Invent. math. 88, 217–224 (1987)

Inventiones mathematicae

Regularity at infinity for area-minimizing hypersurfaces in hyperbolic space

Robert Hardt[1]* and Fang-Hua Lin[2]**

1 School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
2 Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

In the upper half space model, $n+1$ dimensional hyperbolic space is given as the set

$$\mathbb{H}=\{(x, y)\in\mathbb{R}^n\times\mathbb{R}: y>0\}$$

equipped with the hyperbolic metric $y^{-2}(dx^2+dy^2)$. A standard compactification of $\mathbb{H}$ involves adding the boundary $(\mathbb{R}^n\times\{0\})\cup\{*\}$ so that $\bar{\mathbb{H}}$ is simply the one point compactification of the Euclidean closed half-space $\mathbb{R}^n\times[0, \infty)$. Suppose $0\leqq\alpha\leqq 1$ and Γ is a compact $n-1$ dimensional $\mathscr{C}^{1,\alpha}$ smooth submanifold of $\mathbb{R}^n\times\{0\}$. In [A_1], M. Anderson proved that *there exists an n dimensional hyperbolic-area minimizing locally rectifiable current T in $\mathbb{H}$ whose support has Γ as its asymptotic limit.* (See also [A_2].) By the interior regularity theory of geometric measure theory, the support M of any such hyperbolic-area minimizing T is a relatively closed subset of $\mathbb{H}$ which is a real analytic submanifold away from a relatively closed singular set of Hausdorff dimension $n-7$. Anderson's construction gives M for which, in the Euclidean topology, $\bar{M}\sim M=\Gamma$. The question of the behavior of M near Γ was raised in [A_1] and [LR]. Here we prove the "boundary regularity at infinity" result that, *for any such hyperbolic-area minimizing T, the set $M\cup\Gamma$, in the ordinary Euclidean metric, is, near Γ, a finite union of $\mathscr{C}^{1,\alpha}$ submanifolds with boundary Γ; these have disjoint analytic interiors and meet $\mathbb{R}^n\times\{0\}$ orthogonally at Γ.* (For the particular T, constructed by Anderson only one submanifold will occur.) It follows that, for $n\leqq 6$, $M\cup\Gamma$ has finite genus, and, for $n\geqq 7$, any interior singularities of M must remain in a bounded region of hyperbolic space. Near points of Γ (in the Euclidean topology), $M\cup\Gamma$ may thus be described as the graph of a function. This function is the solution of an interesting partial differential equation that becomes degenerate along the part of the boundary corresponding to Γ. The second author has recently established [L] a boundary higher-regularity result for this equation, which implies, in particular, that $M\cup\Gamma$ is $\mathscr{C}^{k,\alpha}$ if Γ is $\mathscr{C}^{k,\alpha}$ for $k=2, 3, \ldots, \infty$. Finally, in case Γ bounds a star-

* Research partially supported by the National Science Foundation
** Research partially supported by the Mathematical Sciences Research Institute

shaped domain in $\mathbb{R}^n \times \{0\}$, we deduce in §4 the uniqueness (up to a constant factor) of T, among stationary currents having asymptotic limit Γ.

§ 1. Preliminaries

For convenience, we adopt the following convention:

All unspecified statements about topology and metric will refer implicitly to the usual Euclidean metric on $\mathbb{R}^n \times \mathbb{R}$, *and the word "hyperbolic" will be stated explicitly when appropriate.*

Throughout, we assume that

$\Gamma = \tilde{\Gamma} \times \{0\}$ *is a fixed compact submanifold of* $\mathbb{R}^n \times \{0\}$ *and that* T *is an* n *dimensional hyperbolic-area minimizing locally rectifiable current in* $\mathbb{H}$ *with* $\Gamma = \bar{M} \sim M$ *where* $M = \operatorname{spt} T$.

We make some preliminary observations on the location of M near Γ. These are all based on the following:

(1.1) *If* $x \in \mathbb{R}^n$ *and* $0 < r < d(x) = \operatorname{dist}(x, \tilde{\Gamma})$, *then* $M \cap \mathbb{B}_r(x, 0) = \emptyset$.

