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A Calculus for Framed Links in S^3

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Dedicated to Arnold Kas

§ 1. A framed link L in S^3 is a finite, disjoint collection of smoothly imbedded circles, $\gamma_1, \ldots, \gamma_r$, knotted or unknotted, with an integer ϕ_i (the framing) associated with each circle. If F is a Seifert surface for a knot K in L, then the zero framing of the normal bundle of K is derived from the normal vector to K pointing inward along F and the normal vector to F. The framing associated with the integer n is obtained by twisting the zero-framing n times in a clockwise (right-handed) direction.

L determines a 4-manifold M_L obtained by adding 2-handles to the 4-ball B^4 along the circles in L using their framings. Note that it makes no difference how we orient the circles; an orientation for M_L and ∂M_L is determined by extending a fixed orientation on B^4 over M_L.

W.B.R. Lickorish [L] and A.D. Wallace showed that any orientable 3-manifold is ∂M_L for some framed link L. We describe below two operations (the calculus) on a framed link and prove that $\partial M_L = \partial M_{L'}$ if and only if we can pass from L to L' by a sequence of these operations.

Operation one ($\mathcal{O}_1$): We may add to or subtract from L an unknotted circle with framing 1 or -1, which is separated from the other circles by an imbedded S^2 in S^3.

This corresponds in M_L to taking connected sum with or splitting off a copy of the complex projective plane CP^2 with its "positive" or "negative" orientation (CP^2 has two orientations, one giving $\langle +1 \rangle$ as the intersection form on $H_2(CP^2; Z)$, the other $\langle -1 \rangle$).

Let γ_0 and γ_1 be two knots in S^3. Let $b\colon I \times I \to S^3$ be an imbedding of $[0,1] \times [0,1]$ for which $b(I \times I) \cap \gamma_i = b(i \times I)$, $i = 0, 1$. Then let $\gamma_0 \underset{b}{\#} \gamma_1 = \gamma_0 \cup \gamma_1 - b(\partial I \times I) \cup b(I \times \partial I)$ and call this the band (over b) connected sum of γ_0 and γ_1.

Operation two ($\mathcal{O}_2$): Given two circles γ_i and γ_j in L, we "add" γ_i to γ_j as follows. First push γ_i off itself (missing L) using the framing ϕ_i, obtaining $\tilde{\gamma}_i$. Now change L by replacing γ_j with $\gamma_j' = \tilde{\gamma}_i \underset{b}{\#} \gamma_j$ where b is any band missing the rest of L.

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This corresponds in M_L to sliding the j^{th} handle over the i^{th} handle via the band b. To compute the framing ϕ_j' of γ_j', it is helpful to study the intersection form on $H_2(M_L; Z)$. If we orient each γ_k, they determine a basis Γ for $H_2(M_L; Z)$, denoted by $\bar{\gamma}_1, \ldots, \bar{\gamma}_r$. The matrix A_L of the intersection form with respect to Γ has the framings ϕ_k down the diagonal and a_{ij} is the algebraic linking number of the oriented γ_i with the oriented γ_j. Operation two corresponds to either adding $\bar{\gamma}_i$ to or subtracting $\bar{\gamma}_i$ from $\bar{\gamma}_j$ depending on whether or not the orientations on γ_i and γ_j correspond under b. Thus $\gamma_j' = \gamma_j + \gamma_i \pm 2a_{ij}$. Beware that the plus or minus is determined only by b and is independent of the orientations chosen on γ_i and γ_j.

Example.

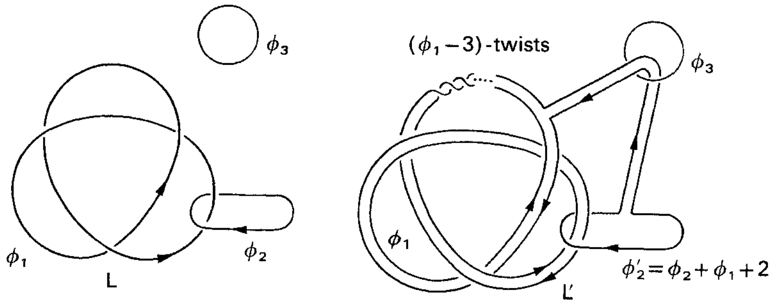


Fig. 1.1

According to the orientations we have subtracted $\bar{\gamma}_1$ from $\bar{\gamma}_2$. There are $\phi_1 - 3$ full twists because the zero framing for the (right-handed) trefoil knot would be drawn with -3 full twists.

Call two framed links L_1 and L_2 ∂-equivalent if we can obtain L_2 from L_1 by a sequence of operations one and two; we write $L_1 \underset{\partial}{\sim} L_2$.

The name boundary equivalent is justified as follows:

Theorem[1]. Given two framed links L_1 and L_2, then $L_1 \underset{\partial}{\sim} L_2 \Leftrightarrow \partial M_{L_1}$ is diffeomorphic to ∂M_{L_2} (preserving orientations).

The proof is given in § 2, using some 4-dimensional Cerf theory which is explained in § 3. Some applications are given in § 4, and § 5 contains miscellaneous remarks including more general versions of the theorem.

As explained in the first remark in § 5, this paper evolved from a topological view of some techniques in complex surface theory. Arnold Kas taught me those techniques in a geometric fashion; in that sense he is the grandfather of this paper. I am also indebted to Andrew Casson, Allen Hatcher, John Morgan, Colin Rourke, and Jack Wagoner for helpful conversations and some simplifications in the proof of the main theorem.

§ 2. Proof of the Theorem: If $L_1 \underset{\partial}{\sim} L_2$, then it is clear from the interpretation of $\mathcal{O}_1$ and $\mathcal{O}_2$ in terms of 4-manifolds that $\partial M_{L_1} = \partial M_{L_2}$.

For the reverse implication, we start by forming the closed, oriented manifold $N^4 = M_{L_1} \cup -M_{L_2} \cup \partial M_{L_1} \times [1, 2]$ where we identify ∂M_{L_i} with $\partial M_{L_i} \times i$.

[1] I understand that Robert Craggs has proven the same theorem [Craggs], although the point of view and the proof are quite different

By adding copies of $\pm CP^2$ to M_{L_1}, we can arrange that index $(N)=0$ (this changes L_1 by $\mathcal{O}_1$; we still call the result L_1). N bounds an oriented, connected 5-manifold W^5.

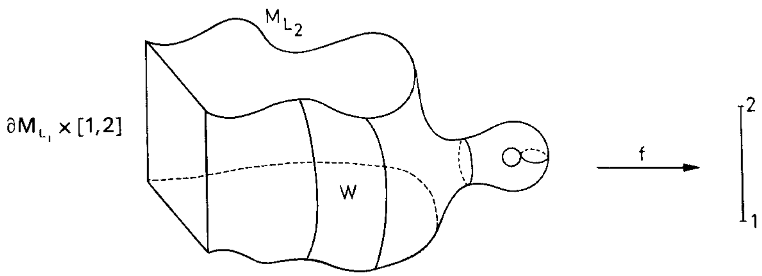


Fig. 2.1

Let f: $W^5 \to [1,2]$ be a Morse function for which $f^{-1}(i)=M_{L_i}$, $i=1,2$, and $f|\partial M_{L_1} \times [1,2]$ is projection. As is true in any dimension, we may cancel the critical points of index 0 and 5 since W^5 is connected.

