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Finite Difference Methods and Their Convergence for a Class of Singular Two Point Boundary Value Problems

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Summary. We discuss the construction of three-point finite difference approximations and their convergence for the class of singular two-point boundary value problems: $(x^\alpha y')' = f(x, y)$, $y(0) = A$, $y(1) = B$, $0 < \alpha < 1$. We first establish a certain identity, based on general (non-uniform) mesh, from which various methods can be derived. To obtain a method having order two for all $\alpha \in (0, 1)$, we investigate three possibilities. By employing an appropriate non-uniform mesh over $[0, 1]$, we obtain a method M_1 based on just one evaluation of f. For uniform mesh we obtain two methods M_2 and M_3 each based on three evaluations of f. For $\alpha = 0$, M_1 and M_2 both reduce to the classical second-order method based on one evaluation of f. These three methods are investigated, their $O(h^2)$-convergence established and illustrated by numerical examples.

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1. Introduction

We consider the class of singular two-point boundary value problem:

$$\begin{gathered} (x^\alpha y')' = f(x, y), \quad 0 < x \leqq 1, \\ y(0) = A, \quad y(1) = B. \end{gathered} \tag{1}$$

Here, $\alpha \in (0, 1)$ and A, B are finite constants. We assume that, for $(x, y) \in \{[0, 1] \, x \, \mathbb{R}\}$: (A) $f(x, y)$ is continuous, $\partial f / \partial y$ exists and is continuous, and $\partial f / \partial y \geqq 0$.

Certain classes of singular boundary value problems have been considered by Jamet [3, 4] and Parter [5], in the linear case only. Jamet studied the application of a standard three-point finite difference scheme with a uniform mesh of size h and has shown that the error in the maximum-norm is $O(h^{1-\alpha})$. Ciarlet et al. [1] used a suitable Rayleigh-Ritz-Galerkin method and improved Jamet's result by showing that the error in the uniform norm for their Galer-

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kin approximation is $O(h^{2-\alpha})$. Gusttafsson [2] gave a numerical method for solving singular boundary value problems by representing the solutions as series expansions on a sub-interval near the singularity and by using difference method for a regular boundary value problem derived for the remaining interval. Reddian [6] and Reddian and Schumaker [7] have studied collocation for the solution of singular two-point boundary value problems. Their methods concern certain projections onto finite dimensional linear spaces of singular non-polynomial splines; these singular splines possess convenient local support basis which have a certain advantage in the numerical computations.

Our object in the present paper is to discuss the construction of three-point finite difference approximations and their convergence, under appropriate conditions, for the class of singular two-point boundary value problems (1). In Section 2 we first establish a certain identity based on general (non-uniform) mesh over [0, 1], from which various methods can be derived. In order to obtain a method having order two for all $\alpha \in (0,1)$, there seem to be three possibilities. By employing an appropriate non-uniform mesh over [0, 1], we obtain our first method M_1 based on just *one* evaluation of f. Alternatively employing uniform mesh, we obtain two methods M_2 and M_3, each based on three evaluations of f. The methods M_1 and M_2 have the property that, for $\alpha = 0$, they reduce to the classical second-order method based on one evaluation of f. In Section 3 these three methods are investigated in detail and, under appropriate conditions, their $0(h^2)$-convergence is established. In Section 4 we consider numerical examples to illustrate these methods and their second-order convergence for various $\alpha \in (0,1)$.

2. The Finite Difference Methods

For a positive integer $N \geqq 2$, consider a general (non-uniform) mesh over [0, 1]: $0 = x_0 < x_1 < \ldots < x_N = 1$. Let $y_k = y(x_k)$, $f_k = f(x_k, y_k)$, etc. We set $z(x) = x^\alpha y'$ and $f(t) = f(t, y(t))$; integrating (1) from x_k to x, dividing by x^α, and then integrating from x_k to x_{k+1} and interchanging the order of integration we obtain

$$y_{k+1} - y_k = J_k z_k + \frac{1}{1-\alpha} \int_{x_k}^{x_{k+1}} (x_{k+1}^{1-\alpha} - t^{1-\alpha}) f(t)\, dt, \tag{2}$$

where we have set

$$J_k = (x_{k+1}^{1-\alpha} - x_k^{1-\alpha})/(1-\alpha).$$

In an analogous manner, we obtain

$$y_k - y_{k-1} = J_{k-1} z_k - \frac{1}{1-\alpha} \int_{x_{k-1}}^{x_k} (t^{1-\alpha} - x_{k-1}^{1-\alpha}) f(t)\, dt. \tag{3}$$

