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Nonlocal Symmetries and the Theory of Coverings: An Addendum to A. M. Vinogradov's 'Local Symmetries and Conservation Laws'

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Abstract. For a system $\mathscr{Y}$ of partial differential equations, the notion of a covering $\tilde{\mathscr{Y}}_\infty \to \mathscr{Y}_\infty$ is introduced where $\mathscr{Y}_\infty$ is infinite prolongation of $\mathscr{Y}$. Then nonlocal symmetries of $\mathscr{Y}$ are defined as transformations of $\tilde{\mathscr{Y}}_\infty$ which conserve the underlying contact structure. It turns out that generating functions of nonlocal symmetries are integro-differential-type operators.

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0. Introduction

In [1] local symmetries and local conservation laws were discussed, i.e., such that are defined by differential operators. For example, any higher infinitesimal symmetry of an equation $\mathscr{Y} \subset J^k(\pi)$ is determined by its generating function ([1], Section 3.5), the latter being, in general, a nonlinear differential operator. As we saw, the local point of view effects in consistent and self-contained theory. Nevertheless, there are certain experimental facts, as well as purely theoretical considerations, which indicate its limitations. First of all it could be seen that the number of local symmetries and conservation laws in cases is just too small. Thus, an evolution equation $u_t = f(u, u_x) + u_{xxx}$, the form of which is quite similar to the Korteweg–de Vries equation, as a rule has no local symmetries other than translations. The KdV equation $U_t = UU_x + U_{xxx}$ itself is not completely integrable in the framework of the local theory.

A natural geometric generalization of the local theory consists of constructing such extensions of objects like $\mathscr{Y}_\infty$ functions on which could be interpreted as some kind of generalized differential operators (e.g., as integro-differential operators). We call symmetries and conservation laws determined by such functions nonlocal.

Nonlocal symmetries of special types were considered in a number of recent publications (see, for example, [2–7]). The simplest examples of such symmetries arise naturally when acting by so-called recursion operators on local symmetries. For instance, by acting by Lenard's recursion operator $D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}$ on KdV scale symmetry $tu_{xxx} + (tu + \frac{1}{3}x)u_x + \frac{2}{3}u$ we shall get a nonlocal symmetry which depends on $\int u\,dx$.

Below, we give a sketch of nonlocal symmetries theory emphasizing the illustrative calculations of particular examples rather than discussing the general theoretical aspects. As in [1], our chief model is Burgers' equation. A more detailed exposition, as well as other applications, will be published elsewhere.

1. Naive Approach

Consider Burgers' equation $Y = \{u_t = uu_x + u_{xx}\}$ and its infinite prolongation $\mathscr{Y}_\infty \in J^\infty(\mathbb{R}^2)$. Choose functions $x_1 = x_2$, $x_2 = t$, $T_{(0)} = u, \ldots, T_{(k)}, \ldots$ as coordinates on $\mathscr{Y}_\infty$ (cf. [1], Section 3.6). Then the algebra Sym $\mathscr{Y}$ of Burgers' equation local symmetries coincides with the kernel of the operator $\bar{l}_F = \bar{D}_1^2 + p_{(0)\bar{D}_1 + p_{(1)}} - \bar{D}_2$, where vector fields

$$\bar{D}_1 = D_x = \frac{\partial}{\partial x} + p_{(1)}\frac{\partial}{\partial p_{(0)}} + \cdots + p_{(k+1)}\frac{\partial}{\partial p_{(k)}} + \cdots,$$

$$\bar{D}_2 = D_t = \frac{\partial}{\partial t} + (p_{(0)}p_{(1)} + p_{(2)})\frac{\partial}{\partial p_{(0)}} + \cdots +$$

$$+ D_x^k(p_{(0)}p_{(1)} + p_{(2)})\frac{\partial}{\partial p_{(k)}} + \cdots$$

determine the contact structure on $\mathscr{Y}_\infty$ (see [1], Section 3.5). Let us extend $\mathscr{Y}_\infty$ up to a new manifold $\tilde{\mathscr{Y}}_\infty$ by introducing another, 'nonlocal', variable $\int p_{(0)}\,dx = \int u\,dx$. Formally speaking, this means that we have added another one to the coordinates described above, namely $p_{(-1)}$, the total derivatives of which with respect to x is equal to $p_{(0)} = u$. It is quite natural to define the total derivative of $p_{(-1)}$ with respect to t as

$$\begin{aligned} D_t(p_{(-1)}) &= D_t(D_x^{-1}(p_{(0)})) = D_x^{-1}(D_t(p_{(0)})) \\ &= D_x^{-1}(p_{(0)}p_{(1)} + p_{(2)}) = \tfrac{1}{2}p_{(0)}^2 + p_{(1)} + c, \end{aligned}$$

where $c = c(t)$ is the 'constant of integration' which may be put as equal to zero. In other words, we extend total derivations D_x and D_t up to operators

$$\tilde{D}_x = D_x + p_{(0)}\frac{\partial}{\partial p_{(-1)}}, \qquad \tilde{D}_t = D_t + (\tfrac{1}{2}p_{(0)}^2 + p_{(1)})\frac{\partial}{\partial p_{(-1)}}.$$

Obviously, we have $[\tilde{D}_x, \tilde{D}_t] = 0$.

Now, the operator $\bar{l}_F$ naturally extends up to the operator $\tilde{l}_F = \tilde{D}_x^2 + p_{(0)}\tilde{D}_x + p_{(1)} - \tilde{D}_t$, and we call a nonlocal symmetry of Burgers' equation depending on $p_{(-1)}$, any solution of the equation

$$\tilde{l}_F(\varphi) = 0 \Leftrightarrow \tilde{D}_t(\varphi) = p_{(1)}\varphi + p_{(0)}\tilde{D}_x(\varphi) + \tilde{D}_x^2(\varphi), \tag{1}$$

where φ is a function on $\tilde{\mathscr{Y}}_\infty$. When φ depends on x, t, $p_{(-1)}$ and $p_{(0)}$ only, Equation (1) transforms into

$$\frac{\partial\varphi}{\partial t} - \frac{1}{2}\frac{\partial\varphi}{\partial p_{(-1)}} = p_{(1)}\varphi + p_{(0)}\frac{\partial\varphi}{\partial x} + \frac{\partial^2\varphi}{\partial x^2} + 2p_{(0)}\frac{\partial^2\varphi}{\partial x\,\partial p_{(-1)}} + 2p_{(1)}\frac{\partial^2\varphi}{\partial x\,\partial p_{(0)}} + p_{(0)}^2\frac{\partial^2\varphi}{\partial p_{(-1)}^2} + 2p_{(0)}p_{(1)}\frac{\partial^2\varphi}{\partial p_{(-1)}\,\partial p_{(0)}} + p_{(1)}^2\frac{\partial^2\varphi}{\partial p_{(0)}^2}. \tag{2}$$

The solutions of this equation are of the form

$$\varphi = \varphi[a] = \left(ap_{(0)} - 2\frac{\partial a}{\partial x}\exp\left(-\tfrac{1}{2}p_{(-1)}\right)\right), \tag{3}$$

where $a = a(x, t)$ is an arbitrary solution of the heat equation $a_t = a_{xx}$.

If we now repeat all the reasonings from [1], Section 3.6, using symmetries $\varphi = \varphi(x, t, p_{(-1)}, p_{(k)})$ instead of classical ones, then we shall see that any solution of (1) is of the form $\varphi = \varphi_l + \varphi_n$ where φ_l is a local symmetry of the Burgers equation, while φ_n is defined by (3).

