6), (8.4.9) and (8.4.17)] then gives

$$\int_0^\infty t^\mu f(t) K_\nu(z t)\, dt = z^{-\mu-1} \sum_{k=1}^n w_k f(t_k/z) + E_n(f; z), \tag{4.4}$$

where z is real and positive,

$$w_k = \frac{A_n}{A_{n-1} p_n'(x_k) p_{n-1}(x_k)} \tag{4.5}$$

and

$$E_n(f; z) = \frac{f^{(2n)}(\xi/z)}{A_n^2 (2n)!\, z^{2n+\mu+1}}, \qquad 0 < \xi < \infty. \tag{4.6}$$

If $f(t)$ is a holomorphic function in $Re\, t > 0$ then, by analytic continuation, (4.4) also holds for complex z as long as $Re\, z > 0$. The following result is a generalization of (2.7).

Theorem 2. *Let $f(t)$ be a holomorphic function in the half plane* $\mathrm{Re}\, t > 0$, *and suppose that $f^{(2n)}(t)$ is continuous in* $\mathrm{Re}\, t \geqq 0$. *If the Bessel transforms in* (1.3) *converge as improper Riemann integrals, then we have*

$$H_1(x) = \frac{2}{\pi} \frac{e^{i(\mu-\nu)\pi/2}}{x^{\mu+1}} \sum_{k=1}^n w_k f\left(i \frac{t_k}{x}\right) + \delta_n^{(1)}(f; x), \tag{4.7}$$

where t_k and w_k are the nodes and weights, respectively, of the Gauss quadrature formula (4.4), *and*

$$\delta_n^{(1)}(f; x) = \frac{2}{\pi} \frac{e^{i(\mu+2n-\nu)\pi/2}}{x^{2n+\mu+1}} \frac{f^{(2n)}(i \xi_1/x)}{A_n^2 (2n)!}, \qquad 0 < \xi_1 < \infty. \tag{4.8}$$

The corresponding formula for $H_2(x)$ is obtained by replacing i by $-i$ in (4.7) *and* (4.8).

Proof. In (4.4), we first put $z = \varepsilon - i x$ and then let $\varepsilon \to 0$. The right-hand side of (4.4) clearly tends to the right-hand side of (4.7), except for the factor $\frac{2}{\pi i} e^{-i\pi\nu/2}$. In view of the connecting formula

$$H_\nu^{(1)}(t) = \frac{2}{\pi i} e^{-i\pi\nu/2} K_\nu(-i t), \qquad 0 < t < \infty, \tag{4.9}$$

the left hand side of (4.4) also tends to the left-hand side of (4.7), provided that the limit (as $\varepsilon \to 0$) can be taken inside the integral sign. The fact that the limit and the integral can indeed be interchanged is justified by the asymptotic expansion [8, p. 250]

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{s=0}^\infty \frac{A_s(\nu)}{z^s}, \tag{4.10}$$

as $z\to\infty$ in $|\arg z|\leqq\frac{3}{2}\pi-\delta$, and the lemma in § 2. The corresponding formula for $H_2(x)$ is obtained by using, instead of (4.9), the connecting formula

$$H_\nu^{(2)}(t)=-\frac{2}{\pi i}e^{i\pi\nu/2}K_\nu(it), \qquad 0<t<\infty, \tag{4.11}$$

and putting $z=\varepsilon+ix$ in (4.4). This completes the proof of Theorem 2.

Remark 3. It is well-known that the Bessel function of the first kind, $J_\nu(t)$, can be written as

$$J_\nu(t)=\tfrac{1}{2}\{H_2^{(1)}(t)+H_\nu^{(2)}(t)\}. \tag{4.12}$$

Hence, the quadrature formulas for $H_1(x)$ and $H_2(x)$ can also be used to numerically compute the Hankel transform

$$\int_0^\infty t^\mu f(t)\,J_\nu(xt)\,dt. \tag{4.13}$$

As a simple check on the validity of (4.7), we take $-\nu-1<\mu<\frac{1}{2}$ and $f(t)=1$. Since from (4.4) we have

$$\sum_{k=1}^n w_k=2^{\mu-1}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right),$$

it follows from (4.12) and Theorem 2 that

$$\int_0^\infty t^\mu J_\nu(xt)\,dt=\frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})\,2^\mu}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2})\,x^{\mu+1}}, \tag{4.14}$$

which agrees with Eq. (19) in [3, p. 49]. (Note that in the case of the Hankel transform (4.13), the condition $\mu\pm\nu>-1$ in (1.3) can be weakened to $\mu+\nu>-1$. The more restrictive assumption is needed only to ensure the convergence of the integrals in (1.3) and (4.2).

Remark 4. The derivative form of the error term in (4.8) can also be expressed in terms of the divided difference of $f(t)$. More explicitly, we have

$$\delta_n^{(1)}(f;x)=\frac{2}{\pi}\frac{e^{i(\mu+2n-\nu)/2}}{A_n^2\,x^{2n+\mu+1}}\,f\left[i\frac{t_1}{x},i\frac{t_1}{x},\ldots,i\frac{t_n}{x},i\frac{t_n}{x},i\frac{\xi}{x}\right] \tag{4.15}$$

for some $\xi\in(0,\infty)$. The use of (4.15) is usually preferable to that of (4.8).