Eliminating z_k from (2) and (3) we obtain the identity:

$$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}}, \qquad k = 1(1)N-1, \tag{4}$$

where we have set

$$I_k^+ = \frac{1}{1-\alpha} \int_{x_k}^{x_{k+1}} (x_{k+1}^{1-\alpha} - t^{1-\alpha}) f(t)\, dt,$$

$$I_k^- = \frac{1}{1-\alpha} \int_{x_{k-1}}^{x_k} (t^{1-\alpha} - x_{k-1}^{1-\alpha}) f(t)\, dt.$$

Identity (4) is our basic result from which methods can be obtained for the singular two-point boundary value problem (1). We are interested here in deriving methods with order two for all $\alpha \in (0,1)$; in the following we give three such methods. By introducing an appropriate non-uniform mesh, in Sect. 2.1 we derive our first method (M_1) based on *one* evaluation of f. Next, employing uniform mesh in Sects. 2.2 and 2.3 we derive two methods (M_2 and M_3) each based on three evaluations of f. For $\alpha = 0$, each of the methods M_1 and M_2 reduce to the classical second-order method based on one evaluation of f. The convergence of these methods, under appropriate conditions, is discussed in Sect. 3.

2.1 First Method (M_1) Based on Non-Uniform Spacing

By Taylor expansion of f we obtain

$$I_k^\pm = A_{0,k}^\pm f_k + A_{1,k}^\pm f_k' + \tfrac{1}{2} A_{2,k}^\pm f''(\xi_k^\pm), \qquad \xi_k^\pm \in (x_k, x_{k\pm 1}), \tag{5}$$

and

$$A_{m,k}^\pm = \frac{1}{m+1} \sum_{j=0}^{m+1} \frac{(-1)^j}{m+2-\alpha-j} \binom{m+1}{j} \cdot x_k^j (x_{k\pm 1}^{m+2-\alpha-j} - x_k^{m+2-\alpha-j}), \qquad m = 0, 1, 2.$$

With the help of (5), from (4) we obtain

$$-\frac{1}{J_{k-1}} y_{k-1} + \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right) y_k - \frac{1}{J_k} y_{k+1} + B_{0,k} f_k + t_k^{(1)} = 0, \qquad k = 1(1)N-1, \tag{6}$$

where

$$t_k^{(1)} = B_{1,k} f_k' + \tfrac{1}{2} B_{2,k} f''(\xi_k), \qquad \xi_k \in (x_{k-1}, x_{k+1}), \tag{7}$$

and

$$B_{m,k} = \frac{A_{m,k}^+}{J_k} + \frac{A_{m,k}^-}{J_{k-1}}, \qquad m = 0, 1, 2.$$

Note that $A_{m,k}^\pm > 0$, $m = 0, 2$, $A_{1,k}^+ > 0$, $A_{1,k}^- < 0$, $J_k > 0$ and $B_{m,k} > 0$, $m = 0, 2$.

Note that each discretization in (6) is based on *one* evaluation of f. In Sect. 3.1 we shall show that a method based on (6), neglecting $t_k^{(1)}$, is $O(h^2)$-convergent for all $\alpha \in (0,1)$ provided we choose the mesh $x_k = (kh)^{1/(1-\alpha)}$.

2.2 *Second Method (M_2) Based on Uniform Spacing*

Here, and in Section 2.3, we assume that the spacing is uniform: $x_{k+1}-x_k=h$. For uniform spacing the method M_1 can be shown to be of order $2-\alpha$. However, in the following we modify (6) so that for uniform spacing the resulting method has order two for all $\alpha\in(0,1)$.