Pick $\varepsilon>0$ so that there is one critical point of index 1 in $f^{-1}([1,1+\varepsilon])$. Then $f^{-1}(1+\varepsilon)=M_{L_1} \# S^1 \times S^3$. But we may achieve this by adding a 3-handle instead of a 1-handle. In this way, we replace all the critical points of index 1 by critical points of index 3 (changing W^5, of course), and similarly we change critical points of index 4 to those of index 2. The new W^5 has a Morse function f: $W^5 \to [1,2]$ with critical points of index 2 belonging to $f^{-1}([1,\frac{3}{2}])$ and critical points of index 3 belonging to $f^{-1}([\frac{3}{2},2])$. Since M_{L_i} is simply connected, it follows that $f^{-1}(\frac{3}{2})$ can be obtained from either M_{L_i} by taking connected sums with $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ (the nontrivial S^2-bundle over S^2). But we shall see, Proposition 2, § 4, that connected summing with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ can be achieved by a sequence of operations one and two. Thus, we have shown so far that each L_i can be altered by $\mathcal{O}_1$ and $\mathcal{O}_2$ so that the new L_i have the property that M_{L_1} is diffeomorphic to M_{L_2} (which is diffeomorphic to $f^{-1}(\frac{3}{2})$). Call this common manifold M^4, and observe that L_1 and L_2 give us two different handlebody structures, or two Morse functions f_i: $M \to [-1,1]$, $i=1,2$, with $f_i^{-1}(-1)$ the only critical point of index 0, $f_i^{-1}(0)=S^3$, $f_i^{-1}(1)=\partial M=\partial M_{L_i}$, and all critical points of index 2 in $f^{-1}(0,1)$.

It follows from the results of the next section, after possibly taking connected sums with $S^2 \times S^2$, that there is a homotopy f_t: $M \to [-1,1]$, $t \in [1,2]$, such that each f_t is a Morse function, except for a finite number of t where two critical points may have the same value. A critical point of index 2 has a descending 2-disk which intersects S^3 in the circle which corresponds to that critical point (or 2-handle). As f_1 deforms to f_2 through f_t, the various descending 2-disks intersected with S^3 describe an isotopy of L_1 to L_2, except for the following possibility. For some t, a descending 2-disk of f_t may not reach S^3 but hit instead a critical point with smaller critical value. This occurs exactly when a 2-handle is being slid over another 2-handle. This is covered by $\mathcal{O}_2$. The proof is finished. □

§ 3. Let M^4 be a DIFF 4-manifold with two boundary components, $\partial_- M^4$ and $\partial_+ M^4$. Let f_0, f_1: $M^4 \to [-K,K]$ be two Morse functions with $f_i(\partial_- M)=-K$ and $f_i(\partial_+ M)=+K$, no critical points near $\partial_\pm M$, and only nondegenerate critical points of index 2 at different levels. (In § 2, we take $\partial_- M^4=S^3$.)

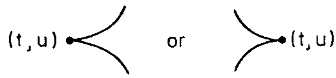
We want to compare, for f_0 and f_1, the descending 2-manifolds of the critical points intersected with $\partial_- M$, i.e., the framed links which determine M. To do this, we will find a nice arc connecting f_0 and f_1 in the space $\mathscr{F}$ of DIFF functions, of M to $[-K, K]$, a procedure pioneered by Cerf.

Following [Cerf] or [H–W], $\mathscr{F}$ has a natural stratification $\mathscr{F} = \mathscr{F}^0 \cup \mathscr{F}^1 \cup \mathscr{F}^2 \cup \cdots$, where $\mathscr{F}^i$ is a stratum of codimension i. $\mathscr{F}^0$ is the set of Morse functions with distinct ctitical values. $\mathscr{F}^1$ consists of two pieces, $\mathscr{F}^1_\alpha$ and $\mathscr{F}^1_\beta$: $\mathscr{F}^1_\alpha$ is the set of Morse functions with distinct critical values except for one critical point which is a birth (or death) point, $\mathscr{F}^1_\beta$ the Morse functions with distinct critical values except for two which coincide. The standard example of a birth point arises from the arc $f_t \colon R \to R$ defined by $f_t(x) = x^3 - t\,x$; at $t = 0$ we witness at $x = 0$ the birth of a cancelling pair of 1 and 0-handles.

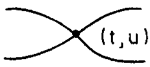
We pick an arc $f = \{f_t\}$, $t \in [0, 1]$ between f_0 and f_1, $f \colon M \times I \to [-K, K]$, which is transverse to the strata $\mathscr{F}^i$ for all i, so that each $f_t \in \mathscr{F}^0 \cup \mathscr{F}^1$ and only finitely many f_t belong to $\mathscr{F}^1$.

The graphic of f is the set of all pairs $(t, u) \in I \times [-K, K]$ for which u is a critical value of f_t. If (t, u) belongs to the graphic then it has a neighborhood in $I \times [-K, K]$ equal to one of the following

if $f_t \in \mathscr{F}^0$ near the critical point.



if $f_t \in \mathscr{F}^1_\alpha$ near the critical point.



if $f_t \in \mathscr{F}^1_\beta$ near the critical point.

A simple graphic for f might be Figure 3.1,

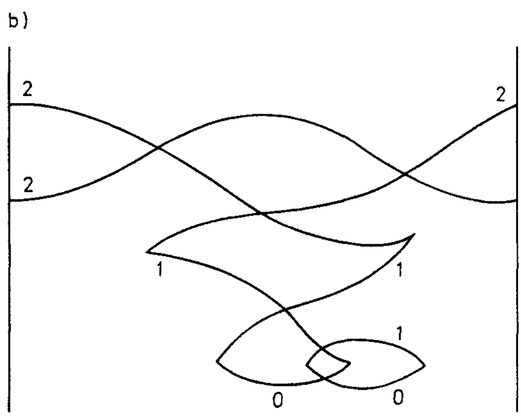


Fig. 3.1

where the integers give the index of the critical point. (The lower left birth point is the graphic of the example above.)

Our first step is to cancel the critical points of index 0 and 4 by deforming f in $\mathscr{F}$ rel f_0 and f_1. For this we need some lemmas from [Cerf].

Triangle Lemma [Cerf, p. 78]: Consider the graphics in Figure 3.2.

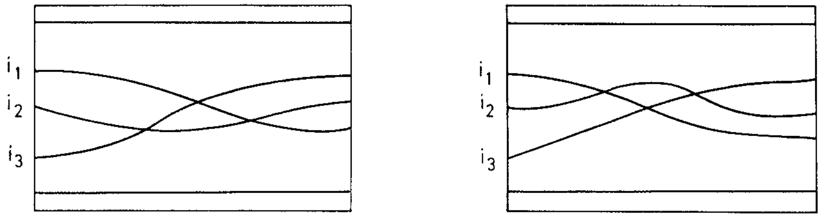


Fig. 3.2

We may change f, rel f_0 and f_1, so that its graphic changes from the first to the second if the indices satisfy $i_1 = i_2 = i_3 \leqq 2$; i.e., we can push the second arc up over the crossing point. We can change from the second graphic to the first if $i_1 = i_2 = i_3 \geqq 2$. (This case follows from the first by changing f to $-f$.)

Beak Lemma [Cerf, p. 83]: In Figure 3.3, we may always change the first graphic to the second if $i_1 > 0$ and the first to the third graphic if $i_2 < 4$.

