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Some New Families of Finite Elements for the Stokes Equations

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Summary. We introduce a way of using the mixed finite element families of Raviart, Thomas and Nedelec [13, 14], and Brezzi et al. [5–7], for constructing stable and optimally convergent discretizations for the Stokes problem.

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1 Introduction

Recently there has been introduced a number of new mixed finite element spaces for the approximation of second order elliptic problems [5–7]. These new methods are of the same type as the methods in the classical Raviart-Thomas-Nedelec families [13, 14], but they differ in that they are more accurate for the same computational cost.

One of the reasons for the increased interest in this field is the success of these methods in certain applications. For some problems in geophysics and semiconductor physics they have been shown to be more efficient than more conventional "displacement methods"; cf. [8, 11, 19] and the references therein.

In this note we will consider the application of these spaces for the approximation of the Stokes equations: Find the velocity $\mathbf{u}$ and the pressure p such that

$$(1.1)\qquad \begin{aligned} -\Delta\mathbf{u}+\nabla p&=\mathbf{f} && \text{in } \Omega,\\ \operatorname{div}\mathbf{u}&=0 && \text{in } \Omega,\\ \mathbf{u}&=\mathbf{u}_0 && \text{on } \partial\Omega, \end{aligned}$$

where for simplicity $\Omega\subset\mathbf{R}^N$, $N=2,3$, is assumed to be a bounded polygonal or polyhedral domain.

Let us assume that $\mathbf{f}\in L^2(\Omega)^N$ and $\mathbf{u}_0\in H^{1/2}(\partial\Omega)^N$ with

$$\int_{\partial\Omega}\mathbf{u}_0\cdot\mathbf{n}\,ds=0$$

so that there is a unique solution to (1.1) (as usual the pressure is normalized to have a zero mean value over Ω).

When deriving optimal L^2-estimates for the velocity we will need the usual regularity assumption for the solution to (1.1) when $\mathbf{u}_0=\mathbf{0}$

$$\|\mathbf{u}\|_2+\|p\|_1 \leqq C\|\mathbf{f}\|_0. \tag{1.2}$$

Below we will show that the spaces of [5–7, 13, 14] can be used as building blocks for producing "Taylor-Hood" type methods for the Stokes problem, i.e. methods with a continuous approximation for the pressure.

Let us here also mention that we in [16] used the same basic finite element spaces for the construction of a family for the discretization of the equations of linear elasticity. The estimates of [16] are uniformly valid with respect to the Poisson ratio, and hence a trivial notational change in the constitutive equation gives yet another family for discretizing Stokes problem.

2 The Finite Element Families

We will give the construction for the triangular and tetrahedral families of [5, 7]. From the presentation it will be evident that the same construction can be done for the triangular and tetrahedral Raviart-Thomas-Nedelec elements [13, 14], and also for all the rectangular and brick elements of [5–7, 13, 14] except the lowest order methods with a piecewise constant approximation for the velocity. For the rectangular and brick elements of [5–7] all estimates obtained are, however, not optimal.

In order to discretize (1.1) we write it as the system

$$\begin{aligned} \boldsymbol{\sigma}-\nabla \mathbf{u}&=\mathbf{0} && \text{in } \Omega,\\ -\operatorname{div} \boldsymbol{\sigma}+\nabla p&=\mathbf{f} && \text{in } \Omega,\\ \operatorname{div} \mathbf{u}&=0 && \text{in } \Omega,\\ \mathbf{u}&=\mathbf{u}_0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\sigma}=\{\sigma_{ij}\}$, $\sigma_{ij}=\partial_j u_i$, $i, j=1, \ldots, N$, and

$$(\operatorname{div} \boldsymbol{\sigma})_i=\sum_{j=1}^{N} \partial_j \sigma_{ij}, \qquad i=1, \ldots, N,$$

$$\operatorname{div} \mathbf{u}=\sum_{j=1}^{N} \partial_j u_j.$$

Let $\mathscr{C}_h$ be the partitioning of $\bar{\Omega}$ into closed triangles or tetrahedrons. The elements of $\mathscr{C}_h$ are assumed to be regular in the usual sense (cf. [10]). The quasiuniformity of the mesh will not be assumed.