With the help of

$$hf_k'=(f_{k+1}-f_{k-1})/2-\frac{h^3}{6}f'''(\sigma_k), \qquad \sigma_k\in(x_{k-1},x_{k+1}),$$

from (6) and (7) we obtain

$$\begin{aligned}-\frac{1}{J_{k-1}}y_{k-1}+\left(\frac{1}{J_k}+\frac{1}{J_{k-1}}\right)y_k-\frac{1}{J_k}y_{k+1}+B_{0,k}f_k&\\ +\frac{1}{2h}B_{1,k}(f_{k+1}-f_{k-1})+t_k^{(2)}=0, \qquad k=1(1)N-1,&\end{aligned}\tag{8}$$

where

$$t_k^{(2)}=\tfrac{1}{2}B_{2,k}f''(\xi_k)-\frac{h^2}{6}B_{1,k}f'''(\sigma_k).\tag{9}$$

Note that the discretization (8) is based on three evaluations of f. In Sect. 3.2 we show that, under suitable conditions, the method M_2 based on (8) is $O(h^2)$-convergent for all $\alpha\in(0,1)$. Since for $\alpha=0$, $B_{1,k}=0$ and hence (8) reduces to the classical second-order method for $y''=f(x,y)$ based on one evaluation of f. Consequently we may regard (8) as the "modified classical second-order method".

2.3 *Third Method (M_3) Based on Uniform Spacing*

Here, in $I_k^{\pm}$ we approximate $f(t)$ by linear interpolation at x_k, $x_{k\pm1}$:

$$I_k^{\pm}=a_{0,k}^{\pm}f_k+a_{1,k}^{\pm}f_{k\pm1}+a_{2,k}^{\pm}f''(\xi_k^{\pm}),\tag{10}$$

where

$$\begin{aligned}a_{0,k}^{\pm}&=\sum_{j=0}^{1}\frac{(-1)^j}{2-\alpha-j}\binom{1}{j}(x_{k\pm1}^{2-\alpha-j}-x_k^{2-\alpha-j})x_k^j\\ &\mp\frac{1}{2h}\sum_{j=0}^{2}\frac{(-1)^j}{3-\alpha-j}\binom{2}{j}(x_{k\pm1}^{3-\alpha-j}-x_k^{3-\alpha-j})x_k^j,\\ a_{1,k}^{\pm}&=\pm\frac{1}{2h}\sum_{j=0}^{2}\frac{(-1)^j}{3-\alpha-j}\binom{2}{j}x_k^j(x_{k\pm1}^{3-\alpha-j}-x_k^{3-\alpha-j}),\end{aligned}$$

and

$$\begin{aligned}a_{2,k}^{\pm}&=\frac{1}{6}\sum_{j=0}^{3}\frac{(-1)^j}{4-\alpha-j}\binom{3}{j}x_k^j(x_{k\pm1}^{4-\alpha-j}-x_k^{4-\alpha-j})\\ &\mp\frac{h}{4}\sum_{j=0}^{2}\frac{(-1)^j}{3-\alpha-j}\binom{2}{j}x_k^j(x_{k\pm1}^{3-\alpha-j}-x_k^{3-\alpha-j}).\end{aligned}$$

Note that $a^{\pm}_{0,k}$, $a^{\pm}_{1,k} > 0$ and $a^{\pm}_{2,k} < 0$. With the help of (10), from the identity (4) we obtain

$$\begin{aligned}-\frac{1}{J_{k-1}} y_{k-1} + \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right) y_k - \frac{1}{J_k} y_{k+1} + b_{0,k} f_k \\ + \frac{a^{+}_{1,k}}{J_k} f_{k+1} + \frac{a^{-}_{1,k}}{J_{k-1}} f_{k-1} + t^{(3)}_k = 0, \qquad k = 1(1)N-1\end{aligned} \tag{11}$$

where

$$t^{(3)}_k = b_{2,k} f''(\xi_k) \tag{12}$$

and

$$b_{m,k} = \frac{a^{+}_{m,k}}{J_k} + (-1)^m \frac{a^{-}_{m,k}}{J_{k-1}}, \qquad m = 0, 2.$$

Note that $b_{0,k} > 0$ and $b_{2,k} < 0$.

