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On the Integral Manifolds of the N-Body Problem

H.E. Cabral (Rio de Janeiro)*

Abstract. Here we make a topological study of the map $I=(E,J)$, where E is the energy and J is the angular momentum of the n-body problem in 3-space. Part of the bifurcation set of I is characterized and some topological information is given on the integral manifolds of negative energy and zero angular momentum.

§ 0. Introduction

In [6] Smale makes a complete study of the topology of the map $I=(E,J)$ where E is the energy and J the angular momentum of the planar n-body problem of celestial mechanics. Explicit descriptions of the topological type of the integral manifolds $I_{cp}=I^{-1}(c,p)$ are given when $n=3$. By using different methods Easton [2] also finds the topological structure of the I_{cp}'s for $n=3$ in the planar case.

Questions of similar nature were already asked by Birkhoff and Wintner concerning the 3-body problem in space of three dimensions. Not much is known in this case and in this paper we give some information on the map I of the n-body problem in 3-space along these lines.

Here the topological characterization of the integral manifolds becomes very hard due to the existence of points x in the configuration space for which the induced linear map J_x is not surjective (see observations in § 3). We obtain partial results in this direction which are stated in Theorems 2 and 3 below. As to the bifurcation question, Theorem 4 gives more complete information.

Recall that we are given n positive real numbers, the masses $m_1, \ldots, m_n$ and consider the configuration space of the n-body problem in 3-space, center of mass at the origin, as the subset $M-\Delta$ of the linear space

$$M=\{(x_1,\ldots,x_n)\in(\mathbb{R}^3)^n \mid \sum m_i x_i=0\},$$

where $\Delta=\bigcup \Delta_{ij}$, $i<j$ and $\Delta_{ij}=\{x\in M \mid x_i=x_j\}$.

The energy E and the angular momentum J are then defined on $(M-\Delta)\times M$ respectively by $E(x,v)=K(v)+V(x)$ and $J(x,v)=\sum m_i x_i\times v_i$, where $K(v)=\frac{1}{2}\sum m_i|v_i|^2$ is the kinetic energy and $V(x)=-\sum_{i<j}\frac{m_i m_j}{|x_i-x_j|}$

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the potential energy. We have an inner product on M given by $K(x,v)=\frac{1}{2}\sum m_i x_i \cdot v_i$ and we denote by S_K the unit sphere of M with respect to the induced norm, $|x|_K=K(x,x)^{\frac{1}{2}}$.

We consider the map $I=(E,J)\colon (M-\Delta)\times M \to \mathbb{R}\times\mathbb{R}^3$ defined by $I(x,v)=(E(x,v),J(x,v))$ and call the spaces $I_{cp}=I^{-1}(c,p)$ the integral manifolds of the n-body problem.

Given a smooth (C^∞) map $f\colon M\to N$ of smooth manifolds, we denote by $\Sigma'(f)$ the set of its critical values and by $\Sigma(f)$ its bifurcation set. We recall that $y\in\Sigma'(f)$ means that for some $x\in f^{-1}(y)$ the derivative $Df(x)$ is not surjective, while $y\notin\Sigma(f)$ means that $f^{-1}(y)$ is a manifold and there exist an open neighborhood U of y in N and a smooth map $g\colon f^{-1}(U)\to f^{-1}(y)$ such that the map $h\colon f^{-1}(U)\to U\times f^{-1}(y)$ is a diffeomorphism, where $h(x)=(f(x),g(x))$. If $y\notin\Sigma(f)$ we say that f is locally trivial at y; in this case, $f=\pi\circ h$ where π is the projection on the first factor, and so it follows not only that y is a regular value of f, i.e., $y\notin\Sigma'(f)$ but also that on U the fiber $f^{-1}(z)$, $z\in U$ does not change the diffeomorphic type.

We now state our results on the topology of the map I.

For the n-body problem in 3-space with arbitrary positive masses $m_1,\dots,m_n$, we have the following theorems.

Theorem 1. (a) *The maps φ and ψ defined below are diffeomorphisms. Therefore $\Sigma(E)=0$*

$$\varphi\colon E^{-1}(\mathbb{R}^-)\to\mathbb{R}^-\times(S_K-\Delta)\times M,\qquad \varphi(x,v)=\left(E(x,v),\frac{x}{|x|_K},v\right)$$

$$\psi\colon E^{-1}(\mathbb{R}^+)\to\mathbb{R}^+\times(S_K-\Delta)\times(M-0),\quad \psi(x,v)=\left(E(x,v),\frac{x}{|x|_K},|x|_K v\right).$$

(b) $\Sigma(J)=0$.

