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On Zeros of Dirichlet's L Series

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In a recent paper in this journal P.X. Gallagher has estimated the least prime in some special arithmetic progressions. It is the aim of the present work to show how his result can be extended to the general case.

Let $q \geqq 3$, $d = \prod_{p/q} p$, $l = \log q(|t|+3)$, $\eta^{-1} = \sqrt{l}(\log 2l)^{\frac{3}{4}}$

$$\vartheta^{-1} = 4 \cdot 10^4 (\log d + (l \log 2l)^{\frac{3}{4}}).$$

We shall prove the following two theorems:

Theorem 1. *Let* $s = \sigma + it$, $\sigma > 1 - \eta$, χ *be a nonprincipal character* $\bmod q$. *Then*

$$|L(s,\chi)| < (d^{\eta} \exp l^{\frac{1}{4}})^{100}.$$

Theorem 2. *There exists at most one character* $\chi \neq \chi_0 \bmod q$ *and a number* ρ *such that*

$$\operatorname{Re} \rho > 1 - \vartheta, \qquad L(\rho, \chi) = 0.$$

If there does exist such a character then it is real and its zero ρ *is real and simple.*

Let $p_{\min}(a,q)$ be the least prime $\equiv a (\bmod q)$, where $(a,q) = 1$. It follows from Theorem 2 and from the estimation

$$N_q(T,\sigma) \ll (qT)^{b(1-\sigma)} \log^A (qT)$$

that

$$p_{\min}(a,q) \underset{\varepsilon,d}{\ll} q^{b+\varepsilon}$$

which extends the result of Gallagher concerning the case $q = p^r$. Also other results of [2] can now be extended without any essential change in the argument. Let us point out that for d small enough comparison with q, Theorem 1 gives a good estimation of $L(s,\chi)$ simultaneously with respect to t and q. This has been achieved by combining an idea of Postnikov (see Lemma 2) with Vinogradov's estimation of trigonometric sums (see Lemmata 3 and 5).

Remark. It has been recently proved by M. Jutila that

$$b \leqq \tfrac{3}{16}(9 + \sqrt{17}) = 2.4605 \cdots$$

(see M. Forti and C. Viola [1]).

I conclude this introduction by expressing my thanks to Professor A. Schinzel for his help in writing this paper

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Lemma 1 (Gallagher). *Let* $L_N = -\sum_{i=1}^{N} \frac{(-x)^i}{i}$. *Then*

$$L_N(x+y+xy) - L_N(x) - L_N(y) = \sum_{N < i+j \leqq 2N} c_{ij} x^i y^j$$

where the numbers $[i,j]$ c_{ij} *are integers.*

Proof. See [2], p. 192.

Lemma 2. (An extension of Postnikov's result). *Let*

$$\tau = (4,q)/(2,q), \quad q^2 | d^N, \quad M_N = \prod_{n \leqq N, (n,q)=1} n,$$

χ — *a primitive character* $\bmod q$. *Then there exists an integer* L *such that* $(L,q)=1$ *and*

$$\chi(1+\tau d u) = e\left(\frac{L M_N L_N(\tau d u)}{q}\right). \tag{1}$$

Proof. Clearly $M_N L_N(du) \in \mathbb{Z}[u]$. It follows from Lemma 1 that

$$M_N L_N(du+dv+d^2uv) \equiv M_N L_N(du) + M_N L_N(dv) \bmod q.$$

Hence, the function $\xi(1+\tau d u) = e\left(\frac{M_N L_N(\tau d u)}{q}\right)$ is a character of the group of residue classes $\bmod q$ congruent to 1 $\bmod \tau d$. It is easy to see that the order of ξ equals $q/\tau d$ i.e. it coincides with the order of the group. Thus, we have for a certain L

$$\chi(1+\tau d u) = e\left(\frac{L M_N L_N(\tau d u)}{q}\right).$$

Since q is the conductor of χ, we have $(L, q/\tau d) = 1$. On the other hand the number L is determined only $\bmod q/\tau d$, thus it can be chosen so that $(L,q)=1$