For those functions φ on $\tilde{\mathscr{Y}}_\infty$ which are of the form $\varphi = \tilde{D}_x(f)$ we can formally define 'evolutionary differentiations'

$$\tilde{\mathfrak{Z}}_\varphi = D_x^{-1}(\varphi)\frac{\partial}{\partial p_{(-1)}} + \varphi\frac{\partial}{\partial p_{(0)}} + \cdots + D_x^k(\varphi)\frac{\partial}{\partial p_{(k)}} + \cdots$$

where $D_x^{-1}(\varphi)$ is the general solution of the equation $\varphi = D_x(f)$. For such functions, an 'extended Jacobi bracket' is defined as:

$$\{\varphi, \psi\}^\sim = \tilde{\mathfrak{Z}}_\varphi(\psi) - \tilde{\mathfrak{Z}}_\psi(\varphi).$$

For example, for any function (3) we have $D_x^{-1}(\varphi[a]) = \alpha - 2a\exp(-\frac{1}{2}p_{(-1)})$ and consequently

$$\{\varphi[a], \varphi[b]\}^\sim = \tfrac{1}{2}\varphi[\beta a - \alpha b] \tag{4}$$

where α and β are arbitrary functions of t.

Which solutions of Burgers' equation are invariant with respect to the nonlocal symmetry (3)? To find them it is necessary to solve Burgers' equation together with the equation

$$\varphi[a] = (ap_{(0)} - 2a_x)\exp(-\tfrac{1}{2}p_{(-1)}) = (au - 2a_x)\exp\left(-\tfrac{1}{2}\int u\,\mathrm{d}x\right) = 0.$$

In particular, it follows that $u = 2a_x/a$, i.e., we get the well-known Cole–Hopf transformation which reduces Burgers' equation to a heat equation. This fact clearly demonstrates the usefulness, of nonlocal symmetries.

2. Criticism of the Naive Viewpoint

The situation which stimulates the introduction of nonlocal symmetries in the manner described above is connected with the so-called recursion operators. Namely, an operator $\mathcal{R}$ is said to be a recursion operator for the equation $\mathcal{Y}$ if $\bar{l}_F \circ \mathcal{R} = \mathcal{R} \circ \bar{l}_F$ holds. From Theorem 6 of [1], Section 3.5, it follows immediately that $\varphi \in \operatorname{Sym} \mathcal{Y}$ implies $\mathcal{R}(\varphi) \in \operatorname{Sym} \mathcal{Y}$ when $\mathcal{R}(\varphi)$ is a smooth function on $\mathcal{Y}_\infty$. The latter is always true when $\mathcal{R}$ is a differential operator. However, all the recursion operators which are now known for the simplest nonlinear equations are not differential ones. For example, such operators for Burgers' and KdV equations are of the forms $\mathcal{R} = D_x + \frac{1}{2}u + \frac{1}{2}u_x D_x^{-1}$ and $\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$ respectively. Therefore, by using the latter operator when acting on, say, the scale symmetry $\varphi = tu_{xxx} + (tu + \frac{1}{3}x)u_x + \frac{2}{3}u$ of a KdV equation, we shall get the function $\psi = \mathcal{R}(\varphi) = tu_{xxxxx} + (\frac{5}{3}tu + \frac{1}{3}x)u_{xxx} + (\frac{10}{3}tu_x + \frac{1}{2}u)u_{xx} + (\frac{5}{6}tu^2 + \frac{1}{3}xu + \frac{1}{9}\int u\,dx)u_x + \frac{4}{9}u^2$. It seems attractive to treat this function as a nonlocal solution of the equation $\bar{l}_F(\varphi) = 0$ and, therefore, as it follows from Theorem 6 of [1], as a nonlocal symmetry of the KdV equation. The reader can find a more precise formulation of this thesis in [1], where we discuss Burgers' equation. Yet such a point of view is inadequate for a number of reasons.

In the first place, there are difficulties of a purely technical nature which hinder the practical realization of this scheme. The reader will understand them if he tries, for instance, to find nonlocal symmetries of Burgers' equation depending on $\int u^2\,dx$.

Secondly, there are no reasons, except for the formal analogies to Theorem 6 of [1], to regard the solutions of the equation $\bar{l}_F(\varphi) = 0$ as nonlocal symmetries. Obviously, here one needs to analyse the very conception of nonlocal symmetry.

At last, the naive point of view is unaesthetical. This can be seen, for example, from the fact that the 'nonlocal' Jacobi bracket $\{\cdot,\cdot\}^{\sim}$ is not defined everywhere and is multivalued (compare with (4)).

For these reasons we shall give up the naive viewpoint and try to arrive at the definition of nonlocal symmetry by analysing the geometry of manifolds $\tilde{\mathcal{Y}}_\infty$ exactly in the same way as was done in [1] when discussing manifolds $\mathcal{Y}_\infty$.

3. Coverings

The reader will find the motivations for the theory of coverings which is fundamental for our approach to the notion of nonlocal symmetries, in [8]. Below we informally describe the basic concepts of this theory. Note that the prolongation structures introduced by Wahlquist and Estabrook (see [9]) are a particular case of coverings. From now on we use the term 'infinitely-dimensional manifold' just in the same sense as was done in [1] with respect to the objects $J^\infty(\pi)$ or $\mathcal{Y}_\infty$. By n we denote the number of independent variables in the equation $\mathcal{Y}$.

Consider an infinitely-dimensional manifold $\tilde{\mathcal{Y}}_\infty$ equipped with an n-dimensional integrable distribution. This means that in the tangent space $Y_y(\tilde{\mathcal{Y}}_\infty)$, which is infinitely-dimensional, at any point $y \in \tilde{\mathcal{Y}}_\infty$, an n-dimensional subspace $\tilde{\mathcal{C}}_y \in T_y(\tilde{\mathcal{Y}}_\infty)$ is determined and that the system $\{\tilde{\mathcal{C}}_y\}$ of these subspaces satisfies the conditions of the classical Frobenius theorem. If, in addition, there is a regular map τ from $\tilde{\mathcal{Y}}_\infty$ onto $\mathcal{Y}_\infty$ inducing an isomorphism of $\tilde{\mathcal{C}}_y$ and $\mathcal{C}_{\tau(y)} \in T_{\tau(y)}(\mathcal{Y}_\infty)$, then we say that $\tilde{\mathcal{Y}}_\infty$ is the covering of the equation $\mathcal{Y}$. This implies that any integral manifold $U \in \tilde{\mathcal{Y}}_\infty$ (i.e., such a manifold that $T_y(U) = \tilde{\mathcal{C}}_y$, $\forall y \in U$) is mapped by τ onto an integrable manifold $V = \tau(U) \in \mathcal{Y}_\infty$; that, is onto a solution of the equation $\mathcal{Y}$.

Consider the coordinate interpretation of the above. Let $W \in \mathbb{R}^N$, $0 < N \leqslant \infty$, be a domain in $\mathbb{R}^n$ and $w_1, w_2 \ldots$ be the standard coordinates in W. Every manifold $\tilde{\mathcal{Y}}_\infty$ can be represented locally as the Cartesian product $\tilde{\mathcal{Y}}_\infty = \mathcal{Y}_\infty \times W$ while the mapping τ is the natural projection $\mathcal{Y}_\infty \times W \to \mathcal{Y}_\infty$. Recall that the contact structure on $\mathcal{Y}_\infty$ is determined by the system consisting of n vector fields $\bar{D}_i$, where $\bar{D}_i$ is the restriction of the total derivative operator on $\mathcal{Y}_\infty$ (see [1], Section 3.5 and 3.6), and, what is more, the equalities $[D_i, D_j] = 0$ hold. Then an n-dimensional contact structure on $\tilde{\mathcal{Y}}_\infty = \mathcal{Y}_\infty \times W$ may be determined by a system of vector fields $\tilde{D}_i = \bar{D}_i + X_i$, $i = 1, \ldots, n$, where $X_i = \Sigma_j X_{ij}\, \partial/\partial w_j$, $X_{ij} \in C^\infty(\tilde{\mathcal{Y}}_\infty)$. Frobenius conditions can now be rewritten as $[\tilde{D}_i, \tilde{D}_j] = 0$, which is equivalent to

$$[\bar{D}_i, X_j] + [X_i, \bar{D}_j] + [X_i, X_j] = 0, \quad i, j = 1, \ldots, b. \tag{5}$$

Relations (5) constitute a system of differential equations with respect to functions X_{ij} which describe all the coverings of the equation $\mathcal{Y}$ with the fibre W. It can be seen from the following examples that we can consider the coordinates w_i as 'nonlocal variables' and the operators $\tilde{D}_i$ as total derivations.