Theorem 2. (a) *I_{c0}, $c<0$ has the same homotopy type of $S_K-\Delta$.*

(b) *In case $n=3$, the homology groups of $S_K-\Delta$ are $\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ and $\mathbb{Z}\oplus\mathbb{Z}$ in dimensions 0, 2 and 4 respectively. The homology is 0 otherwise.*

(c) *For $n=3$ and $c<0$, $\tilde{I}_{c0}=I_{c0}/SO(3)$ is a contractible space, where the rotation group $SO(3)$ acts on I_{c0} by restriction of the action $g\cdot(x,v)=(g\cdot x, g\cdot v)$ on $(M-\Delta)\times M$.*

Theorem 3. *I_{c0}, $c<0$ has the topological type of the space $F(S^2)\times\mathbb{R}^{n-1}$ where $F(S^2)$ is a topological fiber bundle over S^2 with fiber*

$$F=\{(u,y,\lambda)\in(\mathbb{R}^2)^{n-2}\times(\mathbb{R}^2)^{n-2}\times\mathbb{R}^{n-2}\,|\,\textstyle\sum u_i\times y_i=0 \text{ and } (\delta) \text{ holds}\}$$

where in $\mathbb{R}^2$ $a\times b=a^1b^2-a^2b^1$, and the condition (δ) means

$$\begin{aligned} u_i&=u_j \quad \text{for } i\neq j \Rightarrow \lambda_i\neq\lambda_j\\ u_i&=0 \qquad\qquad\;\; \Rightarrow \lambda_i\neq 0 \text{ and } 1.\end{aligned}$$

In particular, if $n \geqq 3$, I_{co}, $c<0$ *is not a manifold. If* $n=2$, I_{co}, $c<0$ *is diffeomorphic to* $S^2 \times \mathbb{R}$.

Theorem 4. (a) *The map* $I=(E, J)$: $(M-\Delta) \times M \to \mathbb{R} \times \mathbb{R}^3$ *is locally trivial on the region* $\mathbb{R}^+ \times (\mathbb{R}^3 - 0)$.

(b) *The set of critical values of* I *is*

$$\Sigma'(I) = (\mathbb{R} \times 0) \cup S$$

where S *is the image, under the action* $g \cdot (c, p) = (c, g \cdot p)$ *of* $SO(3)$ *on* $\mathbb{R} \times \mathbb{R}^3$, *of the set* $\Sigma'(I_2)$ *of the critical values of the map* I_2 *of the planar n-body problem.*

Remarks. (i) J. Palmore has already computed, in the planar case, the homology of $S_K - \Delta$ for all n.

(ii) Comparing Theorem 4 with Theorem D of [6], it is natural to ask whether there is bifurcation on the set $c=0$ and whether the map I is locally trivial at $(c, p) \notin \Sigma'(I)$, $c<0$. We were not able to answer these questions and judging by the planar situation it seems that they may depend very heavily on a knowledge of the topological structure of I_{cp} for $p \neq 0$.

The rotation group $G = SO(3)$ acts naturally on $\mathbb{R}^3$ and hence on M by means of the diagonal action $g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n)$, leaving Δ, V and K invariant. Therefore the induced action on $(M-\Delta) \times M$ given by $g \cdot (x, v) = (g \cdot x, g \cdot v)$ leaves invariant the energy E. The angular momentum J is equivariant with respect to the actions of G on $(M-\Delta) \times M$ and $\mathbb{R}^3$, i.e., $J(g \cdot x, g \cdot v) = g \cdot J(x, v)$, since for $g \in SO(3)$ $\det(g) = 1$. Then for $p \in \mathbb{R}^3$ the isotropy group $G_p = \{g \in SO(3) | g \cdot p = p\}$ acts on I_{cp} and it is the quotient $\tilde{I}_{cp} = I_{cp}/G_p$ that we mean in (c) of Theorem 2.

For $a \in \mathbb{R}^3$ and $y \in M$ write $a \times y$ for $(a \times y_1, \ldots, a \times y_n)$, and similarly for $y \times a$. Then one can easily check the useful relations $J(x, v) \cdot a = 2K(a \times x, v) = 2K(x, v \times a)$, the first of which is Proposition (4.7) of [5]. The angular momentum given by that proposition is $\frac{1}{2}J$ and this accounts for the factor 2 in the above formulae.

For $x = (x_1, \ldots, x_n) \in M$ the following conditions are equivalent:

(i) J_x: $M \to \mathbb{R}^3$ is not surjective.

(ii) $x_1, \ldots, x_n$ are collinear.

(iii) The isotropy group $G_x = \{g \in SO(3) | g \cdot x = x\}$ has positive dimension.

Proof. (i) is equivalent to the existence of a vector $a \in \mathbb{R}^3 - 0$ orthogonal to the subspace $\operatorname{Im} J_x$, that is, a vector $a \neq 0$ such that $2K(a \times x, v) = J(x, v) \cdot a = J_x(v) \cdot a = 0$ for all $v \in M$. Taking $v = a \times x$ we get $a \times x = 0$, i.e., $x_i = \lambda_i a$, for all i.

To prove the equivalence of (ii) and (iii) we consider, for each $a=(a_1, a_2, a_3)\in\mathbb{R}^3$, the 1-parameter subgroup $\exp(tA)$ of $SO(3)$, where

$$A=\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Then $\dim G_x>0$ if and only if there exists $a\neq 0$ such that $\exp(tA)\cdot x=x$ for all t. Since $A\cdot x=a\times x$ we have $\frac{d}{dt}\exp(tA)\cdot x=\exp(tA)\cdot(a\times x)$. The result follows.

We denote by Λ the set of points in M satisfying, equivalently, (i), (ii) or (iii) above. These points are called syzygies. Notice that for a non-zero syzygy x, the map J_x maps M onto the subspace orthogonal to the direction determined by x.

Now a few words on how the paper is organized. It consists of this introduction containing the statements of the theorems and seven more sections, arranged as follows: In Sections 1 and 2 we prove respectively (a) and (b) of Theorem 1, while Section 3 contains the proof of parts (a) of Theorems 2 and 4. In Sections 4 and 5 we prove respectively (b) and (c) of Theorem 2. In Section 6 we give the proof of Theorem 3 and finally in Section 7 we prove (b) of Theorem 4.