Our finite element approximation of (2.1) is now defined as: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, p_h) \in \mathbf{H}_h \times \mathbf{V}_h \times P_h$ such that

$$(2.2)\qquad \begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, \mathbf{u}_h) &= \langle \mathbf{u}_0, \boldsymbol{\tau} \cdot \mathbf{n} \rangle, && \boldsymbol{\tau} \in \mathbf{H}_h, \\ -(\operatorname{div} \boldsymbol{\sigma}_h, \mathbf{v}) + (\nabla p_h, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), && \mathbf{v} \in \mathbf{V}_h, \\ (\mathbf{u}_h, \nabla q) &= 0, && q \in P_h, \end{aligned}$$

where for the index $k \geqq 1$ the finite element spaces are defined through

$$(2.3\text{a})\qquad \mathbf{H}_h = \{\boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}; \Omega) \mid \boldsymbol{\tau}_{|K} \in P_k(K)^{N \times N},\ K \in \mathscr{C}_h\},$$

$$(2.3\text{b})\qquad \mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^N \mid \mathbf{v}_{|K} \in P_{k-1}(K)^N,\ K \in \mathscr{C}_h\},$$

$$(2.3\text{c})\qquad P_h = \{p \in C(\Omega) \cap L_0^2(\Omega) \mid p_{|K} \in P_k(K),\ K \in \mathscr{C}_h\}.$$

As usual $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)^N$ or $L^2(\Omega)^{N \times N}$, whereas $\langle \cdot, \cdot \rangle$ stands for that in $L^2(\partial\Omega)^N$. $L_0^2(\Omega)$ stands for the subspace of $L^2(\Omega)$ consisting of functions with zero mean value over Ω. We point out that for the functions in $\mathbf{H}_h$ the assumption $\boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}; \Omega)$, where

$$\mathbf{H}(\operatorname{div}; \Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^{N \times N} \mid \operatorname{div} \boldsymbol{\tau} \in L^2(\Omega)^N\},$$

is equivalent to the continuity of the normal component of $\boldsymbol{\tau}$ along inter element boundaries. We also note that the spaces (2.3a), (2.3b) are N copies of the spaces of Brezzi et al. [5, 7].

The finite element method (2.3) gives a poor approximation to the velocity, but it is possible to construct a considerably better approximation by various *Element-by-Element Postprocessing Techniques* [1, 5, 7, 17]. Here we will consider a slight variation of a method introduced by us in [17]. We define

$$(2.4)\qquad \mathbf{V}_h^* = \{\mathbf{v} \in L^2(\Omega)^N \mid \mathbf{v}_{|K} \in P_{k+1}(K)^N,\ K \in \mathscr{C}_h\},$$

and calculate a new approximation $\mathbf{u}_h^* \in \mathbf{V}_h^*$ to the velocity separately on each $K \in \mathscr{C}_h$ by solving the following problem

$$(2.5)\qquad \begin{aligned} (\nabla \mathbf{u}_h^*, \nabla \mathbf{v})_K &= (\boldsymbol{\sigma}_h, \nabla \mathbf{v})_K, \qquad \mathbf{v} \in (I - Q_h) \mathbf{V}_{h|K}^*, \\ Q_h \mathbf{u}_{h|K}^* &= \mathbf{u}_{h|K}, \end{aligned}$$

where Q_h: $L^2(\Omega)^N \to \mathbf{V}_h^*$ denotes the L^2-projection and $(\cdot, \cdot)_K$ stands for the inner product in $L^2(K)^{N \times N}$.

Remark. We note that the postprocessing is performed separately for each component of $\mathbf{u}$ and that the local stiffness matrices to invert on each element are the same for all components. □

The analysis technique to be utilized in this paper is the one based on mesh dependent norms, first introduced in [2] for certain mixed approximations of the biharmonic equation and later adapted for mixed methods for second order problems [15]. The norms to be used are the following

$$\|\sigma\|_{0,h}^2 = \|\sigma\|_0^2 + \sum_{T\in\Gamma_h} h_T \int_T |\sigma\cdot\mathbf{n}|^2\,ds, \tag{2.6}$$

for $\sigma\in L^2(\Omega)^{N\times N}$ with $\sigma\cdot\mathbf{n}\in L^2(T)^N$, $T\in\Gamma_h$,

$$\|\mathbf{u}\|_{1,h}^2 = \sum_{K\in\mathscr{C}_h} \|\nabla\mathbf{u}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T^{-1} \int_T |[\mathbf{u}]|^2\,ds, \tag{2.7}$$

for $\mathbf{u}\in L^2(\Omega)^N$ with $\mathbf{u}_{|K}\in H^1(K)^N$, $K\in\mathscr{C}_h$,

and

$$\|\sigma,\mathbf{u}\|_h^2 = \|\sigma\|_{0,h}^2 + \|\mathbf{u}\|_{1,h}^2.$$

Here T stands for a side of an element, $\mathbf{n}$ is the unit normal to T and Γ_h denotes the collection of the sides of the elements of $\mathscr{C}_h$. For element sides in the interior of Ω $[\mathbf{u}]$ denotes the jump in $\mathbf{u}$ whereas for sides on $\partial\Omega$ it denotes the value of $\mathbf{u}$. We note that an integration by part on each $K\in\mathscr{C}_h$ yields