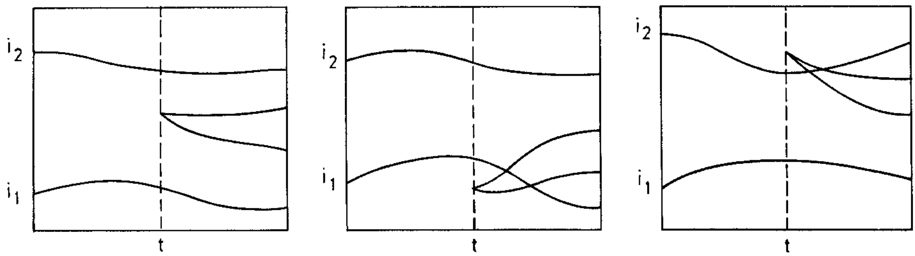


Fig. 3.3

Dovetail Lemma [Cerf, p. 93]: In Figure 3.4, the first graphic can be changed to the second in both cases.

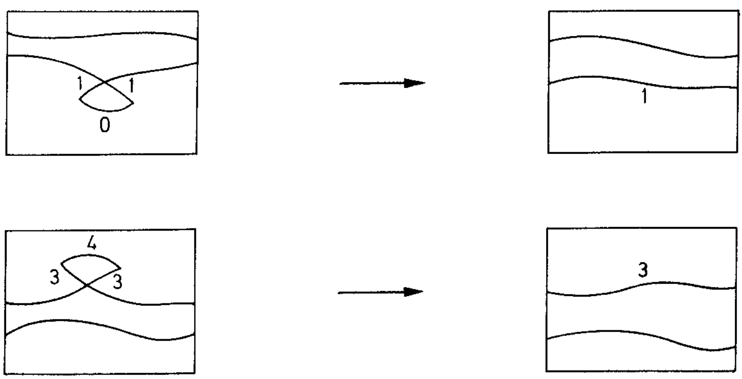


Fig. 3.4

Independent Trajectories Principle [Cerf, p. 255], [H–W, p. 64]: Given two parts of a graphic, suppose that the ascending and descending manifolds of the critical points of one part do not intersect those from the other part. Then we may deform one part relative to the other.

Examples. If $i_1 < i_2$ then we can deform as in Figure 3.5.

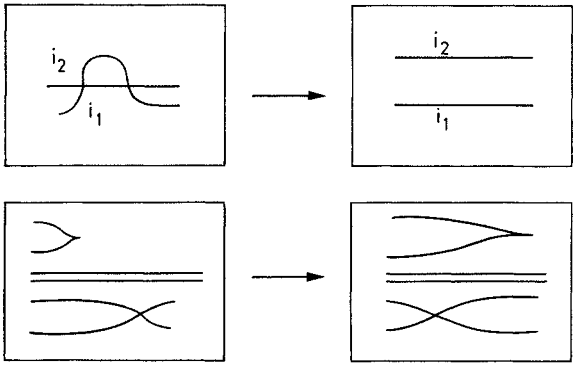


Fig. 3.5

We can add one more feature to the graphic, a dotted vertical line between (t, u) and (t, u'), when $u > u'$, the critical points p and p' associated with u and u' by f_t have the same index, and the descending manifolds of p intersects (in points) the ascending manifold of p'. By transversality, this occurs for only a finite number of t. Figure 3.6 is an example.

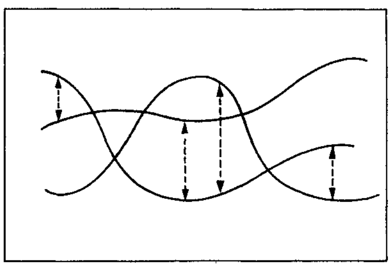


Fig. 3.6

In handle language, the vertical lines indicate that one handle is sliding over the other.

Note that the first three lemmas hold whether or not there are vertical lines present and that the hypothesis of the independent trajectories principle implies that there are no relevant vertical lines present. In particular, we observe that the independent trajectories principle applies as in Figure 3.7 when $i_1 = i_2$.

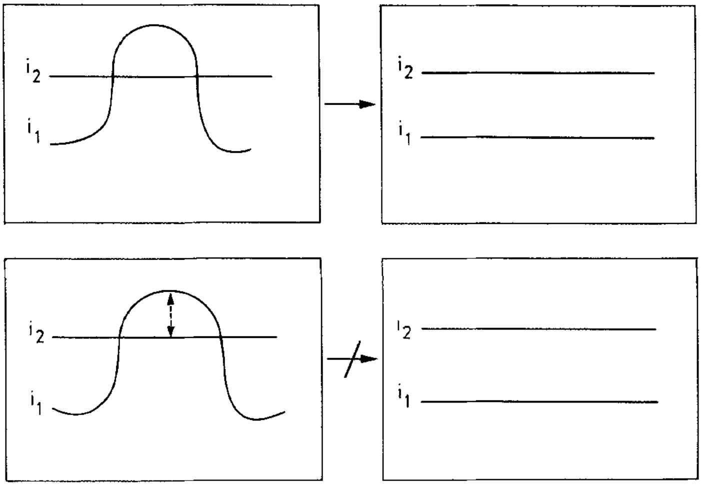


Fig. 3.7

Proposition. The following change in graphics is possible:

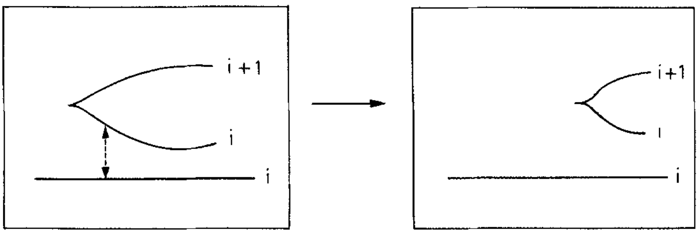


Fig. 3.8

Proof. This is well known, but not stated explicitly in [Cerf] or [H–W]. The $(i+1)$-handle geometrically cancels the i-handle in a neighborhood of the birth point. By easy deformation, we can move the vertical line left under this neighborhood, and then cancel the $(i+1)$- and i-handles in the neighborhood, achieving the graphic on the right.

Now we proceed to change f so that its new graphic has no critical points of index 0 or 4, and equals the old graphic near $\partial I \times [-K, K]$.

Step 1. We change f so that a neighborhood of the index 0 critical points in the graphic looks like Figure 3.9.

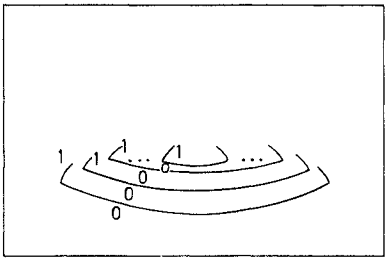


Fig. 3.9

It is easy to arrange the critical points of index 0 as in Figure 3.9, and then to move the critical points of index 1 above those of index 0, by the sequence of graphics in Figure 3.10.

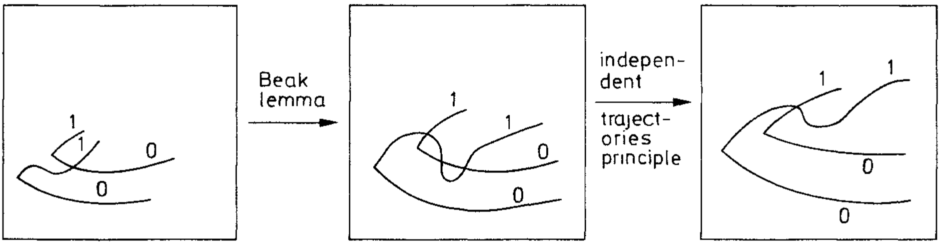


Fig. 3.10

Step 2. We remove the innermost critical point of index zero.