Lemma 3. *Let* $q = \prod p^{\alpha_p}$, $0 < \varepsilon < 1$, $q_\varepsilon = \prod p^{[\varepsilon \alpha_p]} \tau d$, χ — *a primitive character* $\bmod q$. *Then there exists an integer* T *such that* $(T,q)=1$ *and*

$$\chi(1+q_\varepsilon u) = e\left(\frac{T L_{[2/\varepsilon]}(q_\varepsilon u)}{q}\right).$$

Proof. It follows from Lemma 2 that $\chi(1+q_\varepsilon u) = e\left(\frac{L M_N L_N(q_\varepsilon u)}{q}\right)$.

The i-th coefficient of the polynomial $\frac{L M_N}{q} L_N(q_\varepsilon u)$ equals $-\frac{M_N L}{iq}(-q_\varepsilon)^i$, so it is integer for $i > \frac{2}{\varepsilon}$.

Lemma 4 (Vinogradov). *Let* $2 < P < \lambda^{-1} < P^3$, $\Omega = (Q, Q+P)$, $k \geqq 12$, $f(x) \in C^k(\Omega)$ *and*

$$1 \leqq \frac{1}{\lambda k!} |f^{(k)}(x)| \leqq 2^k \quad \textit{for } x \in \Omega.$$

Then

$$\max_{0<R<P} \Big| \sum_{Q<n<Q+R} e(f(n)) \Big| < \gamma_k P^{1-\varepsilon_k}$$

where $\gamma_k = \exp(k \log^2 6k)$, $\varepsilon_k = (3k)^{-2} \log^{-1} 6k$.

Proof. See [4].

Lemma 5 (Vinogradov). *Let* $f(x) = a_1 x + a_2 x^2 + \ldots + a_k x^k \in \mathbb{R}[x]$, $k \geqq 12$, $2 \leqq l \leqq k$, $a_l = a/b$, $(a, b) = 1$, $2 < P < b < P^{l-1}$. *Then*

$$\Big| \sum_{0<n<P} e(f(n)) \Big| < \gamma_k P^{1-\varepsilon_k}.$$

Proof. See [4].

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Lemma 6. *Let* χ *be a primitive character* $\bmod q$, $d^{100} < N < N' < 2N$, $q(|t|+3) = N^z$, $z \geqq \frac{1}{2}$. *Then*

$$\left| \sum_N^{N'} \chi(n) n^{it} \right| < \gamma_{200z} N^{1-\varepsilon_{600z}}. \tag{2}$$

Proof. We can assume that

$$\gamma_{200z} < N^{\varepsilon_{600z}}, \tag{3}$$

since otherwise the estimation (2) is trivial. We consider two cases:

(i) $q+3 \leqq |t|$ and (ii) $q+3 > |t|$.

(i) Put $\delta = \frac{1}{40}$, $\varepsilon = (1-\delta) \log N / \log q$, $P = N/q_\varepsilon$, $P^{k-2} \leqq |t| < P^{k-1}$. Hence

$$N^\delta / \tau d \leqq P < N^\delta, \tag{4}$$

$$11 \leqq 20z + 1 < k < 66z + 2 \leqq 70z, \tag{5}$$

$$\varepsilon k \geqq 2, \tag{6}$$

$$2\pi k < P, \tag{7}$$

$$P < \lambda^{-1} < P^3, \quad \text{where} \quad \lambda = \frac{|t|}{2\pi k} P^{-k}, \tag{8}$$

$$q_\varepsilon \leqq \tau d q^\varepsilon = \tau d N^{1-\delta}. \tag{9}$$

Every integer n from the interval $N \leqq n \leqq N'$ can be represented uniquely in the form $n = a + q_\varepsilon u$, where $0 \leqq a < q_\varepsilon$. Thus

$$\sum_N^{N'} \chi(n) n^{it} = \sum_{\substack{0<a<q_\varepsilon \\ (a,q)=1}} \chi(a) \sum_{\frac{N-a}{q_\varepsilon} \leqq u \leqq \frac{N'-a}{q_\varepsilon}} \chi(1 + q_\varepsilon a^* u)(a + q_\varepsilon u)^{it}, \tag{10}$$

where $a a^* \equiv 1 \pmod{q}$. The last sum is of length $< P$. In virtue of Lemma 3 it can be represented in the form $\sum_u e(f(u))$, where

$$f(u) = \frac{T}{q} L_{[2/\varepsilon]}(q_\varepsilon a^* u) + \frac{t}{2\pi} \log(a + q_\varepsilon u).$$

