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Local Contributions to Global Deformations of Surfaces

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Introduction

In [23], Schiffer and Spencer prove that all small deformations of complex structure on a compact Riemann surface may be realized by altering the complex structure only within an arbitrarily small neighborhood of a point on the surface. It seems interesting in general to consider whether it is possible to construct deformations of an algebraic variety or complex manifold from deformations of neighborhoods of certain subvarieties. Further motivation for trying to understand the role of subvarieties in deformations is suggested by the instability under deformation of the Neron-Severi group, i.e., the group of divisors modulo numerical equivalence.

As an example, one may consider the family of affine surfaces V_t: $x^2+y^2+z^2=t^2$ (t is a parameter; V_0 has a nodal singularity at the origin). This family admits a resolution $\{X_t\} \to \{V_t\}$, with $\{X_t\}$ a smooth family of non-singular surfaces, and each X_t is a minimal resolution of V_t ([4] or [5]). The exceptional curve E in X_0 is a $\mathbb{P}^1$ with self-intersection -2 which does not appear in any X_t, for $t \neq 0$. One may ask whether every smooth surface X with such a curve in it admits a one-parameter family of deformations arising from this local model. Furthermore, if X contains several disjoint such curves, does each one independently contribute one dimension to the moduli of X? The Hartogs' theorem of [14] says that one cannot simply plumb in the local deformation, leaving the structure of X unchanged outside a small neighborhood of E. Moreover, old examples of Segre [26] show that the nodes on certain hypersurfaces V in $\mathbb{P}^3$ are not "independent", i.e., there aren't enough deformations of the resolution X of V to allow for a one-dimensional contribution from each node.

Theorem (3.7) of this paper says that the regularity of the Kuranishi variety of X is sufficient for the deformations of X to realize independently

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the local deformations of disjoint nodal curves, or, more generally, the exceptional curves arising from rational double points. In fact, interpreting "independently" properly, these conditions are equivalent. In conjunction with structure Theorem (2.6) on deformations of X, one readily constructs examples (e.g., those of Segre) of surfaces of general type with obstructed deformations and obtains information on the singularities of the moduli space.

The main technique of the paper is a localization of some deformations of X about an exceptional curve E in cohomology, by means of the local cohomology group $H_E^1(X, \Theta)$, Θ = tangent sheaf of X. It is then shown that nodes are always independent to first order. Once this is known, two types of conclusions follow: (1) The actual independence of certain exceptional curves, given the regularity of the Kuranishi variety; the example of a Kummer $K-3$ surface is worked out in detail (3.9). (2) If we know *a priori* bounds on the dimension of the Kuranishi family, and we know the presence of sufficiently many nodal exceptional curves, then the contribution of $H_E^1(X, \Theta)$ to $H^1(X, \Theta)$ forces the latter to be larger in dimension than the Kuranishi family, and X will have obstructed deformations.

The individual sections may be summarized as follows: §1 and 2 present the formal theory relating global and local deformations of surfaces over infinitesimal bases. §1 shows that $H_E^1(X, \Theta)$ injects into $H^1(X, \Theta)$, if E is a minimal exceptional curve. This local contribution is computed in some cases, especially for E the exceptional curve of a rational double point. In particular, if a surface X contains a non-singular rational E with $E^2 = -2$, then $H^1(X, \Theta) \neq 0$. §2 deals with the relations between the infinitesimal deformation theory of a smooth surface X with exceptional curves E^i and that of the variety V obtained by blowing each E^i down to a point P_i. These theories are also related to deformations of the E^i and of the P_i. The relations are summarized in the structure Theorem (2.6). The methods used here were introduced by Schlessinger [24]. The computations of §1 combine to yield "first order independence" (2.13). §3 extends the previous work to the larger (convergent) analytic category. Here we first have to pick out an appropriate class of neighborhoods of E and consider their deformations. Fortunately, the necessary convergence theorems needed for this approach have recently been proved by several authors (e.g., [1] and [20]). §4 applies the general theory to compute many examples of smooth surfaces with obstructed deformations, obtained by resolving hypersurfaces in $\mathbb{P}^3$ with rational double point singularities. Corollary (2.11) implies the moduli space of such a surface is always a reduced complete intersection. If all singularities are nodes, this space is a hypersurface singularity of multiplicity two, and Corollary (4.3) gives a very geometric characterization of regularity of the Kranishi variety in terms of the position of the nodes in $\mathbb{P}^3$.

The particular form of cohomological localization employed here seems to be of a very special nature, since it applies mainly to nodal exceptional curves (but cf. here (1.16)). We always work over the ground field $\mathbb{C}$.

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§1

(1.1) Let V be a two-dimensional algebraic variety with an (isolated) normal singularity at P. Let $f\colon X \to V$ be a *minimal resolution* of V; i.e., f is a proper birational map, X is non-singular, f is an isomorphism on $X - f^{-1}(P)$, and $f^{-1}(P)$ contains no non-singular rational curves of self-intersection -1. Such a resolution exists and is unique [5]. For any scheme Y, denote by Θ_Y the sheaf of $\mathbb{C}$-derivations, i.e.,

$$\Theta_Y = \mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y).$$

If Y is non-singular, Θ_Y is the (locally free) tangent sheaf.

(1.2) **Proposition.** *There is a natural isomorphism* $f_* \Theta_X \xrightarrow{\sim} \Theta_V$.

Proof. The existence of the natural inclusion $f_* \Theta_X \subset \Theta_V$ follows from the inclusion

$$0 \to \Theta_X \to \mathcal{H}om_{\mathcal{O}_X}(f^* \Omega^1_V, \mathcal{O}_X)$$

and the equalities

$$f_*\left(\mathcal{H}om_{\mathcal{O}_X}(f^* \Omega^1_V, \mathcal{O}_X)\right) = \mathcal{H}om_{\mathcal{O}_V}(\Omega^1_V, f_* \mathcal{O}_X) = \Theta_V.$$

Note $\mathcal{O}_V \xrightarrow{\sim} f_* \mathcal{O}_X$ by normality.

For surjectivity, it suffices to consider an affine open neighborhood Spec R of P in V; let P correspond to a maximal ideal m. We show below that any derivation d of R is such that $d(m) \subset m$. A direct computation then shows d extends to an element of $H^0(B, \Theta_B)$, where $B \to \operatorname{Spec} R$ is the blowing-up of m. A theorem of Seidenberg [27] implies that any derivation of B extends to its normalization $\tilde{B}$. Continue this procedure until d extends to a non-singular variety X' birationally dominating Spec R. Then X is obtained from X' by blowing down, whence we may push the derivation onto X.

To show $d(m) \subset m$, we may pass to the completion S of R at m. Let $\mathbb{C}[\varepsilon] = \mathbb{C}[\varepsilon]/\varepsilon^2$. We may view $1 + \varepsilon d$ as a $\mathbb{C}[\varepsilon]$-automorphism of $S[\varepsilon]$ inducing the identity on S. This extends to a $\mathbb{C}[[t]]$-automorphism $\sigma = e^{td} = 1 + td + \frac{t^2 d^2}{2} + \cdots$ of $S[[t]]$. But σ must send the singular subscheme $V(m) \subset \operatorname{Spec} S[[t]]$ into itself, whence $dm \subset m$ (cf. [32], Lemma 4).