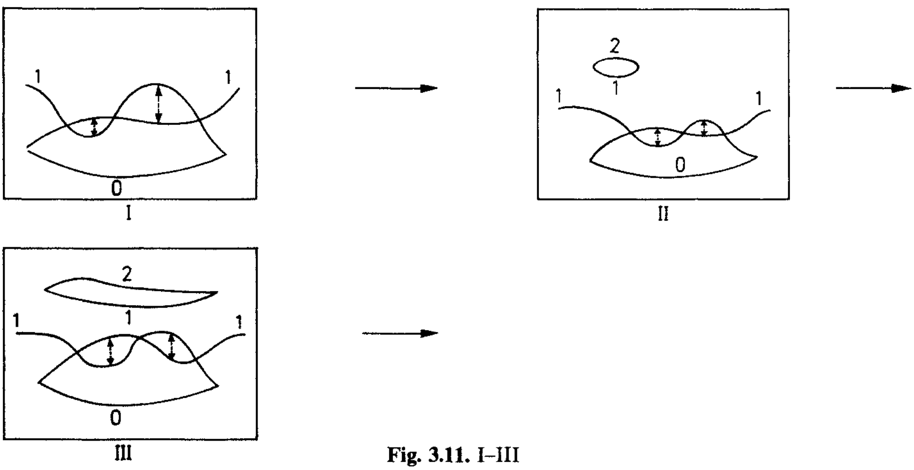


Fig. 3.11. I–III

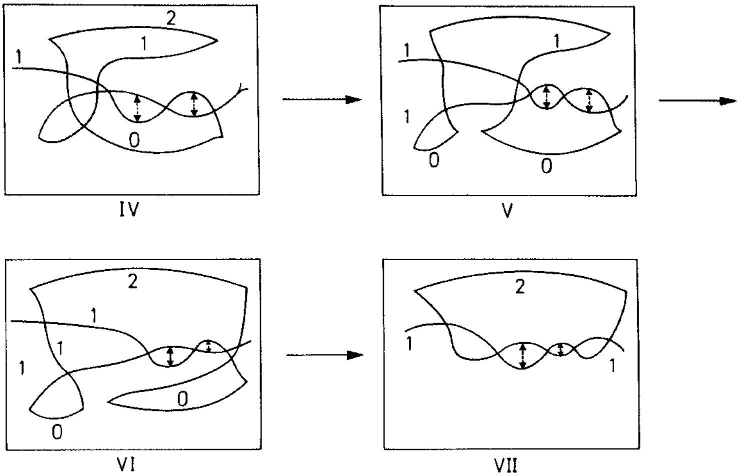


Fig. 3.11. IV–VII

In Figure 3.11, I represents the typical difficulty in removing the innermost index zero critical point; we cannot immediately apply the dovetail lemma. In II we introduce a cancelling pair of 1 and 2-handles. This can be done so that the descending 1-manifold of the new critical point of index 1 intersects the ascending 4-manifold of the critical point of index 0 in exactly one point; thus the new index 1 critical point can be used to cancel the index 0 critical point. We do just that at time t in IV and V after spreading out the pair in III. A sequence of moves using the triangle lemma gets us to VI, and two applications of the dovetail lemma eliminate the innermost index 0 critical point.

The general case differs only in passing from V to VI where there may be more or less applications of the triangle lemma; nothing else is needed since the other index 0 and index 2 critical points are below or above.

Step 3. By the procedure in Step 2, we remove the critical points of index 0 one by one. The same method, turned upside down, eliminates the index 4 critical points.

Next we must eliminate the 1 and 3-handles. Using the beak lemma, we can arrange births and deaths of index 1 and 2 critical points as in Figure 3.12.

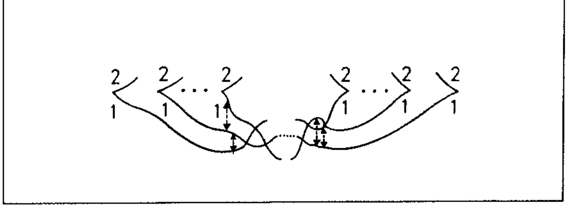


Fig. 3.12

We restrict attention to the smallest value of t for which two index 1 critical points have equal values.

As noted in the above proposition, we may remove the vertical arrow and then interchange the two beaks as in Figure 3.13.

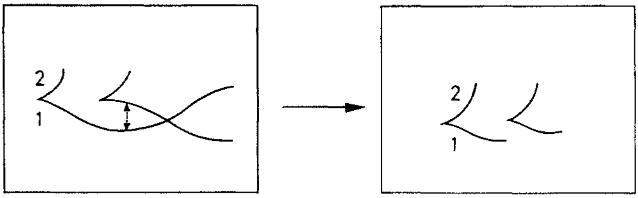


Fig. 3.13

By iterating this simple process we may change Figure 3.12 to Figure 3.14.

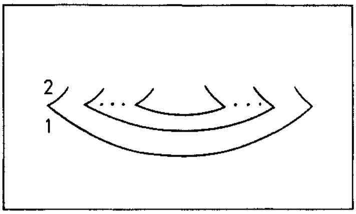


Fig. 3.14

Now move the outermost birth and death points just off the edges.

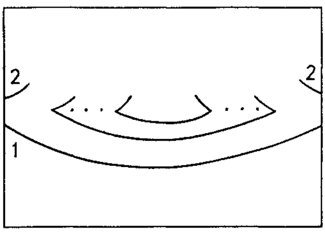


Fig. 3.15

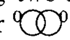
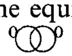
Let D_t denote the descending one-manifold of this index one critical point of f_t. We have arranged that D_t always reaches $\partial_- M^4$. If we connect the end points of D_t by an arc E_t in a collar of $\partial_- M^4$, we get an S_t^1 whose normal bundle is trivial. Since $\pi_1(M^4)=0$, the tangent bundle of M^4 has a unique trivialization (compatible with the orientation of M) over the 1-skeleton. This gives a trivialization of the normal bundle of S_t^1, so that after surgery on S_t^1, M_t is changed by connected sum with $S^2 \times S^2$ (the other trivialization gives connected sum with $S^2 \tilde{\times} S^2$). The new Morse function on $M_t \# S^2 \times S^2$ will have the critical point of index 1 replaced by a critical point of index 2.

Note that the same comments hold if we are dealing with the ascending one-manifold of an index 3 critical point, as we shall be after applying Figures 3.12–15, inverted, to the index 3 critical points.

This surgery must be continuous with respect to t, which means that we must choose E_t continuously. At $t=0$, the critical point of index 2 which cancels our index 1 critical point has a descending 2-manifold; its boundary is a circle, part of which is equal to D_0, and the rest of which can define E_0. We have an isotopy "D_t" of the descending one-manifold, provided by f_t, which extends to an ambient isotopy carrying E_0 to what we define to be E_t. Thus we surger M continuously with respect to t, changing M by $\# S^2 \times S^2$; the problem is to see what happens to the links for $t=0, 1$, or equivalently, where do the two new descending 2-manifolds intersect $\partial_- M^4$.

For $t=0$ in Figure 3.14, let B_p be s small 3-ball in $\partial_- M^4$, centered at some point p, which does not intersect any descending manifolds of critical points. There is an obvious column $B_p \times [-K, K]$ in M which flows down to B_p at $t=0$. When in Figure 3.15 we move our one-two (or two-three) birthpoint just past the left edge, we can assume by transversality that this is done in the column $B_p \times [-K, K]$. Then the descending and ascending manifolds of our cancelling pair of critical points can be inside $B_p \times [-K, K]$, so the surgery is carried out inside $B_p \times [-K, K]$. The crucial point here is that the descending (or ascending for the two-three case) two manifold is in $B_p \times [-K, K]$, so E_0 is also. Then it is clear that the link at $t=0$ is changed by adding  separated by ∂B_p from the rest of the link.