§ 1. Proof of Part (a) of Theorem 1

Identifying $M-\Delta$ with $\mathbb{R}^+\times(S_K-\Delta)$ by means of the diffeomorphism $x\to\left(|x|_K, \frac{x}{|x|_K}\right)$ and writing (t, z, v) for the points of $\mathbb{R}^+\times(S_K-\Delta)\times M$ the maps φ and ψ can be expressed as follows

$$\varphi(t, z, v)=\big(E(tz, v), z, v\big) \quad\text{and}\quad \psi(t, z, v)=\big(E(tz, v), z, tv\big).$$

Since $E^{-1}(\mathbb{R}^-)$ and $E^{-1}(\mathbb{R}^+)$ are open submanifolds of

$$\mathbb{R}^+\times(S_K-\Delta)\times M$$

it is clear that φ and ψ are smooth.

The inverse of φ is clearly the smooth map $\varphi^{-1}(c, z, v)=(t, z, v)$ where t is the positive solution of the equation $E(tz, v)=c$, i.e., $t=\frac{V(z)}{c-K(v)}$.

As to ψ its inverse is the smooth map $\psi^{-1}(c, z, v)=\left(t, z, \frac{1}{t}v\right)$, where t is the positive solution of the equation $E\left(tz, \frac{1}{t}v\right)=c$, i.e., $t=\frac{1}{2c}\{V(z)+\big(V(z)^2+4cK(v)\big)^{\frac{1}{2}}\}$, a smooth function of (c, z, v) in

$\mathbb{R}^+ \times (S_K - \Delta) \times (M - 0)$. For instance, $\psi^{-1}\psi(t, z, v) = \psi^{-1}(E(tz, v), z, tv) = \left(\tau, z, \frac{1}{\tau} tv\right)$ where τ is the positive solution of $E\left(\tau z, \frac{t}{\tau} v\right) = E(tz, v)$, and we must show that $\tau = t$. If, for fixed (t, z, v), we consider the function $g(s) = E\left(sz, \frac{t}{s} v\right) = \frac{1}{s^2} K(tv) + \frac{1}{s} V(z)$ defined for $s > 0$, then elementary calculus shows that it is strictly decreasing on the region where it is positive. Since $g(\tau) = g(t) = E(tz, v) > 0$ it follows that $\tau = t$.

We have thus proved that φ and ψ are diffeomorphisms.

The conclusion $\Sigma(E) = 0$ is clear since the trivializations φ and ψ of E induce diffeomorphisms from $E^{-1}(c)$ to $(S_K - \Delta) \times M$ in case $c < 0$ and to $(S_K - \Delta) \times (M - 0)$ in case $c > 0$, and these spaces are not homeomorphic.

It is interesting to notice that 0 is a regular value of E since $\frac{\partial E}{\partial t} = \frac{-1}{t^2} V(z)$ is never zero.

§ 2. Proof of Part (b) of Theorem 1

We first prove the following result.

(2.1) Proposition. *(z, v) is a critical point of $J: (S_K - \Delta) \times M \to \mathbb{R}^3$ if and only if $z \in \Lambda$, say $z = (\lambda_1 e, \ldots, \lambda_n e)$, $e \in S^2$ and $v_i \times e = 0$ for all i.*

Proof. Since J is the restriction of a bilinear map we have, for

$$(\xi, \eta) \in T_{(z,v)}((S_K - \Delta) \times M) = T_z S_K \times M,$$
$$DJ(z, v) \cdot (\xi, \eta) = J(z, \eta) + J(\xi, v).$$

If $z \notin \Lambda$, then $J_z: M \to \mathbb{R}^3$ is surjective and it follows that $DJ(z, v)$ is surjective.

Suppose now $z \in \Lambda$, say $z = (\lambda_1 e, \ldots, \lambda_n e)$ with $e \in S^2$. Then J_z maps M onto the subspace orthogonal to e. If for some $\xi_0 \in T_z S_K$ $J(\xi_0, v) \notin \operatorname{Im} J_z$, then $\mathbb{R}^3 = \operatorname{Im} J_z \oplus L$ where L is the subspace spanned by $J(\xi_0, v)$. Hence $DJ(z, v)$ is surjective. The converse is obviously true. Therefore (z, v) is a critical point of J, i.e. $DJ(z, v)$ fails to be surjective, if and only if $J(\xi, v)$ is orthogonal to e for any $\xi \in T_z S_K$, that is to say if and only if $K(\xi, v \times e) = \frac{1}{2} J(\xi, v) \cdot e = 0$ for any $\xi \in T_z S_K$. But $K(z, v \times e) = \frac{1}{2} J(z, v) \cdot e = 0$ so that $v \times e \in T_z S_K$ and taking $\xi = v \times e$ we get $v \times e = 0$, i.e., $v_i \times e = 0$ for $i = 1, \ldots, n$.

Note. It is clear that the proposition remains valid if we take the domain of J to be $(M - \Delta) \times M$ or even $(M - 0) \times M$.

It follows from this proposition that $\Sigma'(J) = 0$.

We now prove that J is locally trivial at every $p \neq 0$. Choose an open neighborhood U of $p/|p|$ in S^2 and a smooth map $\beta: U \to SO(3)$ such that $\beta(p') \cdot p/|p| = p'$ for all $p' \in U$, i.e., take a local section β at $p/|p|$ of the map $\alpha: SO(3) \to S^2$, $\alpha(g) = g \cdot p/|p|$.