This follows from a standard argument (see e.g. [A_1, Lemma 5]) using first variation because the sets $\mathbb{H} \cap \partial \mathbb{B}_s(x, 0)$, for $0 < s < r$, are totally geodesic hyperbolic hyperplanes that foliate $\mathbb{H} \cap \mathbb{B}_r(x, 0)$ and serve as barriers. Let v_Γ be a unit normal vectorfield for $\tilde{\Gamma}$ in $\mathbb{R}^n$ and, for $a \in \tilde{\Gamma}$ and $r > 0$,

$$\delta(a, r) = \min\{d(a + r v_\Gamma(a)), d(a - r v_\Gamma(a))\}.$$

We observe that

(i) $r^{-1}\delta(a, r) \to 1$ as $r \to 0$ if Γ is differentiable at $(a, 0)$.

(ii) $\sup_{a \in \Gamma} [1 - r^{-1}\delta(a, r)] \to 0$ as $r \to 0$ if Γ is $\mathscr{C}^1 = \mathscr{C}^{1,0}$.

(iii) $\sup_{a \in \Gamma} [1 - r^{-1}\delta(a, r)] \leqq c_\Gamma r^\alpha$ if Γ is $\mathscr{C}^{1,\alpha}$ for $0 < \alpha < 1$.

(iv) $r^{-1}\delta(a, r) = 1$ for all positive $r < 1/\|\text{max. principle curv.}\|_{L^\infty(\Gamma)}$ if Γ is $\mathscr{C}^{1,1}$.

We will first consider the case when Γ is $\mathscr{C}^1$. The $\mathscr{C}^{1,\alpha}$ cases, with $0 < \alpha \leqq 1$, will be treated in §3. Choose a positive number ρ_Γ so that $r^{-1}\delta(a, r) > \frac{1}{2}$ for all $(a, 0) \in \Gamma$ and $0 < r < 2\rho_\Gamma$. By (1.1),

$M \cap \{y < \rho_\Gamma\}$ *is contained in the set*

(1.2) $$W = [\mathbb{R}^n \times (0, \rho_\Gamma)] \sim \bigcup_{d(x) > 2\rho_\Gamma} \mathbb{B}_{2\rho_\Gamma}(x, 0) \sim \bigcup_{0 < d(x) \leqq 2\rho_\Gamma} \mathbb{B}_{d(x)}(x, 0).$$

Clearly W contains the product $\tilde{\Gamma} \times (0, \rho_\Gamma)$. We will show that W is also asymptotic to $\tilde{\Gamma} \times (0, \rho_\Gamma)$ as $y \to 0$. To see this, suppose that (x_i, y_i) is a sequence of points in W with $y_i \to 0$. Since every point $x \in \mathbb{R}^n \sim \tilde{\Gamma}$ is included in the above definition of W, $d(x_i)$ must approach 0. We now wish to show that $d(x_i)/y_i$ also approaches 0. For this, we may assume that $y_i < d(x_i)^{\frac{3}{4}}$. Choose a point $(a_i, 0) \in \Gamma$

with $d(x_i)=|x_i-a_i|$; hence, $x_i=a_i\pm d(x_i)\nu_\Gamma(a_i)$. Letting $r_i=d(x_i)+d(x_i)^{-1}y_i^2$, we see that the three points, $(a_i,0)$, (x_i,y_i), and $(a_i\pm r_i\nu_\Gamma(a_i),0)$, are vertices of a right triangle. Letting δ_i be the distance between the latter two vertices, we conclude that, as $i\to\infty$.

$$d(x_i)/y_i=[r_i^2-\delta_i^2]^{\frac{1}{2}}/\delta_i=[(r_i/\delta_i)^2-1]^{\frac{1}{2}}\to 0$$

because $r_i\leqq d(x_i)+d(x_i)^{\frac{1}{2}}\to 0$, $\delta(a_i,r_i)\leqq d(a_i\pm r_i\nu_\Gamma(a_i))\leqq\delta_i\leqq r_i$, and $r_i/\delta(a_i,r_i)\to 1$ by (ii). It now follows that

(1.3) *at each edge point $(a,0)\in\Gamma$, the tangent cone of the containing set W equals the vertical half-hyperplane* $\operatorname{Tan}(\tilde{\Gamma},a)\times[0,\infty)$.