$$(\operatorname{div}\sigma,\mathbf{u}) \leqq \|\sigma\|_{0,h}\,\|\mathbf{u}\|_{1,h} \tag{2.8}$$

for each σ and $\mathbf{u}$ for which the corresponding norms are finite. Also, simple scaling arguments (cf. [2]) give the estimates

$$\inf_{\tau\in\mathbf{H}_h} \|\sigma-\tau\|_{0,h} \leqq C h^r |\sigma|_r, \qquad 1/2<r\leqq k+1, \tag{2.9}$$

$$\inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u}-\mathbf{v}\|_{1,h} \leqq C h^{s-1} |\mathbf{u}|_s, \qquad 1\leqq s\leqq k, \tag{2.10}$$

and

$$\|\sigma\|_0 \leqq \|\sigma\|_{0,h} \leqq C\|\sigma\|_0, \qquad \sigma\in\mathbf{H}_h. \tag{2.11}$$

In order to see that the problem (2.2) can be analyzed with Brezzi's theory of saddle-point problems [3] we write it in the following way: Find $(\sigma_h,\mathbf{u}_h,p_h)\in\mathbf{H}_h\times\mathbf{V}_h\times P_h$ such that

$$\begin{aligned} a(\sigma_h,\mathbf{u}_h;\tau,\mathbf{v}) + b(\tau,\mathbf{v};p_h) &= (\mathbf{f},\mathbf{v}) + \langle\mathbf{u}_0,\tau\cdot\mathbf{n}\rangle, \qquad (\tau,\mathbf{v})\in\mathbf{H}_h\times\mathbf{V}_h \\ b(\sigma_h,\mathbf{u}_h;q) &= 0, \qquad q\in P_h, \end{aligned} \tag{2.12}$$

where

$$a(\sigma,\mathbf{u};\tau,\mathbf{v}) = (\sigma,\tau) + (\operatorname{div}\tau,\mathbf{u}) - (\operatorname{div}\sigma,\mathbf{v})$$

and

$$b(\tau,\mathbf{v};p) = (\mathbf{v},\nabla p).$$

Hence, the analysis now consist of the verification of the two following lemmas.

Lemma 2.1. *There is a positive constant C such that*

$$\sup_{\substack{(\sigma,\mathbf{u})\in \mathbf{Z}_h\\ (\sigma,\mathbf{u})\neq(0,0)}} \frac{a(\sigma,\mathbf{u};\tau,\mathbf{v})}{\|\sigma,\mathbf{u}\|_h} \geqq C\|\tau,\mathbf{v}\|_h, \qquad (\tau,\mathbf{v})\in \mathbf{Z}_h,$$

and

$$\sup_{\substack{(\tau,\mathbf{v})\in \mathbf{Z}_h\\ (\tau,\mathbf{v})\neq(0,0)}} \frac{a(\sigma,\mathbf{u};\tau,\mathbf{v})}{\|\tau,\mathbf{v}\|_h} \geqq C\|\sigma,\mathbf{u}\|_h, \qquad (\sigma,\mathbf{u})\in \mathbf{Z}_h,$$

where

$$\mathbf{Z}_h = \{(\sigma,\mathbf{u})\in \mathbf{H}_h\times \mathbf{V}_h \,|\, b(\sigma,\mathbf{u};q)=0,\ q\in P_h\}.$$

Proof. The pair $(\mathbf{H}_h, \mathbf{V}_h)$ is well known to constitute a stable discretization of the (in this case vector) Laplace operator; cf. [7, 5].

Using our mesh dependent norms the stability can be stated as follows: There exist a constant C_1 such that for every $\mathbf{v}\in\mathbf{V}_h$ there is a $\gamma\in\mathbf{H}_h$ satisfying

$$(\operatorname{div}\gamma,\mathbf{v})\geqq C_1\|\mathbf{v}\|_{1,h}^2$$

and

$$\|\gamma\|_{0,h}\leqq \|\mathbf{v}\|_{1,h}.$$

Now, let $(\tau,\mathbf{v})\in\mathbf{Z}_h$, and choose $(\sigma,\mathbf{u})=(\tau-\delta\gamma,\mathbf{v})$ with γ be as above. This gives