When $t=1$, we have a similar column $B_p \times [-K, K]$ containing the ascending and descending manifolds of the cancelling pair of critical points. However our surgery is along $S^1 = D_1 \cup E_1$ and E_1 is not necessarily inside $B_p \times [-K, K]$. In the special case of the Theorem, $\partial_- M^4 = S^3$, so there is an isotopy carrying E_1 inside B_p; then the argument for the $t=0$ case works and the surgery results in the addition of a separated . But in general $\partial_- M^4$ is not simply connected, as with $\partial_+ M^4$.

For the case of a cancelling two-three pair in M^4 $(t=1)$, we observe that the descending two-manifold of the index 2 critical point falls into B_p because the death point is near. This descending two-manifold is not changed by surgery on "the index 3 critical point". After surgery the new descending two-manifold can go anywhere. Thus we change the link at $t=1$ by adding a pair  where one circle is in B_p but the other goes anywhere. But the equivalence in § 4, Proposition 3, shows that this is the same as adding a pair  in B_p.

For the case of a cancelling one-two pair in M^4 $(t=1)$, the two new descending 2-manifolds intersect $\partial_- M^4$ in two circles; one can go anywhere (following E_1) and the other is a very small circle linking the first. This case is not so easy to see as the previous two-three case, but we leave it to the reader for in the case of the Theorem, $\partial_- M^4$ is simply connected and we have already given the proof. (For the case $\pi_1(\partial_- M^4) \neq 0$, see § 5, Remark 4.)

The outcome is that we have eliminated all index one and three critical points and the graphic looks like the one in Figure 3.16, which was required in the proof of the Theorem at the end of § 2.

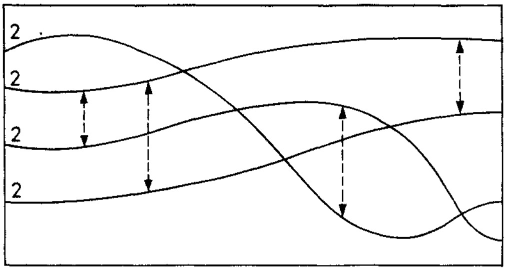


Fig. 3.16

§ 4. Given L_1 and L_2, suppose we guess that $\partial M_{L_1} = \partial M_{L_2}$. Knowing from the theorem that we would then have $L_1 \underset{\partial}{\sim} L_2$, it becomes plausible to try to prove that $\partial M_{L_1} = \partial M_{L_2}$ by proving that $L_1 \underset{\partial}{\sim} L_2$. When trying to show two framed links are ∂-equivalent, it is cumbersome to use only $\mathcal{O}_1$ and $\mathcal{O}_2$; some shortcuts are needed. Here is the principle one, called a K-move in [F–R].

Proposition 1A. If L and L' are identical except for the parts shown in Figure 4.1, then $L \underset{\partial}{\sim} L'$. Here γ_0 is an unknot with framing -1 which disappears in L', and the box denotes a full right hand twist. The framing on γ_i' is given by $\phi_i' = \phi_i + (\lambda(\gamma_0, \gamma_i))^2$. In fact, the linking matrix for L' is gotten from L by adding multiples of γ_0 to the γ_i so that the matrix becomes

$$\begin{array}{cc} & \gamma_0 \\ \gamma_0 & \left(\begin{array}{cc} -1 & 0 \ldots 0 \\ 0 & \\ \vdots & (L') \\ 0 & \end{array}\right) \end{array}$$

and then discarding the first row and column.

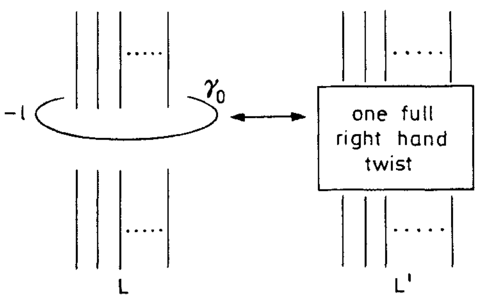


Fig. 4.1

Proposition 1B. The same statement as in Proposition 1A holds when γ_0 has framing $+1$, the box denotes one full left hand twist, and $\phi_i' = \phi_i - (\lambda(\gamma_0, \gamma_i))^2$.

Note that Propositions 1A and 1B imply for one vertical strand the ∂-equivalence in Figure 4.2.

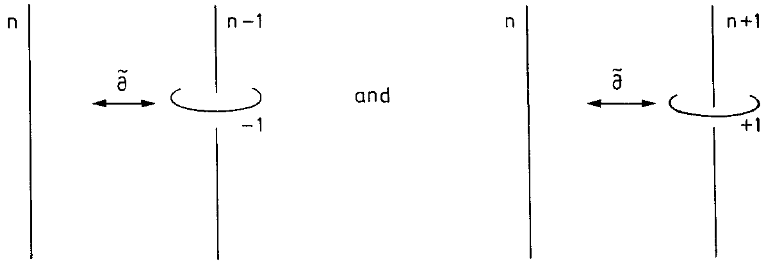


Fig. 4.2

Proof. We prove only Proposition 1A in the following special case, Figure 4.3.

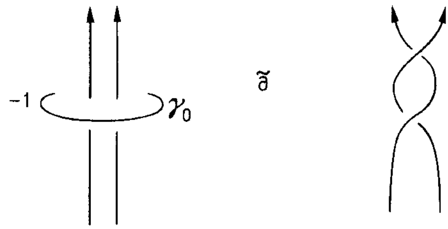


Fig. 4.3

We push off two copies of γ_0 and add one to each vertical strand ($\mathcal{O}_2$) and then eliminate γ_0 ($\mathcal{O}_1$), see Figure 4.4. (One can verify that this does not depend on the orientations; the rule is to follow what the linking matrix indicates.)

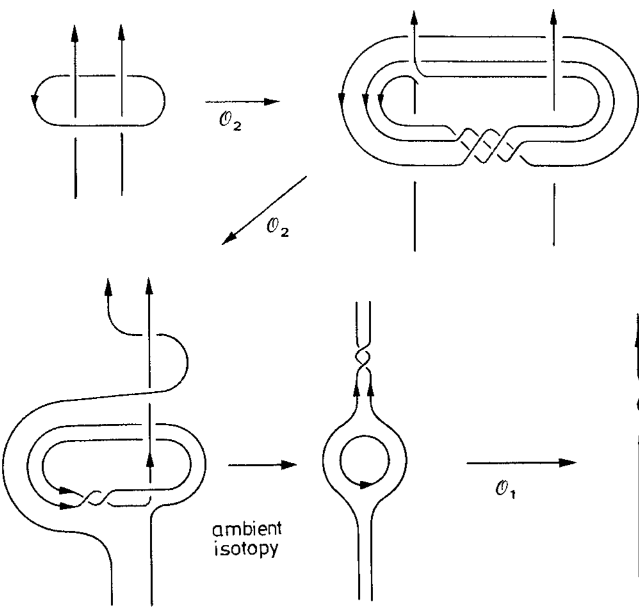


Fig. 4.4

Proposition 2. Given a link L, then we may add to (or substract from) L a copy of either ^{0} ^{0} or ^{0} ^{1} (separated from L by an imbedded 2-sphere) without changing the ∂-equivalence class of L. This corresponds to either of the connected sums $M_L \# S^2 \times S^2$ or $M_L \# S^2 \tilde{\times} S^2$.