EXAMPLE 1. Suppose $n = N = 1$, $\mathcal{Y}_\infty = J^\infty(\mathbb{1}_{\mathbb{R}^1})$, $\mathbb{1}_M$ denotes the trivial one-dimensional bundle over M, while $w = w_1$ is the coordinate in W. Then Equations (5) are obviously trivial and, therefore, any function $X = X(w, x, u, \ldots, p_\sigma, \ldots)$ on $\mathcal{Y}_\infty = J^\infty(\mathbb{1}_{\mathbb{R}^a}) \times W$ determines a covering over $J^\infty(\mathbb{1}_{\mathbb{R}_1})$, the operator $\tilde{D}_1$ being of the form $\tilde{D}_1 = \tilde{D} = X(\partial/\partial w) + D_1$. Hence, from $\tilde{D}(w) = X$ it follows that the function X is the total derivative of the variable w. In other words, if $X = X(x, u, \ldots, p_\sigma, \ldots)$ then w can be understood as the integral $\int X(x, u, \ldots, \partial^\sigma u/\partial x, \ldots)\, dx$.

EXAMPLE 2. Any 'regular' differential operator acting from the sections of a bundle π into the sections of a bundle π' over M induces the mapping $J^\infty(\pi) \to J^\infty(\pi')$ which is a covering over $J^\infty(\pi')$ at its nonsingular points. The operator d/dx, for instance, leads to the covering $J^\infty(\mathbb{1}_{\mathbb{R}^1}) \to J^\infty(\mathbb{1}_{\mathbb{R}^1})$ which, in terms of the previous example, can be described by means of the function $X = u$ (that is $w = \int u\, dx$).

Now, let $V \in \mathcal{Y}_\infty$ be an integral manifold, i.e., a solution of $\tilde{\mathcal{Y}}$. It is easy to see that the restriction of the contact structure over $\mathcal{Y}_\infty$ on the inverse image $\tau^{-1}(V)$ is an integrable n-dimensional distribution. In particular, when $\dim W = 1$ it follows that the manifold $\tau^{-1}(V)$ is foliated by one-dimensional family of integral manifolds. We have a similar but rather more complex situation in the case $\dim W = \infty$. Hence, the entire family of integral manifolds in $\tilde{\mathscr{Y}}_\infty$ corresponds to a single solution of $\mathscr{Y}$. That is why it is pertinent to interpret such manifolds as solutions of $\mathscr{Y}$ with nonlocal parameters. For example, in the covering with the nonlocal variable $w = \int u \, dx$, this nonlocal parameter is a constant of integration, while integral manifolds in $\tilde{\mathscr{Y}}_\infty$ are identified with pairs of the form $(f(x), c)$, where f is a solution of $\mathscr{Y}$ and c is a constant of integration in $\int f(x) \, dx$.

4. Nonlocal Symmetries: The Exact Definition

Consider a differential equation $\mathscr{Y}$ and its covering $\tilde{\mathscr{Y}}_\infty$. A transformation $f\colon \tilde{\mathscr{Y}}_\infty \to \tilde{\mathscr{Y}}_\infty$ is said to be a nonlocal symmetry of $\mathscr{Y}$ is and only if it preserves the contact structure on $\tilde{\mathscr{Y}}_\infty$. In other words, f is a nonlocal symmetry when $f_*(\tilde{\mathscr{C}}_y) = \tilde{\mathscr{C}}_{f(y)}$ for any point $y \in \tilde{\mathscr{Y}}_\infty$.

The grounds for such a definition are as follows: When investigating differential equations we use operators which, in general, are in some sense inverse to differential ones. These operators are nonlocal by their nature. They are multi valued, but become one-valued when some sort of parameters, such as integration constants, are fixed. That is why any actual reasoning concerning the construction of the solutions of $\mathscr{Y}$ deals with the consideration of these constants, explicitly or not. In other words, we do not have to consider 'plain' solutions as such, but rather solutions equipped with 'integration constants' in the sense described above. Thus, nonlocal symmetries transform such solutions into each other. Indeed, as it follows from the previous subsection, integral manifolds in any covering $\tilde{\mathscr{Y}}_\infty$ of the equation $\mathscr{Y}$ could be understood as 'equipped' solutions of $\mathscr{Y}$.

The differential-geometric structure of coverings $\tilde{\mathscr{Y}}_\infty$ is quite analogous to the structure of manifolds $\mathscr{Y}_\infty$. Therefore, the definition of infinitesimal nonlocal symmetries does not differ from that of higher symmetries (cf. [1], Section 3.3). Namely, the Lie factor-algebra

$$\mathrm{Sym}_\tau \mathscr{Y} = \frac{D_{\mathscr{C}}(\tilde{\mathscr{Y}}_\infty)}{\mathscr{C}D(\tilde{\mathscr{Y}}_\infty)}$$

is said to be the algebra of nonlocal symmetries of the type τ for the equation $\mathscr{Y}$. Here $\tau\colon \tilde{\mathscr{Y}}_\infty \to \mathscr{Y}_\infty$ is a covering, $D_{\mathscr{C}}(\tilde{\mathscr{Y}}_\infty)$ consists of such vector fields S on $\tilde{\mathscr{Y}}_\infty$ that $[S, \mathscr{C}D(\tilde{\mathscr{Y}}_\infty)] \in \mathscr{C}D(\tilde{\mathscr{Y}}_\infty)$, while

$$\mathscr{C}D(\tilde{\mathscr{Y}}_\infty) = \left\{ \sum_{i=1}^{n} \varphi_i \tilde{D}_i \mid \varphi_i \in \mathscr{C}^\infty(\tilde{\mathscr{Y}}_\infty) \right\}.$$

This definition could be motivated just in the same way as was done in Sections 3.4 and 3.5 of [1]. Moreover, the elements of $\mathrm{Sym}_\tau \mathscr{Y}$ can be identified with such vector fields

S on $\tilde{\mathscr{Y}}_\infty$, that

$$[S, \tilde{D}_i] = 0, \quad i = 1, \ldots, n, \quad \text{and} \quad S(\varphi) = 0 \quad \text{for any } \varphi = \varphi(x).$$

(Here x denotes the set of independent variables for the equation $\mathscr{Y}$.) The condition $S(\varphi) = 0$, $\varphi = \varphi(x)$, is obviously equivalent to the vanishing of the coefficient of the field S which correspond to $\partial/\partial x_i$, $i = 1, \ldots, n$, components.

EXAMPLE 3. Consider the covering described in Example 1 and corresponding to the function $X = u$. Taking into consideration (6), we shall seek the elements $S \in \mathrm{Sym}_\tau \mathscr{Y}$ as being represented in the form

$$S = \Psi \frac{\partial}{\partial w} + \sum_{i=0}^{\infty} P_k \frac{\partial}{\partial p_{(k)}}, \qquad [S, \tilde{D}] = 0, \quad \tilde{D} = \tilde{D}_1.$$

Then

$$[S, \tilde{D}] = P_0 \frac{\partial}{\partial w} + \sum_{i=0}^{\infty} P_{i+1} \frac{\partial}{\partial p_{(i)}} -$$

$$- \tilde{D}(\Psi) \frac{\partial}{\partial w} - \sum_{i=0}^{\infty} \tilde{D}(P_i) \frac{\partial}{\partial p_{(i)}} = 0,$$

or, equivalently,

$$P_0 = \tilde{D}(\Psi), \; P_{i+1} = \tilde{D}(P_i), \quad i = 0, 1, 2, \ldots .$$

Consequently,

$$S = \varphi \frac{\partial}{\partial w} + \tilde{D}(\varphi) \frac{\partial}{\partial o_0} + \cdots + \tilde{D}^{k+1}(\varphi) \frac{\partial}{\partial p_{(k)}} + \cdots, \tag{7}$$

i.e., (7) is the formula for evolutionary differentiations but 'shifted one position to the left'.