5. More Examples

Example 3. Consider the integral

$$I(x)=\int_0^\infty\frac{J_0(xt)}{1+t}\,dt. \tag{5.1}$$

In the notations of § 4, we have $\mu=\nu=0$ and $f(t)=1/(1+t)$. From (4.12) and Theorem 2 it follows that

$$I(x)=\frac{1}{\pi}\sum_{k=1}^{n} w_k \frac{2x}{x^2+t_k^2}+\delta_n(x). \tag{5.2}$$

On account of (4.15), the remainder $\delta_n(f; x)$ satisfies

$$|\delta_n(x)| \leqq \frac{2}{\pi A_n^2} \frac{1}{x(x^2+t_1^2)\dots(x^2+t_n^2)}. \tag{5.3}$$

The leading coefficients A_n in $p_n(t)$ and the abscissas t_k and weights w_k corresponding to formulas in (5.2) for $2 \leqq n \leqq 5$ are listed (to ten decimal places in Table 3.

Table 3

n	Coefficients	Abscissas	Weights
1	1.0346310752	0.6366197724	1.5707963268
2	0.5839852962	0.3672186882	1.3999512373
		2.8441656971	0.1708450895
3	0.2108566158	0.2609612883	1.2294421665
		1.8802425952	0.3313894656
		5.6269259843	0.009964694674
4	0.05596246029	0.2034678616	1.0948309833
		1.4202585051	0.4377391923
		4.0139876796	0.0377769612
		8.6778718603	0.0004491900176
5	0.01174097062	0.1672481118	0.9888810597
		1.1456294188	0.5037500094
		3.1628739845	0.07510923965
		6.4939278147	0.003038578952
		11.8874874319	0.00001743909733

It is interesting to compare formula (5.2) with the asymptotic approximation [19, Eq. (5.13)]

$$I(x)=\frac{1}{x}-\frac{1}{x^3}+r_3(x), \tag{5.4}$$

where

$$|r_3(x)| \leqq 5.3155\, x^{-7/2}. \tag{5.5}$$

From (5.2), with $n=2$, we have

$$I(10) \doteq 0.0990 \tag{5.6}$$

with truncation error bounded by 0.000019, and hence (5.6) accurate to four decimal places. The asymptotic approximation (5.4) gives the same value as in (5.6). However, the error associated with this approximation is bounded only by 0.0017, and hence does not give any indication concerning the accuracy of the value given in (5.6). Taking $n=5$ in (5.2), we in fact obtain $I(10)=0.0990740$ accurate to seven decimal places.

Example 4. Consider the integral

$$I(x)=\int_0^\infty \frac{J_0(xt)}{\sqrt{t}(1+t)}\,dt. \tag{5.7}$$

Here we have $\mu=-\frac{1}{2}$, $\nu=0$ and $f(t)=1/(1+t)$. From the results in § 4, we have

$$I(x)=\frac{\sqrt{2x}}{\pi}\sum_{k=1}^{n} w_k \frac{x-t_k}{x^2+t_k^2}+\delta_n(x) \tag{5.8}$$

with

$$|\delta_n(x)|\leqq\frac{2}{\pi A_n^2 x^{2n+1/2}}. \tag{5.9}$$

The values of the coefficients A_n, the abscissas t_k and the weights w_k are listed in Table 4.

Table 4

n	Coefficients	Abscissas	Weights
1	1.0429856349	0.2284732905	4.6474760094
2	0.6958420717	0.1272660741	4.4186595245
		2.1828789401	0.2288164849
3	0.2776778397	0.08975811418	4.1825838264
		1.4038905409	0.4551726046
		4.8758067266	0.009719578404
4	0.0791554398	0.06990008100	3.9858407490
		1.04587495139	0.6239374973
		3.4232461358	0.03733775108
		7.8792758079	0.0003600119525
5	0.0175560146	0.05750827921	3.8221033421
		0.8366348329	0.7471552593
		2.6726988390	0.07574750466
		5.8349740709	0.002457777509
		11.0589492082	0.00001212583461

Let us now take $n=2$ in (5.8) and compare it with the asymptotic approximation [16, Eq. (5.22)]

$$I(x)=\frac{\Gamma^2(1/4)}{2\pi\sqrt{x}}-\frac{2\pi}{\Gamma^2(1/4)\,x^{3/2}}+r_2(x), \tag{5.10}$$

where

$$|r_2(x)|\leqq 1.00138\,x^{-2}. \tag{5.11}$$

(The first term on the right-hand side of Eq. (5.22) is missing a factor of $\frac{1}{2}$.) The two-point formula in (5.8) gives $I(50)=0.2944872$ accurate to seven decimal places, whereas the approximation (5.10) gives $I(50)=0.294$ accurate to only three decimal places. The superiority of (5.8) over (5.10) is even more apparent for $x>50$.

Remark 5. There are various ways of constructing quadrature formulas. We have constructed Tables 3 and 4 by using the method based on determinants; see [13, §§2.1 and 2.2]. For a more effective procedure, we refer to a recent article by Gautschi [5, Example 4.10].

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