Proof. For $S^2 \tilde{\times} S^2$, we proceed as follows.

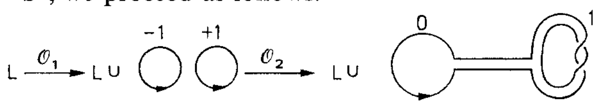


Fig. 4.5

For $S^2 \times S^2$:

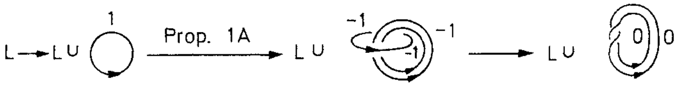


Fig. 4.6

Corollary. Note that in the presence of a "free" ${}^{\pm 1}$, adding ^{0} ^{0} is the same as adding ${}^{+1}$ ${}^{-1}$ in that the 4-manifolds are diffeomorphic.

Proposition 3. If L and L' are the two links in S^3 in Figure 4.7, then $L \underset{\partial}{\sim} L'$.

Fig. 4.7

Proof. The proof should be clear from the sequence in Figure 4.8. We push off copies of γ_1 and add them where needed. This does not change any framing in L_0 because nothing in L_0 links γ_1 and γ_1 has framing zero. Anytime we add γ_1 to γ_2 to unknot γ_2, we also subtract γ_1 from γ_2 in a trivial way, thereby preserving the framing of γ_2.

In principle, this argument is the same as the one showing that any knot K in $S^1 \times S^2$ is unknotted if $K \cap p \times S^2 =$ one point for some $p \in S^1$. If L' and L lie in some non-simply connected Y^3, then γ_2 may be nontrivial in $\pi_1(Y)$, so the argument breaks down.

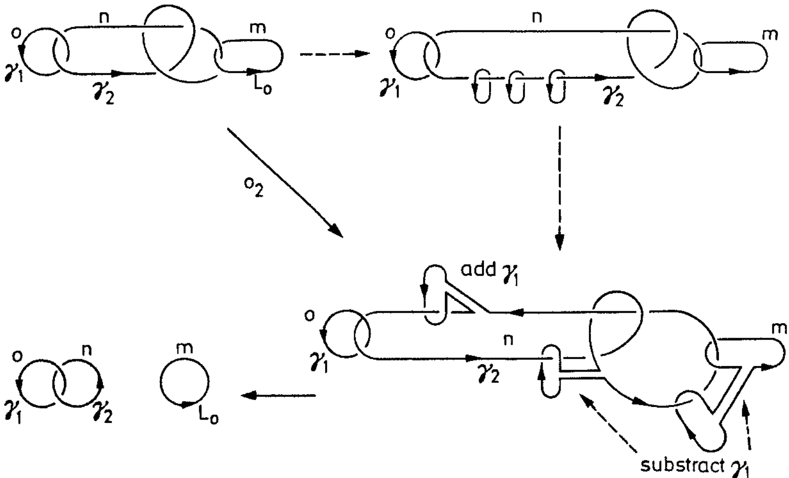


Fig. 4.8

An interesting example involves the homology sphere $\Sigma(2,7,13)$. If L is the left hand (2, 7) torus knot with -1 framing, see Figure 4.9, then ∂M_L has fundamental

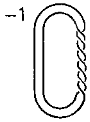
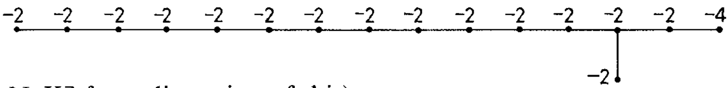


Fig. 4.9

group $\{x, y \mid x^{13} = y^7 = (x^3 y)^2\}$. Also, if we consider the complex variety V given by the solution in C^3 of $z_1^{13} + z_2^7 + z_3^2 = 0$ and intersect V with the unit 5-sphere in C^3, then we get a 3-manifold $\Sigma(2,7,13)$ which is ∂M_L. To prove this, we first resolve the isolated singularity of V at 0, and obtain the 4-manifold Q with $\partial Q = V \cap S^5 = \Sigma(2,7,13)$, gotten by plumbing on



(see [H–N–K] for a discussion of this).

It is not hard to calculate that the intersection form on $H_2(Q; Z)$ is unimodular (so ∂Q is a homology sphere), even, with an index equal to -16. There are two symmetric, even, integral, unimodular bilinear forms of index -16 (see [Milnor-Husemoller, p. 28]); one is the direct sum of two negative copies of the famous E_8, and we have the other here.

Since $\Sigma(2,7,13)$ bounds an even $(\omega_2 = 0)$ 4-manifold of index $= 0$ (16), the Robertello-Rohlin invariant [Robertello] is zero, so it is possible that $\Sigma(2,7,13)$ bounds a homology 4-ball. If so, we get a sought-after, closed, even 4-manifold of index ± 16 and second Betti number 16 (see [Milnor], or [Rohlin] for an incorrect construction). Whether or not $\Sigma(2,7,13)$ bounds a homology ball is a fascinating question.

Q can be described also by the framed link Λ.

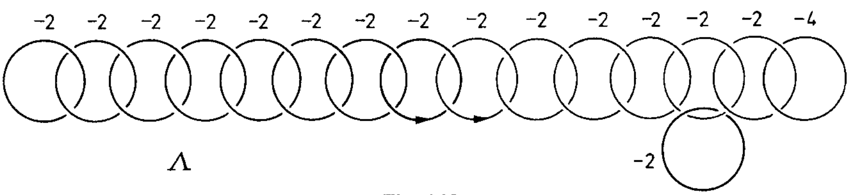


Fig. 4.10

We show that $\Lambda \underset{\partial}{\sim} L$ so that $\Sigma(2,7,13) = \partial M_L$. As in the Corollary to Propositions 1A and 1B, we can change the left end of Λ to

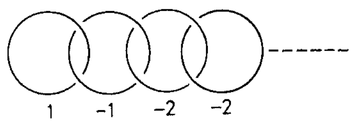


Fig. 4.11

Remove the -1 by Proposition 1A:

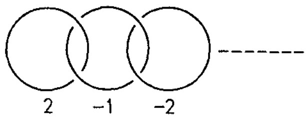


Fig. 4.12

Again, remove -1:

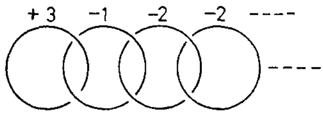


Fig. 4.13

We iterate, obtaining Figure 4.14:

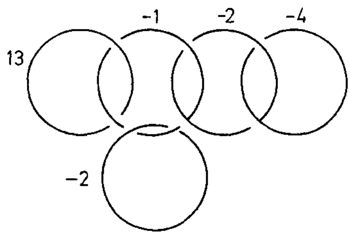


Fig. 4.14

Do the same thing with the -2 circle hanging down:

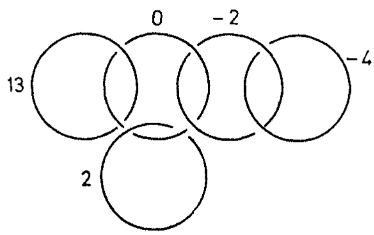


Fig. 4.15

Several applications of Propositions 1A or 1B give:

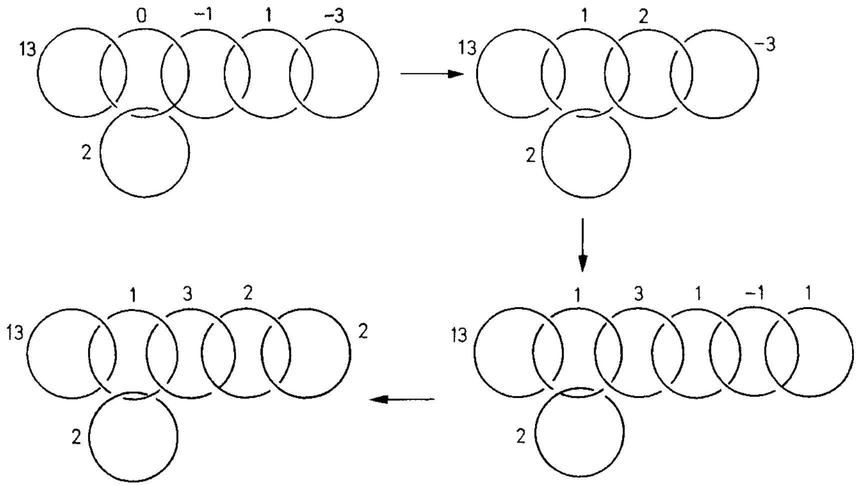


Fig. 4.16

Now remove five +1 circles successively as in Figure 4.17.

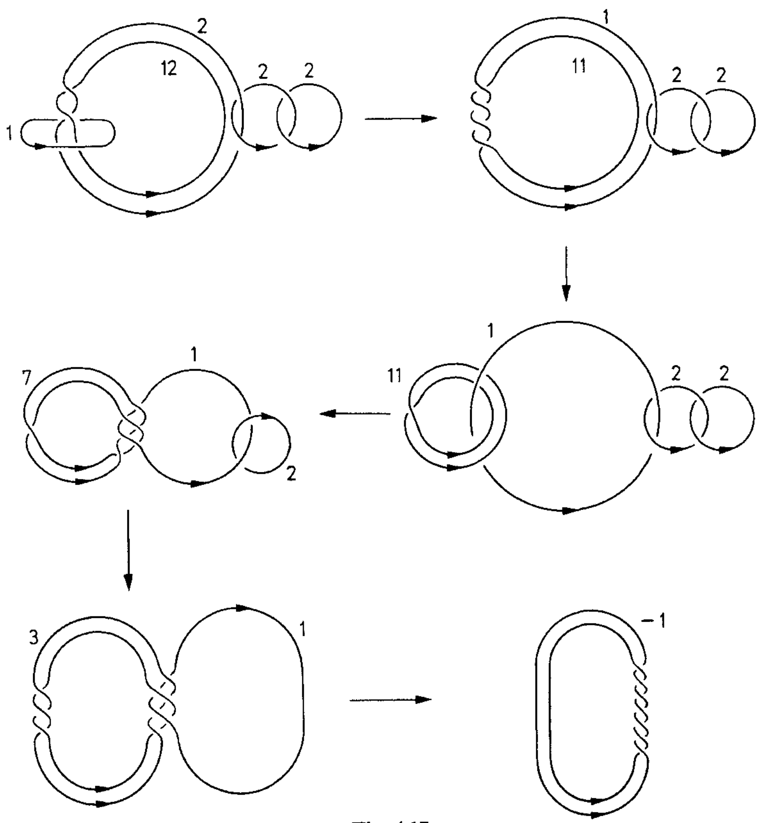


Fig. 4.17

The variety V can be compactified at infinity by adding a complex curve with a cusp which is topologically just a 2-sphere imbedded with a non locally flat point which comes from a cone on the right-handed $(2,7)$ torus knot. We see this by changing the orientation of M_L and adding it to Q. Note that $-L \underset{\partial}{\sim} -\Lambda$, where $-L$ is the right hand $(2,7)$ torus knot with $+1$ framing, and $-\Lambda$ is the link in Figure 4.10, with all framings positive.

§ 5. *Remark 1.* The techniques from complex surface theory which suggested the calculus are two. First, a three manifold may be the link of an isolated singularity of a complex hypersurface in $\mathbb{C}^3$. The resolution of the singularity and the compactification of the hypersurface both involve the two operations in the calculus. For example, if two complex curves intersect at a point p in $\mathbb{C}^2$, one can blow up the (non-singular) point p and separate the two curves. In terms of the calculus, we consider a 4-ball centered at p, and the curves intersect the boundary S^3 in Hopf circles. Blowing up is the same as connected sum with $-CP^2$ which in S^3 amounts to introducing an unknot with -1 framing and adding both Hopf circles to it, thereby unlinking the Hopf circles.

Second, two rational complex surfaces can be shown to be rationally equivalent if certain invariants of the surfaces are equal. Here, a complex surface is rational if it can be obtained by blowing up points on CP^2 or a CP^1 bundle over CP^1, and two surfaces are rationally equivalent if they become isomorphic complex manifolds after possibly blowing up points on each. Again, this is similar to the calculus.

The calculus was suggested by the above considerations plus the knowledge that topologists have long been aware of various cases of the calculus, e.g., see [Hempel], page 211 or [Rolfsen], Chapter 10, and Remark 6 below.

Remark 2. Looking back over the proof of the theorem, we see that we can perform $\mathcal{O}_1$ on both L_1 and L_2 so that the resulting links are equivalent using just $\mathcal{O}_2$. This is because M_{L_1} and M_{L_2} become diffeomorphic after adding enough $\pm CP^2$. In § 3 we show that after adding enough $S^2 \times S^2$ to M_L, the two Morse functions are related by isotopy and $\mathcal{O}_2$. But adding an $S^2 \times S^2$, in the presence of $\bigcirc^{\pm 1}$, is done by adding $\bigcirc^{1}\ \bigcirc^{-1}$ and then sliding handles over handles (see the Corollary to Proposition 2 in § 4).

Remark 3. A framed link L is called even if all the framings are even integers, or equivalently, the intersection form on $H_2(M_L; Z)$ is even (Type II), which is the same as M_L being parallelizable. It is desirable to know an even link L that gives a specific 3-manifold Q^3, $Q^3 = \partial M_L$, for then we can "read off" the Rohlin invariant [Robertello] of Q^3 if Q^3 is a homology 3-sphere, or the μ-invariant [E–K], [C–S], [Gordon], [Hirzebruch] if Q^3 is a $Z/2Z$-homology 3-sphere. Steve Kaplan has found an efficient algorithm for transforming a link L, via $\mathcal{O}_1$ and $\mathcal{O}_2$, to an even link [Kaplan].

Now suppose $Q^3 = \partial M_{L_1} = \partial M_{L_2}$ where both L_1 and L_2 are even. Can we change L_1 to L_2 through even links? That is, can we change L_1 to L_2 using $\mathcal{O}_2$ and, in place of $\mathcal{O}_1$, allowing L_i to change by addition (or subtraction) of the link ${}^{0}\bigcirc\!\!\bigcirc^{0}$, separated from the rest of L_i by an S^2? Of course, it is necessary that index $M_{L_1} =$ index M_{L_2}, or else to change L_i by adding any even link of index $16k$ with boundary S^3. Then the answer is yes, for in § 2, N is almost parallelizable, so W can be spin, and then we can pass from M_{L_i} *to* M^4 by only connected summing with $S^2 \times S^2$ and not $S^2 \tilde{\times} S^2$ ([Wall], p. 147). The remainder of the proof goes through as in the non-even case.