Since $k > [2/\varepsilon]$ we have

$$\lambda \leqq \frac{1}{k!} |f^{(k)}(u)| = \frac{|t|}{2\pi k} \left(\frac{q_\varepsilon}{a + q_\varepsilon u} \right)^k \leqq 2^k \lambda.$$

Thus the sum $\sum e(f(u))$ can be estimated by means of Lemma 4 and we get $|\sum_u e(f(u))| < \gamma_k P^{1-\varepsilon_k}$. Hence

$$\left| \sum_N^{N'} \chi(n) n^{it} \right| < q_\varepsilon \gamma_k P^{1-\varepsilon_k} = \gamma_k N^{1-\varepsilon_k} q_\varepsilon^{\varepsilon_k} < \gamma_k N^{1-\delta\varepsilon_k} (\tau d)^{\varepsilon_k} < \gamma_{200z} N^{1-\varepsilon_{600z}}.$$

(ii) Put $\delta = \frac{1}{90}$, $\varepsilon = \delta \log N / \log q$, $k = \left[\frac{1}{2\varepsilon} \right] + 1$, $H = N^{10\delta}$, $P = H/q_\varepsilon$. Hence

$$N^{9\delta}/\tau d \leqq P < N^{9\delta} \tag{11}$$

$$12 \leqq k < r = \left[\frac{2}{\varepsilon} \right] < \frac{2z}{\delta} = 180z. \tag{12}$$

Let $N \leqq m \leqq 2N$, $m \leqq n \leqq m + H$. Then

$$\begin{aligned} \left(\frac{n}{m} \right)^{it} &= \exp\left(it \log\left(1 + \frac{n-m}{m} \right) \right) \\ &= e\left(\frac{t}{2\pi} L_{k-1}\left(\frac{n-m}{m} \right) \right) + \theta_1 \frac{2t}{k} \left(\frac{H}{m} \right)^k, \end{aligned} \tag{13}$$

where $|\theta_1| < 1$. If $(a, q) = 1$ we solve the congruence $a a^* \equiv 1 \pmod{q}$ and set $Q = \left[\frac{m-a}{q_\varepsilon} \right]$, $f(u) = \frac{T}{q} L_r(q_\varepsilon a^* u) + \frac{t}{2\pi} L_{k-1}\left(\frac{q_\varepsilon u + a - m}{m} \right)$. By Lemma 3 and (13) we get

$$\begin{aligned} \sum_{m<n<m+H} \chi(n) n^{it} &= \sum_{m<n<m+H} \chi(n) e\left(\frac{t}{2\pi} L_{k-1}\left(\frac{n-m}{m} \right) \right) + \theta_2 \frac{2tH}{k} \left(\frac{H}{m} \right)^k \\ &= \sum_{\substack{0<a<q_\varepsilon \\ (a,q)=1}} \chi(a) \sum_{Q<u<Q+P} e(f(u)) + \theta_3 \left(\frac{2tH}{k} \left(\frac{H}{m} \right)^k + q_\varepsilon \right) \\ &= \sum_{\substack{0<a<q_\varepsilon \\ (a,q)=1}} \chi(a) \sum_{0<u<P} e(f(u+Q)) + \theta_4 \left(\frac{2tH}{k} \left(\frac{H}{N} \right)^k + q_\varepsilon \right), \end{aligned}$$

where $|\theta_i| < 1$, $i=2,3,4$. The k-th coefficient of the polynomial $f(u+Q)$ equals

$$a_k = -\frac{T}{kq}(-q_\varepsilon a^*)^k \sum_{i=0}^{r-k} (-q_\varepsilon a^* Q)^i \binom{k+i-1}{i}.$$