(1.3) **Corollary.** *Let $E \subset X$ be the exceptional subset of $f: X \to V$ (i.e., $E = f^{-1}(P)$); then there is an inclusion $H_E^1(\Theta_X) \subset H^1(\Theta_X)$.*

Proof. The morphism above arises in the long exact sequence for local cohomology [12]

$$H^0(X, \Theta_X) \to H^0(X - E, \Theta_X) \to H_E^1(\Theta_X) \to H^1(X, \Theta_X).$$

Let $U = X - E$; of course, $U = V - \{P\}$. So, $H^0(U, \Theta_X) = H^0(U, \Theta_V)$, while the Proposition implies $H^0(X, \Theta_X) \simeq H^0(V, f_* \Theta_X) = H^0(V, \Theta_V)$. The local cohomology sequence on V is

$$H^0(V, \Theta_V) \to H^0(U, \Theta_V) \to H_P^1(\Theta_V).$$

Since V is Cohen-Macaulay at P (being two-dimensional and normal), $\operatorname{depth}_P \Theta_V = 2$, as Θ_V is the dual of a non-zero sheaf ([25], Lemma 1); thus, $H_P^1(\Theta_V) = 0$, whence the result. (Cf. also Artin [1], Cor. 4.5.)

(1.4) *Remark.* The same result shows that if V has several normal singularities, X is its minimal resolution, and E is the union of the exceptional fibres, then $H_E^1(\Theta_X) \subset H^1(\Theta_X)$.

(1.5) In order to compute the local group $H_E^1(\Theta_X)$, we may suppose $V = \operatorname{Spec} R$ is affine. If E is viewed as a reduced divisor on X, then by [12] we have

$$H_E^1(\Theta_X) \xrightarrow{\sim} \varinjlim \operatorname{Ext}_X^1(\mathcal{O}_{nE}, \Theta_X).$$

But

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{nE}, \Theta_X) = 0$$

$$\mathcal{E}xt_{\mathcal{O}_X}^1(\mathcal{O}_{nE}, \Theta_X) = \Theta_X \otimes N_{nE} \quad (\text{where } N_{nE} = \mathcal{O}(nE) \otimes \mathcal{O}_{nE}),$$

hence the standard spectral sequence shows

$$H_E^1(\Theta_X) \xrightarrow{\sim} \varinjlim H^0(\Theta_X \otimes N_{nE}).$$

The surjection $\mathcal{O}_{nE} \to \mathcal{O}_E$ yields

$$0 \to N_{(n-1)E} \to N_{nE} \to \mathcal{O}_E(nE) \to 0.$$

Therefore,

$$H^0(\Theta_X \otimes N_E) \subset H_E^1(\Theta_X).$$

(1.6) Suppose the minimal resolution $X \to \operatorname{Spec} R$ is such that E is a union of non-singular curves E_i, and the only singularities of E are transversal intersections of two components (this happens if $H^1(\mathcal{O}_X) = 0$, i.e., for rational singularities [7]). Then if Z is an effective divisor on X supported on E, one has an exact sequence

$$0 \to \Theta_Z \to \Theta_X \otimes \mathcal{O}_Z \to \bigoplus N_{E_i} \to 0;$$

here, the second map is the sum of the compositions $\Theta_X \otimes \mathcal{O}_Z \to \Theta_X \otimes \mathcal{O}_{E_i} \to N_{E_i}$, and the exactness is easily checked formally. One also has the exact sequence

$$0 \to \Theta_X \otimes \mathcal{O}(-Z) \to \Theta_X \to \Theta_X \otimes \mathcal{O}_Z \to 0.$$

Since for Z sufficiently large, $-Z$ is ample relative to f, we have $H^1(\Theta_X) \xrightarrow{\sim} H^1(\Theta_X \otimes \mathcal{O}_Z)$ for such divisors. (This also follows by the theorem of holomorphic functions.)

(1.7) Recall that a singularity Spec R is *rational* if $H^1(\mathcal{O}_X)=0$ for some (or every) resolution $X \to \operatorname{Spec} R$. Among these are the *rational double points* ([2, 3, 7]), henceforth called R.D.P.'s. These are the singular points such that the E_i's in a minimal resolution are non-singular rational curves of self-intersection -2. They are formally of the following types:

$$A_n(x^2+y^2+z^{n+1}=0),\ n \geqq 1; \qquad D_n(x^2+y(y^{n-2}+z^2)=0),\ n \geqq 4;$$
$$E_6(x^2+y^3+z^4=0); \qquad E_7(x^2+y(y^2+z^3)=0);$$

and

$$E_8(x^2+y^3+z^5=0).$$

A_1 singularities are called *nodes*. The configurations of the exceptional fibres in a minimal resolution are the Dynkin diagrams associated to the Lie algebras A_n, D_n, E_6, E_7, E_8. Finally, the R.D.P.'s are the rational singularities of multiplicity two.

(1.8) Tjurina has proved [29] that $H^1(\Theta_Z)=0$ for all effective divisors Z supported on E, in case Spec R has a rational double or triple point; this is the *tautness* of these singularities. Thus, by (1.6), we have for R.D.P.'s an isomorphism

$$H^1(\Theta_X) \xrightarrow{\sim} \bigoplus H^1(N_{E_i}).$$

Since $N_{E_i} = \mathcal{O}_{E_i}(-2)$, we have

$$\dim H^1(\Theta_X) = \#\ \text{of components } E_i.$$

(1.9) On the other hand, the sequence

$$0 \to \Theta_E \otimes N_E \to \Theta_X \otimes N_E \to \bigoplus N_{E_i} \otimes N_E \to 0$$

yields

$$H^0(\Theta_E \otimes N_E) \subset H^0(\Theta_X \otimes N_E) \subset H^1_E(\Theta_X) \subset H^1(\Theta_X).$$

But one sees easily that

$$\Theta_E \simeq \bigoplus \Theta_{E_i}(-t_i) = \bigoplus \mathcal{O}_{E_i}(2-t_i),$$

where t_i is the number of intersections of E_i with the other E_j; in other words, derivations of the (reduced) scheme E consist of derivations on every component which vanish on the intersections. But $E \cdot E_i = t_i - 2$, whence

$$\Theta_E \otimes N_E = \bigoplus \mathcal{O}_{E_i},$$

and $\dim H^0(\Theta_E \otimes N_E) = \#$ of components E_i. Putting all this together and doing one final computation, we have

(1.10) **Proposition.** *Let $X \to \operatorname{Spec} R$ be the minimal resolution of a rational double point, given formally by $g(x, y, z) = 0$ (1.7). Then there is a natural isomorphism $H^1_E(\Theta_X) \xrightarrow{\sim} H^1(\Theta_X)$, where each space has dimension equal to the number of components in E. That number is the subscript k in A_k, D_k, or E_k, and is also equal to the dimension of*

$$T^1_R = \mathbb{C}[[x, y, z]]/(g, g_x, g_y, g_z),$$

the space of first-order deformations of the singularity [15].

(1.11) **Corollary.** *Let X be a non-singular projective surface, and let $E^1, \ldots, E^s$ be disjoint configurations of curves on X which are those of R.D.P.'s. Then* $\dim H^1(\Theta_X) \geqq k_1 + \cdots + k_s$, *where k_i is the integer associated to an R.D.P. of Proposition* (1.10).

Proof. Artin has shown [3] that the E^i are exceptional fibres in a (minimal) resolution $f: X \to V$ of a normal algebraic surface. The global computation of Corollary (1.3) plus the local computation of (1.10) now suffice.