Remark 4. What happens to the calculus if we consider links in an arbitrary 3-manifold Y. The Theorem continues to hold if $Y \cong S^3$ (same proof) or if the diffeomorphism from ∂M_{L_1} to ∂M_{L_2} and the identity from Y to Y extend to a homotopy equivalence from M_{L_1} to M_{L_2}. A proof of the latter (and generalizations) is given in [F–R_2]. But the Theorem fails if Y is not simply connected.

An easy counterexample occurs when $Y = S^1 \times S^2$. Think of Y as the boundary of M_{L_0} where L_0 is the unknot with framing 0. Then add a pair of unknots in two different ways to get L_1 and L_2 (see Fig. 5.0). Then $\partial_+ M_{L_1} \cong \partial_+ M_{L_2} \cong S^1 \times S^2$. But there is no way in the calculus to pass from L_1 to L_2 because one cannot link L_0 with anything except by sliding L_0 over another handle, which is not allowed. This also shows that there is no relative version of the theorem stating that we can pass from L_1 to L_2 by the calculus while fixing a common sublink L_0.

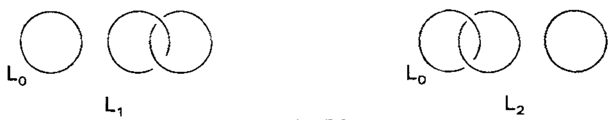


Fig. 5.0

Remark 5. The greatest value of the theorem is not in its corollaries (of which I know none), but that it suggests a practical method for trying to show that two oriented 3-manifolds are diffeomorphic, which is bound to work if in fact they are diffeomorphic, e.g. see [Lickorish_2].

Three manifold have traditionally been studied using our knowledge of 2-manifolds, e.g., via Heegard decompositions and their homeomorphisms of surfaces, or via imbedded surfaces in the 3-manifold. The philosophy we encourage here is to study 3-manifolds by studying some 4-manifolds they bound.

Along these lines, we have already mentioned Kaplan's work in Remark 2. Selman Akbulut and I have used the calculus to fairly efficiently compute the Casson-Gordon invariant of some non-rational knots; see [C–G] and [A–K_1]. Also we have investigated, [A–K_2], a conjecture of Zeeman. The conjecture [Zeeman, Conjecture (5), p. 357]: If we add a 2-handle to $S^1 \times B^3$ along the curve J (see Fig. 5.1) with framing n and t full twists, does the curve K bound a PL imbedded 2-ball in the resulting contractible 4-manifold. The crossings may vary and the integers t and n may vary over the integers.

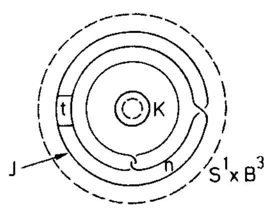


Fig. 5.1

Akbulut and I have tried to find a homology 3-sphere with Rohlin (or μ-invariant zero which does not bound a contractible manifold, or even a homology ball. We conjecture that the (2, 3, 11)-homology sphere does not. It is the link of the hypersurface $x^2+y^3+z^{11}=0$ in C^3 (with fundamental group $\{a, b, c \mid a^2=b^3=c^{11}$ and $a=b c^2\}$), and is also described by the links in Figure 5.2.

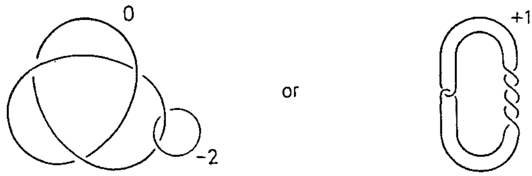


Fig. 5.2

Remark 6. There is a description in [Rolfsen], Chapter 10, of a related calculus for Dehn surgeries on a link in S^3. Briefly, a knot and an associated rational number $\frac{q}{p}$ describe a 3-manifold obtained by removing a solid torus (the thickened knot) from S^3 and sewing it back in so that the meridian goes to p times the longitude and q times the meridian, where the longitude lies on the Seifert surface of the knot. This "Dehn surgery" corresponds to ordinary surgery when $p=1$ and then q is just our framing. Rolfsen gives a calculus for these links with associated fractions which shows how to change to a link where all fractions are integers. Then we have the obvious corollary that two links with fractions give the same 3-manifold iff they are equivalent in Rolfsen's calculus to framed links which are ∂-equivalent in our calculus.

Remark 7. R.A. Fenn and C.P. Rourke have shown [F–R] that the operation in Figure 4.1 (and its companion) is equivalent to our $\mathcal{O}_1$ and $\mathcal{O}_2$. Propositions 1A and 1B prove half this equivalence and here is Fenn and Rourke's argument for the other half.

Their operation, which they call a K-move, consists of introducing or removing an unknotted circle with framing ± 1, twisting left or right all arcs passing through a spanning disk of the circle, and changing the framings as explained in Propositions 1A and 1B. Thus $\mathcal{O}_1$ is just a K-move with no arcs passing through the spanning disk.

What remains is to show that we can achieve an $\mathcal{O}_2$ by a sequence of K-moves. Suppose we want to "add" γ_i to γ_j. If γ_i is unknotted with framing -1, then two K-moves achieve the addition, as in Figure 5.3. If there are other strands going through γ_i, including arcs of γ_j, then we may gather them into a tube represented by the dotted line in Figure 5.3. When we "blow down" γ_i the framings of the strands are changed, but they are changed back when γ_i is "blown up" again.

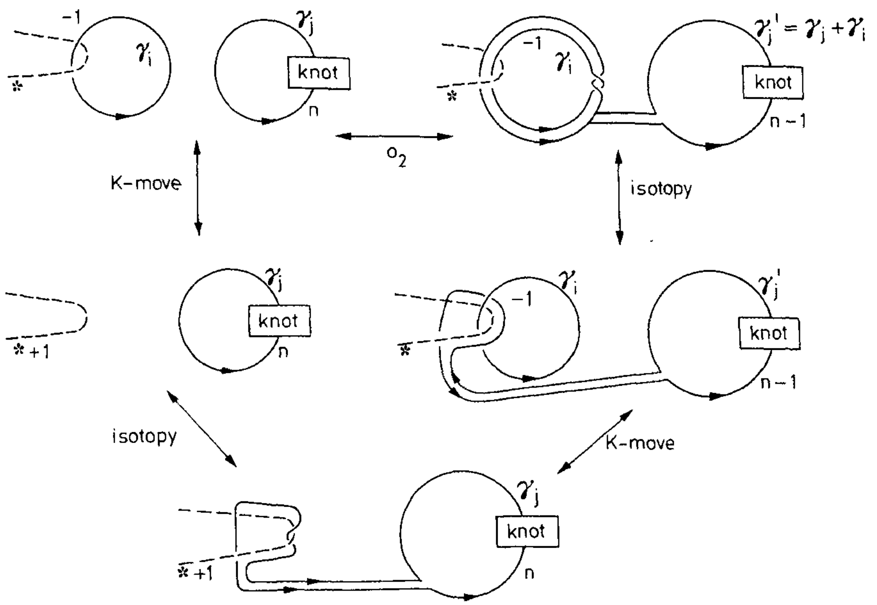


Fig. 5.3

In general, we can make γ_i unknotted with framing -1 by a series of K-moves, "blowing up" circles to change crossings and unknot γ_i (e.g., $\times \longrightarrow$ -1) and blowing up circles to make the framing -1. We proceed as in Figure 5.3 and then reverse this process, "blowing down" the circles to restore γ_i to its original knottedness and framing.

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