§ 2. $\mathscr{C}^1-\mathscr{C}^1$ regularity at ∞

We will sometimes identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1}\times\mathbb{R}$ and use the projection

$$p\colon(\mathbb{R}^{n-1}\times\mathbb{R})\times\mathbb{R}\to\mathbb{R}^{n-1}\times\mathbb{R},\qquad p((w,z),y)=(w,y).$$

2.1. Lemma. (Interior regularity.) *For any nonnegative integer κ and positive number ε, there exists a positive δ so that if S is an n dimensional hyperbolic-area minimizing rectifiable current in* $\mathbb{H}$ *with*

$$\operatorname{spt}S\subset\mathbb{B}_2^{n-1}(0)\times[-\delta,\delta]\times[\tfrac{1}{2},4],\qquad \operatorname{spt}\partial S\subset p^{-1}\partial(\mathbb{B}_2^{n-1}(0)\times[\tfrac{1}{2},4]),$$

and $p_*S=\kappa[\![\mathbb{B}_2^{n-1}(0)\times[\tfrac{1}{4},4]]\!]$, *then* $S\llcorner p^{-1}(\mathbb{B}_1^{n-1}(0)\times[\tfrac{1}{2},2])$ *is the sum of κ oriented graphs (which are pairwise either disjoint or identical) of real analytic functions $z=u_j(w,y)$ with $\|u_j\|_{\mathscr{C}^{1,1}}\leqq\varepsilon$.*

(If $\kappa=0$, then $S\llcorner p^{-1}(\mathbb{B}_1^{n-1}(0)\times[1,2])=0$.)

Proof. Here $\kappa[\![\mathbb{B}_2^{n-1}(0)\times[\tfrac{1}{4},4]]\!]$ denotes κ times the current corresponding to oriented integration, with respect to the hyperbolic metric, over the set $\mathbb{B}_2^{n-1}(0)\times[\tfrac{1}{4},4]$. Hyperbolic area may be described by the parametric integrand [F, 5.1.1] $\Psi((x,y),\xi)=y^{-n}|\xi|$. On the region $\mathbb{R}^n\times(\tfrac{1}{4},4)$, this integrand is elliptic [F, 5.1.2] with ellipticity bound 4^n. As in the proof of [F, 5.3.18] one may find a decomposition $S=\bigcup_{j=-\infty}^{\infty}S^{(j)}$ where each nonzero $S^{(j)}$ satisfies the hypothesis with $\kappa\in\{-1,0,+1\}$. Then one deduces, as in [F, 5.1.1], a mass estimate for each $S^{(j)}$ from the Ψ-minimality and the small height.

To prove the lemma one may now argue by contradiction, and consider, for any positive ε, a sequence of positive numbers δ_i approaching 0 and a corresponding sequence $S_i=\bigcup_{j=-\infty}^{\infty}S_i^{(j)}$ as above satisfying the hypothesis but not the conclusion of 2.2. By the lower density bound [F, 5.1.6], the support of each $S_i^{(j)}$ with zero projection does not meet $p^{-1}(\mathbb{B}_1^{n-1}(0)\times[\tfrac{1}{2},2])$ for i large. Using the mass bounds, the BV compactness theorem, the convergence of supports, and the regularity theory [F, 5.3.14], one finds that, for i large, the support of each $S_i^{(j)}$ with nonzero projection, is, on the smaller cylinder

$p^{-1}(\mathbb{B}^{n-1}_{3/2}(0)\times[\frac{1}{3},3])$, an oriented graph of a $\mathscr{C}^{1,\frac{1}{2}}$ function $z=u_{i,j}(w,y)$. Moreover, a simple cutting and pasting argument shows that, for each such i, these all have parallel orientation. Thus, for each such i, there are precisely κ currents $S_i^{(j)}$ with $p_\# S_i^{(j)}\neq 0$, and each has $p_\# S_i^{(j)}=+[\![\mathbb{B}^{n-1}_2(0)\times[\frac{1}{4},4]]\!]$. Finally each $u_{i,j}$ is analytic by [M, 5.8], and

$$\lim_{i\to\infty}\|u_{i,j}\|_{\mathscr{C}^{1,1}(\mathbb{B}^{n-1}_1(0)\times[\frac{1}{2},2])}=0$$

by interior estimates [GT, 15.2] because $\lim_{i\to\infty}\delta_i=0$. This now contradicts, for i sufficiently large, the choice of S_i. □