$$\begin{aligned} a(\sigma,\mathbf{u};\tau,\mathbf{v}) &= a(\tau-\delta\gamma,\mathbf{v};\tau,\mathbf{v}) = \|\tau\|_0^2 - \delta(\gamma,\tau) + \delta(\operatorname{div}\gamma,\mathbf{v})\\ &\geqq \|\tau\|_0^2 + \delta C_1\|\mathbf{v}\|_{1,h}^2 - \delta\|\gamma\|_0\|\tau\|_0 \geqq \|\tau\|_0^2 + \delta C_1\|\mathbf{v}\|_{1,h}^2 - \delta\|\mathbf{v}\|_{1,h}\|\tau\|_0\\ &\geqq \left(1-\frac{\delta}{2\varepsilon}\right)\|\tau\|_0^2 + \delta\left(C_1-\frac{\varepsilon}{2}\right)\|\mathbf{v}\|_{1,h}^2\\ &\geqq C(\|\tau\|_0^2+\|\mathbf{v}\|_{1,h}^2), \end{aligned}$$

when choosing $0<\delta<2\varepsilon<4C_1$.

Since

$$\|\sigma\|_{0,h}+\|\mathbf{u}\|_{1,h}\leqq\|\tau\|_{0,h}+(1+\delta)\|\mathbf{v}\|_{1,h}$$

and $(\sigma,\mathbf{u})\in\mathbf{Z}_h$, the first asserted estimate follows from above and (2.11).

The second estimate is proven similarly. □

Lemma 2.2. *There is a positive constant C such that*

$$\sup_{\substack{(\sigma,\mathbf{u})\in \mathbf{H}_h\times\mathbf{V}_h\\ (\sigma,\mathbf{u})\neq(0,0)}} \frac{b(\sigma,\mathbf{u};q)}{\|\sigma,\mathbf{u}\|_h} \geqq C\|q\|_0, \qquad q\in P_h.$$

Proof. Since for $q\in P_h$ it holds $\nabla q\in\mathbf{V}_h$, we can choose $\mathbf{u}$ through

$$\mathbf{u}_{|K}=h_K^2\,\nabla q_{|K}, \qquad K\in\mathscr{C}_h.$$

With $\boldsymbol{\sigma}=\mathbf{0}$ this gives

$$b(\boldsymbol{\sigma}, \mathbf{u}; q) = \sum_{K \in \mathscr{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 .$$

Scaling arguments yield

$$\|\mathbf{u}\|_{1,h}^2 \leqq C \sum_{K \in \mathscr{C}_h} h_K^{-2} \|\mathbf{u}\|_{0,K}^2 = C \sum_{K \in \mathscr{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 .$$

By combining the above estimates we get

$$\sup_{\substack{(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}_h \times \mathbf{V}_h \\ (\boldsymbol{\sigma}, \mathbf{u}) \neq (0,0)}} \frac{b(\boldsymbol{\sigma}, \mathbf{u}; q)}{\|\boldsymbol{\sigma}, \mathbf{u}\|_h} \geqq C \Big(\sum_{K \in \mathscr{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \Big)^{1/2} .$$

The assertion now follows from an argument by Verfürth [18, Proposition 3.3]. We remark that by using the above locally weighted norm for the pressure, the quasiuniformity assumption (i.e. $h_K \geqq C h \quad \forall K \in \mathscr{C}_h$) of [18] can be avoided. □

We now get the following error estimates.

Theorem 2.1. *For the solution* $\mathbf{u}_h^* \in \mathbf{V}_h^*$ *and* $p_h \in P_h$ *to* (2.5) *and* (2.2) *we have*

$$\|\mathbf{u} - \mathbf{u}_h^*\|_{1,h} + \|p - p_h\|_0 \leqq C h^{k+1} (|\mathbf{u}|_{k+2} + |p|_{k+1}). \tag{2.13}$$

Moreover, if (1.2) *is valid we have*

$$\|\mathbf{u} - \mathbf{u}_h^*\|_0 \leqq C h^{k+2} (|\mathbf{u}|_{k+2} + |p|_{k+1}) \quad \text{for } k \geqq 2, \tag{2.14a}$$

$$\|\mathbf{u} - \mathbf{u}_h^*\|_0 \leqq C h^2 (|\mathbf{u}|_3 + |p|_2) \quad \text{for } k = 1. \tag{2.14b}$$

If in addition $\mathbf{f} \in \mathbf{V}_h$, *then* (2.14a) *is also valid for* $k=1$.

Proof. Let us split the proof into four steps.