Thus a_k represented as a fraction in its reduced form, has the denominator contained in the interval $(q/q_\varepsilon^k, kq) \subset (P, P^{k-1})$. Therefore the sum $\sum e(f(u+Q))$ can be estimated by means of Lemma 5 and we get

$$\Big|\sum_{m<n<m+H} \chi(n) n^{it}\Big| < q_\varepsilon \gamma_r P^{1-\varepsilon_r} + \frac{2|t|H}{k}\left(\frac{H}{N}\right)^k + q_\varepsilon,$$

whence

$$\left|\sum_N^{N'} \chi(n) n^{it}\right| < \frac{N}{H} q_\varepsilon \gamma_r P^{1-\varepsilon_r} + \frac{2|t|N}{k}\left(\frac{H}{N}\right)^k + NP^{-1} + H$$

$$< \gamma_r NP^{-\varepsilon_r} + 1 + NP^{-1} + H < \gamma_{200z} N^{1-\varepsilon_{600z}}.$$

By Lemma 6 we get

Lemma 7. *Let* χ *be a primitive character* $\bmod q$, $d^{100} < N < N' \leqq 2N$, $q(|t|+3) = N^z$, $z \geqq \frac{1}{2}$ *and* $\operatorname{Re} s > 0$. *Then*

$$\left|\sum_N^{N'} \chi(n) n^{-s}\right| < \gamma_{200z} N^{1-\operatorname{Re}s-\varepsilon_{600z}}.$$

Lemma 8. *If* χ *is a nonprincipal primitive character* $\bmod q$ *then in the region* $\operatorname{Re} s > 1-\eta$ *we have*

$$\eta\,|L(s,\chi)| < d^{100\eta} + \exp(60\, l^{\frac{3}{4}}).$$

Proof. Clearly $q \geqq 3$, $\eta < \frac{1}{2}$, $|R(x)| = |\sum_{n\leqq x} \chi(n)| \leqq q/2$,

$$\sum_{n>N} \chi(n) n^{-s} = s\int_N^\infty \frac{R(x)}{x^{s+1}}\,dx - \frac{R(N)}{N^s} \qquad (\operatorname{Re} s > 0).$$

Hence, setting $\log Z = 2l$, $\log Y = 60(l\log 2l)^{\frac{3}{4}}$ we get

$$\Big|\sum_{n>Z} \chi(n) n^{-s}\Big| < q(|t|+3)\, Z^{\eta-1} < 1,$$

$$\Big|\sum_{n<Y} \chi(n) n^{-s}\Big| < \sum_{n<Y} n^{\eta-1} < \eta^{-1}(Y^\eta - 1) + 1.$$

If $Y > Z$ there is nothing more to prove. Assume therefore that $Y \leqq Z$ and put $z_1 = l/\log Y$. An easy computation shows that

$$\eta < \varepsilon_{600z_1} \quad\text{and}\quad 3l\,\gamma_{200z_1} Y^{\eta-\varepsilon_{600z_1}} < 1.$$

It follows from Lemmata 6 and 7 that if $d^{100} < N < N' \leqq 2N$, $Y \leqq N \leqq Z$ we have $\left|\sum_{N}^{N'} \chi(n) n^{-s}\right| < \gamma_{200 z_1} Y^{\eta - \varepsilon_{600 z_1}} < \frac{1}{3l}$, hence

$$\left|\sum_{Y}^{Z} \chi(n) n^{-s}\right| < \frac{2l}{\log 2} \cdot \frac{1}{3l} + \sum_{1 < n \leqq d^{100}} n^{\eta - 1} < (d^{100\eta} - 1)\eta^{-1} + 1$$

and the proof is complete

Lemma 9. *Let χ be a nonprincipal character* $\bmod q$, $M = (d^{\eta} \exp l^{\frac{1}{4}})^{100}$. *Then in the region* $\operatorname{Re} s > 1 - \eta$ *we have* $|L(s, \chi)| < M$.