2.2. Theorem. *If Γ, T, and M are as in § 1, and Γ is $\mathscr{C}^1$, then there exists a positive $\rho<\rho_\Gamma$ so that $(M\cup\Gamma)\cap\{y<\rho\}$ is a finite union of $\mathscr{C}^1$ submanifolds with boundary; these have disjoint analytic interiors and meet $\mathbb{R}^n\times\{0\}$ orthogonally at Γ.*

Proof. We will first verify the statement:
for some positive ρ, the set $M\cap\{y<\rho\}$ has no interior singularities and

$$\nu_M(x_i,y_i)\to(\nu_\Gamma(a),0)\qquad \textit{whenever}\ (x_i,y_i)\in M\to(a,0)\in\Gamma.$$

If the statement were false, then there would exist a sequence (x_i,y_i) in M approaching a point $(a,0)$ in Γ so that

either $(x_i,y_i)\in\operatorname{Sing}(M)$ for all i

or $\lim_{i\to\infty}|\nu_M(x_i,y_i)-(\nu_\Gamma(a_i),0)|>0$

for any choice of orienting normal field ν_M. Since a rotation or translation of $\mathbb{R}^n$ induces a hyperbolic isometry of $\mathbb{H}$, we may assume, for convenience, that $a=0$ and that $\nu_\Gamma(a)=(0,1)\in\mathbb{R}^{n-1}\times\mathbb{R}$.

By the $\mathscr{C}^1$ smoothness of Γ, we may choose, for each positive ρ, a positive σ so that p projects $\Gamma\cap[\mathbb{B}^{n-1}_{2\sigma}(0)\times(-\sigma\rho,+\sigma\rho)\times\{0\}]$ $\mathscr{C}^1$-diffeomorphically onto $\mathbb{B}^{n-1}_{2\sigma}(0)\times\{0\}$. It then follows from the discussion of the set W in § 1 that

$$M\cap[\mathbb{B}^{n-1}_\sigma(0)\times\{\pm\sigma\rho\}\times(0,\tau)]=\emptyset$$

for some sufficiently small positive τ. By the constancy theorem [F, 4.1.7] (which holds in any metric),

$$p_\#(T\llcorner[\mathbb{B}^{n-1}_\sigma(0)\times(-\sigma\rho,+\sigma\rho)\times(0,\tau)])=\kappa[\![\mathbb{B}^{n-1}_\sigma\times(0,\tau)]\!]$$

for some integer κ. Replacing T by $-T$ if necessary, we may assume that κ is nonnegative.

Next we wish to scale T appropriately. To do this note that, for each positive r, the homothety map that sends (x,y) to (rx,ry) induces a hyperbolic isometry of $\mathbb{H}$. Thus, for each i, the expression

$$\Phi_i(x,y)=y_i^{-1}(x-x_i,y),$$

defines a hyperbolic isometry, and the current

$$S_i=\Phi_{i\#}(T\llcorner[\mathbb{R}^{n-1}\times(-\sigma\rho,+\sigma\rho)\times(0,\infty)])\llcorner p^{-1}(\mathbb{B}^{n-1}_2(0)\times[\tfrac{1}{4},4])$$

is hyperbolic-area minimizing. Moreover, for any positive δ, we easily verify, using (1.2) and (1.3), that S_i satisfies the hypotheses of Lemma 1.1 for all i sufficiently large. We conclude that, for such i, (x_i, y_i) is a regular point of M because $((0,0),1)=\Phi_i(x_i,y_i)$ is, by 2.1, a regular point of $\operatorname{spt} S_i$. Moreover, letting $u_{i,j}$ denote the functions provided by the conclusion of 2.1 and letting δ_i correspond in 2.1 to a sequence of positive ε_i approaching 0, we may, by (1.2) and (1.3), pass to a subsequence to have