Step 1. Let us prove the estimate

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,h} + \|\mathbf{u}_h - Q_h \mathbf{u}\|_{1,h} + \|p - p_h\|_0 \leqq C h^{k+1} (|\boldsymbol{\sigma}|_{k+1} + |p|_{k+1}). \tag{2.15}$$

Using [3, Proposition 2.1], Lemmas 2.1 and 2.2 imply the existence of a triple $(\boldsymbol{\tau}, \mathbf{v}, q) \in \mathbf{H}_h \times \mathbf{V}_h \times P_h$ such that

$$\|\boldsymbol{\tau}\|_{0,h} + \|\mathbf{v}\|_{1,h} + \|q\|_0 \leqq C \tag{2.16}$$

and

$$\begin{aligned} \|\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}\|_{0,h} &+ \|\mathbf{u}_h - Q_h \mathbf{u}\|_{1,h} + \|p_h - \tilde{p}\|_0 \\ &\leqq a(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}, \mathbf{u}_h - Q_h \mathbf{u}; \boldsymbol{\tau}, \mathbf{v}) + b(\boldsymbol{\tau}, \mathbf{v}; p_h - \tilde{p}) + b(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}, \mathbf{u}_h - Q_h \mathbf{u}; q), \end{aligned} \tag{2.17}$$

where $\tilde{\sigma}$ and $\tilde{p}$ are the interpolants to σ and p, respectively. Using (2.1) and (2.2) we get

$$\begin{aligned}(2.18)\quad & a(\sigma_h-\tilde{\sigma}, \mathbf{u}_h-Q_h\mathbf{u}; \tau, \mathbf{v})+b(\tau, \mathbf{v}; p_h-\tilde{p})+b(\sigma_h-\tilde{\sigma}, \mathbf{u}_h-Q_h\mathbf{u}; q)\\ &=a(\sigma-\tilde{\sigma}, \mathbf{u}-Q_h\mathbf{u}; \tau, \mathbf{v})+b(\tau, \mathbf{v}; p-\tilde{p})+b(\sigma-\tilde{\sigma}, \mathbf{u}-Q_h\mathbf{u}; q)\\ &=(\sigma-\tilde{\sigma}, \tau)+(\mathbf{u}-Q_h\mathbf{u}, \operatorname{div}\tau)-(\mathbf{v}, \operatorname{div}(\sigma-\tilde{\sigma}))\\ &\quad+(\mathbf{v}, \nabla(p-\tilde{p}))+(\mathbf{u}-Q_h\mathbf{u}, \nabla q).\end{aligned}$$

Let us now estimate the terms on the right hand side of (2.18). Since $\nabla q\in\mathbf{V}_h$ and $\operatorname{div}\tau\in\mathbf{V}_h$ the definition of Q_h yields

$$(2.19)\qquad (\mathbf{u}-Q_h\mathbf{u}, \operatorname{div}\tau)=0$$

and

$$(2.20)\qquad (\mathbf{u}-Q_h\mathbf{u}, \nabla q)=0.$$

Further, standard interpolation theory gives

$$(2.21)\qquad (\sigma-\tilde{\sigma}, \tau)\leqq\|\sigma-\tilde{\sigma}\|_0\,\|\tau\|_0\leqq C\,h^{k+1}|\sigma|_{k+1},$$

and using (2.8) and (2.9) we get

$$(2.22)\qquad (\mathbf{v}, \operatorname{div}(\sigma-\tilde{\sigma}))\leqq C\,\|\sigma-\tilde{\sigma}\|_{0,h}\,\|\mathbf{v}\|_{1,h}\leqq C\,h^{k+1}|\sigma|_{k+1}.$$

Finally, the last term is estimated as follows

$$\begin{aligned}(2.23)\quad (\mathbf{v}, \nabla(p-\tilde{p}))&=-\sum_{K\in\mathscr{C}_h}(\operatorname{div}\mathbf{v}, p-\tilde{p})_K+\sum_{T\in\Gamma_h}\int_T([\mathbf{v}\cdot\mathbf{n}])(p-\tilde{p})\,ds\\ &\leqq\Big(\sum_{K\in\mathscr{C}_h}\|\operatorname{div}\mathbf{v}\|_{0,K}^2+\sum_{T\in\Gamma_h}h_T^{-1}\int_T|[\mathbf{v}\cdot\mathbf{n}]|^2\,ds\Big)^{1/2}\\ &\quad\cdot\Big(\|p-\tilde{p}\|_0^2+\sum_{T\in\Gamma_h}h_T\int_T|p-\tilde{p}|^2\,ds\Big)^{1/2}\\ &\leqq C\,\|\mathbf{v}\|_{1,h}\Big(\|p-\tilde{p}\|_0^2+\sum_{T\in\Gamma_h}h_T\int_T|p-\tilde{p}|^2\,ds\Big)^{1/2}\\ &\leqq C\,h^{k+1}|p|_{k+1},\end{aligned}$$

where we in the last step used an interpolation estimate easily obtained by scaling; cf. [2, Lemma 3].