Thus in the case of uniform spacing, a second method of order two, for all $\alpha \in (0,1)$. can be based on the discretization (11). The $O(h^2)$-convergence of the method M_3, under appropriate conditions, is proved in Sect. 3.3.

3. Convergence of the Methods M_1, M_2, M_3

We next discuss the convergence of the methods M_1 M_2 and M_3 showing that, under suitable conditions, each of these methods is $O(h^2)$-convergent for all $\alpha \in (0,1)$. For the purpose, it is convenient to introduce matrix notation and we shall describe all the three methods together. In each case the differential equation is discretized at x_k for $k = 1(1)N-1$ and $y_0 = A$, $y_N = B$.

Let $D = (d_{ij})$ denote the tridiagonal matrix with

$$d_{k,k-1} = -\frac{1}{J_{k-1}}, \qquad d_{k,k} = \frac{1}{J_k} + \frac{1}{J_{k-1}}, \qquad d_{k,k+1} = -\frac{1}{J_k}.$$

Let $P = (p_{ij})$ denote the tridiagonal matrix and let $Q = (q_1, 0, \ldots, 0, q_{N-1})^T$, where for the method M_1:

$$p_{k,k} = B_{0,k}, \; p_{k,k\pm 1} = 0; \qquad q_1 = \frac{A}{J_0}, \; q_{N-1} = \frac{B}{J_{N-1}};$$

for the method M_2:

$$p_{k,k} = B_{0,k}, \quad p_{k,k\pm 1} = \pm\frac{1}{2h} B_{1,k};$$

$$q_1 = \frac{A}{J_0} + \frac{1}{2h} B_{1,1} f_0, \quad q_{N-1} = \frac{B}{J_{N-1}} - \frac{1}{2h} B_{1,N-1} f_N;$$

for the method M_3:

$$p_{k,k} = b_{0,k}, \quad p_{k,k+1} = \frac{a^{+}_{1,k}}{J_k}, \quad p_{k,k-1} = \frac{a^{-}_{1,k}}{J_{k-1}};$$

$$q_1 = (A - a^{-}_{1,1} f_0)/J_0, \quad q_{N-1} = (B - a^{+}_{1,N-1} f_N)/J_{N-1}$$

Also, let $F(Y)=(f_1,\ldots,f_{N-1})^T$, $Y=(y_1,\ldots,y_{N-1})^T$ and let $T=(t_1,\ldots,t_{N-1})^T$, where $t_k=t_k^{(m)}$, $m=1,2,3$ corresponding to the three methods M_1, M_2 and M_3. Then, each of the discretizations (6), (8) and (11) can be expressed in the matrix form:

$$DY+PF(Y)+T=Q. \tag{13}$$

The method M_1, M_2 or M_3 now consists of finding an approximation $\tilde{Y}$ for Y by solving the $(N-1)\times(N-1)$ system:

$$D\tilde{Y}+PF(\tilde{Y})=Q. \tag{14}$$

Let $E=\tilde{Y}-Y$. We may write $F(\tilde{Y})-F(Y)=ME$ where $M=\operatorname{diag}\{U_1,\ldots,U_{N-1}\}$, (note that $U_k\geqq 0$) and then from (13) and (14) we obtain the error equation:

$$(D+PM)E=T. \tag{15}$$

For the discussion in the following, we make here some of the common definitions needed in the following sections. Let $Z=(1,\ldots,1)^T$, and let $S=(S_1,\ldots,S_{N-1})^T=(D+PM)Z$ denote the vector of row-sums of $D+PM$. Similarly, let $S^*=(S_1^*,\ldots,S_{N-1}^*)^T=DZ$ denote the vector of row-sums of D. Also, let $V=(V_1,\ldots,V_{N-1})^T$ where $V_j=\exp(2)-\exp(x_j)$, and let $R=(R_1,\ldots,R_{N-1})^T=DV$.