(1.12) **Corollary.** *Let X be a non-singular projective surface for which $H^1(\Theta_X) = 0$. Then X contains no non-singular rational curves E of self-intersection -2.*

(1.13) **Corollary.** *Let X be a surface of general type* (i.e., *for n large, nK_X defines a birational map). If $H^1(\Theta_X) = 0$, then X is a minimal model and K_X is ample.*

Proof. First, note that if Y is a non-singular surface, and $g: B \to Y$ is the blowing-up at a point P, there is an exact sequence

$$0 \to g_* \Theta_B \to \Theta_Y \to N_P \to 0,$$

where N_P = normal bundle of P in Y, and where $R^1 g_* \Theta_B = 0$. Now, for a surface of general type, $H^0(\Theta_Y) = 0$ [17]; of course, $\dim H^0(N_P) = 2$, so $H^1(\Theta_B) = H^1(g_* \Theta_B) \neq 0$. Thus, X in the corollary must be a minimal model.

Finally, observe that on a minimal model, either K is ample or $K \cdot E = 0$ for some non-singular rational curves E with $E^2 = -2$ [19].

(1.14) *Remarks.* We shall see in § 4 that the first order deformations of X arising from $H^1_E(\Theta_X)$ may be obstructed.

(1.15) If $X \to \operatorname{Spec} R$ is the minimal resolution of a rational singularity, then $H^1_E(\Theta_X)$ may be thought of as those first order deformations of X

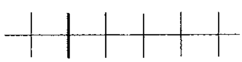
which blow down (see (2.3)) to the trivial deformation over Spec $R[\varepsilon]$. To see this, consider the diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_E(X, \Theta_X) & \longrightarrow & H^1(X, \Theta_X) & \longrightarrow & H^1(X-E, \Theta_X) \\ & & & & & & \| \\ & & 0 & \longrightarrow & T^1_R & \longrightarrow & H^1(\operatorname{Spec} R - m, \Theta_R) \end{array}$$

The bottom inclusion is Lemma 2 of [25]. It is easy to check that the α-construction of (2.3) is compatible with these maps.

(1.16) In the analytic case (i.e., if we consider analytic spaces with isolated normal singularities), all preceding results remain true, especially (1.3) and (1.10) (see § 3, and in particular the proof of Theorem (3.7)).

(1.17) If $E \subset X$ is a negative definite configuration of non-singular curves intersecting transversally, then E can be blown down (at least analytically) to a normal singularity, by Grauert's result [10]. Therefore, if E contains no curves exceptional of the first kind, then $H^1_E(\Theta_X)$ gives a contribution to the first order deformations of X ((1.3) and (1.16)). If E is one non-singular curve, then $H^1_E(\Theta_X)=0$ unless E is rational and $E^2=-1$ or -2. If E is the (minimal) configuration of a rational singularity, then $H^1_E(\Theta_X)=0$ if and only if $E_i^2<-2$, all i [31]. On the other hand, one must in general look at higher order terms $H^0(\Theta_X \otimes N_{nE})$, $n>1$, to get hold of all of $H^1_E(\Theta_X)$. For instance, a (non-rational) configuration



where all curves are rational with self-intersection -3, has $H^1_E(\Theta_X)=\mathbb{C}$, hence would contribute a first-order deformation to any surface on which it lies ([31]).

§ 2

(2.1) Let X be a scheme of finite type over $\mathbb{C}$, and let $\mathbf{C}$ be the category of Artin local $\mathbb{C}$-algebras. We will do infinitesimal deformation theory over this category [24, 30]. For each $A \in \mathbf{C}$, let $D(A)$ be the set of deformation classes of X over Spec A, i.e., diagrams

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $\bar{X} \to \operatorname{Spec} A$ is flat and $X \to \bar{X} \times_A \mathbb{C}$ is an isomorphism, modulo equivalence. Schlessinger's theory [24] shows that if $D(\mathbb{C}[\varepsilon])$ is finite dimensional (e.g., $H^1(\Theta_X)$ is finite-dimensional and X has only isolated singularities), then D is (formally) *versal;* that is, one has a complete local $\mathbb{C}$-algebra R and a morphism φ: $h_R = \mathrm{Hom}(R, \ ) \to D$ of functors on $\mathbf{C}$ such that

(i) $\varphi(\mathbb{C}[\varepsilon])$, the tangent map, is a bijection
(ii) φ is smooth, i.e., if $A' \to A$ is surjective, so is

$$h_R(A') \to h_R(A) \underset{D(A)}{\times} D(A').$$

If for instance $H^0(\Theta_X) = 0$, then D is *universal*, i.e., φ is a bijection. (We are abusing terminology somewhat in that it is usually a deformation, not a functor, which is called versal; however, no other adjective seems available.) Thus, there is a compatible sequence of deformation classes over Spec R/m_R^n which "represents" D on the category $\mathbf{C}$; we call R the *formal moduli space* of X. If X is a non-singular projective variety, then R is the complete local ring of the Kuranishi space of X. In this case, the obstructions to the smoothness of D (i.e., of R) lie in $H^2(\Theta_X)$. We say X has *obstructed deformations*, or *is obstructed*, if R is not regular, that is, if $\dim R < \dim H^1(\Theta_X)$.

(2.2) If X is an algebraic variety, we consider the subfunctor $D' \subset D$ of deformation classes which are locally trivial (either formally or in the Zariski topology – see [30], 2.1.5). Thus, the singularities remain analytically unchanged during deformation in D'. If $D(\mathbb{C}[\varepsilon])$ is finite-dimensional, then D' is formally versal, essentially by the smoothness of the local automorphism functor in characteristic 0 ([30], 1.3).

(2.3) **Proposition.** *Let f: $X \to V$ be the minimal resolution of a surface with only rational singularities. Then there is a morphism α: $D'_V \to D_X$ and a "blowing down" morphism β: $D_X \to D_V$ such that $\beta \circ \alpha$ is the inclusion $D'_V \subset D_V$.*

Proof. Let $\{V_i\}$ be an affine open cover of V, at most one singular point in each V_i. A locally trivial deformation $\bar{V}$ to Spec A induces deformations $\bar{V}_i$ which can be trivialized via maps $\bar{V}_i \xrightarrow{\sim} V_i \times \mathrm{Spec}\, A$. Using these, if $X_i = f^{-1}(V_i)$, one constructs maps $\bar{X}_i \to \bar{V}_i$ which must patch to give a deformation $\bar{X}$. To define α, one must show the independence of $[\bar{X}]$ on the choice of trivializations. But we claim two such maps $\bar{X}^i \to \bar{V}$ $(i = 1, 2)$ built in this way are uniquely $\bar{V}$-isomorphic. Since this is true away from the singular points, one need only find some $\bar{V}$-isomorphism locally near the singularities. Thus, suppose p_i: $X \times A \to \mathrm{Spec}\, R \otimes A$, $i = 1, 2$, differ by an A-automorphism σ of $\mathrm{Spec}\, R \otimes A$, i.e., $p_1 = \sigma \cdot p_2$. The condition $f_* \Theta_X = \Theta_R$ implies σ lifts to an A-automorphism $\bar{\sigma}$ of $X \times A$. But one then has $p_1 = p_2 \cdot \bar{\sigma}$, as desired.