$$\lim_{i\to\infty} \|u_{i,j}\|_{\mathscr{C}^{1,1}}=0.$$

In particular

$$\lim_{i\to\infty} |\nu_{\mathbf{M}}(x_i,y_i)-(\nu_{\boldsymbol{\Gamma}}(a),0)| = \lim_{i\to\infty} |\nu_{\mathbf{M}}(x_i,y_i)-((0,1),0)|=0$$

because $\lim_{i\to\infty} \nabla u_{i,j}(0)=0$. This now contradicts the choice of (x_i, y_i) and a, and completes the proof of the statement.

Defining $\nu_{\mathbf{M}}(a,0)=(\nu_{\boldsymbol{\Gamma}}(a),0)$ for $(a,0)\in\Gamma$, we infer from the statement, the continuity of $\nu_{\boldsymbol{\Gamma}}$ on Γ, and the continuity of $\nu_{\mathbf{M}}$ on $M\cap\{y<\rho\}$, that $\nu_{\mathbf{M}}$, so extended, is continuous on $\Gamma\cup[M\cap\{y<\rho\}]$.

Let Γ_k be a component of Γ, and consider a component M_j of $M\cap\{y<\rho\}$ with $\Gamma_k\subset\bar{M}_j$. Then M_j is, by interior regularity, an analytic submanifold, and $\Gamma_k\cup M_j$ is $\mathscr{C}^1$ regular at Γ_k. The M_j are disjoint by the maximum principle. By orienting each $n-1$ plane $\operatorname{Tan}(\Gamma_k, a)$ continuously in $a\in\Gamma_k$, we may define a continuous family of currents $[\![\operatorname{Tan}(\Gamma_k,a)\times(0,\rho)]\!]$. Arguing as above, there is an integer κ_k, independent of $a\in\Gamma_k$, so that the orthogonal projection onto $\operatorname{Tan}(\Gamma_k,a)\times\mathbb{R}$ locally projects $T\llcorner\{y<\rho\}$ to a piece of $\kappa_k[\![\operatorname{Tan}(\Gamma_k,a)\times(0,\rho)]\!]$. Again by minimality and cutting and pasting, we see that the currents $T\llcorner M_j$ all must induce the same orientations under these projections. Thus there are at most κ_k such M_j. In particular, the number of components of $M\cap\{y<\rho\}$ is finite. □

§ 3. $\mathscr{C}^{1,\alpha}-\mathscr{C}^{1,\alpha}$ regularity at ∞

If Γ is $\mathscr{C}^1$, then, we may, as in 2.2, change coordinates to view each component of $M\cap\{y<\rho\}$ locally, near $(0,0)\in\Gamma$, as lying in the graph of a function $z=u(w,y)$ that is $\mathscr{C}^1$ on a region $\mathbb{B}^{n-1}_\rho(0)\times[0,\rho)$. For any compact $K\subset\mathbb{B}^{n-1}_\rho(0)\times(0,\rho)$, the hyperbolic area of the graph of $u|K$ is

$$\int_{\mathbf{K}} (1+u_w^2+u_y^2)^{\frac{1}{2}}\, y^{-n}\, dw\, dy.$$

The minimality of T leads to the Euler-Lagrange equation

$$\sum_{j=1}^{n-1} [(1+u_w^2+u_y^2)^{-\frac{1}{2}} u_{w_j}]_{w_j} + [(1+u_w^2+u_y^2)^{-\frac{1}{2}} u_y]_y - \left(\frac{n}{y}\right)[(1+u_w^2+u_y^2)^{-\frac{1}{2}} u_y]=0.$$

We find it more convenient and natural to view M using polar graphs over regions in a hemisphere centered in $\mathbb{R}^n\times\{0\}$.