5. Coverings over Burgers' Equation

Now we return to Burgers' equation $\mathscr{Y} = \{u_t = uu_x + u_{xx}\}$. In this section, we shall describe a class of its coverings which subsequently will be used for the calculation of nonlocal symmetries. We preserve the notations for the coordinates and operators from Section 1.

Let $\tau : \tilde{\mathscr{Y}}_\infty = \mathscr{Y}_\infty \times W \to \mathscr{Y}_\infty$ be the natural projection, w_i, $i = 1, 2, \ldots$, coordinates in W, and $\tilde{D}_1 = \tilde{D}_x = D_x + X$, $\tilde{D}_2 = \tilde{D}_t = D_t + T$, where

$$X = \sum_{i \geqslant 1} X_i \frac{\partial}{\partial w_i}, \qquad T = \sum_{i \geqslant 1} T_j \frac{\partial}{\partial w_j}, X_i, \quad T_j \in \mathscr{C}^\infty(\tilde{\mathscr{Y}}_\infty).$$

Then according to (5), the fields X and T should satisfy the equation

$$[D_x, T] + [X, D_t] + [X, T] = 0. \tag{8}$$

We shall find all the solutions of the latter equation for which the functions X_i, T_j do not depend on the variables x, t and $p_{(k)}$, $k > 1$. For such X and T (8) can be rewritten as

$$p_{(1)} \frac{\partial T}{\partial p_{(0)}} + p_{(2)} \frac{\partial T}{\partial p_{(1)}} - (p_{(2)} + p_{(0)} p_{(1)}) \frac{\partial X}{\partial p_{(0)}} - \\ - (p_{(3)} + p_{(1)}^2 + p_{(0)} p_{(2)}) \frac{\partial X}{\partial p_{(1)}} + [X, T] = 0, \tag{9}$$

where the symbol $\partial Y/\partial z$ for the field $Y = \Sigma_i Y_i(\partial/\partial y_i)$ means $\Sigma_i(\partial Y_i/\partial z)\,\partial/\partial y_i$. The left-hand side of (9) is a 'vector-valued' polynomial in variables $p_{(2)}$ and $p_{(3)}$. Hence, its coefficients vanish, and so it follows that functions X_i do not depend on $p_{(1)}$ and

$$\frac{\partial T}{\partial p_{(1)}} = \frac{\partial X}{\partial p_{(0)}} \Leftrightarrow T = p_{(1)} \frac{\partial X}{\partial p_{(0)}} + R, \tag{10}$$

where the coefficients of the fields R do not depend on $p_{(1)}$. Now, substituting (10) into (9) we get

$$p_{(1)}^2 \frac{\partial^2 X}{\partial p_{(0)}^2} + p_{(1)} \left(\frac{\partial R}{\partial p_{(0)}} - p_{(0)} \frac{\partial X}{\partial p_{(0)}}\right) + \left[X, \frac{\partial X}{\partial p_{(0)}}\right] + [X, R] = 0.$$

As X and R do not depend on $p_{(1)}$, this equation is equivalent to the following system

$$\begin{aligned} &\frac{\partial^2 X}{\partial p_{(0)}^2} = 0, \\ &\frac{\partial R}{\partial p_{(0)}} = p_{(0)} \frac{\partial X}{\partial p_{(0)}} - \left[X, \frac{\partial X}{\partial p_{(0)}}\right] \\ &[X, R] = 0 \end{aligned} \tag{11}$$

From the first of these equations it follows that

$$X = p_{(0)} A + B, \tag{12}$$

where the coefficients of the fields A and B depend on the variables w_i only. In other words, A and B are fields on W. With regard to (12), the second equation in (11) transforms into

$$\frac{\partial R}{\partial p_{(0)}} = p_{(0)} A + [A, B],$$

which is equivalent to

$$R = \tfrac{1}{2}p_{(0)}^2 A + p_{(0)}[A, B] + C. \tag{13}$$

Here C is a field on W. At last, substituting (12) and (13) into the last of Equations (14) we shall get

$$p_{(0)}^2([A,[A,B]] + \tfrac{1}{2}[B,A]) + p_{(0)}([A,C] + [B,[A,B]]) + [B,C] = 0$$

or, equivalently,

$$\begin{aligned} &[A,[A,B]] = \tfrac{1}{2}[A,B],\\ &[B,[B,A]] = [A,C],\\ &[B,C] = 0. \end{aligned} \tag{14}$$

Thus, we have proved the following statement.

THEOREM 1. *Any covering of Burgers' equation in which the coefficients of the fields X and T do not depend on x, t and $p_{(k)}$, $k > 1$, are determined by the fields of the form*

$$\begin{aligned} &\tilde{D}_x = D_x + p_{(0)}A + B,\\ &\tilde{D}_t = D_t + (p_{(1)} + \tfrac{1}{2}p_{(0)}^2)A + p_{(0)}[A,B] + C \end{aligned} \tag{15}$$

where A, B and C are fields on W which satisfy (14).

Here the following remarkable fact should be noted. Consider an abstract Lie algebra $\mathfrak{G}$ generated (as a Lie algebra) by the elements a, b, c and which satisfy the relations (14) where $A \mapsto a$, $B \mapsto b$, $C \mapsto c$. Then the Lie algebra of the vector fields generated by the fields A, B and C is a representation of the algebra $\mathfrak{G}$ in the algebra of the vector fields on W. Thus, from Theorem 1 it follows that any covering of Burgers' equation of the type considered here is uniquely determined by some representation of the algebra $\mathfrak{G}$ as an algebra of the vector fields. By this reason we shall call the algebra $\mathfrak{G}$ universal.

6. Nonlocal Symmetries of Burgers' Equation

Now we shall consider the problem of calculating of Burgers' equation nonlocal symmetries in the coverings described in Theorem 1. According to (6), we identify the elements of $\mathrm{Sym}_\tau \mathcal{Y}$ with such fields S on $\tilde{\mathcal{Y}}_\infty$, that $S = P + \Phi$, where $P = \Sigma_{i \geqslant 0} P_i(\partial/\partial p_{(i)})$, $\Phi = \Sigma_{j \geqslant 1} \Phi_j(\partial/\partial w_j)$, $P_i, \Phi_j \in C^\infty(\tilde{\mathcal{Y}}_\infty)$, and $[S, \tilde{D}_x] = [S, \tilde{D}_t] = 0$.