Let W be the open subset of $\mathbb{R}^3$ consisting of all points $q \neq 0$ such that $q/|q| \in U$ and consider the positive real-valued function defined on W by $a(q) = |p|/|q|$. For $(x, v) \in J^{-1}(W)$ write $a(x, v) = a(J(x, v))$ and $\beta(x, v) = \beta(J(x, v)/|J(x, v)|)$.

Then, noticing that $J^{-1}(p)$ is a smooth manifold and considering the action on $(M - \Delta) \times M$ of the Introduction, we get a smooth map g: $J^{-1}(W) \to J^{-1}(p)$ defined by $g(x, v) = \beta(x, v)^{-1} \cdot (x, a(x, v)\, v)$ and such that the map h: $J^{-1}(W) \to W \times J^{-1}(p)$ is a diffeomorphism, where $h(x, v) = (J(x, v), g(x, v))$. Its inverse is the map $h^{-1}(q; x, v) = \beta(q/|q|) \cdot (x, a(q)^{-1} v)$ which is smooth since it is smooth as a map on $W \times (M - \Delta) \times M$. This completes the proof of (b) of Theorem 1.

Remark. It is clear that the above proof works equally well if we consider the induced action on $(S_K - \Delta) \times M$, so that the map J: $(S_K - \Delta) \times M \to \mathbb{R}^3$ is locally trivial at every $p \neq 0$. It is this situation that we will meet in the next section.

§ 3. Proofs of Part (a) of Theorem 2 and Part (a) of Theorem 4

To prove (a) of Theorem 2 we notice that the diffeomorphism φ of Theorem 1 induces a homeomorphism from I_{c0} onto the space

$$J_0 = \{(z, v) \in (S_K - \Delta) \times M \,|\, J(z, v) = 0\}.$$

Now J_0 has $S_K - \Delta$ as a strong deformation retract. Indeed the map H: $S_K \times M \times [0, 1] \to S_K \times M$ given by $H(z, v, s) = (z, (1 - s)\, v)$ is continuous and maps $J_0 \times [0, 1]$ into J_0. Since $H(z, v, 0) = (z, v)$, $H(z, v, 1) = (z, 0)$ and $H(z, 0, s) = (z, 0)$ for all s, it follows that H is a strong deformation retraction from J_0 to $(S_K - \Delta) \times 0$ and this space can be identified with $S_K - \Delta$.

To prove (a) of Theorem 4 consider the smooth map

$$\mathrm{id} \times J\colon\ \mathbb{R} \times (S_K - \Delta) \times M \to \mathbb{R} \times \mathbb{R}^3, \qquad (t, z, v) \to (t, J(z, v)).$$

Then $I = (\mathrm{id} \times J) \circ \psi$ where ψ is the diffeomorphism of Theorem 1. By the remark at the end of § 2, J is locally trivial on $\mathbb{R}^3 - 0$, hence $\mathrm{id} \times J$ is locally trivial on $\mathbb{R}^+ \times (\mathbb{R}^3 - 0)$ and the same is true of the map I since ψ is a diffeomorphism.

Observations. 1) If for each $z \in S_K - \Delta$ the linear map $J_z\colon M \to \mathbb{R}^3$ were surjective, then we would have an exact sequence of vector bundles

$$(S_K - \Delta) \times M \to (S_K - \Delta) \times \mathbb{R}^3 \to 0$$

where the first map is $h = \pi \times J$, π being the projection on the first factor. Hence the kernel J_0 of the bundle homomorphism h would be a vector space bundle over $S_K - \Delta$. This is actually the case in the planar n-body problem (see [6]). Due to the existence of syzygies in $S_K - \Delta$ we do not have such a situation in 3-space. In fact, here J_0 is not even a manifold (cf. Theorem 3).

2) For $c>0$ and $p \neq 0$, the map ψ of Theorem 1 induces a diffeomorphism from I_{cp} onto the manifold

$$J_p = \{(z, v) \in (S_K - \Delta) \times M \mid J(z, v) = p\}$$

given by $(t, z, v) \to (z, t v)$. But, because of the sysygies in $S_K - \Delta$, it is still very difficult to study the manifold J_p. Compare with the planar situation where the map $(z, v) \to (z, v - \alpha_p(z))$ gives a diffeomorphism from J_p onto J_0. Here α_p is the vectorfield on $M - \Delta$ considered in [5].

§ 4. Proof of Part (b) of Theorem 2

Here and in the next two sections we use the same symbol Δ to denote the generalized diagonal in a product of copies of $\mathbb{R}^3$.

We first exhibit the structure of $S_K - \Delta$.

(4.1) Proposition. *$S_K - \Delta$ is diffeomorphic to $Q(S^2)$, a smooth fiber bundle over S^2, with fiber*

$$Q_{\eta_1} = \{y \in (\mathbb{R}^3 - 0)^{n-2} \mid y \notin \Delta \text{ and } y_i \neq \eta_1, \text{ all } i\} \quad \text{for } \eta_1 \in S^2.$$

Proof. Consider the isomorphism $T\colon M \to (\mathbb{R}^3)^{n-1}$ given by $T(x_1, \ldots, x_n) = (x_1 - x_n, \ldots, x_{n-1} - x_n)$ and notice that $x \in \Delta$ if and only if $T(x) \in \Delta$ or $(T(x))_i = 0$, for some $i = 1, \ldots, n-1$.