Proof. We get from Lemma 8

$$|L(s, \chi)| < \eta^{-1} \prod_{p/d} (1 + p^{\eta - 1})\left(d^{100\eta} + \exp(60\, l^{\frac{1}{4}})\right) < M.$$

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Lemma 10. *Let $F(s)$ be a function regular in the circle $|s - s_0| < r$ and satisfying there the inequality $|F(s)| \leqq M\, |F(s_0)|$. Then*

$$\operatorname{Re} \frac{F'}{F}(s_0) \geqq -\frac{4}{r} \log M + \operatorname{Re} \sum_{\rho} \frac{1}{s_0 - \rho}$$

where ρ runs over the zeros of $F(s)$ in the circle $|s - s_0| \leqq r/2$ consted with their multiplicities.

Proof. This is a simple consequences of the maximum principle, see [3], p. 384.

Lemma 11. *Let χ be a nonprincipal character* $\bmod q$, $\vartheta = \eta/400 \log M$.
If the function $L(s, \chi)$ has a zero in the region $\operatorname{Re} s > 1 - \vartheta$ *then it is unique, simple and real. The character χ is then real.*

Proof. For $\sigma > 1$ we have

$$3 \operatorname{Re} \frac{L'}{L}(\sigma, \chi_0) + 4 \operatorname{Re} \frac{L'}{L}(\sigma + it, \chi) + \operatorname{Re} \frac{L'}{L}(\sigma + 2it, \chi^2) \leqq 0. \tag{14}$$

For $s = \sigma + it$, $9\,|t| < 1 < \sigma$ we have

$$\left|\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right| < 3, \qquad \left|\frac{L'}{L}(s, \chi_0) - \frac{\zeta'}{\zeta}(s)\right| < \sum_{p/q} \frac{\log p}{p-1} < 2 \log \log 3d.$$

Hence

$$\operatorname{Re} \frac{L'}{L}(s, \chi_0) \geqq \frac{1 - \sigma}{(1 - \sigma)^2 + t^2} - 2 \log 5l. \tag{15}$$

The proof will be completed in two steps.

(i) Put

$$\Gamma=\begin{cases}\{s;\ \mathrm{Re}\, s>1-\vartheta\} & \text{if } \chi^2 \neq \chi_0,\\ \{s;\ \mathrm{Re}\, s>1-\vartheta,\ 4\,|\mathrm{Im}\, s|>\eta\} & \text{if } \chi^2=\chi_0,\end{cases}$$

and suppose that for certain $\rho=\beta+i\gamma\in\Gamma$ we have $L(\rho,\chi)=0$. Put also $\sigma_0=1+5\vartheta$, $s_0=\sigma_0+i\gamma$, $s_1=\sigma_0+2i\gamma$. It follows from Lemma 9 that in the circle $|s-s_0|<\eta$ we have

$$|L(s,\chi)\,L^{-1}(s_0,\chi)|<M\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\sigma_0}}<M/5\vartheta.$$

Hence by Lemma 10 we get

$$\mathrm{Re}\,\frac{L'}{L}(s_0,\chi)\geqq-\frac{4}{\eta}\log(M/5\vartheta)+\frac{1}{\sigma_0-\beta}. \tag{16}$$

The function $L(s,\chi^2)$ is regular in the circle $|s-s_1|<\eta/2$ and satisfies there the inequality $|L(s,\chi^2)\,L^{-1}(s_1,\chi^2)|<2M/5\vartheta$. Hence by Lemma 10 we get

$$\mathrm{Re}\,\frac{L'}{L}(s_1,\chi^2)\geqq-\frac{8}{\eta}\log(2M/5\vartheta). \tag{17}$$

The formulae (14)–(17) give

$$\frac{3}{1-\sigma_0}+\frac{4}{\sigma_0-\beta}\leqq 6\log 5l+\frac{16}{\eta}\log(M/5\vartheta)+\frac{8}{\eta}\log(2M/5\vartheta)\leqq\frac{1}{15\vartheta},$$

hence $\beta\leqq 1-\vartheta$. This contradiction shows that the function $L(s,\chi)$ does not vanish in Γ.