Let $\mathbb{S}_+^n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 + y^2 = 1,\ y > 0\}$. For a region $\Omega \subset \mathbb{S}_+^n$ and a positive $\mathscr{C}^2$ function u on Ω, the polar graph of u is the set

$$\operatorname{graph}(u) = \{u(\omega)\omega : \omega \in \Omega\}.$$

We want to calculate the hyperbolic area of $\operatorname{graph}(u|K)$ for any compact $K \subset \mathbb{S}_+^n$. For this purpose, we represent points in $\mathbb{S}_+^n \sim \{(0, 0, \dots, 1)\}$ using pairs $(\theta, \eta) \in \mathbb{S}^{n-1} \times (0, \frac{1}{2}\pi]$ where $\theta = x/|x|$ and $\eta = \sin^{-1} y$ (i.e., $\eta = \frac{1}{2}\pi - \varphi$). In these coordinates, $\mathbb{S}_+^n$ has the standard metric $d\eta^2 + \cos^2 \eta\, d\theta^2$. The hyperbolic area is

$$\int_K [(u^2 + |\nabla u|^2) u^{2(n-1)}]^{\frac{1}{2}} (u \cdot \sin \eta)^{-n} \cos \eta\, d\theta\, d\eta$$
$$= \int_K (1 + u^{-2} |\nabla u|^2)^{\frac{1}{2}} (\sin \eta)^{-n} \cos \eta\, d\theta\, d\eta$$

where $|\nabla u|^2 = |\nabla_\theta u|^2 + |\nabla_\eta u|^2$. Note that, as expected, the graphs of $u|K$ and $\lambda u|K$, for $\lambda > 0$, have the same hyperbolic area.

3.1. Theorem. *If $0 < \alpha \leqq 1$ and Γ is $\mathscr{C}^{1,\alpha}$, then each of the manifolds in the conclusion of Theorem 2.1 are $\mathscr{C}^{1,\alpha}$ regular at their boundaries.*

Proof. We first consider the case $\alpha = 1$. Suppose that N is one of the components of $M \cap \{y < \rho\}$ and $(a, 0) \in \Gamma \cap \bar{N}$. After applying hyperbolic isometries induced by a suitable translation and rotation of $\mathbb{R}^n \times \{0\}$, we may assume that $a = (0, \dots, 0, 1)$ and the $v_\Gamma(a) = (0, \dots, 0, 1, 0)$. By Theorem 2.1, there is a positive δ so that $N \cup \Gamma$ is, near a, the polar graph of a positive $\mathscr{C}^1$ function u on a region $\Omega \subset \mathbb{S}_+^n$ defined by $|\theta - a| < \delta$ and $0 \leqq \eta < \delta$. Here $u(a, 0) = 1$ and $\nabla u(a, 0) = 0$.

The function $v(\theta, \eta) = \log(u(\theta, \eta))$ is also $\mathscr{C}^1$ on Ω. It locally minimizes the integral

$$\int (1 + |\nabla v|^2)^{\frac{1}{2}} (\sin \eta)^{-n} \cos \eta\, d\theta\, d\eta,$$

and has $v(a, 0) = 0$ and $\nabla v(a, 0) = 0$. Since Γ is $\mathscr{C}^{1,1}$ there is, as in § 1(iv), a positive r, depending only on Γ, so that the two closed balls in $\mathbb{R}^n \times \mathbb{R}$,

$$\bar{\mathbb{B}}_r(0, \dots, 0, 1 - r, 0) \quad \text{and} \quad \bar{\mathbb{B}}_r(0, \dots, 0, 1 + r, 0),$$

which are tangent to Γ at a, do not intersect M. Since these have quadratic contact with the unit sphere $\mathbb{S}^n$,

$$|v(\theta, \eta)| \leqq c(\Gamma)(|\theta - a|^2 + \eta^2) \qquad \text{for all } (\theta, \eta) \in \Omega,$$

where $c(\Gamma)$ depends only on Γ. Letting $\gamma_\theta(\lambda)$ be the shortest constant speed geodesic on $\mathbb{S}^n$ going from $\gamma_\theta(0) = a$ to $\gamma_\theta(1) = \theta$, it follows that, for $0 < \lambda < 1$, the scaled function v_λ, defined by $v_\lambda(\theta, \eta) = \lambda^{-1} v(\gamma_\theta(\lambda), \lambda\eta)$, satisfies