By collecting (2.17)–(2.23), using the triangle inequality and (2.9) we get (2.15) which contains the asserted estimate for the pressure.

Step 2. Let us derive an estimate for $\|Q_h(\mathbf{u}_h^*-\mathbf{u})\|_0$.

We first note that $Q_h\mathbf{u}_h^*=\mathbf{u}_h$.

Suppose that (1.2) is valid so that the solution $(\mathbf{z},\gamma,q)\in H_0^1(\Omega)^N\times L^2(\Omega)^{N\times N}\times L_0^2(\Omega)$ to

$$\begin{aligned} \gamma-\nabla\mathbf{z}&=\mathbf{0} && \text{in } \Omega,\\ -\operatorname{div}\gamma+\nabla q&=\mathbf{u}_h-Q_h\mathbf{u} && \text{in } \Omega,\\ \operatorname{div}\mathbf{z}&=0 && \text{in } \Omega,\\ \mathbf{z}&=\mathbf{0} && \text{on } \partial\Omega, \end{aligned}\tag{2.24}$$

satisfies

$$\|\mathbf{z}\|_2+\|\gamma\|_1+\|q\|_1\leqq C\|\mathbf{u}_h-Q_h\mathbf{u}\|_0.\tag{2.25}$$

Let $\tilde{\gamma}$ and $\tilde{q}$ be the interpolants to γ and q. Using the relations

$$(\operatorname{div}\tilde{\gamma},\mathbf{u})=(\operatorname{div}\tilde{\gamma},Q_h\mathbf{u})\quad\text{and}\quad(\nabla\tilde{q},\mathbf{u})=(\nabla\tilde{q},Q_h\mathbf{u})$$

standard arguments gives

$$\begin{aligned}(2.26)\quad &\|\mathbf{u}_h-Q_h\mathbf{u}\|_0^2\\ &=(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\gamma}-\gamma)+(\operatorname{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathbf{z}-Q_h\mathbf{z})-(\nabla(p-p_h),\mathbf{z}-Q_h\mathbf{z})\\ &\quad+(\operatorname{div}\gamma,Q_h\mathbf{u}-\mathbf{u}_h)-(\operatorname{div}\tilde{\gamma},\mathbf{u}-\mathbf{u}_h)-(\nabla q,Q_h\mathbf{u}-\mathbf{u}_h)+(\nabla\tilde{q},\mathbf{u}-\mathbf{u}_h)\\ &=(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\gamma}-\gamma)+(\operatorname{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathbf{z}-Q_h\mathbf{z})-(\nabla(p-p_h),\mathbf{z}-Q_h\mathbf{z})\\ &\quad+(\operatorname{div}(\gamma-\tilde{\gamma}),Q_h\mathbf{u}-\mathbf{u}_h)-(\nabla(q-\tilde{q}),Q_h\mathbf{u}-\mathbf{u}_h)\\ &\leqq C(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,h}+\|\mathbf{u}_h-Q_h\mathbf{u}\|_{1,h}+\|p-p_h\|_0+(\sum_{T\in\Gamma_h}h_T\int_T|p-p_h|^2\,ds)^{1/2})\\ &\quad\cdot(\|\gamma-\tilde{\gamma}\|_{0,h}+\|\mathbf{z}-Q_h\mathbf{z}\|_{1,h}+\|q-\tilde{q}\|_0+(\sum_{T\in\Gamma_h}h_T\int_T|q-\tilde{q}|^2\,ds)^{1/2})\\ &\leqq C h^{k+1}(|\boldsymbol{\sigma}|_{k+1}+|p|_{k+1})\\ &\quad\cdot(\|\gamma-\tilde{\gamma}\|_{0,h}+\|\mathbf{z}-Q_h\mathbf{z}\|_{1,h}+\|q-\tilde{q}\|_0+(\sum_{T\in\Gamma_h}h_T\int_T|q-\tilde{q}|^2\,ds)^{1/2}).\end{aligned}$$

For $k\geqq 2$, (2.9), (2.10) and the interpolation estimate of [2, Lemma 3] give

$$\begin{aligned}(2.27)\quad &(\|\gamma-\tilde{\gamma}\|_{0,h}+\|\mathbf{z}-Q_h\mathbf{z}\|_{1,h}+\|q-\tilde{q}\|_0+(\sum_{T\in\Gamma_h}h_T\int_T|q-\tilde{q}|^2\,ds)^{1/2})\\ &\leqq C h(|\gamma|_1+|q|_1+|\mathbf{z}|_2).\end{aligned}$$