(i) Let χ-be a real character, $\Delta=\{s;\ \mathrm{Re}\, s>1-\vartheta,\ 4\,|\mathrm{Im}\, s|\leqq\eta\}$. Suppose that $\rho=\beta+i\gamma$ is a zero of $L(s,\chi)$ in Δ with the greatest imaginary part and that if ρ is real and simple $L(s,\chi)$ has in the region $\mathrm{Re}\, s>1-\vartheta$ still another zero ρ^*.

Put

$$\rho_1=\begin{cases}\bar\rho & \text{if } \rho\neq\bar\rho,\\ \rho & \text{if } \rho=\bar\rho \text{ is a multiple zero},\\ \rho^* & \text{if } \rho=\bar\rho \text{ is a simple zero}.\end{cases}$$

Hence $\rho_1=\beta_1+i\gamma_1$, where $\beta_1>1-\vartheta$, $|\gamma_1|\leqq|\gamma|\leqq\eta/4$. Set $\sigma_0=1+5\vartheta$, $s_0=\sigma_0+i\gamma$. Thus the numbers ρ and ρ_1 are in the circle $|s-s_0|<\eta/2$ and are zeros of $L(s,\chi)$ (if $\rho=\rho_1$ then ρ is a multiple zero). It follows from Lemma 9 that in the circle $|s-s_0|<\eta$ we have $|L(s,\chi)\,L^{-1}(s_0,\chi)|<M/5\vartheta$. Hence by Lemma 10 we get

$$\mathrm{Re}\,\frac{L'}{L}(s_0,\chi)\geqq-\frac{4}{\eta}\log(M/5\vartheta)+\frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta_1}{(\sigma_0-\beta_1)^2+(\gamma-\gamma_1)^2}. \tag{18}$$

Since

$$\frac{4(\sigma_0-\beta_1)}{(\sigma_0-\beta_1)^2+(\gamma-\gamma_1)^2} \geqq \frac{4(\sigma_0-\beta_1)}{(\sigma_0-\beta_1)^2+(2\gamma)^2} \geqq \frac{\sigma_0-1}{(\sigma_0-1)^2+\gamma^2},$$

we get by (14), (15) and (18)

$$\frac{3}{1-\sigma_0}+\frac{4}{\sigma_0-\beta} \leqq 8 \log 5l + \frac{16}{\eta} \log(M/5\vartheta) < \frac{1}{15\vartheta},$$

hence $\beta < 1-\vartheta$. The obtained contradiction completes the proof of the Lemma

In order to prove Theorem 2 it remains to show.

Lemma 12. *Let* $\chi_i\,(i=1,2)$ *be distinct non-principal real characters* mod q, β_i *real zeros of* $L(s,\chi_i)$. *Then*

$$\min(\beta_1,\beta_2) \leqq 1-\vartheta.$$

Proof. The character $\chi_3=\chi_1\chi_2$ is not principal. Put $\sigma_0=1+5\vartheta$. It follows from Lemma 9 that in the circle $|s-\sigma_0|<\eta$ we have

$$|L(s,\chi_i)\,L^{-1}(\sigma_0,\chi_i)| < M/5\vartheta \qquad \text{for } i=1,2,3.$$

Suppose that $\min(\beta_1,\beta_2) > 1-\vartheta$. Hence by Lemma 10 we get

$$0 \leqq -\sum_{i=0}^{3} \frac{L'}{L}(\sigma_0,\chi_i) < 2\log 5l + \frac{12}{\eta}\log(M/5\vartheta) + \frac{1}{\sigma_0-1}$$

$$-\frac{1}{\sigma_0-\beta_1} - \frac{1}{\sigma_0-\beta_2} < \frac{2}{15\vartheta} + \frac{1}{5\vartheta} - \frac{1}{6\vartheta} - \frac{1}{6\vartheta} = 0.$$

The contradiction obtained completes the proof.

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