3.1 Convergence of the Method M_1

For the method M_1, $x_k=(kh)^{1/(1-\alpha)}$, therefore, $J_k=h/(1-\alpha)$, $k=0(1)N-1$, and it is easy to see that $D+PM$ is irreducible. Now, since $B_{0,k}>0$ and $U_k\geqq 0$, therefore S_1, $S_{N-1}>0$ and $S_k\geqq 0$, $k=2(1)N-2$, and $D+PM$ is also monotone. Therefore, $(D+PM)^{-1}$ exists and $(D+PM)^{-1}\geqq O$. Since D is also irreducible and monotone, and $PM\geqq O$, therefore $(D+PM)^{-1}\leqq D^{-1}$. In order to establish convergence we next obtain bounds for $D^{-1}=(d_{ij}^{-1})$.

Since for M_1, $S_1^*=S_{N-1}^*=(1-\alpha)/h$, with the help of $D^{-1}S^*=Z$ we obtain

$$d_{i,1}^{-1}\leqq 1/S_1^*=h/(1-\alpha), \qquad d_{i,N-1}^{-1}\leqq 1/S_{N-1}^*=h/(1-\alpha), \qquad i=1(1)N-1. \tag{16}$$

Next, to obtain bounds for the rest of d_{ij}^{-1} we consider the vector R. It is easy to see that for M_1, R_1, $R_{N-1}>0$, and for sufficiently small h,

$$R_k>\frac{h\alpha}{1-\alpha}x_k^{2\alpha-1}, \qquad k=2(1)N-2. \tag{17}$$

Since $D^{-1}R=V$ and since $V_i\leqq e^2-1$, $i=0(1)N$, with the help of (17) we obtain

$$\frac{h\alpha}{1-\alpha}\sum_{k=2}^{N-2} d_{ik}^{-1}x_k^{2\alpha-1}<e^2-1, \qquad i=1(1)N-1. \tag{18}$$

We next obtain bounds for the local truncation error $t_k^{(1)}$ for the method M_1. We assume that, for $x\in(0,1]$, $x^\alpha|f'|\leqq N_1$ and $x^{\alpha+1}|f''|\leqq N_2$ for suitable constants N_1, N_2. Since for sufficiently small h we have

$$B_{1,k}<\frac{\alpha h^3}{2(1-\alpha)^3}x_k^{3\alpha-1},\qquad B_{2,k}<\frac{h^3}{3(1-\alpha)^3}x_k^{3\alpha},\qquad k=1(1)N-1,$$

from (7) we obtain for sufficiently small h,

$$|t_k^{(1)}|\leqq\frac{h^3}{6(1-\alpha)^3}x_k^{2\alpha-1}(3\alpha N_1+N_2),\qquad k=1(1)N-1. \tag{19}$$

Now, since $(D+PM)^{-1}\leqq D^{-1}$, from (15) it follows that $\|E\|\leqq\|D^{-1}T\|$, and with the help of (16), (18) and (19) we obtain the following result.

Theorem 1. *Assume that f satisfies (A); further, let $f''\in C\{(0,1]\times\mathbb{R}\}$, $x^\alpha f'$, $x^{\alpha+1}f''\in C\{[0,1]\times\mathbb{R}\}$. Then, for the method M_1 based on (6) with $x_k=(kh)^{1/(1-\alpha)}$, we have for sufficiently small h, and for all $\alpha\in(0,1)$, $\|E\|=O(h^2)$.*

3.2 Convergence of the Method M_2

For the method M_2, $x_k=kh$, $k=0(1)N$. Since for fixed x_k and for $h\to0$, $J_k\sim hx_k^{-\alpha}$ and $B_{1,k}\sim-\alpha h^3/(12x_k)$, it is easy to see that for sufficiently small h, $D+PM$ is irreducible. Again, since for $h\to0$, $B_{0,k}\sim h$, we obtain for sufficiently small h,

$$S_1>1/(2J_0),\qquad S_{N-1}>1/J_{N-1},\qquad S_k>B_{0,k}U_k/2,\qquad k=2(1)N-2. \tag{20a}$$