Recalling that $Q_h\mathbf{u}_h^*=\mathbf{u}_h$, (2.25)–(2.27) give

$$\|Q_h(\mathbf{u}_h^*-\mathbf{u})\|_0\leqq C h^{k+2}(|\boldsymbol{\sigma}|_{k+1}+|p|_{k+1})\qquad\text{for } k\geqq 2.\tag{2.28}$$

For the case $k=1$ and $\mathbf{f}\in\mathbf{V}_h$ we have

$$(\operatorname{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\nabla(p-p_h),\mathbf{z}-Q_h\mathbf{z})=(\mathbf{f}-\operatorname{div}\boldsymbol{\sigma}_h+\nabla p_h,\mathbf{z}-Q_h\mathbf{z})=0,$$

and (2.28) is still valid.

If $\mathbf{f} \notin \mathbf{V}_h$ we only get

$$\|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0 \leqq C h^2(|\boldsymbol{\sigma}|_2 + |p|_2) \quad \text{for } k=1. \tag{2.29}$$

Step 3. Next we estimate $\|(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_0$ and $\|(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{1,h}$.

First, since $(I-Q_{h|K})\mathbf{v}_{|K} = \mathbf{0}$ for $\mathbf{v} \in P_0(K)^N$, we have

$$\|(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{0,K} \leqq C h_K |(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \tag{2.30}$$

Next we write

$$\begin{aligned} |(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}^2 &= (\nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}), \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &= (\nabla(\mathbf{u}_h^* - \mathbf{u}), \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &\quad - (\nabla Q_h(\mathbf{u}_h^* - \mathbf{u}), \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K. \end{aligned} \tag{2.31}$$

Now, (2.1) and (2.5) give

$$\begin{aligned} (\nabla(\mathbf{u}_h^* - \mathbf{u}), \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K &= (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &\leqq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K} |(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \end{aligned} \tag{2.32}$$

Further, a local inverse inequality yields

$$\begin{aligned} &(\nabla Q_h(\mathbf{u}_h^* - \mathbf{u}), \nabla(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &\quad \leqq |Q_h(\mathbf{u}_h^* - \mathbf{u})|_{1,K} |(I-Q_h)(\mathbf{u}_h^* - \mathbf{u}))|_{1,K} \\ &\quad \leqq C h_K^{-1} \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_{0,K} |(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \end{aligned} \tag{2.33}$$

Combining (2.30)–(2.33) and summing over all $K \in \mathscr{C}_h$ gives

$$\|(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_0 \leqq C(h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0). \tag{2.34}$$

Also, from (2.31), (2.32), (2.33) and the inequality (proven by scaling)

$$h_K^{-1} \int_{\partial K} |(I-Q_h)\mathbf{v}|^2 \, ds \leqq C|\nabla(I-Q_h)\mathbf{v}|_{0,K}^2, \qquad \mathbf{v} \in \mathbf{V}_{h|K}^*,$$

we get

$$\|(I-Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{1,h} \leqq C(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_{1,h}). \tag{2.35}$$

Step 4. By combining (2.28), (2.29), (2.34) and (2.15) we obtain the asserted estimates for $\|\mathbf{u} - \mathbf{u}_h^*\|_0$, and that for $\|\mathbf{u} - \mathbf{u}_h^*\|_{1,h}$ follows from (2.15) and (2.35). □

3 The Implementation by Hybridization

Let us close the paper by briefly considering the solution of the discrete system (2.2). The method advocated in [1, 5–7] is the Fraijs de Veubeke hybridization technique. In that the condition $\operatorname{div} \boldsymbol{\sigma} \in L^2(\Omega)^N$ for $\boldsymbol{\sigma} \in \mathbf{H}_h$, which is equivalent to the continuity of $\boldsymbol{\sigma}\cdot\mathbf{n}$ along interelement boundaries, is enforced by introducing Lagrange multipliers. More precisely, define

$$\mathbf{M}_h=\{\mathbf{m}\,|\,\mathbf{m}_{|T}\in P_k(T)^N,\ T\in\Gamma_h,\ \mathbf{m}_{|T}=\mathbf{0}\text{ for }T\subset\partial\Omega\},$$

and let $\hat{\mathbf{H}}_h$ be as in (2.3a) with the condition $\operatorname{div}\boldsymbol{\sigma}\in L^2(\Omega)^N$ dropped. The modified discretization now reads: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h,\mathbf{m}_h,p_h)\in\hat{\mathbf{H}}_h\times\mathbf{V}_h\times\mathbf{M}_h\times P_h$ such that