To define β, let $[\bar{X}]\in D_X(A)$, and let $\mathcal{O}_{\bar{X}}$ be the sheaf of algebras on (the topological space) X. We claim $f_*(\mathcal{O}_{\bar{X}})=\mathcal{O}_{\bar{V}}$ defines a deformation of V to A. This requires checking only near the singularities, so we may consider $X\to \operatorname{Spec} R$. Then $\mathcal{O}_{\bar{V}}$ is the sheaf for $\operatorname{Spec}\Gamma(\mathcal{O}_{\bar{X}})$, which we must show is a deformation of $\operatorname{Spec} R$ (note $R=\Gamma(\mathcal{O}_X)$ by normality). But $H^1(\mathcal{O}_X)=0$ (the definition of rationality) implies $\Gamma(\mathcal{O}_{\bar{X}})\to R$ is surjective (e.g., [30], 3.1.3), hence $A\to\Gamma(\mathcal{O}_{\bar{X}})$ is flat (by the infinitesimal criterion of flatness – see [31]). This process defines β, and it is clear that $\beta\circ\alpha$ is the natural inclusion.

(2.4) **Proposition.** *Let $f\colon X\to \operatorname{Spec} R$ be a resolution of an isolated normal singularity, $\hat{R}$ the completion of R at the singular point, and $\hat{X}$ the formal scheme obtained by completing X along the exceptional fibre E. Then the natural morphisms $D_R\to D_{\hat{R}}$ and $D_X\to D_{\hat{X}}$ are smooth and bijective on the tangent spaces; thus, the formal moduli spaces depend only on the completions.*

Proof. Let

$$\begin{array}{ccc} \bar{R} & \longrightarrow & R \\ \uparrow & & \vdots \\ A & \longrightarrow & C \end{array}$$

define a deformation of R, with $\bar{m}\subset R$ the pull-back of the maximal ideal. Then it is easy to check that $\varprojlim \bar{R}/\bar{m}^n$ is flat over A, hence is a deformation of $\hat{R}$; this defines $D_R\to D_{\hat{R}}$. The definition of $D_X\to D_{\hat{X}}$ is similar; one takes the completion of a deformation of X along the pull-back of the sheaf of ideals defining E. We must show the morphisms are bijective on the tangent spaces and injective on the obstruction spaces. In the first case, these spaces are, respectively, T^1 and T^2; see [15] and [30] for definitions and the desired result. In the second case, the spaces for D_X are $H^1(\Theta_X)=H^1(\Theta_X\otimes\mathcal{O}_Z)$ and $H^2(\Theta_X)=H^2(\Theta_X\otimes\mathcal{O}_Z)$, where Z is some divisor supported on the exceptional fibre; but we clearly get the same spaces for $D_{\hat{X}}$.

(2.5) Let V be a projective surface whose only singularities $P_1,\dots,P_t$ are rational. Let $\{V_i\}$ be an affine open cover as in the proof of (2.3). Let $f\colon X\to V$ be the minimal resolution and $X_i=f^{-1}(V_i)$; let $L_i=D_{X_i}$ and $D_i=D_{V_i}$. By Proposition (2.4), L_i and D_i essentially depend only on the singularity P_i, and not on the particular choices of neighborhoods V_i and X_i. Finally, let $\bar{L}=\prod_{i=1}^{t} L_i$ and $\bar{D}=\prod_{i=1}^{t} D_i$. Note that all functors are formally versal, and $\bar{L}$ is smooth.

(2.6) **Theorem.** *Under the above assumptions, there is a cartesian diagram of functors*

$$\begin{array}{ccc} D_X & \longrightarrow & \bar{L} \\ \downarrow & & \downarrow \\ D_V & \longrightarrow & \bar{D} \end{array}$$

where the horizontal maps are restrictions, and the vertical maps are defined by β.

Proof. For the injectivity of $D_X \to \bar{L} \times_{\bar{D}} D_V$, suppose two deformations $\bar{X}$, X' of X blow down to isomorphic deformations $\bar{V}$, V' of V, and yield isomorphic deformations $\bar{X}_i$, X_i' of the X_i. Given an isomorphism $\bar{V} \xrightarrow{\sim} V'$, one uses $f_* \Theta_X \xrightarrow{\sim} \Theta_V$ as in the proof of (2.3) to find for each i a commutative diagram

$$\begin{array}{ccc} \bar{X}_i & \xrightarrow{\sim} & X_i' \\ \cap & & \cap \\ \bar{X} & & X' \\ \downarrow & & \downarrow \\ \bar{V} & \xrightarrow{\sim} & V' \end{array}$$

It is then easy to construct an A-isomorphism of $\bar{X}$ into X'. One shows surjectivity similarly.

(2.7) The morphism $D_X \to D_V$ has been considered in another guise by Brieskorn [6] and Artin [1]. In the algebraic case, let $\pi: \mathscr{V} \to T$ be a flat surjective map of finite type of algebraic spaces, whose fibres V_t are normal surfaces and whose singular locus is finite over T. Then Artin proves the existence of a (not necessarily separated) T-algebraic space $\mathscr{R} = \mathrm{Res}(\mathscr{V}/T)$, a smooth morphism π': $\mathscr{X} \to \mathscr{R}$, and a commutative diagram

$$\begin{array}{ccc} \mathscr{X} & \longrightarrow & \mathscr{V} \\ \downarrow & & \downarrow \\ \mathscr{R} & \longrightarrow & T \end{array}$$

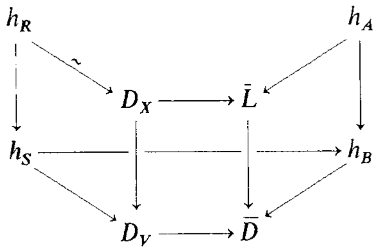
such that for all $r \in \mathscr{R}$, $X_r \to V_{t(r)}$ is a minimal resolution; furthermore, π' represents the functor of such resolutions of $\mathscr{V}$ over T. Suppose π induces a formal versal deformation of $V_0 = \pi^{-1}(0) = V$, and that the V_t have only rational singularities. Then if $0' \in \mathscr{R}$ maps to 0, Lemma (3.3) of [1] shows π': $\mathscr{X} \to \mathscr{R}$ induces a formal versal deformation of $X_0 = \pi'^{-1}(0') = X$. Thus, the map of the completion of T at 0 to the completion of $\mathscr{R}$ at $0'$ is (up to isomorphism) the map on formal moduli spaces associated to $D_X \to D_V$. In the local situation, the morphism $L_i \to D_i$ is well-understood for rational double points; in fact, on the scheme level, it is a Galois covering whose group is the Weyl group of the associated Dynkin diagram [8]. In particular, if V as in (2.5) has only nodes, then $\bar{L} \to \bar{D}$ is given on the ring level by:

$$C[[X_1, \ldots, X_t]] \to C[[Y_1, \ldots, Y_t]]$$
$$X_i \mapsto Y_i^2.$$

(2.8) **Corollary.** *Suppose D_X is universal* (e.g., $H^0(\Theta_X)=0$), *and the versality of the preceding functors is given by smooth morphisms $h_R \xrightarrow{\sim} D_X$, $h_s \xrightarrow{\sim} D_V$, $h_A \to \bar{L}$, $h_B \to \bar{D}$. Then we have an isomorphism*

$$S \hat{\otimes}_B A \xrightarrow{\sim} R.$$

Proof. Using the definition of smoothness, one constructs a commutative diagram:



Since $\bar{L}$ is smooth, A is smooth, so $h_A \to \bar{L} \times_{\bar{D}} h_B$ (an isomorphism on the tangent spaces) is smooth. Using this, one can find a morphism $h_R \to h_A$ making all squares above commutative. The maps in

$$h_R \to h_S \times_{h_B} h_A \to D_V \times_{\bar{D}} \bar{L} \xleftarrow{\sim} D_X \xleftarrow{\sim} h_R$$

are all injective on the tangent spaces, hence injective, thus bijective, so

$$h_R \xrightarrow{\sim} h_S \times_{h_B} h_A = h_{S \otimes_B A}.$$

(2.9) **Corollary.** *Suppose D_X is universal and $\bar{D}$ is smooth* (e.g., *every singularity is a rational double or triple point*). *Then R is a complete intersection over S, and* Spec $R \to$ Spec S *is finite, flat, and surjective. If each P_i is a node, then R is generated over S by elements of degree* 2.