Then T induces a diffeomorphism

$$\tau\colon S_K - \Delta \to \{\xi \in S^{3n-4} - \Delta \mid \xi_i \neq 0, \text{ all } i\},$$

defined by $z \to T(z)/|T(z)|$.

We further get a diffeomorphism σ from this manifold onto

$$Q(S^2) = \{\eta \in S^2 \times (\mathbb{R}^3 - 0)^{n-2} \mid \eta \notin \Delta\},$$

by mapping $\xi = (\xi_1, \ldots, \xi_{n-1})$ into $\rho(\xi) = \xi / |\xi_1|$.

Now $Q(S^2)$ is a smooth fiber bundle over S^2 with projection map $\pi(\eta) = \eta_1$, and fiber, the space Q_{η_1}, described in the proposition. To show the local triviality of π choose, for each $\eta_1 \in S^2$, an open neighborhood U of η_1 in S^2 and a smooth map $\beta\colon U \to SO(3)$ such that $\beta(\eta_1') \cdot \eta_1 = \eta_1'$ for $\eta_1' \in U$. Then define the diffeomorphism $h\colon \pi^{-1}(U) \to U \times Q_{\eta_1}$, by

$$h(\eta_1', \eta_2, \ldots, \eta_{n-1}) = (\eta_1', \beta(\eta_1')^{-1} \cdot \eta_2, \ldots, \beta(\eta_1')^{-1} \cdot \eta_{n-1}).$$

In case $n=3$, the fiber is $F = \mathbb{R}^3 - 2$ points which has the homotopy type of the space $S^2 \vee S^2$. Indeed, let ρ be the continuous map described in Fig. 1. Then ρ is a retraction of F to $S^2 \vee S^2$ and the homotopy $H(q,t) = q + t(\rho(q) - q)$, $0 \leqq t \leqq 1$ is a strong deformation retraction of F to $S^2 \vee S^2$. Therefore the homology of F is $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ in dimensions 0 and 2, respectively, and 0 otherwise.

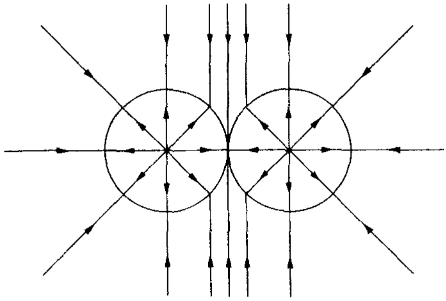

Fig. 1

We can now apply the Wang homology sequence

$$\to H_r(Q) \to H_{r-2}(F) \to H_{r-1}(F) \to H_{r-1}(Q) \to H_{r-3}(F) \to$$

in order to get the homology of $Q = Q(S^2)$. For example, taking $r=3$, we get

$$H_1(F) \to H_2(F) \to H_2(Q) \to H_0(F) \to H_1(F)$$

or

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_2(Q) \to \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is a free module this exact sequence yields $H_2(Q) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The other cases are handled just as easily.

§ 5. Proof of Part (c) of Theorem 2

(5.1) Proposition. *Fix $e \in S^2$. Then $\tilde{I}_{co}$, $c<0$ has the homotopy type of the space Q_e/S^1, where*

$$Q_e = \{y \in (\mathbb{R}^3 - 0)^{n-2} \mid y \notin \Delta \text{ and } y_i \neq e, \text{ all } i\}$$

and $S^1 = \{g \in SO(3) \mid g \cdot e = e\}$ acts on Q_e by restriction of the diagonal action on $(\mathbb{R}^3)^{n-2}$.

Proof. The homeomorphism from I_{co} onto J_0 induced by the map φ of Theorem 1, the strong deformation retraction H of J_0 to $S_K - \Delta$ given in § 3 and the diffeomorphisms σ and τ of § 4, all these maps are equivariant with respect to the actions of $G = SO(3)$ on these spaces (the action on $J_0 \times [0,1]$ being $g \cdot (z, v, t) = (g \cdot z, g \cdot v, t)$ and restrictions of diagonal action elsewhere).

Therefore, $\tilde{I}_{co}$, $c<0$ is homeomorphic to J_0/G, which has $(S_K-\Delta)/G$ as strong deformation retract, which is in turn homeomorphic to $Q(S^2)/G$. Hence $\tilde{I}_{co}$, $c<0$ has the homotopy type of $Q(S^2)/SO(3)$.

To conclude the proof we will now show that this space is homeomorphic to Q_e/S^1. Brackets will be used to denote equivalence classes, i.e., elements in the quotient spaces. The map

$$F:\ Q(S^2)\to Q_e/S^1, \qquad F(\eta, y)=[g\cdot y]$$

where $y\in Q_\eta$, $\eta\in S^2$ and $g\in SO(3)$ is such that $g\cdot\eta=e$ is well-defined and continuous. To see the continuity at (η, y), let U be an open neighborhood of η in S^2 and β: $U\to SO(3)$ a smooth map such that $\beta(\xi)\cdot\xi=e$, all $\xi\in U$.

Then, on $\pi^{-1}(U)\subset Q(S^2)$, F is the composition $y'\in Q_\xi\to\beta(\xi)\cdot y'\to [\beta(\xi)\cdot y']$, which shows that F is continuous.