We now assume that $\partial f/\partial y>0$. Let $U_*=\min\partial f/\partial y$; then $U_*>0$, and for sufficiently small h,

$$S_k>(h/4)\,U_*. \tag{20b}$$

Since $(D+PM)^{-1}S=Z$, with the help of (20) it follows that for sufficiently small h,

$$\begin{aligned}&(D+PM)_{i1}^{-1}\leqq1/S_1<\frac{2h^{1-\alpha}}{1-\alpha},\qquad (D+PM)_{i,N-1}^{-1}\leqq1/S_{N-1}<2h^{1-\alpha},\\ &\sum_{k=2}^{N-2}(D+PM)_{i,k}^{-1}\leqq1/\min_{2\leqq k\leqq N-2}S_k\leqq4/(hU_*),\qquad i=1(1)N-1.\end{aligned} \tag{21}$$

To obtain bounds for the truncation error $t_k^{(2)}$ for the method M_2, we assume that $|f''|\leqq\bar N_1$, $|f'''|\leqq\bar N_2$, $0<x\leqq1$, for suitable constants $\bar N_1$ and $\bar N_2$. Since for fixed x_k and $h\to0$, $B_{2,k}\sim h^3/6$, it follows that for sufficiently small h, $B_{2,k}<h^3/3$. With this result and with $|B_{1,k}|<\alpha h^3/(6x_k)$, from (9) we obtain for sufficiently small h,

$$|t_k^{(2)}|\leqq(h^3/36)(6\bar N_1+\alpha\bar N_2). \tag{22}$$

Now, since $\|E\|\leqq\|(D+PM)^{-1}T\|$, with the help of (21) and (22) we obtain the following result.

Theorem 2. *Assume that f satisfies (A); further, let $f'''\in C\{[0,1]\times\mathbb{R}\}$, and let $\partial f/\partial y>0$. Then, for the method M_2 based on (8) with $x_k=kh$ we have for sufficiently small h, and for all $\alpha\in(0,1)$, $\|E\|=O(h^2)$.*

3.3 Convergence of the Method M_3

The arguments here are similar to those given for the convergence of the method M_1. For M_3, $x_k=kh$, $k=0(1)N$, and since for fixed x_k and for $h\to 0$, $a^{\pm}_{1,k}\sim(h^2/6)x_k^{-\alpha}$, it is easy to see that $D+PM$ is irreducible. Again, since $b_{0,k}>0$, $a^{\pm}_{1,k}>0$, by the row-sum criterion it follows that $D+PM$ is also monotone. Therefore, $(D+PM)^{-1}$ exists and $(D+PM)^{-1}\geqq O$. Since D is also irreducible and monotone and since $PM\geqq O$, therefore $(D+PM)^{-1}\leqq D^{-1}$. We next obtain bounds for D^{-1}.

Since $S_1^*=1/J_0$ and $S_{N-1}^*=1/J_{N-1}$, with the help of $D^{-1}S^*=Z$ we obtain

$$\begin{aligned} d_{i,1}^{-1}&\leqq 1/S_1^*=h^{1-\alpha}/(1-\alpha),\\ d_{i,N-1}^{-1}&\leqq 1/S_{N-1}^*<2h^{1-\alpha}, \qquad i=1(1)N-1. \end{aligned} \tag{23}$$

To obtain bounds for the rest of d_{ij}^{-1} we consider the vector R. It is easy to see that for M_3, R_1, $R_{N-1}>0$, and that for sufficiently small h,

$$R_k>(\alpha h/2)x_k^{\alpha-1}, \qquad k=2(1)N-2. \tag{24}$$

Since $D^{-1}R=V$, with the help of (24) we obtain for sufficiently small h,

$$(\alpha h/2)\sum_{j=2}^{N-2} d_{kj}^{-1}x_k^{\alpha-1}\leqq V_0=e^2-1, \qquad k=1(1)N-1. \tag{25}$$

To obtain bounds for the truncation error $t_k^{(3)}$, we assume that $x^{1-\alpha}|f''|\leqq\bar{\bar{N}}_1$, $0<x\leqq 1$, for a suitable constant $\bar{\bar{N}}_1$. Since for fixed x_k and $h\to 0$, $b_{2,k}\sim -h^3/12$, we have for sufficiently small h, $|b_{2,k}|<h^3/24$, and then from (12) it follows that for sufficiently small h,

$$|t_k^{(3)}|\leqq(h^3/24)x_k^{\alpha-1}\bar{\bar{N}}_1, \qquad k=1(1)N-1. \tag{26}$$

Now, since $(D+PM)^{-1}\leqq D^{-1}$, from (15) we have $\|E\|\leqq\|D^{-1}T\|$, and with the help of (23), (25) and (26) we obtain the following result.