Proof. It follows by [1], Theorem 3, that Spec$A \to$ Spec$B$ is finite, flat, and surjective, whence A is a complete intersection over B (as both are smooth). Since $B \to A$ is finite, we may replace the $\hat{\otimes}$ in Corollary (2.8) by $\otimes$, and conclude the results on Spec$R \to$ Spec$S$. For the remark on nodes, recall (2.7).

(2.10) *Remark.* If P_i is an R.D.P., then $L_i \to D_i$ is not a surjective map of functors (it is the zero map on the tangent spaces, by (1.10) and (1.15)), although the corresponding scheme map is surjective.

(2.11) **Corollary.** *Let $V \subset \mathbb{P}^3$ be a hypersurface of degree $n \geqq 5$ with only R.D.P.'s as singularities, and let $f: X \to V$ be the minimal resolution. Then the formal moduli space of X is a reduced complete intersection of Krull dimension* $\binom{n+3}{3} - 16$.

Proof. Since the canonical bundle is $K_X = f^* \mathcal{O}_V((n-4)H)$ [3], X is a surface of general type, hence $H^0(\Theta_X)=0$. Let $L = f^* \mathcal{O}_V(H)$. Now $R^1 f_* L^{\otimes r} = 0$, $r \geqq 0$ (by rationality), so $H^1(X, L^{\otimes r}) = H^1(V, \mathcal{O}_V(rH)) = 0$. Now, K_X lifts under deformations of X (uniquely, since $H^1(\mathcal{O}_X)=0$), hence, by obstruction theory, so does L; further, $H^1(L)=0$ implies all sections of L lift as well. Thus, for every deformation of X, one can lift L and four global sections to get a map into $\mathbb{P}^3$. The image of $D_X \to D_V$ lands therefore in the subfunctor $D_1 \subset D_V$ of hypersurface deformations modulo projective equivalence; D_1 is a subfunctor since $H^0(\Theta_V)=0$ implies two hypersurface deformations are isomorphic if and only if they are projectively equivalent. One easily sees D_1 is smooth and universal, of dimension $\binom{n+3}{3} - 16$; it may be constructed by taking a non-singular subvariety of the Hilbert scheme of V transversal at V to the orbit of $PGL(3)$ (again since $H^0(\Theta_V)=0$). Theorem (2.6) and its corollaries go through for the subfunctor $D_1 = h_{S_1}$ of D_V; so, if $h_R \xrightarrow{\sim} D_X$, then R is reduced if and only if it is generically reduced (i.e., (0) has no embedded components). But $\operatorname{Spec} A \to \operatorname{Spec} B$ (Corollary (2.9)) is generically étale, hence so is $\operatorname{Spec} R \to \operatorname{Spec} S_1$, whence R is generically reduced.

(2.12) *Remark.* In Theorem (4.2), we give a cohomological necessary and sufficient condition for D_X to be smooth.

(2.13) We can now use the computations of § 1, especially Proposition (1.10), to conclude the following:

(2.14) **Theorem.** *Let $f: X \to V$ be the minimal resolution of a projective surface with only R.D.P.'s at $P_1, \ldots, P_t$, and let*

$$\gamma: D_X \to \bar{L} = \prod D_{X_i}.$$

Then

(i) *γ_ε is surjective.*

(ii) *Via α: $D'_V \subset D_X$, we have D'_V is the "fibre" of γ,* i.e., $D'_V(A) = \{[\mathcal{X}] \in D_X(A) \mid \gamma_A([\mathcal{X}])$ *is the trivial element*$\}$.

(iii) *If D_X is smooth, then γ and D'_V are smooth.*

(iv) *If $H^0(\Theta_X)=0$, D_X is smooth if and only if $D_V \to \bar{D}$ is smooth.*

Proof. γ_ε is the map $H^1(\Theta_X) \to \bigoplus_{i=1}^{t} H^1(X_i, \Theta_{X_i})$. Let $E^i = f^{-1}(P_i)$, and $E = \bigcup E_i$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^1_E(\Theta_X) & \xrightarrow{\sim} & \bigoplus H^1_{E_i}(\Theta_{X_i}) \\ \downarrow & & \downarrow \\ H^1(\Theta_X) & \longrightarrow & \bigoplus H^1(\Theta_{X_i}). \end{array}$$

Since the second vertical map is an isomorphism by (1.10), γ_ε is surjective. For (ii), one uses the construction of α and β (2.3) or Theorem (2.6). If D_X is smooth and γ_ε is surjective, γ is smooth (by the "implicit function theorem"); then, the smoothness of D'_V follows from (ii), whence (iii) is proved. For (iv), first recall the exact sequence [24]:

$$0 \to H^1(\Theta_V) \to D_V(\mathbb{C}[\varepsilon]) \to T^1_V = \bar{D}(\mathbb{C}[\varepsilon]).$$

Suppose $h_R \xrightarrow{\sim} D_X$, $h_S \xrightarrow{\sim} D_V$. Via Corollary (2.9), we have

$$\dim R = \dim S \leqq \dim D_V(\mathbb{C}[\varepsilon]) \leqq \dim H^1(\Theta_V) + \dim T^1_V = \dim H^1(\Theta_X).$$

The last equality follows by (ii) and (1.10). (iv) follows readily.

(2.15) *Remark.* The Theorem says that the smoothness of D_X is equivalent to both the independent behavior of the exceptional curves (i.e., the smoothness of γ) and the independent behavior of the singular points on V. In particular, if X is unobstructed, then D_V is smooth, and one can independently smooth the singular points on V. (See (4.3) below.)

§3

(3.1) This § is devoted to convergent analytic consequences of the formal theory of §§1–2. One of our first tasks will be to discuss the convergent version of the map β in (2.3). Such a blowing down map should be local in nature about the subvarieties to be blown down, and we describe below the proper choice of localization for our purposes. This will consist of choosing "1-convex tubes" about an exceptional subvariety, so that we might apply the 1-convex relative analogues of Grauert's coherence and blowing down theorems. Intuitively, such a tube is an analytic version of a family of resolutions of affine varieties with isolated singularities.

Let $\pi: \mathcal{U} \to T$ be a smooth, 1-convex map of complex manifolds. By definition, this means that there is a differentiable function $\varphi: \mathcal{U} \to \mathbb{R}$ with the properties

(1) π restricted to $\varphi^{-1}((-\infty, c]) = \mathcal{U}^c$ is proper, for every $c \in \mathbb{R}$.

(2) φ restricted to $\mathcal{U}_{c_0}=\varphi^{-1}([c_0,\infty))$ is strictly plurisubharmonic, for some $c_0\in\mathbb{R}$. That is, the complex Hessian $\frac{\partial^2\varphi}{\partial z_i\,\partial\bar{z}_j}$ is a positive definite hermitian form, a condition independent of the choice of local coordinates $\{z_i\}$ on $\mathcal{U}$. (In particular, each fiber of π is a 1-convex manifold.)