Taking into consideration (15), we get the equations

$$[S, \tilde{D}_x] = \sum_i (P_{i+1} - \tilde{D}_x(P_i)) \frac{\partial}{\partial p_{(i)}} + P_0 A + [\Phi, \tilde{D}_x] = 0,$$

$$[S, \tilde{D}_t] = \sum_i \left(\sum_k P_k \frac{\partial}{\partial p_{(k)}} (\tilde{D}_x^k(p_{(2)} + p_{(0)}p_{(1)})) - \tilde{D}_t(P_i) \right) \frac{\partial}{\partial p_{(i)}} +$$

$$+ (P_1 + p_{(0)}P_0)A + P_0[A,B] + [\Phi, \tilde{D}_t] = 0.$$

The coefficients of the fields $[\Phi, \tilde{D}_x]$ and $[\Phi, \tilde{D}_t]$ corresponding to the components $\partial/\partial p_{(i)}$ vanish for all i, that is, these fields are vertical with respect to the projection τ. Hence, the first of these equations is equivalent to

$$P_{i+1} = \tilde{D}_x(P_i), \quad i = 0, 1, \ldots, P_0 A + [\Phi, \tilde{D}_x] = 0,$$

from which it follows that $P_i = \tilde{D}_x^i(\psi)$, where ψ now denotes P_0. Similarly, the second equation is equivalent to

$$(P_1 + p_{(0)} P_0) A + P_0[A, B] + [\Phi, \tilde{D}_t] = 0,$$

$$\sum_k \tilde{D}_x^k(\psi) \frac{\partial}{\partial p_{(k)}} (\tilde{D}_x^i (p_{(2)} + p_{(0)} p_{(1)})) = \tilde{D}_t \tilde{D}_x^i(\psi). \tag{16i}$$

Let

$$\tilde{\partial}_\psi = \sum_k \tilde{D}_x^k(\psi) \frac{\partial}{\partial p_{(k)}}.$$

Then obviously $[\tilde{\partial}_\psi, \tilde{D}_x] = 0$. Since (16i) can be rewritten in the form

$$(\tilde{\partial}_\psi \circ \tilde{D}_x^i)(p_{(2)} + p_{(0)} p_{(1)}) = (\tilde{D}_x^i \circ \tilde{D}_t)(\psi)$$

it follows that it can be obtained by applying the operator $\tilde{D}_x^i$ to the equality (16_0):

$$\tilde{\partial}_\psi(p_{(2)} + p_{(0)} p_{(1)}) = \tilde{D}_t(\psi)$$

or

$$\tilde{D}_x^2(\psi) + p_{(0)} \tilde{D}_x(\psi) + p_{(1)} \psi = D_t(\psi).$$

Note, that the last equation can be rewritten as $\tilde{l}_F(\psi) = 0$, where $F = p_{11} + p_0 p_1 - p_2$ (see Section 1). Thus, we see how the exact theory from Section 4 correlates with the naive approach. Namely, the equation $\tilde{l}_F(\psi) = 0$ is not the sole condition which must be satisfied by the function ψ. More precisely, the results of the previous calculations lead to the following:

PROPOSITION. *Any nonlocal symmetry of Burgers' equation in the covering* (15) *is of the form* $S = \tilde{\partial}_\psi + \Phi$, *where* $\Phi = \sum_i \Phi_i(\partial/\partial w_i)$, $\psi, \Phi_i \in C^\infty(\mathcal{Y}_\infty)$ *while the function* ψ *and the field* Φ *satisfy the system of the following differential equations*

$$\begin{aligned} &\psi A = [\tilde{D}_x, \Phi], \\ &(\tilde{D}_x(\psi) + p_{(0)}\psi) A + \psi[A, B] = [\tilde{D}_t, \Phi], \\ &\tilde{l}_F(\psi) \equiv \tilde{D}_x^2(\psi) + p_{(0)} \tilde{D}_x(\psi) + p_{(1)}\psi - \tilde{D}_t(\psi) = 0. \end{aligned} \tag{17}$$

Note that when the covering is trivial, i.e., $A = B = C = 0$, the first two equations in (17) are satisfied in a trivial way, and so this system is reduced to the equation $l_F(\psi) = 0$. Thus, the local theory of symmetries is a natural part of the nonlocal one.

Further analysis of system (17) should be based on particular realizations of the universal algebra $\mathfrak{G}$. Below we shall consider several examples.

7. One-Dimensional Coverings

Suppose $W = \mathbb{R}$. First consider the case $A = 0$. Then the system (15) transforms into a single equation $[B, C] = 0$. If the field B is nonzero, then we can choose a coordinate w on W in such a way that $B = \partial/\partial w$. Then it follows that $C = \gamma(\partial/\partial w)$ where $\gamma = \text{const}$. The case $B = 0$, $C \neq 0$ can be treated in a similar way. Thus, it is always possible to choose a coordinate w in such a way that $B = \beta(\partial/\partial w)$ and $C = \gamma(\partial/\partial w)$, $\beta, \gamma = \text{const}$. In this case we have

$$\tilde{D}_x = D_x + \beta \frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + \gamma \frac{\partial}{\partial w}.$$

Suppose $v = w - \beta x - \gamma t$. Then $\tilde{D}_x(v) = \tilde{D}_t(v) = 0$, hence, in $v, x, t, \dots, p_{(k)}, \dots$ coordinate system the equalities $\tilde{D}_x = D_x, \tilde{D}_t = D_t$ hold. Thus, in this case the two-dimensional distribution determined by the fields $\tilde{D}_t$, $\tilde{D}_x$ on $\tilde{\mathcal{Y}}_\infty$ is tangent to the surfaces $\prod_\lambda = \{v = \lambda \mid \lambda = \text{const}\}$. Therefore, this distribution induces two-dimensional integrable distributions on $\prod_\lambda$. The surface $\prod_\lambda$, together with such a distribution, is isomorphic to the pair $(\mathcal{Y}_\infty, \mathcal{C})$ where $\mathcal{C}$ is the Cartan distribution on $\mathcal{Y}_\infty$. This isomorphism is induced by the projection τ. In other words, in this case $\tilde{\mathcal{Y}}_\infty$ equipped with the corresponding contact structure can be represented as a one-parameter family of the manifolds $\mathcal{Y}_\infty$, v being the parameter. Thus, any nonlocal symmetry in such a covering is simply a v-parameter family of local symmetries. Obviously, such coverings are of no interest and so we call them trivial.

Now we shall suppose that $A \neq 0$. Choose a coordinate w on W in such a way that $A = \partial/\partial w$ locally. In this case $B = b(w)\,\partial/\partial w$, $C = c(w)\,\partial/\partial w$, and the system (14) acquires the form

$$b'' = \tfrac{1}{2}b', \quad (b')^2 - b''b = c', \; b'c = bc',$$

when $f' = \mathrm{d}f/\mathrm{d}w$. Solving this system we get two kinds of solutions: $b = \text{const}$, $c = \text{const}$ and

$$b = \mu \exp(w/2) + \nu, \qquad c = -\nu/2b, \quad \mu = \text{const} \neq 0, \quad \nu = \text{const}. \tag{18}$$

In the first case, in accordance to (15), we have

$$\tilde{D}_x = D_x + (p_{(0)} + b)\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + (p_{(1)} + \tfrac{1}{2}p_{(0)}^2 + c)\frac{\partial}{\partial w}.$$

Using the coordinate $v = w - bx - ct$ instead of w, we get

$$\tilde{D}_x = D_x + p_{(0)}\frac{\partial}{\partial w}, \; \tilde{D}_t = D_t + (p_{(1)} + \tfrac{1}{2}p_{(0)}^2)\frac{\partial}{\partial w}.$$

In other words, we may assume b and c to be equal to zero. In this case (17) transforms into

$$\psi = \tilde{D}_x(\varphi), \qquad \tilde{D}_x(\psi) + p_{(0)}\psi = \tilde{D}_t(\varphi), \qquad \tilde{l}_F(\psi) = 0,$$

where $\Phi = \varphi(\partial/\partial v)$. It is easy to see that the third equation in this system is the consequence of the first two and so it can be transformed into

$$\psi = \tilde{D}_x(\varphi), \qquad \tilde{D}_x^2(\varphi) + p_{(0)}\tilde{D}_x(\varphi) = \tilde{D}_t(\varphi). \tag{19}$$

REMARK. Denoting v by $p_{(-1)}$ we see that the covering considered here is identical to the extended manifold $\tilde{\mathcal{Y}}_\infty$ which was introduced in Section 1. Then Equation (1) is the consequence of Equations (19) but not vice-versa. This again points out the drawback of the naive viewpoint.