Now the map F_1: $Q_e\to Q(S^2)/SO(3)$, $F_1(y)=[e, y]$ is continuous as composition of continuous maps $y\to(e, y)\to[e, y]$.

It is now easily checked that

$$\tilde{F}^{-1}=\tilde{F}_1:\ Q_e/S^1\to Q(S^2)/SO(3)$$

so that $\tilde{F}$ is a homeomorphism.

For $n=3$, $Q_e=\{y\in\mathbb{R}^3\,|\,y\neq 0 \text{ and } y\neq e\}$, hence $Q_e/S^1=\{(s,t)\in\mathbb{R}^2\,|\,s\geqq 0\}-\{(0,0),(0,1)\}$, which is a contractible space.

§ 6. Proof of Theorem 3

In addition to the isomorphism T: $M\to(\mathbb{R}^3)^{n-1}$ of § 4, we consider the one given by $S(v_1,\dots,v_n)=(m_1 v_1,\dots,m_{n-1}v_{n-1})$. Then, one easily checks that $J_1(T(x), S(v))=J(x,v)$, where

$$J_1(\eta, w)=\sum_{i=1}^{n-1}\eta_i\times w_i, \qquad \text{for } \eta,\ w\in(\mathbb{R}^3)^{n-1}.$$

By considering the diffeomorphisms σ and τ of § 4, we get a diffeomorphism

$$(S_K-\Delta)\times M\to\{(\eta,w)\in(S^2\times(\mathbb{R}^3-0)^{n-2})\times(\mathbb{R}^3)^{n-1}\,|\,\eta\notin\Delta\}$$

defined by $(z,v)\to(\sigma\tau(z), S(v))$, which maps

$$J_0=\{(z,v)\in(S_K-\Delta)\times M\,|\,J(z,v)=0\}$$

onto the topological space

$$N=\{(\eta,w)\in(S^2\times(\mathbb{R}^3-0)^{n-2}-\Delta)\times(\mathbb{R}^3)^{n-1}\,|\,J_1(\eta,w)=0\}.$$

We now decompose η_i, $i=2,\dots,n-1$ and w_j, $j=1,2,\dots,n-1$ along η_1 and the orthogonal complement $\eta_1^\perp$ of η_1. From now on in this section, when we write the subscripts i and j, we always mean these ranges. We have

$$\begin{aligned} \eta_i &= \lambda_i \eta_1 + \eta_i', & \lambda_i &= \eta_i \cdot \eta_1 \\ w_j &= \mu_j \eta_1 + w_j', & \mu_j &= w_j \cdot \eta_1 . \end{aligned} \tag{$*$}$$

A straightforward computation shows that

$$J_1(\eta, w) = \eta_1 \times \left(w_1' + \sum (\lambda_i w_i' - \mu_i \eta_i')\right) + \sum \eta_i' \times w_i',$$

so that the condition $J_1(\eta, w) = 0$ is equivalent to the conditions

$$\begin{aligned} & w_1' = \sum (\mu_i \eta_i' - \lambda_i w_i') \\ & \sum \eta_i' \times w_i' = 0 . \end{aligned} \tag{$**$}$$

The assignment $(\eta, w) \to (\eta_1, \eta_i', w_i', \lambda_i, \mu_j)$ clearly defines a smooth map f from $S^2 \times (\mathbb{R}^3)^{n-2} \times (\mathbb{R}^3)^{n-1}$ into $S^2 \times (\mathbb{R}^3)^{n-2} \times (\mathbb{R}^3)^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-1}$. Conversely, given a point $(\eta_1, \eta_i', w_i', \lambda_i, \mu_j)$ in the second space, we take w_1' as defined by the first of the equations $(**)$ and then use the first set of equations $(*)$ to determine a point (η, w), which obviously depends smoothly on the given point; this defines a smooth map g which goes on the opposite direction to f.

For the decomposition $\eta_i = \lambda_i \eta_1 + \eta_i'$, the conditions $\eta \notin \Delta$ and $\eta_j \neq 0$ for all j are equivalent to the conditions

$$\begin{aligned} & \eta_r' = \eta_s' \quad \text{with} \quad r \neq s \Rightarrow \lambda_r \neq \lambda_s \\ & \eta_r' = 0 \qquad\qquad\quad\;\; \Rightarrow \lambda_r \neq 0 \text{ and } 1. \end{aligned} \tag{δ}$$

Now consider the space

$$F(S^2) = \{(\eta_1, \eta', w', \lambda) \in S^2 \times (\mathbb{R}^3)^{n-2} \times (\mathbb{R}^3)^{n-2} \times \mathbb{R}^{n-2} \mid \eta_i', w_i' \in \eta_1^\perp, \\ \textstyle\sum \eta_i' \times w_i' = 0 \,\&\, (\delta) \text{ holds}\}.$$

Then restricted to N and $F(S^2) \times \mathbb{R}^{n-1}$, the maps f and g are inverse to each other. Therefore these spaces are homeomorphic, which means that J_0, hence I_{c0}, $c<0$ (cf. § 3), is homeomorphic to $F(S^2) \times \mathbb{R}^{n-1}$.