If $0\in T$ is a base point, and $U=\pi^{-1}(0)$, then we are considering a deformation of U. Henceforth, all maps to and from T will be map germs about 0. Recall that the exceptional subvariety of a 1-convex manifold U is the maximal compact subvariety of U, all of whose components are positive dimensional. Such subvarieties may be blown down to isolated normal singular points [10]. In the situation of π as above, Riemenschneider has proved the following theorem [20]:

(3.2) **Theorem.** *Suppose $H^1(U,\mathcal{O}_U)=0$. Then there exists a complex space $\mathcal{V}$ and a commutative diagram of holomorphic maps:*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ \tau\ } & \mathcal{V} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & T & \end{array}$$

Here we have:

(1) τ is a proper, surjective map which blows down the exceptional subvarieties of the fibers of π.

(2) ω is a flat map with normal fibers.

The existence of such a $\mathcal{V}$ and the maps was already shown by Markoe-Rossi [16], and Siu [28]. The rationality $H^1(U,\mathcal{O}_U)=0$ is used to show flatness and normality by methods formally the same as those in algebraic geometry, given the coherence and semi-continuity theorems of [28] and [21].

Note that, in particular, if the exceptional subvariety of the 1-convex manifold U is connected, then $V=\omega^{-1}(0)$ has at most one singular point p, and ω defines a deformation of the germ of V at p.

(3.3) Now consider a deformation $\pi\colon\mathcal{X}\to T$ of a compact analytic manifold $X=\pi^{-1}(0)$, i.e., π is a proper, smooth map. In (3.3) and (3.4), we allow T to be singular, but reduced. Let $E\subset X$ be a connected exceptional subvariety, and let U be a 1-convex neighborhood of E [10]. Let V denote U with E blown down to a point, a normal analytic space with one possible singular point P. We want to construct a deformation of the local singularity (V,p) from the deformation of X over T, by means of (3.2). To this end, let us cut out of $\mathcal{X}$ over T the germ of a 1-convex tube about E as follows:

Shrinking T about 0 if necessary, we may assume that $\mathcal{X}$ is diffeomorphic to $X\times T$. Since U is 1-convex, by definition there exists a differentiable function φ_0: $U \to \mathbb{R}$ with properties (1) and (2) of (3.1) above (π maps U to a point). Let $\mathscr{U} = U \times T \subset X \times T$, considered as an open subset of $\mathscr{X}$; φ_0 can be considered as a function from $\mathscr{U}$ to $\mathbb{R}$. Since φ_0 is proper on U, and strict plurisubharmonicity is defined by a strict differential inequality, by shrinking T about 0, we may assume that φ_0 restricted to any $U \times \{t\}$ is strictly plurisubharmonic on

$$U \times \{t\} \cap \varphi_0^{-1}([c_1, c_2]).$$

Here $c_1 < c_2$ are two real constants, independent of $t \in T$. Shrinking T further, we assume T is strongly pseudo-convex with exhaustion function $\varphi_1 : T \to \mathbb{R}$. Let $\mathscr{U}^c = \varphi_0^{-1}((-\infty, c))$, where $c_1 < c < c_2$, and still denote by π the restriction of π to $\mathscr{U}^c$. Then the function $\varphi = \pi^* \varphi_1 - \log(c - \varphi_0)$ exhibits $\pi: \mathscr{U}^c \to T$ as a 1-convex smooth morphism.

(3.4) *Definition.* An open set $\mathscr{U} \subset \mathscr{X}$ such that π restricted to $\mathscr{U}$ is a 1-convex smooth map, and such that $\mathscr{U} \cap X$ contains E as connected maximal exceptional subvariety, will be called a 1-convex tube about E in $\mathscr{X}$.

Two 1-convex tubes $\mathscr{U}_1$, $\mathscr{U}_2$ about E are considered equivalent if there exists a third 1-convex tube $\mathscr{U}_3$ about E in $\mathscr{X}$ such that $\mathscr{U}_3 \subset \mathscr{U}_1 \cap \mathscr{U}_2$, perhaps over a smaller neighborhood of 0. A germ of 1-convex tube about E in $\mathscr{X}$ is an equivalence class of such tubes. The remarks in (3.3) show that a connected exceptional E in X determines a germ of 1-convex tube about E in $\mathscr{X}$.

Suppose U is a 1-convex neighborhood of E in X, and suppose $H^1(U, \mathcal{O}_U) = 0$. This condition is independent of U, for U sufficiently small. V denotes U with E blown down to a point P.

(3.5) **Proposition.** *Let E be as above in $\mathscr{X} \xrightarrow{\pi} T$, with T non-singular. If $H^1(U, \mathcal{O}_U) = 0$, then $\pi: \mathscr{X} \to T$ determines a germ of deformation $\mathscr{V}$ of the germ of V at P.*

Proof. One only has to apply (3.2) to a 1-convex tube about E in $\mathscr{X}$. The uniqueness is obvious, by the comments after (3.4).

(3.6) Consider now a deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} \mathbb{C} = 0 & \longrightarrow & T \end{array}$$

of a smooth, compact analytic surface X over the non-singular base T. Assume that X contains disjoint, connected exceptional curves $E^1, \ldots, E^k$, which blow down to rational singular points $P_1, \ldots, P_k$, respectively. Let $\mathscr{U}^1, \ldots, \mathscr{U}^k$ be 1-convex tubes about $E^1, \ldots, E^k$, respectively, and let $S_1, \ldots, S_k$ be the base spaces of the versal deformations of $P_1, \ldots, P_k$, respectively [9]. Let $S = \prod S_i$. There is a holomorphic map Φ: $T \to S$, classifying the product of the various local deformations $\mathscr{V}^i$, determined by the $\mathscr{U}^i$'s, via (3.5) [9].

Artin [1] shows that there exists an algebraic space R_i, of finite type over $\mathbb{C}$, over each S_i, classifying resolutions of deformations of the singularity V^i (U^i with E^i blown down). Thus, our map Φ must lift to a map Ψ: $T \to R = \prod_i R_i$. By the definition of a holomorphic map into the algebraic spaces R_i, there is, for each i, an analytic space Z_i, and a map Ψ_i: $T \to Z_i$ with the following properties:

(i) over Z_i, there is a deformation $\mathscr{V}_{Z_i}$ of V^i, together with a resolution $\mathscr{U}_{Z_i}$ of that deformation.

(ii) we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{U}^i & \xrightarrow{\ \sim\ } & \Psi_i^* \mathscr{U}_{Z_i} \\ \downarrow & & \downarrow \\ \mathscr{V}^i & \xrightarrow{\ \sim\ } & \Psi_i^* \mathscr{V}_{Z_i} \end{array}$$

(Thus, Z_i is versal for deformations of neighborhoods of the exceptional curve E^i.) We also denote by Ψ the map Ψ: $T \to Z = \prod_i Z_i$, the product of the Ψ_i's.

Consider the case where π: $\mathscr{X} \to T$ is the Kuranishi family of X, where all the singularities $P_1, \ldots, P_k$ are assumed R.D.P.'s, and where the E^i's are the exceptional curves of minimal resolutions. Note that we are assuming that T is nonsingular.