The second of the relations (19) is of the form $l_G(\varphi) = 0$, where $G = v_{xx} + \frac{1}{2}v_x^2 - v_t$. This gives an idea that the manifold $\tilde{\mathcal{Y}}_\infty$ is of the form $\mathcal{Y}'_\infty$, where $\mathcal{Y}'$ is the equation $v_t = v_{xx} + \frac{1}{2}v_x^2$. In fact, we can identify $\mathcal{Y}'_\infty$ and $\tilde{\mathcal{Y}}_\infty$ together with respective contact structures by patting $x = x'$, $t = t'$, $p_{(k)} = p'_{(k+1)}$, $v = p'_{(0)}$, where x', t', $p'_{(k)}$ are the standard coordinates on $\mathcal{Y}'_\infty$. Then it follows that $\mathrm{Sym}_\tau\,\mathcal{Y} = \mathrm{Sym}\,\mathcal{Y}'$. Calculating the algebra $\mathrm{Sym}\,\mathcal{Y}'$ by the same schema as was used in [1], we find that this algebra is generated by the elements $\varphi_{-\infty} = \alpha(x, t)\exp(-\frac{1}{2}v)$, where $\alpha_{xx} = \alpha_t$

$$\varphi^0_{-1} = 1$$

$$\varphi^i_k = t^i p_{(k)} + \tfrac{1}{2}((k+1)\,t^i p_{(0)} + ixt^{i-1})\,p_{(k-1)} + \dots,$$

$$k = 0, 1, \dots, i = 0, 1, \dots, k+1.$$

It should be noted that the form of the system (17) depends essentially on a coordinate system in $\mathbb{R}^1$, i.e., on a representation of the field A. If $A = a(w)\partial/\partial w$, then in the case considered, Equations (17) transform into

$$\psi = \frac{1}{a}\,(\tilde{D}_x(\varphi) - p_{(0)}\,a'\varphi),$$

$$\tilde{D}_x^2(\varphi) + (p_{(0)} - 2a'\,p_{(0)})\,\tilde{D}_x(\varphi) + \\ + [(a')^2 - aa'' - \tfrac{1}{2}a']\,p_{(0)}^2\varphi = \tilde{D}_t(\varphi).$$

The latter equation is of the form $L(\varphi) = 0$, where

$$L = \tilde{D}_x^2 - (1 - 2a')\,p_{(0)}\tilde{D}_x - \tilde{D}_t + [(a')^2 - aa'' - \tfrac{1}{2}a']\,p_0^2.$$

The simpliest form of the operator L could be obtained when killing its component with $\tilde{D}_x$. For this, it suffices to have $2a' = 1$, or $a = \frac{1}{2}w$. Then $L = \tilde{D}_x^2 - \tilde{D}_t$.

For the same reasons as above, we conclude that $\tilde{\mathcal{Y}}_\infty = \mathcal{Y}''_\infty$, where $\mathcal{Y}'' = \{w_{xx} = w_t\}$ is the heat equation. Thus, the covering considered here is of the form $\mathcal{Y}''_\infty \to \mathcal{Y}_\infty$, where $\mathcal{Y}''$ is the heat equation. Therefore Burgers' equation is the factor equation of heat equation $w_{xx} = w_t$ with respect to the one-parameter transformation group $A_\varepsilon = \{w \mapsto \varepsilon w\}$, ε being the parameter. Note that all linear equations admit such a group of transformations.

In the coordinate system on $\tilde{\mathscr{Y}}_\infty$ considered here (i.e., when $A = \frac{1}{2}w(\partial/\partial w)$ the algebra $\mathrm{Sym}_\tau\,\mathscr{Y} = \mathrm{Sym}\,\mathscr{Y}''$ is of the form

$$\varphi_{-\infty} = \alpha(x, t), \qquad \frac{\partial^2\alpha}{\partial x^2} = \frac{\partial\alpha}{\partial t}, \qquad \varphi^0_{-1} = w,$$

$$\varphi^i_k = w[t^i p_{(k)} + \tfrac{1}{2}((k+1)\,t^i p_{(0)} + it^{i-1}x)\,p_{(k-1)} + \dots,$$
$$k = 0, 1, \dots, i = 0, 1, \dots, k+1.$$

Now we shall consider the coverings which are described by functions (18). Obviously, there is such a coordinate transformation $w' = w + \mathrm{const}$ which results in $|\mu| = 1$, i.e., $b = \pm\exp(\frac{1}{2}w) + v$. It is easy to show that all such coverings are pair-wise different.

For the coverings considered, system (17) can easily be transformed into

$$\square(\varphi) = 0, \qquad \psi = \tilde{D}_x(\varphi) - b'\varphi, \tag{20}$$

where $S = \varphi(\partial/\partial w) + \tilde{\mathfrak{Z}}_\psi$, $S \in \mathrm{Sym}_\tau\,\mathscr{Y}$, while

$$\square = \tilde{D}_x^2 + p_{(0)}\tilde{D}_x - b'(p_{(0)} + b) - \tilde{D}_t. \tag{21}$$

System (22) (in fact, the equation $\square(\varphi) = 0$) can be solved in the same way as was done for the equation $\bar{l}_F = 0$, where $F = 0$ was Burgers' equation (cf. [1]). Furthermore, the algebra $\mathrm{Sym}_\tau\,\mathscr{Y}$ happens to be isomorphic to $\mathrm{Sym}\,\mathscr{Y}$. For example, the functions φ which are the solution of (20) depending on x, t, w, $p_{(0)}$, $o_{(1)}$ only, are the linear combinations of the following

$$\varphi^0_1 = p_{(0)} + b,$$
$$\varphi^1_1 = t(p_{(0)} + b) - (b')^{-1},$$

$$\varphi^0_2 = p_{(1)} + \tfrac{1}{2}p^2_{(0)} + b'\,p_{(0)} - \frac{v}{2}\,b,$$

$$\varphi^1_2 = t(p_{(1)} + \tfrac{1}{2}p^2_{(0)}) + (tb' + \tfrac{1}{2}x)p_{(0)} + \tfrac{1}{2}b(x - vt) + $$
$$+ 1 + \tfrac{1}{2}v(b')^{-1},$$
$$\varphi^2_2 = t^2(p_{(1)} + \tfrac{1}{2}p^2_{(0)}) + (t^2b' + tx)p_{(0)} + \tfrac{1}{2}b(2tx - vt^2) + $$
$$+ 2t + (vt - x)(b')^{-1}.$$

Respectively, the functions ψ are of the form

$$\psi^0_1 = p_{(1)},$$
$$\psi^1_1 = tp_{(1)} + \tfrac{1}{2}(p_{(0)} + v)(b')^{-1} + 2,$$
$$\psi^0_2 = p_{(2)} + p_{(0)}p_{(1)},$$
$$\psi^1_2 = t(p_{(2)} + p_{(0)}p_{(1)}) + \tfrac{1}{2}[xp_{(1)} + p_{(0)} - \tfrac{1}{4}v(p_{(0)} + b)(b')^{-1}]$$
$$\psi^2_2 = t^2(p_{(2)} + p_{(0)}p_{(1)}) + t[xp_{(1)} + p_{(0)} - \tfrac{1}{2}v(p_{(0)} + b)(b')^{-1}] + $$
$$+ [\tfrac{1}{2}x(p_{(0)} + b + 2b') - 1](b')^{-1}.$$

Let S^i_j be the symmetry corresponding to the function φ^i_j. Then, from this list of symmetries it follows that S^0_1 and S^0_2 are local symmetries while S^1_2 is essentially nonlocal when $\nu \neq 0$ and S^1_1, S^2_2 are always essentially nonlocal symmetries.