$$(3.1)\qquad \begin{aligned}
(\boldsymbol{\sigma}_h,\boldsymbol{\tau})+\sum_{K\in\mathscr{C}_h}\{(\operatorname{div}\boldsymbol{\tau},\mathbf{u}_h)_K-\langle\boldsymbol{\tau}_h\cdot\mathbf{n}_K,\mathbf{m}_h\rangle_{\partial K}\}&=\langle\boldsymbol{\tau}\cdot\mathbf{n},\mathbf{u}_0\rangle, && \boldsymbol{\tau}\in\hat{\mathbf{H}}_h,\\
-\sum_{K\in\mathscr{C}_h}(\operatorname{div}\boldsymbol{\sigma}_h,\mathbf{v})_K+(\nabla p_h,\mathbf{u})&=(\mathbf{f},\mathbf{v}), && \mathbf{v}\in\mathbf{V}_h,\\
(\mathbf{u}_h,\nabla q)&=0, && q\in P_h,\\
\sum_{K\in\mathscr{C}_h}\langle\boldsymbol{\sigma}_h\cdot\mathbf{n}_K,\mathbf{l}\rangle_{\partial K}&=0, && \mathbf{l}\in\mathbf{M}_h.
\end{aligned}$$

Above $\mathbf{n}_K$ stands for the unit outward normal to ∂K.

Clearly, the three first components of (2.2) and (3.1) coincide. The physical meaning of the new unknown $\mathbf{m}_h$ is that of an approximation to the velocity $\mathbf{u}$ along the inter element boundaries.

For $\mathbf{m}_h$ we have the following error estimate which is needed if any of the postprocessing methods of [1, 5, 7] is used.

Lemma 3.1. *For $T\in\Gamma_h$ let I_T: $L^2(T)^N\to\mathbf{M}_{h|T}$ be the L^2-projection. Then we have*

$$\|\mathbf{m}_h-I_T\mathbf{u}\|_{0,T}\leq C\{h_K^{1/2}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,K}+h_K^{-1/2}\|Q_h\mathbf{u}-\mathbf{u}_h\|_{0,K}\}.$$

for $T\subset\partial K$, $K\in\mathscr{C}_h$. □

For the proof of this result we refer to [7, Lemma 4.1].

The algebraic equations generated by (3.1) are of the form

$$\begin{aligned}
\mathscr{A}\Sigma+\mathscr{B}U-\mathscr{C}M&=0,\\
-\mathscr{B}^T\Sigma+\mathscr{D}P&=F,\\
\mathscr{D}^TU&=0,\\
\mathscr{C}^T\Sigma&=0,
\end{aligned}$$

where Σ, U, M and P are the vectors for the degrees of freedom for $\boldsymbol{\sigma}_h$, $\mathbf{u}_h$, $\mathbf{m}_h$ and p_h, respectively.

Now the matrix $\mathscr{A}$ is positively definite and block diagonal. Hence, Σ can be eliminated separately on each element which gives the system

$$\begin{aligned}
\hat{\mathscr{A}}\hat{U}+\hat{\mathscr{D}}P&=\hat{F},\\
\hat{\mathscr{D}}^T\hat{U}&=0,
\end{aligned}$$

where $\hat{U}^T=(U, M)^T$, $\hat{\mathscr{D}}^T=(\mathscr{D}^T, 0)$, $\hat{F}^T=(F, 0)^T$ and

$$\hat{\mathscr{A}}=\begin{pmatrix} \mathscr{B}^T\mathscr{A}^{-1}\mathscr{B}^T & -\mathscr{B}^T\mathscr{A}^{-1}\mathscr{C} \\ -\mathscr{C}^T\mathscr{A}^{-1}\mathscr{B} & \mathscr{C}^T\mathscr{A}^{-1}\mathscr{C} \end{pmatrix},$$

which is positively definite. Hence the system is of the same form as for more conventional Talor-Hood type discretizations of Stokes problem. Thus any of the methods for solving linear systems of this kind (cf. [12]) can be utilized.

Further, $\mathscr{B}^T\mathscr{A}^{-1}\mathscr{B}^T$ is also positively definite and block diagonal, and hence U can be eliminated on each element separately. This results in a system of the form

$$\mathscr{E}M+\mathscr{G}P=G,$$
$$\mathscr{G}^T M-\mathscr{H}P=H,$$

where both $\mathscr{E}$ and $\mathscr{H}$ are symmetric and positively definite. Now the system has the same form as that of some recent stabilized mixed methods [4, 9].

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