But one can easily verify that $F(S^2)$ is a fiber bundle over S^2, with projection map $\pi(\eta_1, \eta', \omega', \lambda) = \eta_1$ and fiber, the topological space

$$F = \{(u, y, \lambda) \in (\mathbb{R}^2)^{n-2} \times (\mathbb{R}^2)^{n-2} \times \mathbb{R}^{n-2} \mid \textstyle\sum u_i \times y_i = 0 \,\&\, (\delta) \text{ holds}\}$$

where the condition (δ) is with respect to u and λ, and this completes the proof of the first part of Theorem 3.

Now let $n \geqq 3$ and take $\lambda^0 \in \mathbb{R}^{n-2}$ such that $1 < \lambda_1^0 < \cdots < \lambda_{n-2}^0$. Then there exists an open ball B in $\mathbb{R}^{n-2}$ centered at λ^0 and such that $1 < \lambda_1 < \cdots < \lambda_{n-2}$, for all $\lambda \in B$. Therefore, $\{(u, y, \lambda) \in F \mid \lambda \in B\} = C \times B$, where C is the cone

$$C = \{(u, y) \in (\mathbb{R}^2)^{n-2} \times (\mathbb{R}^2)^{n-2} \mid \textstyle\sum u_i \times y_i = 0\}.$$

Therefore, in a neighborhood of some point, the space I_{c0}, $c<0$ has the topological structure of $C \times \mathbb{R}^{2n-1}$ and so it is not a manifold.

Finally, if $n=2$, then

$$N=\{(\eta, \omega)\in S^2 \times \mathbb{R}^3 \mid \eta \times \omega = 0\},$$

which is diffeomorphic to $S^2 \times \mathbb{R}$ under the map $(\eta, \omega) \to (\eta, \eta \cdot \omega)$; this is a trivialization of the line bundle N over S^2.

§ 7. Proof of Part (b) of Theorem 4

(7.1) Proposition. *$(x, v)\in(M-\Delta)\times M$ is a critical point of $I=(E,J)$ if and only if $DJ(x,v)$ is not surjective or (x,v) is a critical point of $E_p = E\,|\,J^{-1}(p)$, where $p=J(x,v)$.*

Proof. Assume $DJ(x,v)$ surjective and write $M\times M = T_{(x,v)}J^{-1}(p)\oplus P$ so that $DJ(x,v)\colon P\to\mathbb{R}^3$ is an isomorphism. Then

$$DE_p(x,v)\colon\ T_{(x,v)}J^{-1}(p)\to\mathbb{R}$$

is surjective if and only if

$$(DE(x,v), DJ(x,v))\colon\ T_{(x,v)}J^{-1}(p)\oplus P\to\mathbb{R}\times\mathbb{R}^3$$

is surjective.

(7.2) Corollary. $\Sigma'(I)=(\mathbb{R}\times 0)\cup\{(c,p)\in\mathbb{R}\times\mathbb{R}^3 \mid p\neq 0\ \&\ c\in\Sigma'(E_p)\}$.

Proof. This follows from (7.1) and the fact that $\Sigma'(J)=0$.

To study the structure of $\Sigma'(E_p)$ for $p\neq 0$ we consider the direct sum decomposition $M=M_1\oplus M_2$, where

$$M_1=\{x^1\in M \mid x_i^1\times p=0\} \quad\text{and}\quad M_2=\{x^2\in M \mid x_i^2\cdot p=0\}.$$

Let E^2 be the subspace of $\mathbb{R}^3$ orthogonal to p. Then

$$M_2=\{(x_1^2,\dots,x_n^2)\in(E^2)^n \mid \textstyle\sum m_i x_i^2=0\}.$$

Considering $M_2-\Delta_2$ as the configuration space of the planar n-body problem with masses $m_1,\dots,m_n$, then the energy E_2 and the angular momentum J_2 are the restrictions to $(M_2-\Delta_2)\times M_2$ of the energy E and the angular momentum J.

Let L denote the map J with domain $M\times M$ and L_2 the map J_2 with domain $M_2\times M_2$. Then, since L is an antisymmetric bilinear map, we see that for $x=x^1+x^2$, $v=v^1+v^2$ in M_1+M_2, we have

(7.3) Lemma. *$L(x,v)=p$ if and only if* (i) *$L_2(x^2,v^2)=p$ and* (ii) *$L(x^1,v^2)+L(x^2,v^1)=0$.*

By condition (i) the projection $\pi: M \to M_2, \pi(x) = x^2$ induces a smooth map from the manifold $L^{-1}(p)$ onto the manifold $L_2^{-1}(|p|) = L^{-1}(p) \cap (M_2 \times M_2)$, defined by $\pi(x, v) = (x^2, v^2)$.

(7.4) Proposition. *π: $L^{-1}(p) \to L_2^{-1}(|p|)$ is a vector bundle with fiber*

$$F = \{(x^1, v^1) \in M_1 \times M_1 \mid L(x^1, v^2) + L(x^2, v^1) = 0\},$$

for $(x^2, v^2) \in L_2^{-1}(|p|)$.

Proof. Let $\varphi(x, v) = L(x^1, v^2) + L(x^2, v^1)$. Then, for fixed $(x^2, v^2) \in L_2^{-1}(|p|)$ the linear map $\varphi_{(x^2, v^2)}$: $M_1 \times M_1 \to E^2$ is surjective. Indeed, if not let $e \in E^2 - 0$ be orthogonal to the image of this map so that $2K(x^1, v^2 \times e) + 2K(e \times x^2, v^1) = L(x^1, v^2) \cdot e + L(x^2, v^1) \cdot e = 0$, for all $x^1, v^1 \in M_1$. Since $v^2 \times e$, $e \times x^2 \in M_1$, we get $|v^2 \times e|_K^2 + |e \times x^2|_K^2 = 0$, hence $x_i^2 = \lambda_i e$, $v_i^2 = \mu_i e$, all i and so $L_2(x^2, v^2) = 0$, a contradiction.