The existence of an isomorphism between the algebras $\mathrm{Sym}_\tau\, \mathcal{Y}$ and $\mathrm{Sym}\, \mathcal{Y}$ gives an idea that the contact manifold $\tilde{\mathcal{Y}}_\infty$ is diffeomorphic to $\mathcal{Y}'_\infty$, where $\mathcal{Y}'$ is Burgers' equation. In order to verify this hypothesis, we shall try to represent $\tilde{\mathcal{Y}}_\infty$ in the form $\mathcal{Y}'_\infty$, where $\mathcal{Y}'$ is some differential equation. Assuming x and t to be independent variables in $\mathcal{Y}'$ and w to be a dependent one, we shall introduce the coordinates $x, t, w = \prod_0, \dots, \prod_i = \tilde{D}^i_x(w)$ on $\tilde{\mathcal{Y}}_\infty$. Now, it is natural to assume that $\mathcal{Y}'$ is an evolutionary equation. Then it must be of the form $w_t = \tilde{D}_t(w)$. Calculating $\tilde{D}_t(w)$ in terms of the variables $w_{x,\dots,x,} = \prod_i(i \text{ times})$, we shall find that $\mathcal{Y}'$ is

$$w_{xx} + \tfrac{1}{2}w_x^2 - bw_x - w_t = 0. \tag{22}$$

Here, as it should be, the operator of the universal linearization for (22) coincides with the operator (21).

If we now try to transform (22) into Burgers' equation by the coordinate transformation $v = f(w)$, we shall immediately get that $v = -b$. Thus, by choosing the coordinates $x, t, v = \pi_0, \dots, \pi_i = \tilde{D}^i_x(v), \dots$ on $\mathcal{Y}$ we discover that $\tilde{\mathcal{Y}}_\infty \approx \mathcal{Y}_\infty$. Hence, the mapping $\tau\colon \tilde{\mathcal{Y}}_\infty \to \mathcal{Y}_\infty$ can be treated as the mapping of Burgers' equation into itself. An explicit formula for this mapping can be found from the expression $v_x = \tilde{D}_x(v) = -b(u + b) = \frac{1}{2}(v + \nu)(u - v)$. Then it follows that

$$u = \frac{2v_x}{v + \nu} + v. \tag{23}$$

Thus, we have come to the following remarkable result: if v is a solution of Burgers' equation, then the function u defined by (22) is also a solution of Burgers' equation for any value of parameter ν.

8. One Infinitely-Dimensional Covering

In conclusion, we shall consider the following infinitely-dimensional covering of Burgers' equation:

$$A = \frac{\partial}{\partial w_1},$$

$$B = \exp(\tfrac{1}{2}w_1)\,\frac{\partial}{\partial w_2} + w_2\,\frac{\partial}{2\partial w_3} + \dots + w_i\,\frac{\partial}{\partial w_{i+1}} + \dots, \tag{24}$$

$$C = \exp(\tfrac{1}{2}w_1)\,\frac{\partial}{\partial w_3} + w_2\,\frac{\partial}{2\partial w_4} + \dots + w_i\,\frac{\partial}{\partial w_{i+2}} + \dots.$$

The fields A, B and C obviously satisfy Equations (15). Let $S = \tilde{\partial}_\psi + \Sigma_{i=1}^{\infty} \Phi(\partial/\partial w_i)$ be a symmetry in the covering (24), $\Phi_i \in C^\infty(\tilde{\mathscr{Y}}_\infty)$. Then taking into account the relation $[A, B] = \frac{1}{2}\exp(\frac{1}{2}w_1)\partial/\partial w_2$ we can rewrite system (17) in the form

$$
\begin{array}{ll}
\psi = \tilde{D}_x(\Phi_1), & p_{(1)}\psi + p_{(0)}\tilde{D}_x(\psi) + \tilde{D}_x^2(\psi) = \tilde{D}_t(\psi),\\
\Phi_1 = 2\exp(-\frac{1}{2}w_1)\tilde{D}_x(\Phi_2), & p_{(0)}\psi + \tilde{D}_x(\psi) = \tilde{D}_t(\Phi 1),\\
\Phi_2 = \tilde{D}_x(\Phi_3), & \psi + \frac{1}{2}p_{(0)}\Phi_1 = 2\exp(-\frac{1}{2}w_1)\tilde{D}_t(\Phi_2),\\
\Phi_3 = \tilde{D}_x(\Phi_4), & \Phi_1 = 2\exp(-\frac{1}{2}w_1)\tilde{D}_t(\Phi_3),\\
\dots & \Phi_2 = \tilde{D}_t(\Phi_4),\\
\Phi_{k-1} = \tilde{D}_x(\Phi_k), & \dots\\
\dots & \Phi_{k-2} = \tilde{D}_t(\Phi_k)\\
\dots & \dots
\end{array}
$$

This system, as can be easily verified, is equivalent to the following:

$$
\begin{array}{lll}
\psi = \tilde{D}_x(\Phi_1), & p_{(1)} + p_{(0)}\tilde{D}_x(\psi) + \tilde{D}_x^2(\psi) = \tilde{D}_t(\psi), & (25_0)\\
\Phi_1 = 2\exp(-\frac{1}{2}w_1)\tilde{D}_x(\Phi_2), & p_{(0)}\tilde{D}_x(\Phi_1) + \tilde{D}_x^2(\Phi_1) = \tilde{D}_t(\Phi_1), & (25_1)\\
\Phi_2 = \tilde{D}_x(\Phi_3), & \tilde{D}_x^2(\Phi_2) = \tilde{D}_t(\Phi_2), & (25_2)\\
\Phi_3 = \tilde{D}_x(\Phi_4), & \tilde{D}_x^2(\Phi_3) = \tilde{D}_t(\Phi_3), & (25_3)\\
\dots & \dots & \\
\Phi_{k-1} = \tilde{D}_x(\Phi_k), & \tilde{D}_x^2(\Phi_k) = \tilde{D}_t(\Phi_k), & (25_k)
\end{array}
$$

Before solving this infinite system of differential equations, let us introduce a filtration into the algebra $C^\infty(\tilde{\mathscr{Y}}_\infty)$ by putting for any $\varphi \in C^\infty(\tilde{\mathscr{Y}}_\infty)$ deg $\varphi \leqslant k$ when $\partial\varphi/\partial p_{(i)} = 0$ for all $i > k$ and deg $\varphi \leqslant -k$ when $\partial\varphi/\partial w_i = 0$ for all $j < k$ and $\partial\varphi/\partial p_{(i)} = 0$ for all $i \geqslant 0$. Now it should be noted that if the equations in the left-hand column of (25) are satisfied and if some of Equation (25_k) is satisfied, then all Equations (25_i) for $i > k$ are satisfied too. Two cases are to be considered: (a) deg $\Phi_2 \geqslant 4$; (b) deg $\Phi_2 < 4$. Consider case (a) first: Since deg $\tilde{D}_x(\varphi) = \deg\varphi + 1$ for all $\varphi \in C^\infty(\tilde{\mathscr{Y}}_\infty)$ then from the left-hand column of (25), it follows that there is such a number k that for all $j \geqslant k$ functions Φ_j depend on the variables t, x, w_4, $w_5, \dots$ only, i.e., deg $\Phi_j \leqslant -4$. Let $(\psi, \Phi_1, \Phi_2, \dots, \Phi_k, \dots)$ be a solution of (25) and k be such that

$$\Phi_k = \Phi_k(t, x, w_4, w_5, \dots, \dots, w_r), \qquad \frac{\partial\Phi_k}{\partial w_4} \neq 0, \quad k \geqslant 2, \tag{26}$$

i.e., deg $\Phi_k = -4$. For the functions (26) Equation (25_k) transforms into

$$\frac{\partial^2\Phi_k}{\partial x^2} + 2\sum_{i=4}^{r} w_{i-1}\frac{\partial^2\Phi_k}{\partial x\,\partial w_i} + \sum_{i,j=1}^{r} w_{i-1}w_{j-1}\frac{\partial^2\Phi_k}{\partial w_i\,\partial w_j} = \frac{\partial\Phi_k}{\partial t}.$$