$$|v_\lambda(\theta, \eta)| \leqq c(\Gamma) \lambda (|\theta - a|^2 + \eta^2).$$

Moreover, since v_λ locally minimizes an integral of the type

$$\int (1 + |\nabla v|^2)^{\frac{1}{2}} \lambda^n (\sin \lambda\eta)^{-n} \cos \eta\, d\theta\, d\eta,$$

we infer from the interior gradient estimates [GT, 15.2] that

$$|\nabla v_\lambda(\theta,\eta)| \leqq c(n,\Gamma)\lambda \quad \text{and} \quad |\nabla^2 v_\lambda(\theta,\eta)| \leqq c(n,\Gamma)\lambda$$

for $0<\lambda<\frac{1}{2}\delta$, $|\theta-a|\leqq 1$, and $\frac{1}{2}\leqq\eta\leqq 1$. For such λ, θ, η, this means that

$$|\nabla v(\lambda\theta,\lambda\eta)| \leqq c(n,\Gamma)\lambda \quad \text{and} \quad |\nabla^2 v(\lambda\theta,\lambda\eta)| \leqq c(n,\Gamma),$$

hence,

$$|\nabla v(\theta,\eta)| \leqq C(n,\Gamma)(|\theta-a|+\eta) \quad \text{and} \quad |\nabla^2 v(\theta,\eta)| \leqq C(n,\Gamma)$$

for $|\theta-a|\leqq\frac{1}{2}\delta$, and $\frac{1}{2}|\theta-a|\leqq\eta\leqq\frac{1}{2}\delta$. Since these estimates are independent of a, the function v, and hence u, is $\mathscr{C}^{1,1}$ in a neighborhood of $(a,0)$ in Ω, and $\Gamma\cup N$ is $\mathscr{C}^{1,1}$ near $(a,0)$.

To treat the $\mathscr{C}^{1,\alpha}$ case with $0<\alpha<1$, we again choose u and v as above. Now by § 1 the two closed balls in $\mathbb{R}^n\times\mathbb{R}$,

$$\mathbb{B}_{b(a,r)}(0,\ldots,0,1-r,0) \quad \text{and} \quad \mathbb{B}_{b(a,r)}(0,\ldots,0,1+r,0),$$

which are no longer tangent to Γ at a, do not intersect M. Taking the envelopes of these balls over all small positive r, we obtain functions $v_\pm$ on Ω which have $v_-\leqq v\leqq v_+$ and which satisfy, by § 1 (iii) and the argument in § 1, estimates

$$\sup_{|\theta-a|+|\eta|<r} |v_\pm(\theta,\eta)| \leqq c(\Gamma)r^{1+\alpha}.$$

Following the $\mathscr{C}^{1,1}$ argument above, we now infer that

$$|\nabla v(\theta,\eta)| \leqq C(n,\Gamma)(|\theta-a|+\eta)^\alpha$$

and

$$|\nabla^2 v(\theta,\eta)| \leqq C(n,\Gamma)(|\theta-a|+\eta)^{\alpha-1}$$

for $|\theta-a|\leqq\frac{1}{2}\delta$, and $0\leqq\eta\leqq\frac{1}{2}\delta$. Now, v, and hence u, is $\mathscr{C}^{1,\alpha}$ in a neighborhood of $(a,0)$ in Ω, and $\Gamma\cup N$ is $\mathscr{C}^{1,\alpha}$ near $(a,0)$. □

§ 4. Star-shaped domains at ∞

4.1. Theorem. *Suppose Ω is a bounded $\mathscr{C}^1$ star-shaped domain in $\mathbb{R}^n$ and $\Gamma=\partial\Omega\times\{0\}$. There exists a unique, up to multiplicity, n dimensional hyperbolic-stationary locally rectifiable current T in $\mathbb{H}$ such that $\Gamma=\bar{M}\sim M$ where $M=\operatorname{spt} T$. Moreover, T is hyperbolic-area minimizing, and M is the polar graph of a function defined on an open hemisphere centered in $\Omega\times\{0\}$.*

Proof. As noted in the introduction, M. Anderson showed, in [A_1, Theorem 3], the existence of a hyperbolic-area minimizing T with $\Gamma=\bar{M}\sim M$ where $M=\operatorname{spt} T$. We will first verify that this particular M is a completely regular polar graph.