Therefore, if $h = \varphi \times \pi_2$ where π_2 is the projection on the second factor, the sequence of vector bundles

$$(M_1 \times M_1) \times L_2^{-1}(|p|) \xrightarrow{h} E^2 \times L_2^{-1}(|p|) \to 0$$

is exact. Hence its kernel, which by Lemma (7.3) is $L^{-1}(p)$, is a vector space bundle over $L_2^{-1}(|p|)$.

A configuration $x \in M - \Delta$ is said to be a relative equilibrium if there exists a 1-parameter subgroup g_t of $SO(3)$ which acting on x induces a motion of the system, i.e.,

$$m \frac{d^2}{dt^2} g_t \cdot x = -\operatorname{grad} V(g_t \cdot x).$$

A solution of relative equilibria is always planar (see [8]). Therefore, if the angular momentum along the trajectory is $p \neq 0$, then the configurations have to lie in the plane orthogonal to p.

The set R_e of relative equilibria is invariant under the action of $SO(3)$ on $M - \Delta$. Indeed, let $x(t) = g_t \cdot x$, $x \in R_e$ and set $h_t = g g_t g^{-1}$. Then, for $y = g \cdot x$ we have $y(t) = h_t \cdot y = g \cdot x(t)$ so that

$$m y'' = g \cdot m x'' = -g \cdot \operatorname{grad} V(x(t)) = -\operatorname{grad} V(g \cdot x(t)) = -\operatorname{grad} V(y),$$

hence $y \in R_e$.

(7.5) Proposition. *$(x, v) = (x^1, v^1) + (x^2, v^2) \in J^{-1}(p)$ is a critical point of $E_p = E|J^{-1}(p)$ if and only if $x^1 = 0$, $v^1 = 0$ and (x^2, v^2) is a critical point of $E_{2,p} = E_2|J_2^{-1}(|p|)$.*

Proof. Since $J^{-1}(p) = ((M - \Delta) \times M) \cap L^{-1}(p)$ is an open submanifold of the vector bundle $L^{-1}(p)$ of Proposition (7.4), we can write

$$DE_p(x, v) = DE(x, v)|F + DE(x, v)|\, T_{(x^2, v^2)} J_2^{-1}(|p|).$$

Now,

$$DE(x, v)|F \cdot (\xi^1, \eta^1) = 2K(v^1, \eta^1) + \frac{\partial V}{\partial x^1}(x) \cdot \xi^1$$

and

$$\frac{\partial V}{\partial x^1}(x) \cdot \xi^1 = \sum_{i<k} \frac{m_i m_k}{|x_i - x_k|^3} (x_i^1 - x_k^1)(\xi_i^1 - \xi_k^1).$$

Since $(x^1, v^1) \in F$ and $\sum m_i x_i^1 = 0$, we have $DE(x, v)|F = 0 \Leftrightarrow x^1 = 0$ and $v^1 = 0$. Therefore, both factors in the above expression for $DE_p(x, v)$ vanish if and only if $x^1 = 0$, $v^1 = 0$ and (x^2, v^2) is a critical point of $E_{2,p}$.

(7.6) Corollary. $\Sigma'(E_p) = \Sigma'(E_{2,p})$.

Now fix $e \in S^2$, say $e = (0, 0, 1)$ and take $g \in SO(3)$ such that $g \cdot p = |p|\, e$. Let $M_2^{(0)}$ be the configuration space of the planar n-body problem determined by e. Then the isomorphism $M_2 \to M_2^{(0)}, x \to g \cdot x$ induces a bijection between the sets $R_e^{(p)}$ and $R_e^{(0)}$ of relative equilibria in M_2 and $M_2^{(0)}$, which shows that the following equality holds:

$$\{-V(|p|\, z)^2 | z \in R_e^{(p)}, K(z) = 1\} = \{-V(|p|\, z)^2 | z \in R_e^{(0)}, K(z) = 1\}.$$

This means that (see Section 2 of [6]) $\Sigma'(E_{2,p}) = \Sigma'(E_{2,|p|e})$.

Therefore if $S = \{(c, p) \in \mathbb{R} \times \mathbb{R}^3 | p \neq 0$ and $c \in \Sigma'(E_p)\}$ is the set of Corollary (7.2), then by Corollary (7.6)

$$(c, p) \in S \Leftrightarrow p \neq 0 \quad \text{and} \quad c \in \Sigma'(E_{2,|p|e}) \Leftrightarrow (c, |p|\, e) \in \Sigma'(I_2)$$

where I_2 is the map I of the n-body problem with configuration space $M_2^{(0)}$ (notice that in the planar case, Proposition (7.1) also holds and the angular momentum has no critical points).

Therefore,

$$S = G \cdot \Sigma'(I_2),$$

where $G = SO(3)$ acts on $\mathbb{R} \times \mathbb{R}^3$, trivially on $\mathbb{R}$, naturally on $\mathbb{R}^3$.

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H.E. Cabral
Instituto de Matematica Pura e Aplicada
Rua Luiz de Camões, 68
Rio de Janeiro, Brasil

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