It is easy to prove by the induction that any solution of this equation is of the form

$$\Phi_k = \varphi_{k4}w_4 + \varphi_{k5}w_5 + \dots + \varphi_{kr}w_r + \varphi_{k0}\,.$$

where φ_{ki}, $i = 4, \ldots, r$, are functions in t and x only which satisfy the following system of differential equations

$$\begin{aligned}
&\frac{\partial \varphi_{k4}}{\partial x} = 0,\\
&\frac{\partial^2 \varphi_{k4}}{\partial x^2} + 2\,\frac{\partial \varphi_{k5}}{\partial x} = \frac{\partial \varphi_{k4}}{\partial t},\\
&\ldots\\
&\frac{\partial^2 \varphi_{kr-1}}{\partial x^2} + 2\,\frac{\partial \varphi_{kr}}{\partial x} = \frac{\partial \varphi_{kr-1}}{\partial t},\\
&\frac{\partial^2 \varphi_{kr}}{\partial x^2} = \frac{\partial \varphi_{kr}}{\partial t}
\end{aligned} \tag{27}$$

while $\varphi_{k0} = \varphi_{k0}(t, x)$ is an arbitrary solution of the heat equation $\partial^2 \varphi_{k0}/\partial x^2 = \partial \varphi_{k0}/\partial t$. It is easy to show that all the solutions of (27) are polynomials in t and x while φ_{k4} is a polynomial of the degree $r - 4$ depending on t only. For the calculation of the symmetries it suffices to know the solutions of (27) determined up to such solutions for which $\varphi_{k4} = 0$. The dimension of this space is equal to $r - 3$ and its basis is determined by the functions $\varphi_{k4} = 1, t, \ldots, t^{r-4}$.

Suppose $\Phi_k = \varphi_{k4} w_4 + \ldots + \varphi_{kr} w_r$ is a solution of (25_k). Now to find the corresponding symmetry of Burgers' equation, it suffices to construct such functions Φ_{k+i}, $i = 1, 2, \ldots$ that $\Phi_{k+i+1} = \tilde{D}_x(\Phi_{k+i})$ and Φ_{k+i} satisfies Equation (25_{k+i}). Let

$$\Phi_{k+1} = \sum_{j \geqslant 4,\, \alpha > 0}^{r} (-1)^{\alpha+1} \frac{\partial^{\alpha-1} \varphi_{kj}}{\partial x^{\alpha-1}} w_{j+\alpha}.$$

Since all φ_{kj} are polynomials in x, the function Φ_{k+1} is well defined. From (27), it follows that $\tilde{D}_x^2(\Phi_{k+1}) = \tilde{D}_t(\Phi_{k+1})$. Just in the same way the functions Φ_{k+i} are constructed which satisfy (25_{k+i}).

The second case, i.e., $\Phi_2 = \Phi_2(t, x, x_5, \ldots, w_r)$, is treated analogously to the first one.

Denote by Φ_k^i, $k \geqslant -2$, $i \geqslant 0$, the symmetry of Burgers' equation for which the function Φ_{k+4} is of the form $\Phi_{k+4} = t^i w_4 + \Psi(t, x, w_5, \ldots, w_r)$ and by Φ_k^i, $k < -2$, $i \geqslant 0$, the symmetry for which $\Phi_2 = t^i w_{2-k} + \Psi(t, x, w_{2-k+1}, \ldots, w_r)$. Then the following result holds.

THEOREM 2. *Lie algebra of the symmetries of Burgers' equation in the covering* (24) *is additively generated by the functions* Φ_k^i, $k = 0, \pm 1, \pm 2, \ldots$, $i = 0, 1, 2, \ldots$, *and by the functions* $\Phi_2 = \varphi(t, x)$, *where* φ *is an arbitrary solution of the heat equation* $\partial^2 \varphi/\partial x^2 = \partial \varphi/\partial t$.

It should noted that the local symmetries $\psi_k^i = t^i p_{(k)} + \ldots$ correspond to the functions Φ_k^i with $k > 0$ and $i < k$, while all the other generators have no local analogs. The correspondence between the generators of the algebra $\mathrm{Sym}_\tau \mathscr{Y}$ and the 'generating functions' ψ can be obtained from the left-hand column of system (25). Some examples of such correspondence are adduced in Table I.

Table I

A generator of $\mathrm{Sym}_\tau \mathscr{Y}$	A generating function
Φ^0_{-2}	$\psi = (2w_2 - w_3 p_{(0)}) \exp(-\frac{1}{2}w_1)$
Φ^0_{-1}	$\psi = 1 - w_2 p_{(0)} \exp(-\frac{1}{2}w_1)$
Φ^1_1	$\psi = tp_{(1)} - 2w_2 p_{(0)}) \exp(-\frac{1}{2}w_1) + 5$
$\Phi_2 = \varphi(t, x)$, $\varphi_{xx} = \varphi_t$	$\psi = \left(2 \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x} p_{(0)}\right) \exp(-\frac{1}{2}w_1)$

9. Concluding Remarks

The facts above, as well as other examples we know, convince us that the nonlocal symmetries theory has its natural origin in covering theory. Therefore, its further development will greatly depend on progress in the latter.

It is now clear that all remarkable phenomena in soliton-type equations (the Bäcklund transformations, the Cole–Hopf substitution, the Miura transformation, the inverse scattering equations, etc.) may be naturally and rather simply explained in terms of the covering theory. Moreover, the latter one leads to some efficient algorithms for searching these and other similar properties for equations of particular interest. Also, coverings are quite important for conservation laws theory. The relevant techniques (the $\mathscr{C}$-spectral sequence and its generalizations) allow us to treat, along the same lines, such different-looking topics as characteristic classes, the Gelfand–Fucks cohomology, conservation laws, Lagrangian formalism, etc.

We will discuss these topics in more details in a later publication.

References

1. Vinogradov, A. M.: 'Local Symmetries and Conservation Laws', *Acta Appl. Math.* **2** (1984), 21–78 (this issue).
2. Vinogradov, A. M. and Krasilchchik, I. S.: 'A Method of Computing Higher Symmetries of Nonlinear Evolution Equations and Nonlocal Symmetries. *Doklady AN SSSR*, **253** (1980), 1289–1293 (in Russian).
3. Kaptsov, O. V.: 'An Extension of Symmetries of Evolution Equations', *Doklady AN SSSR*, **265** (1982), 1056–1059 (in Russian).
4. Olver, P. J.: 'Evolution Equations Possessing Infinitely Many Symmetries', *J. Math. Phys.*, **18** (1977), 1212–1215.
5. Fushchich, V. I.: 'On Additional Invariance of Vector Fields', *Doklady AN SSSR*, **257** (1981), 1105–1109 (in Russian).
6. Ibragimov, N. Kh. and Shabat, A. B.: On Infinite Algebras of Lie–Bäcklund', *Funct. Anal. Appl.* **14** (1980), 79–80 (in Russian).

7. Konopelchenko, B. G. and Mokhnakov, V. G.: 'On the Group Theoretical Analysis of Differential Equations', *J. Phys. A: Math. Gen.*, **13** (1980), 3113–3124.
8. Vinogradov, A. M.: 'The Category of Nonlinear Differential Equations', in *Equations on manifolds*, 1982, pp. 26–51 (in Russian).
9. Wahlquist, H. D. and Estabrook, F. B.: 'Prolongation Structures of Nonlinear Evolution Equations', *J. Math. Phys.* **16** (1975), 1–7.