To see this, we first apply a suitable translation to get Ω to be star-shaped about the origin. We may choose positive numbers r and s so that

$$\mathbb{B}_r^{n-1}(0)\subset\Omega\subset\bar{\Omega}\subset\mathbb{B}_s^{n-1}(0).$$

Then, by (1.1), $M \subset \mathbb{B}_s(0) \sim \bar{\mathbb{B}}_r(0)$; hence,

$$\left(\frac{r}{s}\right) M \cap M = \emptyset = M \cap \left(\frac{s}{r}\right) M.$$

We claim that $(\lambda M) \cap \mathrm{M} = \emptyset$ for all positive $\lambda \neq 1$. Otherwise, we could choose λ with either $\frac{r}{s} < \lambda < 1$ or $1 < \lambda < \frac{s}{r}$ so that the sets of regular points, $\operatorname{Reg}(M)$ and $\operatorname{Reg}(\lambda M)$, intersect transversally. Near $\mathbb{R}^n \times \{0\}$ the sets M and λM are disjoint because Ω is star-shaped about the origin. One may then, by minimality, replace a piece of T by the piece of $\mu_{\lambda\#} T$ that is cut off by T and still have a minimizing current. But then the singular set of the resulting current would include the intersection $\operatorname{Reg}(M) \cap \operatorname{Reg}(\lambda M)$, whose $n-1$ dimensionality contradicts the $n-7$ dimensional bound on the singular set. From this claim, it follows that M is a polar graph of a function u on $\mathbb{S}^n_+$. Moreover, by the small size of $\operatorname{Sing}(M)$, all the singularities of the function u are, as in the proof of [HS, 2.1], removable, and u is analytic on $\mathbb{S}^n_+$.

Next we assume that S is any n dimensional hyperbolic-stationary locally rectifiable current in $\mathbb{H}$ with $\overline{\operatorname{spt} S} \sim \operatorname{spt} S = \Gamma$. By the argument in § 1, we again find that

$$\operatorname{spt} S \subset \mathbb{B}_s(0) \sim \bar{\mathbb{B}}_r(0).$$

Let $\lambda_+ = \inf\{\lambda\colon \lambda M \cap \operatorname{spr} S \neq \emptyset\}$. If λ_+ were strictly greater than one, then we could (because Ω is star-shaped about the origin) find a point $b \in \lambda M \cap \operatorname{spt} S$. By the regularity and connectedness of M and the argument of [H, 4.4], we would find that $\lambda M \subset \operatorname{spt} S$, contradicting that $\overline{\lambda M} \sim \lambda M \subset \lambda \Gamma$ and $\lambda \Gamma \cap \Gamma = \emptyset$. Thus $\lambda_+ \leqq 1$. Similarly, $\sup\{\lambda\colon \lambda M \cap \operatorname{spt} S \neq \emptyset\} \geqq 1$. Thus $\operatorname{spt} S \subset M$, and S is, by the constancy theorem [F, 4.1.4], a multiple of T. □

References

[A_1] Anderson, M.: Complete minimal varieties in hyperbolic space. Invent. math. **69**, 477–494 (1982)

[A_2] Anderson, M.: Complete minimal hypersurfaces in hyperbolic n-manifolds. Comment. Math. Helv. **83**, 264–290 (1983)

[F] Federer, H.: Geometric measure theory. Berlin, Heidelberg, New York: Springer 1969

[GT] Gilbarg, D., Trudinger, N.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1977

[H] Hardt, R.: On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand. Comment. P.D.E. **2**, 1163–1232 (1977)

[HS] Hardt, R., Simon, L.: Area minimizing hypersurfaces with isolated singularities. J. Reine Angew. Math. **362**, 102–129 (1985)

[L] Lin, F.-H.: On the Dirichlet problem for minimal graphs in hyperbolic space. Preprint, MSRI

[LR] Levitt, G., Rosenberg, H.: Symmetry of constant mean curvature hypersurfaces in hyperbolic space. Duke Math. J. **52**, 53–59 (1985)

[M] Morrey, C.B., Jr.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966

Oblatum 14-VII-1986