Theorem 3. *Assume that f satisfies (A); further, let $f''\in C\{(0,1]\times\mathbb{R}\}$, $x^{1-\alpha}f''\in C\{[0,1]\times\mathbb{R}\}$. Then, for the method M_3 based on (11) with $x_k=kh$ we have for sufficiently small h, and for all $\alpha\in(0,1)$, $\|E\|=O(h^2)$.*

4. Numerical Illustrations

We next illustrate our methods M_1, M_2 and M_3 and show that each of these methods is $O(h^2)$-convergent for $\alpha\in(0,1)$. For the purpose we consider the following two singular two-point boundary value problems:

$$(x^\alpha y')'=\beta x^{\alpha+\beta-2}((\alpha+\beta-1)+\beta x^\beta)y, \qquad y(0)=1,\ y(1)=e, \tag{27}$$

Table 1. The Boundary Value Problem (27)

N	Method M_1	Method M_2	Method M_3
	$\alpha=0.50$, $\beta=4.0$	$\alpha=0.50$, $\beta=5.0$	$\alpha=0.50$, $\beta=4.0$
16	4.3(−2)	1.8(−2)	1.2(−2)
32	1.1(−2)	4.7(−3)	3.0(−3)
64	2.9(−3)	1.2(−3)	7.3(−4)
128	7.2(−4)	3.0(−4)	1.8(−4)
	$\alpha=0.75$, $\beta=3.75$	$\alpha=0.75$, $\beta=4.75$	$\alpha=0.75$, $\beta=3.75$
16	1.4(−1)	1.8(−2)	1.2(−2)
32	4.1(−2)	4.6(−3)	2.9(−3)
64	1.1(−2)	1.2(−3)	7.2(−4)
128	2.7(−3)	2.9(−4)	1.8(−4)

Table 2. The Boundary Value Problem (28)

N	Method M_1	Method M_2	Method M_3
	$\alpha=0.50$, $\beta=5.0$	$\alpha=0.50$, $\beta=5.0$	$\alpha=0.50$, $\beta=4.0$
16	1.5(−3)	7.5(−4)	3.9(−4)
32	3.7(−4)	1.9(−4)	9.7(−5)
64	9.3(−5)	4.7(−5)	2.4(−5)
128	2.3(−5)	1.2(−5)	6.1(−6)
	$\alpha=0.75$, $\beta=3.75$	$\alpha=0.75$, $\beta=4.75$	$\alpha=0.75$, $\beta=3.75$
16	7.3(−3)	1.0(−3)	6.2(−4)
32	1.8(−3)	2.6(−4)	1.6(−4)
64	4.5(−4)	6.4(−5)	3.9(−5)
128	1.1(−4)	1.6(−5)	9.7(−6)

with the exact solution $y(x)=\exp(x^\beta)$, and

$$\begin{aligned}&(x^\alpha y')'=\beta x^{\alpha+\beta-2}(\beta x^\beta e^y-(\alpha+\beta-1))/(4+x^\beta),\\&y(0)=\ln(1/4),\qquad y(1)=\ln(1/5),\end{aligned}\tag{28}$$

with the exact solution $y(x)=\ln(1/(4+x^\beta))$.

We solved the boundary value problems (27) and (28) for a few selected values of α and β, and for $N=2^k$, $k=4(1)7$. The error-norms obtained for the three methods are shown in Tables 1 and 2; the numerical results verify second-order convergence for all the three methods for the sets of values of α and β considered. It may also be noted that the method M_3 is far superior, in accuracy, than the method M_1.

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