(3.7) **Theorem.** *The map Ψ: $T \to Z$ is locally a submersion at $0 \in T$. In particular, the exceptional curves $E^1, \ldots, E^k$ have independent behavior in deformations of X.*

Proof. Of course, this is nothing more than saying that the differential of Ψ is surjective at $0 \in T$. To see this, consider the following commutative diagram, with $\Theta = \Theta_X$ or Θ_{U^i}, as the context dictates:

$$\begin{array}{ccccc} H^1(X, \Theta) & \xrightarrow{\ r\ } & \prod_i H^1(U^i, \Theta) & \xrightarrow{\ \mathrm{id}\ } & \prod_i H^1(U^i, \Theta) \\ & \nwarrow^{\sim} & \uparrow{\scriptstyle \rho_2} & & \uparrow{\scriptstyle \rho_3} \\ & & T_0(T) & \xrightarrow{\ d\Psi_*\ } & T_{\Psi(0)}(Z) \end{array}$$

Here all the vertical arrows are Kodaira-Spencer characteristic maps, r is the restriction map, and id comes from identifying the fiber over $\Psi_i(0)$ in $\mathscr{V}_{Z_i}$ and V^i. The commutativity of the square comes from (3.6)(ii), and the naturality of the characteristic map; that of the triangle from naturality. The map ρ_1 is an isomorphism since T is the Kuranishi family.

Now, r is surjective, by an analytic analogue of (2.13), where r is called γ_ε, which we shall prove directly below. Thus ρ_3 is also surjective. Since $T_{\Psi(0)}(Z)$ and $\prod_i H^1(U^i, \Theta)$ have the same dimension, as noted in (1.9) and [1], ρ_3 is an isomorphism. Hence, $d\Psi_*$ is surjective.

To check that r is surjective, we go back to the proof of (2.13) and see that it is enough to show that $H^1_{E^i}(U^i, \Theta) \to H^1(U^i, \Theta)$ is an isomorphism, for each i. We'll show this by comparison with the algebraic case proved in (2.13). Now each of these cohomology groups is independent of the choice of 1-convex U^i, provided E^i is the only exceptional curve of U^i. Thus, since p_i is an algebraic singularity, we may assume U^i is the minimal resolution X^i of an affine algebraic variety V^i with one singular point p_i. Let us agree to drop the superscript i from the above, and denote by X the algebraic variety corresponding to X^i, and X^{an} the corresponding complex manifold. Similarly for coherent algebraic and analytic sheaves, maps, etc. It is easy to check (by integrating vector fields) that (1.2) above remains true in the analytic category. Using the corresponding long exact sequences for local cohomology, we obtain:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1_E(X, \Theta) & \xrightarrow{\sim} & H^1(X, \Theta) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{E^{\mathrm{an}}}(X^{\mathrm{an}}, \Theta^{\mathrm{an}}) & \longrightarrow & H^1(X^{\mathrm{an}}, \Theta^{\mathrm{an}}) \end{array}$$

The vertical maps are the functorial ones introduced by Serre in *GAGA*. The top isomorphism is by (2.13). We want to show the bottom row is an isomorphism. To see this, consider the commutative diagram arising from the Leray spectral sequences for τ and τ^{an}, where $\tau\colon X \to V$ is the resolution map:

$$\begin{array}{ccccc} H^1(X, \Theta) & \xrightarrow{\sim} & H^0(V, R^1\tau_*\Theta) & \xrightarrow{\sim} & (R^1\tau_*\Theta)_p \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X^{\mathrm{an}}, \Theta^{\mathrm{an}}) & \xrightarrow{\sim} & H^0(V^{\mathrm{an}}, R^1\tau_*^{\mathrm{an}}\Theta^{\mathrm{an}}) & \xrightarrow{\sim} & (R^1\tau_*\Theta)_p^{\mathrm{an}} \end{array}$$

Here we've used $R^1\tau_*^{\mathrm{an}}\Theta^{\mathrm{an}} \xrightarrow{\sim} (R^1\tau_*\Theta)^{\mathrm{an}}$, see [11]. Since $R^1\tau_*\Theta_p$ is of finite length, the right vertical arrow is an isomorphism, and we are done.

(3.8) *Remarks.* As in (2.13), we may note that the fiber of Ψ over $\Psi(0)$ consists of all small deformations of X, which induce locally trivial deformations of $V = X$ with $E^1, \ldots, E^k$ all blown down.

The proof of (3.7) is a bit long, but we prefer to keep explicit the relation between the algebraic and analytic categories.

The theorem on blowing down in families is true for deformations of resolutions of R.D.P.'s over singular T as well, and the surjectivity of $d\Psi_*$ follows functorially, as above. However, the local R.D.P.'s, or rather, their exceptional curves, will not, in general, be independent if T is singular, since the independence (in the sense of surjectivity of Ψ) depends on the implicit function theorem, invalid for a map from a singular space.

It does not seem that enough is known at present about Kuranishi families for deformations of V, or locally trivial deformations of V, to make more precise statements fully analogous to (2.7), except in special cases. Given the existence of the appropriate Kuranishi families, the convergent analogue of (2.7) would follow directly from the formal theory leading to (2.7).

(3.9) *Example.* Let A be a 2-dimensional complex torus, and let -1 denote the automorphism of A sending a to $-a$, the group inverse of a in A. Let $V = A/(\pm 1)$ be the quotient of A by the $\mathbb{Z}/2\mathbb{Z}$ action generated by -1, and let $X = V$ with the 16 nodes of V blown-up. Then X is a Kummer $K-3$ surface [22], and its Kuranishi space is non-singular, 20-dimensional. The above theory applied to this example shows that the Kuranishi variety of X fibers over the 16-dimensional family of local deformations corresponding to the 16 exceptional, nodal $\mathbb{P}^1$'s in X. It is easy to see that the fiber over $\Psi(0)$, in the notation of (3.7), is the family of Kummer deformations of X, i.e., the deformations of X gotten by allowing the A in the Kummer construction to vary in its 4-dimensional Kuranishi family. It also happens in this case that $K_X = 0$, and so $H^1(X, \Omega^1_X)$ is naturally isomorphic to $H^1(X, \Theta)$, the isomorphism determined up to a non-zero scalar by the choice of a nowhere vanishing holomorphic two-form ω on X. If $E^1, \ldots, E^{16}$ are the 16 exceptional curves in X, then $N_i = \Omega^1_{E^i}$ on E^i, where N_i is the normal bundle of the imbedding $E^i \hookrightarrow X$. Let $[E^i]^*$ denote the class in $H^1(X, \Omega^1_X)$ determined by E^i via Serre duality. Then the $[E^i]^*$'s determine a splitting of γ_ε in (2.13).

In fact, using ω, we obtain a commutative diagram

(3.10)
$$\begin{array}{ccc} \Theta & \xrightarrow{r_1} & \bigoplus_i N_i \\ \wr\downarrow & & \downarrow\wr \\ \Omega^1_X & \xrightarrow{r_2} & \bigoplus_i \Omega^1_{E^i} \end{array}$$

Here the horizontal arrows are the natural restrictions and the left vertical arrow sends a germ of vector field ξ to $i(\xi)\,\omega$, the contraction of ω and ξ. We have to check that this map induces an isomorphism $N_i \xrightarrow{\sim} \Omega^1_{E^i}$. Check this locally, in coordinates (z_1, z_2) where E^i is given by $z_1=0$. Then the kernel of r_1 consists of vector fields of the form $\xi = a z_1 \frac{\partial}{\partial z_1} + b \frac{\partial}{\partial z_2}$, where a, b are arbitrary holomorphic functions; the kernel of r_2 consists of forms $\eta = a\,dz_1 + b z_1\,dz_2$, with a, b arbitrary. Locally, $\omega = h\,dz_1 \wedge dz_2$, where $h \neq 0$, and

$$i\left(f\frac{\partial}{\partial z_1} + g\frac{\partial}{\partial z_2}\right)\omega = h(f\,dz_2 - g\,dz_1),$$

and our check of (3.10) is complete.

From (3.10) we get:

(3.11)
$$\begin{array}{ccc} H^1(X,\Theta) & \xrightarrow{r_1=\gamma_\varepsilon} & \bigoplus_i H^1(E^i, N_i) \\ \wr\downarrow & & \wr\downarrow \\ H^1(X,\Omega_X^1) & \xrightarrow{r_2} & \bigoplus_i H^1(E^i, \Omega_{E^i}^1) \end{array}$$

To check, finally, the claim about the $[E^i]^*$'s determining a section of γ_ε, it will suffice to compute $r_2([E^i]^*)$ for each i. Denote by $[E^i]$, $[X]$ the fundamental cycles in homology of E^i, X, respectively. Then:

$$\begin{aligned} \langle r_2([E^i]^*), [E^j]\rangle &= \langle [E^i]^* \cup [E^j]^*, [X]\rangle \\ &= E^i \cdot E^j \\ &= -2\delta_{ij} \quad (\delta_{ij} = \text{Kronecker index}). \end{aligned}$$

Here $\langle\ ,\ \rangle$ denotes natural homology-cohomology pairings. The first equality comes from viewing $[E^i]^*$ in $H^2(X, \mathbb{C})$, via Poincare duality. Note that this also proves the surjectivity of γ_ε at the same time. Such a geometric proof of surjectivity also works for E^i's in a $K-3$ surface associated to other R.D.P.'s, using the negative definiteness of intersection matrices among the components of an E^i.

§ 4

(4.1) Let $V \subset \mathbb{P}^3$ be a hypersurface of degree $n \geqq 5$ with only R.D.P.'s, and $f\colon X \to V$ its minimal resolution. We derive in Theorem (4.2) a necessary and sufficient cohomological criterion for the unobstructedness of X. In particular, a nodal hypersurface yields unobstructed X if and only if the nodes are in "general position" in $\mathbb{P}^3$ (Corollary (4.3)). We know of no other moduli problem where such a cohomological condition is necessary as well as sufficient. Examples of obstructed moduli exist for all $n \geqq 5$. One should keep in mind that the X as above are diffeomorphic to non-singular hypersurfaces of the same degree (Atiyah [4], Brieskorn [5]). The only previously known obstructed surfaces were some elliptic surfaces (Kas [13]), and some surfaces of general type (E. Horikawa [33]) based on a complicated example of Mumford [18]; in neither instance is anything known about the order of obstructedness. However, in case V is nodal, the moduli space of X is defined by hypersurfaces of degree 2.

(4.2) **Theorem.** *Let $V \subset P = \mathbb{P}^3$ be a hypersurface of degree $n \geqq 5$ with only R.D.P.'s as singularities, with $V = V(F)$. Let $I = (F_X, F_Y, F_Z, F_W)$ be the (invariantly defined) homogeneous polynomial ideal defined by the partial derivatives, with $T \subset \mathbb{P}^3$ the "scheme of singularities" defined by the associated sheaf $\tilde{I} \subset \mathcal{O}_P$. Let $f\colon X \to V$ be the minimal resolution, and R the formal moduli space of X. Then R is a reduced complete intersection, and*

$$\dim m_R/m_R^2 - \dim R = \dim H^1(\mathbb{P}^3, \tilde{I}(n)).$$

Thus, R is regular if and only if

$$H^0(\mathcal{O}_P(n)) \to H^0(\mathcal{O}_T) \quad \textit{is surjective.}$$

Proof. The first assertion is in (2.11). Letting $h^i = \dim H^i$, and using (1.10), we have

$$\dim m_R/m_R^2 = h^1(\Theta_X) = h^1(\Theta_V) + \dim T_V^1.$$

Consider the exact sequences (where $i\colon V \subset \mathbb{P}^3$):

$$\begin{aligned} &0 \to \Theta_V \longrightarrow i^* \Theta_P \to N'_V \to 0,\\ &0 \to N'_V \longrightarrow N_V \quad \to T_V^1 \to 0,\\ &0 \to \mathcal{O}_P \xrightarrow{F} \tilde{I}(n) \quad \to N'_V \to 0. \end{aligned}$$

(Cf. [30]; these sequences are easily verified using $N_V = \mathcal{O}_V(n)$. Recall $N'_V \subset N_V$ is the sheaf of germs of locally trivial relative deformations of V in $\mathbb{P}^3$.) Note that $h^0(\Theta_V) = 0$, $h^0(i^* \Theta_P) = 15$ (by the standard presentation of Θ_P); since all locally trivial deformations of V take place in $\mathbb{P}^3$ (use the morphism α and the proof of (2.11)), $H^0(N'_V) \to H^1(\Theta_V)$ is surjective. Thus,

$$\begin{aligned} h^1(\Theta_V) &= h^0(N'_V) - 15\\ &= h^0(N_V) - \dim T^1 - 15 + h^1(N'_V)\\ &= \binom{n+3}{3} - 16 - \dim T^1 + h^1(\tilde{I}(n)). \end{aligned}$$

This and the sequence

$$0 \to \tilde{I}(n) \to \mathcal{O}_P(n) \to \mathcal{O}_T \to 0$$

yield the theorem.

(4.3) **Corollary.** *Let $V \subset \mathbb{P}^3$ be of degree n with exactly d nodes at the subset $T \subset \mathbb{P}^3$. Then X is unobstructed if and only if the set T is n-independ-*

ent, i.e., *for any partition* $T=T'\cup T''$, *one may find a hypersurface of degree* n *containing* T' *and missing* T''.

(4.4) *Example.* Let F be a non-singular quartic form, G and H non-singular m-tics ($m\geqq 4$), such that $V(F,G,H)$ consists of $4m^2$ distinct points. If $J=(F,G,H)$, then the general form of degree $2m$ in J^2 is irreducible with exactly $4m^2$ nodes (Bertini's Theorems – cf. [30], 3.5); further, the associated sheaf $\tilde{I}$ (as in Theorem (4.2)) is equal to $\tilde{J}$. The standard Koszul resolution shows $\dim H^1(\tilde{J}(2m))=1$. Thus, we have for every even dimension $n\geqq 8$, a nodal hypersurface of n^2 nodes such that R for the resolving surface X is a reduced hypersurface with a singularity of multiplicity two.

(4.5) *Example.* If we choose F, G, H to be non-singular m-tics ($m\geqq 4$), intersecting in m^3 distinct points, we may use the generic form of degree $2m$ or $2m+1$ in J^2 ($J=(F,G,H)$) to get nodal hypersurfaces of all degrees $n\geqq 8$, such that the resolving X are obstructed, and the number of obstructed deformations of X goes to infinity as $n^3/48$.

(4.6) *Example.* In order for $H^0(\mathcal{O}_P(n))\to H^0(\mathcal{O}_T)$ to be surjective, it is of course necessary that $d=\dim H^0(\mathcal{O}_T)\leqq\binom{n+3}{3}$. On the other hand, Segre [26] has given examples of surfaces of all even degrees n with $(n^3-n^2)/4$ nodes. For such V, the resolving X is of course obstructed.

(4.7) *Example.* Let X, Y, Z, W be coordinates in $\mathbb{P}^3$, let $L_1,\dots,L_n$ ($n\geqq 5$) be general linear forms in X, Y, and Z, and let

$$F_n=W^n-\prod_{i=1}^{n}L_i.$$

Then a computation shows $V(F_n)$ gives obstructed surfaces for $n\geqq 5$. In particular, $V(F_5)$ is a quintic with ten A_4-singularities. This is the best possible degree for a counterexample, since a $V\subset\mathbb{P}^3$ of degree $\leqq 4$ with only R.D.P.'s has $H^2(\Theta_X)=0$.

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